PART VII

THEORETICAL PAPERS AND SUMMARY

Martin Schwarzschild in the Lecture Hall, National Academy of Sciences.

https://doi.org/10.1017/S007418090014570X Published online by Cambridge University Press

THE VOGT-RUSSELL THEOREM, AND NEW RESULTS ON AN OLD PROBLEM

Helmuth Kähler

Hamburger Sternwarte

1. THE VOGT-RUSSELL THEOREM

Half a centry ago Henry Norris Russell and Heinrich Vogt independently made a conjecture concerning the structure of spherical stars which are in hydrostatic and thermal equilibrium (Russell, 1927; Vogt, 1926). This conjecture has later come to be known as the Vogt-Russell theorem and is usually formulated as follows: The structure of a star is uniquely determined by the mass and the composition. In other words, the statement claims the existence and uniqueness of a stellar equilibrium configuration for given parameters mass and composition, and you may find what is called a mathematical proof in many textbooks on stellar structure.

In the last decade, however, there have been found many counter-examples which disprove the theorem. In fact, both the existence and the uniqueness part of the theorem are violated. Let me give you some examples. Consider first the existence part: For a given mass and composition there should exist an equilibrium model. But you certainly can't find a model composed of iron with a mass of two solar masses: Since iron doesn't burn there could be at best a cold degenerate model, but for such models there is an upper mass limit at about 1.2 solar masses, the so-called Chandrasekhar limiting mass.

Consider next the uniqueness part: For given parameters there should be only one model. In numerical calculations however, up to 9 different models have been encountered. Let me give you an example of such multiple solutions which is closely connected with stellar evolution. The HR diagram in Fig. 1 shows the post-main-sequence

A. G. Davis Philip and D. S. Hayes (eds.), The HR Diagram, 303-311.

https://doi.org/10.1017/S007418090014570X Published online by Cambridge University Press

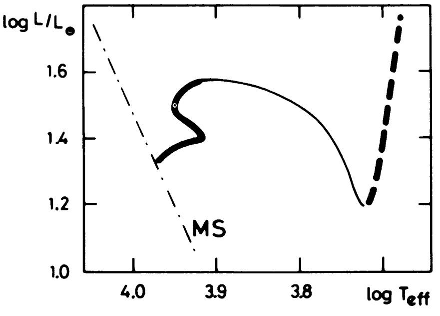

Fig. 1. Evolution of a star of two solar masses (see text).

evolution of a star of two solar masses (Roth, 1973). The heavy solid part of the evolutionary path corresponds to a slow phase in which the star evolves on a nuclear time scale. Thereafter the evolution speeds up and the star moves on a thermal time scale through the Hertzsprung gap, until it goes again through a slow phase near the Hayashi line which is indicated by a dashed line. During these phases the star has a burnt-out helium core and a hydrogen-burning shell source. Consider now the corresponding equilibrium models. Fig. 2a shows a sequence of models with different values of the core mass (Roth, 1973). I have plotted the core radius against the core mass. You can see that for a certain range of the core mass there are three different solutions, in contradiction to the uniqueness theorem. In the HR diagram (Fig. 2b) this sequence of equilibrium models appears as a curve which is similar to the evolutionary path, and indeed there is a simple connection. The equilibrium models on the solid part of the curve have an isothermal He-core and simulate stars in the slow phase just after leaving the main sequence. During evolution the core mass is growing. Correspondingly, in Fig. 2a the stars move to the right until, at the point SC, the Schönberg-Chandrasekhar limit for isothermal cores is reached. Now the star can no longer stay in thermal equilibrium but has to move on a thermal time scale, until

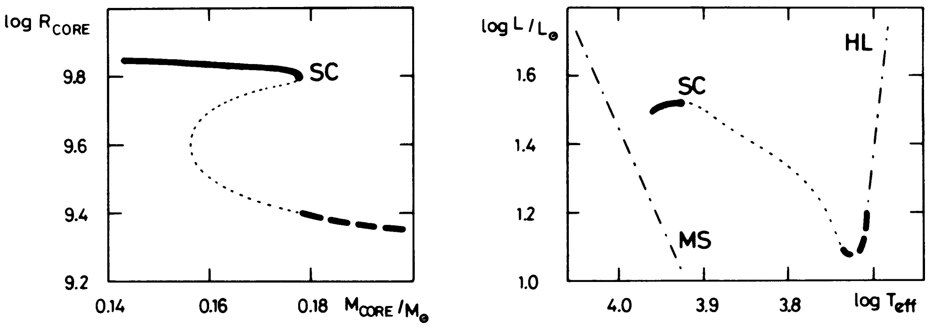

Fig. 2a and b. Equilibrium models of two solar masses (see text).

https://doi.org/10.1017/S007418090014570X Published online by Cambridge University Press

it finds a new equilibrium state on the dashed part of the sequence. This dashed part indeed simulates evolutionary models in the slow phase near the Hayashi line. The rapid crossing of the Hertzsprung gap can therefore be interpreted as a transition between two different equilibrium configurations for the same mass and composition. This example shows therefore not only the existence of multiple solutions, but also their relevance for the interpretation of evolutionary calculations.

