COMPOSITIO MATHEMATICA

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Compositio Math. **158** (2022), 1850–1877.

doi:10.1112/S0010437X22007709

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https://doi.org/10.1112/S0010437X22007709 Published online by Cambridge University Press

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The Hanna Neumann conjecture for surface groups

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Dedicated to Warren Dicks, for his generosity

Abstract

The Hanna Neumann conjecture is a statement about the rank of the intersection of two finitely generated subgroups of a free group. The conjecture was posed by Hanna Neumann in 1957. In 2011, a strengthened version of the conjecture was proved independently by Joel Friedman and by Igor Mineyev. In this paper we show that the strengthened Hanna Neumann conjecture holds not only in free groups but also in non-solvable surface groups. In addition, we show that a retract in a free group and in a surface group is inert. This implies the Dicks–Ventura inertia conjecture for free and surface groups.

1. Introduction

Let G be a group. We say that G satisfies the *Howson property* if the intersection of two finitely generated subgroups of G is again finitely generated. This property was introduced by Howson [How54] where he proved that it holds for free groups. In fact, Howson gave an effective bound for the number of generators of the intersection which was improved few years later by H. Neumann [Neu56].

Let $d(G)$ denote the number of generators of a group G. H. Neumann showed that if U and W are non-trivial finitely generated subgroups of a free group, then

$$d(U \cap W) - 1 \leq 2(d(U) - 1)(d(W) - 1)$$

and she conjectured that, in fact, the factor 2 can be omitted. This conjecture became known as the *Hanna Neumann conjecture*.

In 1980, W. Neumann improved the result of H. Neumann. For a group G we put $\overline{d}(G) = \max\{0, d(G) - 1\}$. W. Neumann showed that if U and W are finitely generated subgroups of a free group F, then

$$\sum_{x \in U \backslash F / W} \overline{d}(U \cap xWx^{-1}) \leq 2\overline{d}(U)\overline{d}(W)$$

and he also conjectured that again the factor 2 can be omitted. This conjecture became known as the *strengthened Hanna Neumann conjecture*. It was proved independently by Friedman [Fri14] and by Mineyev [Mine12] in 2011. These were also the first proofs of the Hanna Neumann

Received 19 November 2021, accepted in final form 17 June 2022, published online 12 October 2022.
2020 Mathematics Subject Classification 20F67, 20E18 (primary), 20J05, 20C07 (secondary).
Keywords: surface groups, limit groups, L2-Betti numbers, the Hanna Neumann conjecture, Lück's approximation.
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conjecture. Dicks presented two simplifications of the previous proofs (see [Dic11] and [Fri14, appendix]).

In [Jai17], the second author proved the strengthened Hanna Neumann conjecture for free pro-p groups. The proofs of Friedman and Mineyev used combinatorial and geometric aspects of free groups. This kind of techniques are not available (or probably not enough developed) in the world of pro-p groups and, therefore, Jaikin-Zapirain's proof used a homological approach. It turned out that this new method (with suitable modifications) gave also a new proof of the original strengthened Hanna Neumann conjecture for free groups. In [JS19], Jaikin-Zapirain and Shusterman developed further the pro-p part of [Jai17] and showed that the strengthened Hanna Neumann conjecture holds for non-solvable Demushkin pro-p groups (the Demushkin pro-p groups are the Poincaré duality pro-p groups of cohomological dimension 2).

By a *surface group* we mean the fundamental group of a compact closed surface of negative Euler characteristic. In the orientable case, surface groups admit presentations of the form $\langle x_1, \dots, x_n, y_1, \dots, y_n | \ [x_1 y_1] \cdots [x_n, y_n] = 1\rangle$ $(n \geq 2)$; and in the non-orientable closed case it is $\langle x_1, \dots, x_n | \ x_1^2 \cdots x_n^2 = 1\rangle$ $(n \geq 3)$. Although free groups arise as fundamental groups of non-closed surfaces of negative Euler characteristic, we do not consider free groups as surface groups.

We note that all surface groups but $\langle a, b, c \mid a^2 b^2 c^2 = 1\rangle$ are limit groups, and the latter has an index two subgroup that is a limit group. The class of virtually limit groups plays an important role throughout this work.

In this paper, we develop the discrete part of [Jai17], and we prove the strengthened Hanna Neumann conjecture for surface groups.

Theorem 1.1. *Let G be a surface group. Then for any finitely generated subgroups U and W of G,*

$$\sum_{x \in U \backslash G / W} \overline{d}(U \cap xWx^{-1}) \leq \overline{d}(U)\overline{d}(W).$$

In the context of the Hanna Neumann conjecture, the best previous bound when G is an orientable surface group was obtained by Soma in [Som90, Som91]: $\overline{d}(U \cap W) \leq 1161 \cdot \overline{d}(U)\overline{d}(W)$.

Theorem 1.1 is obtained from the following generalization of the strengthened Hanna Neumann conjecture. Let Γ be a virtually FL-group. Then we define its Euler characteristic as

$$\chi(\Gamma) = \frac{1}{|\Gamma : \Gamma_0|} \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma_0, \mathbb{Q}),$$

where Γ_0 is an FL-subgroup of Γ of finite index.

Let $\overline{\chi}(\Gamma) = \max\{0, -\chi(\Gamma)\}$. Observe that for a non-trivial finitely generated free group Γ, $\overline{\chi}(\Gamma) = \overline{d}(\Gamma)$, for a surface group Γ we have $\overline{\chi}(\Gamma) = \overline{d}(\Gamma) - 1$ and for a finitely generated virtually abelian group Γ, $\overline{\chi}(\Gamma) = 0$.

Theorem 1.2. *Let G be a surface group. Then for every two finitely generated subgroups U and W of G,*

$$\sum_{x \in U \backslash G / W} \overline{\chi}(U \cap xWx^{-1}) \leq \overline{\chi}(U)\overline{\chi}(W).$$

We conjecture that the previous theorem holds in a greater generality. Recall that the class of limit groups coincides with the class of constructible limit groups, and from that one can deduce that they are fundamental groups of finite CW-complexes [Wil09, Corollary 4.11], and, thus, they are FL. As finitely generated subgroups of limit groups are limit groups, and limit

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groups satisfy Howson's property [Dah03], we see that the family of finitely generated subgroups of a limit group is a family of FL-subgroups closed under intersections and conjugations. We believe that Theorem 1.2 can be extended further and we propose the following conjecture.