We have therefore seen in explicit examples that both the existence and uniqueness parts of the so-called Vogt-Russell theorem are violated. On the other hand proofs of the theorem have been published. What can we say about these proofs? In fact there has been no proof but in essence only the following plausibility argument: We have a system of four differential equations for stellar structure together with four boundary conditions (two at the stellar center and two at the surface). Since the number of differential equations is equal to the number of boundary conditions, the solution should be completely determined. This classical arqument however applies only to linear equations, and even then it may fail as we shall see.

In order to make this point quite clear, let us comprise the parameters (i.e. mass and composition) in the symbol p, and a particular stellar model (i.e. a particular solution of the structure equations) in the symbol S. So far we have asked for the global existence and uniqueness of a solution: Is there always, for given p, one and only one stellar model S?

<u>global</u> existence and uniqueness: $p \rightarrow S$?

The answer is no, the classical argument cannot say anything because the corresponding equations are highly nonlinear. If however we confine ourselves to a small neighborhood of a given solution, we may work with linearized equations, and then the classical argument will usually be applicable. So we are led to the problem of the local existence and uniqueness of a given solution: Do small changes of the parameters lead uniquely to small changes of the model?

<u>local</u> existence and uniqueness: $\Delta p \rightarrow \Delta S$?

The answer will usually be yes because this is a linear problem.

At this point let us stop for a moment and recall what Russell actually claimed about the manifold of stellar models. He wrote in his book "that a star of given mass and composition will usually be in equilibrium for only one value of the radius..." (as an example he considered main sequence stars), but he admitted that "in more complicated cases there might be two or more different configurations". Accordingly Russell did not claim the global uniqueness as a strict theorem. He admitted exceptions in complicated cases; today we would add that most cases are complicated indeed. Summarizing, Russell's original conjecture is cautious, but the modified version which was later formulated in lectures and textbooks as a so-called theorem is definitely wrong and did Russell no service.

Next let me report about recent work which has given an exact local formulation of the Vogt-Russell theorem (Kähler, 1972; Kähler and Weigert, 1973). The problem is to give precise conditions for the local existence and uniqueness of stellar models, in the vicinty of a given model. First let me illustrate the problem with an example. Consider again a star with a burnt-out He-core (Fig. 2a). We have already observed that within a certain range of the core mass there are three different solutions. With growing core mass two of them approach each other, and finally they merge. At this point (where the Schönberg-Chandrasekhar limit has been reached) the local existence and uniqueness is violated. For a somewhat larger core mass there is no neighboring solution at all, and for a somewhat smaller core mass there are two different neighboring solutions. It will become clear later that the solution at the turning point itself should be considered as a double solution.

Let us now obtain a criterion for the local existence and uniqueness. For given parameters p, a model is asserted to be completely determined by the values of luminosity and effective temperature. Starting with trial values for L and T_{eff}, we may use the surface conditions for stars which are provided by the theory of stellar atmospheres, and may then perform an inward integration of the differential equations for the stellar interior. In this way we obtain a solution of the differential equations which satisfies the outer boundary conditions, and which depends on p, L and T_{eff}. This solution gives a stellar model if and only if the two boundary conditions at the stellar center are also satisfied. It can be shown that these two conditions can be written in the form $g_1 = 0$ and $g_2 = 0$, where the g_i are functions of p, L and T_{eff}. Summarizing, a stellar equilibrium model corresponds to a common zero of two functions g_1 and g_2 which are defined by the differential equations and boundary conditions for stellar structure, and which depend on the arguments mass, composition, luminosity and effective temperature.

Let a model now be given, and consider the existence and uniqueness of neighboring models. An arbitrary but small variation of the mass or the composition should yield one and only one neighboring model. This means that an infinitesimal change δp should lead to a unique solution δL, δT_{eff} of the equations $\delta g_i = 0$. By linearization we find the following vector equation

$$\begin{pmatrix} \partial g_1/\partial p \\ \partial g_2/\partial p \end{pmatrix} \delta p + \begin{pmatrix} \partial g_1/\partial L & \partial g_1/\partial T_{eff} \\ \partial g_2/\partial L & \partial g_2/\partial T_{eff} \end{pmatrix} \cdot \begin{pmatrix} \delta L \\ \delta T_{eff} \end{pmatrix} = 0$$

in which the first vector stands for the change in the stellar parameters, and the last vector shows the reaction of the star in the HR diagram. Let the 2x2 matrix be called G. If the determinant of G is non-zero, the reaction of the star is uniquely determined:

$$|G| = 0: \qquad \begin{pmatrix} \delta L \\ \delta T_{eff} \end{pmatrix} = -G^{-1} \cdot \begin{pmatrix} \partial g_1/\partial p \\ \partial g_2/\partial p \end{pmatrix} \cdot \delta p$$