CONJECTURE 1 (The geometric Hanna Neumann conjecture). *Let G be a limit group. Then for every two finitely generated subgroups U and W of G,*

$$\sum_{x\in U\backslash G/W} \overline{\chi}(U\cap xWx^{-1}) \le \overline{\chi}(U)\overline{\chi}(W).$$

Note that by Theorem 9.4, the left-hand side of the above inequality is known to be finite.

The L^2-*independence* and L^2-*Hall* properties are two new technical notions that we introduce in this paper (see § 4 for definitions). In this paper, we prove that retracts in free and surface groups are L^2-independent. In particular, this implies that surface groups are L^2-Hall. Further understanding of these new concepts would help to make progress on Conjecture 1. For example, the proof of L^2-Hall property for limit groups would lead to the solution of the conjecture in the case of hyperbolic limit groups.

THEOREM 1.3. *Let G be a hyperbolic limit group. Assume that G satisfies the L^2-Hall property. Then the geometric Hanna Neumann conjecture holds for G.*

Recall that a subgroup U of a group G is called *inert* if for every subgroup H of G, $d(H\cap U) \le d(H)$. In addition, to Theorem 1.1, the consideration of L^2-independence helps us to show that a retract in a free or a surface group is inert.

THEOREM 1.4. *Let G be either a free or a surface group. Then any retract in G is inert.*

As a consequence we obtain the Dicks–Ventura inertia conjecture for free groups [DV96, Problem 5], [Ven02, Conjecture 8.1] and the analogous result for surface groups. This conjecture has its origin in an influential paper of Bestvina and Handel [BH92], where a conjecture of Scott was proved: the subgroup of elements of a free group of rank n fixed by a given automorphism has rank at most n.

COROLLARY 1.5 (The Dicks–Ventura inertia conjecture). *Let G be either a free or a surface group and let $\mathcal{F}$ be a finite collection of endomorphisms of G. Then*

$$\mathrm{Fix}(\mathcal{F}) = \{g\in G:\ \phi(g)=g \ \ \textit{for all } \phi\in\mathcal{F}\}$$

is inert in G. In particular, $d(\mathrm{Fix}(\mathcal{F}))\le d(G)$.

Proof. Assume first that G is a finitely generated free group. The fact that the inertia conjecture follows from inertia of retracts (i.e. our Theorem 1.4) is well-known (see the discussion of [Ven02, Conjecture 81]) and we reproduce it for the convenience of the reader.

As the intersection of inert subgroups is inert, without loss of generality, we can assume that $\mathcal{F}$ consists of a single endomorphism ϕ. The case when ϕ is injective was proved in [DV96, Theorem IV.5.5]. Consider an arbitrary endomorphism ϕ. Let

$$\phi^\infty(G) = \cap_{i=0}^\infty \phi^i(G).$$

Then by [Tur96, Theorem 1], $\phi^\infty(G)$ is a retract in G, and, thus, by Theorem 1.4, we only have to show that $\mathrm{Fix}(\phi)$ is inert in $\phi^\infty(G)$. By [IT89, Theorem 1], the restriction of ϕ on $\phi^\infty(G)$ is an automorphism. Thus, [DV96, Theorem IV.5.5] gives us the desired result.

Ventura has pointed out to us that the same reduction argument works in the case of a surface group G.

If ϕ is not an automorphism, then $\phi(G)$ has infinite index and, hence, it is free. In particular, $\phi^\infty(G)$ is still a retract of G and the argument applies verbatim. The only difference is when

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ϕ is an automorphism. However, this case was proved already by Wu and Zhang in [WZ14, Corollary 1.5]. □

Let us briefly describe the structure of the paper. In § 2 we include main definitions and facts that we use in the paper. In § 3 we introduce L^2-Betti numbers $\beta_k^{K[G]}(M)$ for $K[G]$-modules M with K a subfield of $\mathbb{C}$ and explain the Atiyah and Lück approximation conjectures. The L^2-independence and L^2-Hall properties are discussed in § 4. In § 5 we prove Theorem 1.4. In § 6 we introduce an auxiliary ring $L_\tau[G]$ which already played an important role in Dicks' simplification of Freidman's proof. We finish the proof of Theorems 1.2 and 1.1 in § 7. In § 8 we reformulate the geometric Hanna Neumann conjecture in terms of an inequality for $\beta_1^{\mathbb{Q}[G]}$. A key step of our proof of Theorem 1.3 is to find a specific submodule of $K[G/U] \otimes K[G/W]$ with trivial $\beta_1^{K[G]}$. This is done in § 11. However, previously we present two auxiliary properties. In § 9 we prove a generalization of Howson property for quasi-convex subgroups of hyperbolic groups and for subgroups of limit groups and in § 10 we prove the Wilson–Zalesskii property for quasi-convex subgroups of hyperbolic virtually compact special groups. We finish the proof of Theorem 1.3 in § 12 and we describe also some limitations of our methods in order to extend them to more cases of Conjecture 1.

Remark 1.6. Theorems 1.2 and 1.4 and Corollary 1.5 hold also for fundamental groups of surfaces of non-negative Euler characteristic (i.e. the trivial group, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}^2$ and $\langle a, b \mid a^2b^2\rangle$, the fundamental group of a Klein bottle). However, the results are either trivial, or use simple arguments specific for these cases. On the other hand, it is easy to produce a counter-example of Theorem 1.1 when G is virtually $\mathbb{Z}^2$.

2. Preliminaries

Although our main result is about surface groups, many steps of our proof hold in more general contexts of word hyperbolic, limit or virtually special compact groups. In this section, we recall all the relevant definitions and facts about these groups.

Let Y be a geodesic metric space. A subset $Z \subseteq Y$ is called *quasi-convex* if there exists $\epsilon \geq 0$ such that for any points $z_1, z_2 \in Z$, any geodesic joining these points is contained in the closed ϵ-neighborhood of Z.

A geodesic metric space Y is called *(Gromov) hyperbolic* if there exists a constant $\delta \geq 0$ such that for any geodesic triangle Δ in Y, any side of Δ is contained in the closed δ-neighborhood of the union of other sides. A finitely generated group G is said to be *hyperbolic* if its Cayley graph with respect to some finite generating set is a hyperbolic metric space. Quasi-convex subgroups of G are very important in the study of hyperbolic groups. Such subgroups are themselves hyperbolic and are quasi-isometrically embedded in G (see, for example, [ABCF+90]). Moreover, for finitely generated subgroups of hyperbolic groups, being quasi-isometrically embedded in G is equivalent to be quasi-convex. The intersection of two quasi-convex subgroups in a hyperbolic group is quasi-convex by a result of Short [Sho91].

For a subgroup $H \leq G$, we write $H^g = gHg^{-1}$. A subgroup H of a group G is called malnormal if for every $x \in G \setminus H$, $H^x \cap H = \{1\}$.

A finitely generated group G is a *limit* group if, for any finite subset X of G, there exists a homomorphism $f\colon G \to F$ to a free group so that the restriction of f on X is injective. By a result of Wilton [Wil08], a finitely generated subgroup of a limit group is a virtual retract. Therefore, in a limit group all finitely generated subgroups are quasi-isometrically embedded and, in particular, in hyperbolic limit groups finitely generated subgroups are quasi-convex.

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A *right-angled Artin group (RAAG)* is a group which can be given by a finite presentation, where the only defining relators are commutators of the generators. To construct such a group, one usually starts with a finite graph Γ with vertex set V and edge set E. One then defines the corresponding RAAG $A(\Gamma)$ by the following presentation:

$$A(\Gamma) = \langle V \mid uv = vu \text{ whenever } \{u, v\} \in E\rangle.$$

We always view $A(\Gamma)$ as a metric space with respect to the word metric induced by V when considering quasi-convexity of subgroups.

Special cube complexes were introduced in [HW08]. A group is called *(compact) special* if it is the fundamental group of a non-positively curved (compact) special cube complex. If G is the fundamental group of X, a (compact) special cube complex, then $\tilde{X}$, the universal cover of X, is a CAT(0) cubical complex where G acts. By a quasi-convex subgroup of G we mean a subgroup of H with a quasi-convex orbit of vertices in $\tilde{X}$ with respect to the combinatorial metric. A nice group theoretic characterization of these groups is that a group is (compact) special if and only if it is a (quasi-convex) subgroup of a RAAG (see [Hag08, HW08]). By [Hag08, Theorem F], quasi-convex subgroups of RAAGs are virtual retracts. Thus, we have the following theorem that will be used several times.

Theorem 2.1. *Any quasi-convex subgroup of a virtually compact special group is a virtual retract.*

By a result of Wise [Wis12, Corollary 16.11], a limit group is virtually compact special.

In this paper, we explore profinite properties of virtually compact special groups. Recall that if $\mathcal{S} = \{S_1, \dots, S_k\}$ is a family of disjoint subsets of a group G, we say that a normal subgroup N, of G *separates* $\mathcal{S}$ if $S_1N, \dots, S_kN$ are disjoint. The family $\mathcal{S}$ is *separable* if there exists a normal subgroup of G of finite index that separates $\mathcal{S}$. A subset S of G is *separable* if for every $g \in G \setminus S$, the family $\{S, g\}$ is separable. A group is *residually finite* if the trivial element is separable. For example, since a quasi-convex subgroup H of a virtually compact special group G is virtually a retract and G is residually finite, any finite family $\{x_1H, \dots, x_kH\}$ of left cosets of H is separable. For hyperbolic groups this is the property *GFERF* introduced in [Mina06] and we use the following theorem.

Theorem 2.2 [Mina06, Theorem 1.1 and Remark 2.2]. *Assume G is a virtually compact special hyperbolic group (or, more generally, a GFERF hyperbolic group), and $H_1, \dots, H_s$ are quasi-convex subgroups of G, $s \in \mathbb{N}$ and $g_0, \dots, g_s \in G$. Then the product $g_0H_1g_1 \dots g_{s-1}H_sg_s$ is separable in G.*

3. L^2-Betti numbers, the strong Atiyah conjecture and the Lück approximation

Let G be a discrete group and let $l^2(G)$ denote the Hilbert space with Hilbert basis the elements of G; thus, $l^2(G)$ consists of all square summable formal sums $\sum_{g\in G} a_g g$ with $a_g \in \mathbb{C}$ and inner product

$$\left\langle \sum_{g\in G} a_g g, \sum_{h\in G} b_h h \right\rangle = \sum_{g\in G} a_g \overline{b_g}.$$

The left- and right-multiplication action of G on itself extend to left and right actions of G on $l^2(G)$. The right action of G on $l^2(G)$ extends to an action of $\mathbb{C}[G]$ on $l^2(G)$ and so we obtain that the group algebra $\mathbb{C}[G]$ acts faithfully as bounded linear operators on $l^2(G)$. The ring $\mathcal{N}(G)$ is the ring of bounded operators on $l^2(G)$ which commute with the left action of G. We often consider $\mathbb{C}[G]$ as a subalgebra of $\mathcal{N}(G)$. The ring $\mathcal{N}(G)$ satisfies the left and right Ore conditions

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(a result proved by Berberian in [Ber82]) and its classical ring of fractions is denoted by $\mathcal{U}(G)$. The ring $\mathcal{U}(G)$ can be also described as the ring of densely defined (unbounded) operators which commute with the left action of G.

The computations of L^2-Betti numbers have been algebraized through the seminal works of Lück [Lüc98a, Lüc98b]. The basic observation is that one can use a dimension function $\dim_{\mathcal{U}(G)}$, which is defined for all modules over $\mathcal{U}(G)$ and compute the kth L^2-Betti number of a $\mathbb{C}[G]$-module M using the following formula:

$$\beta_k^{\mathbb{C}[G]}(M) = \dim_{\mathcal{U}(G)} \operatorname{Tor}_k^{\mathbb{C}[G]}(\mathcal{U}(G), M).$$

We recommend the book [Lüc02] for the definition of $\dim_{\mathcal{U}(G)}$ and its properties.