If on the other hand the determinant vanishes, there may be no solution at all, or there may be several solutions; this means that the local existence and uniqueness is violated. Thus we have arrived at a rigorous criterion: The local existence and uniqueness of stellar models, in the vicinity of a given model, is equivalent to the non-vanishing of the determinant of G. This statement may be considered to be the Vogt-Russell theorem in a local sense.

What is the physical meaning of this criterion? It can be shown that the vanishing of the determinant of G is equivalent to the occurrence of a zero eigenvalue in the stellar stability problem. From the viewpoint of stellar evolution, a model can be stable only if all eigenvalues have a negative real part, which excludes zero eigenvalues. We thus deduce that the existence and uniqueness of neighboring models holds automatically for each model which is stable. Finally it can be shown that in numerical calculations the sign of the determinant of G is closely connected with the sign of the Henyey determinant. This provides a simple means for testing computed models.

2. GLOBAL STATEMENTS ON THE MANIFOLD OF STELLAR MODELS

We have already observed that classical arguments cannot say anything about the total number of equilibrium models for the given parameters, and that indeed global uniqueness statements are out of the question. Accordingly it seemed to be hopeless to look for general statements on the number and the stability of the different models. Surprisingly, recent work has shown that such statements are nevertheless possible when appropriate algebraical and topological methods are applied (Kähler, 1975). For this purpose two new properties of equilibrium models have been defined, the multiplicity m and the charge c.

Consider first the multiplicity. If we ask for the number of models, we have to think about cases of mathematical degeneracy in which several solutions coincide. For example, we have seen that at the Schönberg-Chandrasekhar limit two solutions have merged, so we might speak of a double solution. In order to give a corresponding definition we recall that, for given parameters p, a stellar model is completely determined by its location in the HR diagram, and that the values of L and T_{eff} satisfy the two equations $g_i = 0$. Each of these equations defines a curve in the HR diagram, and the model corresponds to an intersection of the two curves. Let us define the multiplicity m of the model as the corresponding intersection number in the sense of algebraic geometry. The multiplicity so-defined has a simple physical meaning. A model which is locally unique has m = 1. This is the usual case; such a model reacts uniquely to parameter changes. If on the other hand the local uniqueness is violated, we have m > 1. In this case m solutions coincide. This mathematical degeneracy can be resolved by a small change of a parameter; up to m different models can thereby be obtained. In the HR diagram, the usual case of m = 1 corresponds to a simple intersection of the two curves, as is shown in Fig. 3a. A higher multiplicity may for example correspond to a contact of the two curves (Fig. 3b), or to the occurrence of singular points (Fig. 3c).

Next, let us consider the charge of a stellar model. We have observed that the vanishing of the determinant of G is accompanied by the occurrence of a zero eigenvalue in the stellar stability problem. Hence we may expect that the sign of this determinant (or equivalently of the Henyey determinant) is important for stellar stability. This has indeed been shown by Paczynski (1972). The charge of a model is now defined as a property which allows for this sign. The definition involves topological properties of

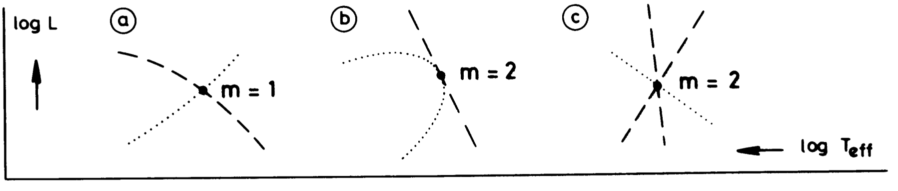

Fig. 3. Definition of the multiplicity (see text). The dashed and dotted lines represent the curves $g_1 = 0$ and $g_2 = 0$, respectively.

the functions g_i; let me omit details. The charge c is equal to the sign of det G if this sign is defined, that means if the determinant is non-zero. Otherwise the charge is usually equal to either 0 or ±1, depending on whether m is even or odd. You may ask why this property of a model has been called, in particular, charge. The reason is that the total charge of all models in a given region in the HR diagram depends only on the boundary of the region and can be expressed by a certain line integral. This indicates an analogy to electrostatics where the total electric charge in a given volume can be determined from the electric field on the surface.