The ring $\mathcal{U}(G)$ is an example of a $*$-regular ring. Already in the case $G = \langle t \rangle \cong \mathbb{Z}$ it is quite complicated as a ring (it is isomorphic to $L^1(S^1)$). Therefore, sometimes, it is more convenient to consider a smaller object $\mathcal{R}_{\mathbb{C}[G]}$ introduced by Linnell and Schick [LS12].

Let K be a subfield of $\mathbb{C}$. We define $\mathcal{R}_{K[G]}$ as the $*$-regular closure of $K[G]$ in $\mathcal{U}(G)$, i.e. $\mathcal{R}_{K[G]}$ is the smallest $*$-regular subring of $\mathcal{U}(G)$ that contains $K[G]$. We can also define a dimension function $\dim_{\mathcal{R}_{K[G]}}$ on $\mathcal{R}_{K[G]}$-modules and use it in order to define the L^2-Betti numbers (see [Jai19a, Jai19b]). If M is a $K[G]$-module, then its L^2-Betti numbers are computed using the formula

$$\beta_k^{K[G]}(M) = \dim_{\mathcal{R}_{K[G]}} \operatorname{Tor}_k^{K[G]}(\mathcal{R}_{K[G]}, M).$$

The object $\mathcal{R}_{K[G]}$ is much simpler that $\mathcal{U}(G)$. For example, in the case $G = \langle t \rangle \cong \mathbb{Z}$, $\mathcal{R}_{K[G]}$ is isomorphic to $K(t)$ and $\dim_{\mathcal{R}_{K[G]}}$ is the dimension of $K(t)$-vector spaces. More generally, the strong Atiyah conjecture (see [Lüc02]) predicts that if G is torsion-free, then all numbers $\beta_k^{K[G]}(M)$ are integers, $\mathcal{R}_{K[G]}$ is a division algebra and $\dim_{\mathcal{R}_{K[G]}}$ is the dimension of $\mathcal{R}_{K[G]}$-vector spaces.

In this paper, we use the solution of the strong Atiyah conjecture in the case where G is a torsion-free virtually compact special group.

PROPOSITION 3.1 [DLMS$^+$03, Sch14, Jai19a]. *Let G be a torsion-free virtually compact special group, and let K be a subfield of $\mathbb{C}$. Then all numbers $\beta_k^{K[G]}(M)$ are integers and $\mathcal{R}_{K[G]}$ is a division algebra.*

Another important conjecture about L^2-Betti numbers is the Lück approximation conjecture (see [Lüc02]). In this paper, we use the solution of this conjecture in the case of approximation by sofic groups.

PROPOSITION 3.2 [Lüc94, DLMS$^+$03, ES05, Jai19a]. *Let G be a group and let $G > G_1 > G_2 > \ldots$ be a chain of normal subgroups with trivial intersection such that G/G_i are sofic. Let K be a subfield of $\mathbb{C}$ and let M be a finitely presented $K[G]$-module. Then*

$$\dim_{\mathcal{R}_{K[G]}}(\mathcal{R}_{K[G]} \otimes_{K[G]} M) = \lim_{i\to\infty} \dim_{\mathcal{R}_{K[G/G_i]}}(\mathcal{R}_{K[G/G_i]} \otimes_{K[G]} M).$$

In this paper, we consider only the fields K which are subfields of $\mathbb{C}$. Let M be a $K[G]$-module. By [Jai19a, Corollary 1.7], if G is sofic, then $\beta_k^{K[G]}(M)$ does not depend on the embedding of K into $\mathbb{C}$. Thus, in what follows, if the group G is sofic, we do not indicate the embedding of K into $\mathbb{C}$.

Recall that the kth L^2-Betti number of a group G is defined as $b_k^{(2)}(G) = \dim_{\mathcal{U}(G)} H_k(G; \mathcal{U}(G))$. Thus, we obtain that

$$b_k^{(2)}(G) = \dim_{\mathcal{U}(G)} \operatorname{Tor}_k^{\mathbb{Z}[G]}(\mathcal{U}(G), \mathbb{Z}) = \dim_{\mathcal{R}_{K[G]}} \operatorname{Tor}_k^{K[G]}(\mathcal{R}_{K[G]}, K) = \beta_k^{K[G]}(K),$$

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where K is arbitrary subfield of $\mathbb{C}$. In the case when G is a virtually limit group, we have a good control of its L^2-Betti numbers.

Proposition 3.3. *Let G be a virtually limit group and K a subfield of $\mathbb{C}$. Then*

$$\beta_k^{K[G]}(K) = b_k^{(2)}(G) = 0 \quad \textit{if } k \geq 2.$$

In particular,

$$\overline{\chi}(G) = b_1^{(2)}(G) = \beta_1^{K[G]}(K).$$

Proof. See, for example, [BK17, Corollary C]. □

If U is a subgroup of a group G, then $\mathcal{R}_{K[G]}$ is a flat right $\mathcal{R}_{K[U]}$-module and for every left $\mathcal{R}_{K[U]}$-module M,

$$\dim_{\mathcal{R}_{K[G]}} \mathcal{R}_{K[G]} \otimes_{\mathcal{R}_{K[U]}} M = \dim_{\mathcal{R}_{K[U]}} M.$$

This implies the following result.

Proposition 3.4. *Let U be a subgroup of a group G and let M be a left $K[U]$-module, then for every k,*

$$\beta_k^{K[U]}(M) = \beta_k^{K[G]}(K[G] \otimes_{K[U]} M).$$

4. The L^2-Hall property for surface groups

Let U be a subgroup of G. The embedding of U into G induces the corestriction map

$$\mathrm{cor} : H_1(U;\mathcal{U}(G)) \to H_1(G;\mathcal{U}(G)).$$

We say that U is L^2-*independent in G* if

$$\dim_{\mathcal{U}(G)} \ker(\mathrm{cor}) = 0.$$

We say that the group G is L^2-*Hall*, if for every finitely generated subgroup U of G, there exists a subgroup H of G of finite index containing U such that U is L^2-independent in H.