So far we have defined two properties of a model, multiplicity and charge, which are small integers and which might be considered to be quantum numbers for the model. Next let us ask for applications. Consider first the stability of a model with given quantum numbers. If $m > 1$, the model has a zero eigenvalue and is therefore unstable from the viewpoint of stellar evolution. Stellar stability, therefore, requires $m = 1$. Furthermore, it can be shown that stability requires a definite value of the charge. This value turns out to be unity. Summarizing, a model can be stable only if $m = c = 1$. This condition is necessary but not sufficient.

As a second application of this formalism let us ask for the models which can be obtained from a given model by continuous changes of the parameters. (In other words, let us consider a linear series of stellar models.) Suppose we have, for the parameter p, a model of multiplicity m' and charge c'. What are the total number and the total charge of the different neighboring models obtained by a small change δp? The general result is that the number of models (defined as the sum over the multiplicities) is either conserved or it decreases by an even number, and that the total charge is always conserved:

	p	$p+\delta p$
Σm	m'	$m' - 2r$ (r integer ≥ 0)
Σc	c'	c'

This result may be considered to be a selection rule for the quantum numbers m and c. As an example consider again stars with a burnt-out helium core and a total mass of two solar masses (Fig. 4a). The number of models with given core mass is either 1 or 3; it may change by an even number. But the total charge of the different models is thereby conserved. A more complicated example is the bifurcation point in Fig. 4b which has been reported by Paczynski (1972).

We are now ready to ask for a global statement on the manifold of equilibrium models with given mass and composition. These basic parameters are again comprised in the symbol p. Let C_1 denote the total charge of all models which do exist for $p = p_1$. Similarly, let C_2 be the total charge of all models which do exist for $p = p_2$. Take a continuous sequence of parameters which connects p_1 and p_2. Along this sequence, the total charge of all models is conserved provided that no model escapes to 'infinity', which means to extreme values of pressure or temperature for which no local thermodynamic equilibrium is possible for the given composition. This assumption is strong but physically plausible if we confine ourselves to

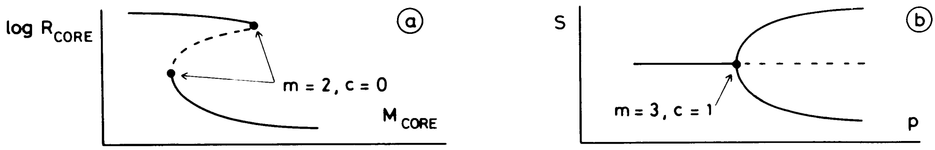

Fig. 4. Examples of linear series of stellar models (see text). Solid and dashed lines denote models with m = c = 1 and m = -c = 1, respectively.

compositions consisting mainly of lighter elements (say, up to carbon). Let us make the above assumption. The conservation of charge implies then $C_1 = C_2$. In other words, the total charge C of all models does not depend on the mass and the composition. The value of C can therefore be obtained from a simple special case, the limit of very small stellar mass. In this limit there is only one degenerate model (white dwarf or planet) which has c = 1. Accordingly the total charge C of all models has the universal value +1.

Consider now the total number N of models for given mass and composition. We may write N as the sum of the contributions N_+ and N_- which refer to models with positive and negative charge, respectively. The requirement that the total charge be unity implies the following equations:

$$N_+ = (N + 1)/2, \text{ and } N_- = (N - 1)/2.$$

These equations show that N is odd, Accordingly $N \geq 1$, and we arrive at an existence theorem: For given mass and composition there is always one equilibrium model, but additional models may appear in pairs of opposite charge. Moreover we obtain some information about the stability of the models. Since stable models are of positive charge, up to (N+1)/2 models can be stable but the remaining ones are necessarily unstable.

Summarizing, we have made a plausible assumption which excludes stars consisting of heavy elements. Under this assumption, a general statement has been derived which concerns the old problem Russell dealt with. The statement gives some information about the number and the stability of the different equilibrium configurations, and in particular it guarantees the existence of at least one model.

REFERENCES

Kähler, H. (1972). Astron. Astrophys. **20**, 105.
Kähler, H. and Weigert, A. (1973). Astron. Astrophys. **30**, 431.
Kähler, H. (1975). Astron. Astrophys. **43**, 443.
Paczyński, B. (1972). Acta Astron. **22**, 163.
Roth, M.L. (1973). Thesis, Hamburg.

Russell, H.N. (1927). In Astronomy, Vol. 2, by H.N. Russell, R.S. Dugan and J.Q. Stewart, Ginn and Co., Boston, p. 910.

Vogt, H. (1926). Astron. Nachr. 226, 301.

https://doi.org/10.1017/S007418090014570X Published online by Cambridge University Press