The L^2-independence can also be characterized in terms of $\mathcal{R}_{K[G]}$.

Lemma 4.1. *Let G be a group and K a subfield of $\mathbb{C}$. Then a subgroup U of G is L^2-independent if and only if*

$$\dim_{\mathcal{R}_{K[G]}} \ker(\mathrm{cor}) = 0,$$

where $\mathrm{cor} : H_1(U;\mathcal{R}_{K[G]}) \to H_1(G;\mathcal{R}_{K[G]})$ is the corestriction.

Proof. As $\mathcal{R}_{K[G]}$ is von Neumann regular, $\mathcal{U}(G)$ is a flat $\mathcal{R}_{K[G]}$-module and we are done. □

If $\mathcal{R}_{K[G]}$ is a semi-simple algebra, Lemma 4.1 implies that in order to show that U is L^2-independent in G, one has to prove that $\ker(\mathrm{cor}) = \{0\}$. In the case of virtually limit groups, we can give also the following description.

Proposition 4.2. *Let G be a virtually limit group and let $H_1 \leq H_2$ be two finitely generated subgroups of G. Let K be a subfield of $\mathbb{C}$. Consider the exact sequence*

$$1 \to M \to K[G/H_1] \to K[G/H_2] \to 0.$$

Then H_1 is L^2-independent in H_2 if and only if $\beta_1^{K[G]}(M) = 0$.

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Proof. We have the following exact sequence of Tor-functors:

$$\operatorname{Tor}_2^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_2]) \to \operatorname{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, M)$$
$$\to \operatorname{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_1]) \xrightarrow{\alpha} \operatorname{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_2]).$$

By Proposition 3.3, $\operatorname{Tor}_2^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_2]) \cong \operatorname{Tor}_2^{K[H_2]}(\mathcal{R}_{K[G]}, K) = 0$. In addition, the Shapiro lemma provides canonical isomorphisms

$$\gamma_i : \operatorname{Tor}_1^{K[G]}(\mathcal{R}_{K[G]}, K[G/H_i]) \to \operatorname{Tor}_1^{K[H_i]}(\mathcal{R}_{K[G]}, K) = H_1(H_i; R_{K[G]}) \quad (i = 1, 2)$$

such that $\operatorname{cor} = \gamma_2 \circ \alpha \circ \gamma_1^{-1}$. Thus, $\dim_{\mathcal{R}_{K[G]}} \ker(\operatorname{cor}) = 0$ if and only if $\dim_{\mathcal{R}_{K[G]}} \ker \alpha = 0$ if and only if $\beta_1^{K[G]}(M) = 0$. □

Corollary 4.3. *Let U be a finitely generated subgroup of a virtual limit group G and K a subfield of $\mathbb{C}$. Then U is L^2-independent in G if and only if $\beta_1^{K[G]}(I_G/I_U^G) = 0$.*

Here I_G (respectively, I_U) is the augmentation ideal of $K[G]$ (respectively, $K[U]$) and I_U^G is the left ideal of $K[G]$ generated by I_U.

Proof. Use Proposition 4.2 and take into account that $K[G/U] \cong K[G]/I_U^G$. □

In this section, we show that surface groups are L^2-Hall.

Theorem 4.4. *Finitely generated free groups and surface groups are L^2-Hall.*

Let P be a pro-p group. We denote by $d(P)$ the minimal cardinality of a topological generating set of P. If P is finitely generated and L is an $\mathbb{F}_p[[P]]$-module, then the functions $\beta_k^{\mathbb{F}_p[[P]]}(L)$ are defined in the following way. Fix a chain $P_1 > P_2 > P_3 > \dots$ of open normal subgroups of P with trivial intersection and we put

$$\beta_k^{\mathbb{F}_p[[P]]}(L) = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} H_k(P_i, L)}{|P : P_i|},$$

assuming that all $\dim_{\mathbb{F}_p} H_k(P_i, L)$ are finite. The limit always exists and it does not depend on the chain (see [Jai19b, Proposition 11.2]).

An (infinite) *Demushkin* pro-p group is a Poincaré duality pro-p group of cohomological dimension 2. For the purposes of this paper, it is enough to know that the fundamental group of a closed surface is residually finite 2-group and its pro-2 completion is Demushkin. First, let us present the following result whose proof is essentially contained in the proof of [JS19, Proposition 7.2].

Proposition 4.5. *Let P be an infinite Demushkin pro-p group and let H be a proper closed subgroup of P such that the map $H_1(H; \mathbb{F}_p) \to H_1(P; \mathbb{F}_p)$ is injective. Let L be the kernel of the map $\mathbb{F}_p[[P/H]] \to \mathbb{F}_p$. Then*

$$\beta_1^{\mathbb{F}_p[[P]]}(L) = 0 \quad \text{and} \quad \beta_0^{\mathbb{F}_p[[P]]}(L) = d(P) - d(H) - 1.$$

Proof. As H is a proper subgroup of P and $H_1(H; \mathbb{F}_p) \to H_1(P; \mathbb{F}_p)$ is injective, H is of infinite index, and, thus, H is a free pro-p group. Moreover, because the map $H_1(H; \mathbb{F}_p) \to H_1(P; \mathbb{F}_p)$ is injective, H is finitely generated.

In the proof of [JS19, Proposition 7.2], it is shown that L is an one-relator $\mathbb{F}_p[[P]]$-module. Thus, we can produce an exact sequence

$$0 \to C \to \mathbb{F}_p[[P]]^d \to L \to 0,$$

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where C is a non-trivial cyclic $\mathbb{F}_p[[P]]$-module. As $\mathbb{F}_p[[P]]$ is a domain, $C \cong \mathbb{F}_p[[P]]$. By [JS19, Corollary 6.2], $\beta_1^{\mathbb{F}_p[[P]]}(L) = 0$. Hence, $\beta_0^{\mathbb{F}_p[[P]]}(L) = d - 1 = \chi_P(L)$, where $\chi_P(L)$ is the Euler characteristic of L as a $\mathbb{F}_p[[P]]$-module.

On the other hand, using the exact sequence

$$0 \to L \to \mathbb{F}_p[[P/H]] \to \mathbb{F}_p \to 0,$$

we obtain that

$$\begin{aligned} \chi_P(L) &= \chi_P(\mathbb{F}_p[[P/H]]) - \chi_P(\mathbb{F}_p) = \chi_H(\mathbb{F}_p) - \chi_P(\mathbb{F}_p) \\ &= 1 - d(H) - (2 - d(P)) = d(P) - d(H) - 1. \end{aligned}$$

In the penultimate equality we have used that H is free and P is Demushkin. □

The previous proposition leads to a criterion for L^2-independence of a subgroup of a free or a surface group.

Proposition 4.6. *Let G be a finitely generated free group or a surface group and U a retract of G. Then U is L^2-independent in G.*

Proof. Without loss of generality, we assume that U is non-trivial and proper. Thus, G is infinite and U is a free group. First consider the case where G is a surface group.

As U is a retract, $H_1(U;\mathbb{F}_2) \to H_1(G;\mathbb{F}_2)$ is injective. Let P be the pro-2 completion of G. As we have mentioned, G is a Demushkin pro-2 group. Let $P_1 > P_2 > P_3 > \cdots$ be a chain of open normal subgroups of P with trivial intersection. We put $G_i = G \cap P_i$. Let H be the closure of U in P. As U is a retract of G, H is a free pro-2 group, and, thus, it is a proper subgroup of P.

The condition $H_1(U;\mathbb{F}_2) \to H_1(G;\mathbb{F}_2)$ is injective implies that $H_1(H;\mathbb{F}_2) \to H_1(P;\mathbb{F}_2)$ is injective and $d(H) = \dim_{\mathbb{F}_p} H_1(U;\mathbb{F}_2)$ (and, thus, $d(H) = d(U)$).

Consider two exact sequences

$$0 \to M \to \mathbb{Z}[G/U] \to \mathbb{Z} \to 0 \quad \text{and} \quad 0 \to L \to \mathbb{F}_2[[P/H]] \to \mathbb{F}_2 \to 0.$$

Tensoring the first sequence with $\mathbb{F}_2$ over $\mathbb{Z}$, we obtain another exact sequence of $\mathbb{F}_2[G]$-modules,

$$0 \to \mathbb{F}_2 \otimes_{\mathbb{Z}} M \to \mathbb{F}_2[G/U] \to \mathbb{F}_2 \to 0.$$

Put $\overline{M} = \mathbb{F}_2 \otimes_{\mathbb{Z}} M$. As G is pro-2 good (see [GJPZ14]),

$$H_1(G;\mathbb{F}_2[[P]]) = \operatorname{Tor}_1^{\mathbb{F}_2[G]}(\mathbb{F}_2[[P]],\mathbb{F}_2) = 0.$$

Therefore, the sequence

$$0 \to \mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} \overline{M} \to \mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} \mathbb{F}_2[G/U] \to \mathbb{F}_2 \to 0$$

is also exact. As $\mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} \mathbb{F}_2[G/U] \cong \mathbb{F}_2[[P/H]]$, we obtain that $L \cong \mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} \overline{M}$ as $\mathbb{F}_2[[P]]$-modules. In particular,

$$\dim_{\mathbb{F}_2} H_0(G_i;\overline{M}) = \dim_{\mathbb{F}_2} H_0(P_i;\mathbb{F}_2[[P]] \otimes_{\mathbb{F}_2[G]} \overline{M}) = \dim_{\mathbb{F}_2} H_0(P_i, L).$$

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Thus,

$$\begin{aligned}\lim_{i\to\infty}\frac{\dim_{\mathbb{Q}}\operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}\otimes_{\mathbb{Z}}M)}{|G:G_i|} &= \lim_{i\to\infty}\frac{\dim_{\mathbb{Q}}\operatorname{Tor}_0^{\mathbb{Z}[G_i]}(\mathbb{Q},M)}{|G:G_i|}\\ &\le \lim_{i\to\infty}\frac{\dim_{\mathbb{F}_2}\operatorname{Tor}_0^{\mathbb{Z}[G_i]}(\mathbb{F}_2,M)}{|G:G_i|} = \lim_{i\to\infty}\frac{\dim_{\mathbb{F}_2}H_0(G_i;\overline{M})}{|G:G_i|}\\ &= \lim_{i\to\infty}\frac{\dim_{\mathbb{F}_2}H_0(P_i,L)}{|P:P_i|} \overset{\text{Proposition 4.5}}{=} d(P)-d(H)-1\\ &= d(G)-d(U)-1.\end{aligned}$$

Consider again the exact sequence $0\to M\to\mathbb{Z}[G/U]\to\mathbb{Z}\to 0$. It induces the exact sequence

$$0\to\mathbb{Q}\otimes_{\mathbb{Z}}M\to\mathbb{Q}[G/U]\to\mathbb{Q}\to 0.$$

The long exact sequences of Tor-functors implies that

$$\begin{aligned}\dim_{\mathbb{Q}}\operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}\otimes_{\mathbb{Z}}M) \le \dim_{\mathbb{Q}}\operatorname{Tor}_2^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}) &+ \dim_{\mathbb{Q}}\operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}[G/U])\\ -\dim_{\mathbb{Q}}\operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}) &+ \dim_{\mathbb{Q}}\operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}\otimes_{\mathbb{Z}}M)\\ -\dim_{\mathbb{Q}}\operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}[G/U]) &+ \dim_{\mathbb{Q}}\operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}).\end{aligned}$$

Observe that

$$\dim_{\mathbb{Q}}\operatorname{Tor}_2^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}) = \dim_{\mathbb{Q}}\operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}) = 1,$$

$$\lim_{i\to\infty}\frac{\dim_{\mathbb{Q}}\operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}[G/U])}{|G:G_i|} = d(U)-1,\quad \lim_{i\to\infty}\frac{\dim_{\mathbb{Q}}\operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q})}{|G:G_i|} = d(G)-2$$

and, because we assume that U is not trivial,

$$\lim_{i\to\infty}\frac{\dim_{\mathbb{Q}}\operatorname{Tor}_0^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}[G/U])}{|G:G_i|} = 0.$$

Putting all limits together, we obtain that

$$\beta_1^{\mathbb{Q}[G]}(\mathbb{Q}\otimes_{\mathbb{Z}}M) \overset{\text{Proposition 3.2}}{=} \lim_{i\to\infty}\frac{\dim_{\mathbb{Q}}\operatorname{Tor}_1^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}\otimes_{\mathbb{Z}}M)}{|G:G_i|} = 0.$$

By Proposition 4.2, U is L^2-independent in G.

The remaining case is the case where G is a finitely generated free group. The proof works verbatim just bearing in mind that in Proposition 4.5 one has to change P to be a free pro-p group in the hypothesis, and in the conclusion $\beta_0^{\mathbb{F}_p[[P]]}(L) = d(P)-d(H)$. In addition, in the proof of Proposition 4.6, one has that the groups G_i are free and, hence, $\dim_{\mathbb{Q}}\operatorname{Tor}_2^{\mathbb{Q}[G_i]}(\mathbb{Q},\mathbb{Q}) = 0$. □

Proof of Theorem 4.4. Let G be a finitely generated free group or a surface group and U a finitely generated subgroup of G. There exists a subgroup S of finite index in G, containing U and such that U is a retract of S (see [Hal49, Sco78]). Now, we can apply Proposition 4.6. □

5. The proof of Theorem 1.4

In this section, we prove Theorem 1.4. A similar argument is used later in our proof of Theorem 1.2. A key observation is the following proposition.

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Proposition 5.1. *Let G be a surface group or a free group and H a subgroup of G. Let K be a subfield of $\mathbb{C}$.*

(1) *Any $K[G]$-submodule of a $K[G]$-module of projective dimension 1 is also of projective dimension 1.*
(2) *If M is a $K[G]$-module of projective dimension 1 and $\beta_1^{K[G]}(M)=0$. Then $\beta_1^{K[H]}(M)=0$.*

Proof. Part (1) is clear when G is free, because $K[G]$ is of global dimension 1. If G is not free, then $K[G]$ is of global dimension 2 and for such rings a submodule of a module of projective dimension 1 is also of projective dimension 1.

In order to show part (2) we have to prove that $\operatorname{Tor}_1^{K[H]}(\mathcal{R}_{K[H]},M)=0$. By Shapiro's lemma,

$$\operatorname{Tor}_1^{K[H]}(\mathcal{R}_{K[H]},M)\cong \operatorname{Tor}_1^{K[G]}(\mathcal{R}_{K[H]}\otimes_{K[H]}K[G],M).$$

Observe that $\mathcal{R}_{K[H]}\otimes_{K[H]}K[G]$ is naturally embedded in $\mathcal{R}_{K[G]}$ (see, for example, the discussion after [Lin06, Problem 4.5]). As M is of projective dimension 1 and $\operatorname{Tor}_1^{K[G]}(\mathcal{R}_{K[G]},M)=0$, $\operatorname{Tor}_1^{K[G]}(\mathcal{R}_{K[H]}\otimes_{K[H]}K[G],M)=0$ as well. □

Proposition 5.2. *Let G be a free group or a surface group and U an L^2-independent subgroup of G. If H is a finitely generated subgroup of G, then $H\cap U$ is L^2-independent in H. In particular, $d(H\cap U)\le d(H)$.*

Proof. Without loss of generality we may assume that $G\neq U$. Hence, G is infinite and U is free. By Corollary 4.3, because U is an L^2-independent subgroup of G, $\beta_1^{K[G]}(I_G/I_U^G)=0$.

As U is free, $K[G/U]\cong K[G]/I_U^G$ is of projective dimension 1. By Proposition 5.1(1), I_G/I_U^G is also of projective dimension 1. Therefore, by Proposition 5.1(2), $\beta_1^{K[H]}(I_G/I_U^G)=0$ as well.

Put $M=I_G/I_U^G$ and $L=I_H/I_{U\cap H}^H$. In the previous paragraph we have obtained that

$$\operatorname{Tor}_1^{K[H]}(\mathcal{R}_{K[H]},M)=0.$$

As $I_{U\cap H}^H=I_H\cap I_U^G$, L is a $K[H]$-submodule of M. Let $T\subset G$ be a set of representatives of the double (H,U)-cosets in G and assume that $1\in T$. Consider the $K[H]$-module M/L. Then we have that

$$M/L\cong K[G/U]/K[H/(U\cap H)]\cong \oplus_{t\in T\setminus\{1\}}K[H/(U^t\cap H)].$$

As $U^t\cap H$ are free groups, M/L is of projective dimension 1 as a $K[H]$-module, and, thus,

$$\operatorname{Tor}_2^{K[H]}(\mathcal{R}_{K[H]},M/L)=0.$$

Thus, from the exact sequence

$$\operatorname{Tor}_2^{K[H]}(\mathcal{R}_{K[H]},M/L)\to \operatorname{Tor}_1^{K[H]}(\mathcal{R}_{K[H]},L)\to \operatorname{Tor}_1^{K[H]}(\mathcal{R}_{K[H]},M)$$

we obtain that $\operatorname{Tor}_1^{K[H]}(\mathcal{R}_{K[H]},L)=0$ and $\beta_1^{K[H]}(L)=\beta_1^{K[H]}(I_H/I_{U\cap H}^H)=0$. Thus, $H\cap U$ is L^2-independent in H by Corollary 4.3. □

Proof of Theorem 1.4. Let U be a retract of G and H a subgroup of G. By Proposition 4.6, U is L^2-independent in G. Thus, the theorem follows from Proposition 5.2. □

A subgroup U of G is called *compressed* if $d(U)\le d(H)$ for every subgroup H of G containing U. Dicks and Ventura conjectured that every compressed subgroup of a free group is also inert. We finish this section with the following natural question.

Question 2. Is any compressed subgroup of a free group also L^2-independent?

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6. The structure of acceptable $L_\tau[G]$-modules

Let L be a field and let $\tau\colon G \to \mathrm{Aut}(L)$ be a homomorphism. We denote by $L_\tau[G]$ the twisted group ring: its underlying additive group coincides with the ordinary group ring $L[G]$, but the multiplication is defined as follows:

$$\left(\sum_{i=1}^{n} k_i f_i\right)\left(\sum_{j=1}^{m} l_j g_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} k_i \tau(f_i)(l_j) f_i g_j, \quad k_i, l_j \in L,\ f_i, g_j \in G.$$

The main advantage of working with $L_\tau[G]$-modules instead of $L[G]$-modules is stated in the following lemma.

Lemma 6.1 [Jai17, Claim 6.3]. *Let G be a group and L a field. Let $\tau\colon G \to \mathrm{Aut}(L)$ and $H = \ker\tau$. Assume that H is of finite index in G. Then:*

(1) *L is an irreducible $L_\tau[G]$-module if we define*

$$\left(\sum_{i=1}^{k} l_i f_i\right)\cdot l = \sum_{i=1}^{k} l_i \tau(f_i)(l)\ (l, l_i \in L,\ f_i \in G);$$

(2) *up to isomorphism, L is the unique irreducible $L_\tau[G]$-module on which H acts trivially.*

Our next task is to prove a version of the strong Atiyah conjecture for $L_\tau[G]$-modules where G is a torsion-free virtually compact special group. We use the fact that a torsion-free virtually compact special group G has the *factorization property*. This means that any map from G to a finite group factors through a torsion-free elementary amenable group. This was proved by Schreve (see Corollary 2.6, Lemma 2.2 and the proof of Theorem 1.1 in [Sch14]).

Proposition 6.2. *Let G be a torsion-free virtually compact special group, L a subfield of $\mathbb{C}$ and $\tau : G \to \mathrm{Aut}(L)$. Assume that $H \le \ker\tau$ is of finite index in G. Let M be an $L_\tau[G]$-module with finite $\beta_k^{L[H]}(M)$. Then $|G : H|$ divides $\beta_k^{L[H]}(M)$.*

Remark. In order to understand better the significance of this proposition, consider the case when τ sends all elements of G to the identity automorphism ($L_\tau[G] = L[G]$ in this case). Then, by the multiplicative property of L^2-Betti numbers,

$$\beta_k^{L[H]}(M) = |G : H| \cdot \beta_k^{L[G]}(M),$$

and so what we want to prove is that $\beta^{L[G]}(M)$ is an integer number. This is the strong Atiyah conjecture for G (see Proposition 3.1).

The idea of the proof of the proposition for general τ is to define $\beta_k^{L_\tau[G]}(M)$ by

$$\beta_k^{L_\tau[G]}(M) = \frac{\beta_k^{L[H]}(M)}{|G : H|},$$

and using the Lück approximation, show, in a similar way as in [DLMS+03], that $\beta_k^{L_\tau[G]}(M)$ is an integer.

Proof. Recall that G is residually finite. Using the factorization property, we can construct a chain $G \ge H > T_1 > T_2 > \cdots$ of normal subgroups of G with trivial intersection such that for each i, $A_i = G/T_i$ is torsion-free elementary amenable.

As τ sends the elements of T_i to the trivial automorphism of L, abusing slightly the notation we can construct $L_\tau[A_i]$. By a result of Moody [Moo89] (see also [KLM88] and [Lin98, Corollary 4.5]), $L_\tau[A_i]$ has no non-trivial zero-divisors. As A_i is amenable and $L_\tau[A_i]$ is a domain,

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$L_\tau[A_i]$ satisfies the left Ore condition. Thus, $L_\tau[A_i]$ has the classical division ring of fractions $\mathcal{Q}(L_\tau[A_i])$.

Let $B_i = H/T_i$. As B_i is of finite index in A_i, $\mathcal{Q}(L_\tau[A_i])$ is isomorphic to the Ore localization of $L_\tau[A_i]$ with respect to non-zero elements of $L[B_i]$. Thus,

$$\mathcal{Q}(L_\tau[A_i]) \cong \mathcal{Q}(L[B_i]) \otimes_{L[B_i]} L_\tau[A_i] \tag{1}$$

as $(\mathcal{Q}(L[B_i], L_\tau[G]))$-bimodules. Equivalently, $\mathcal{Q}(L_\tau[A_i])$ is isomorphic to a crossed product $\mathcal{Q}(L[B_i]) * (A_i/B_i)$.

Let M be a finitely presented $L_\tau[G]$-module and let

$$M_i = M/(T_i - 1)M \cong L_\tau[A_i] \otimes_{L_\tau[G]} M.$$

Then from (1) we obtain that

$$\mathcal{Q}(L[B_i]) \otimes_{L[B_i]} M_i \cong (\mathcal{Q}(L[B_i]) \otimes_{L[B_i]} L_\tau[A_i]) \otimes_{L_\tau[G]} M \cong \mathcal{Q}(L_\tau[A_i]) \otimes_{L_\tau[G]} M.$$

In particular, again taking (1) into account, we conclude that

$$\dim_{\mathcal{Q}(L[B_i])}(\mathcal{Q}(L[B_i]) \otimes_{L[B_i]} M_i) = |G:H| \dim_{\mathcal{Q}(L_\tau[A_i])}(\mathcal{Q}(L_\tau[A_i]) \otimes_{L_\tau[G]} M). \tag{2}$$

The groups B_i are torsion-free elementary amenable groups, and, thus, they satisfy the strong Atiyah conjecture [Lin93]. Hence, the rings $\mathcal{R}_{L[B_i]}$ are division rings. Therefore, by Proposition 3.2, there exists i such that

$$\beta_0^{L[H]}(M) = \dim_{\mathcal{R}_{L[B_i]}}(\mathcal{R}_{L[B_i]} \otimes_{L[H]} M) = \dim_{\mathcal{R}_{L[B_i]}}(\mathcal{R}_{L[B_i]} \otimes_{L[B_i]} M_i).$$

Observe that $\mathcal{R}_{L[B_i]}$ is isomorphic to the classical division ring of fractions $\mathcal{Q}(L[B_i])$ of $L[B_i]$ as $L[B_i]$-ring (see, for example, [Lin93] and [Jai19b, Corollary 9.4]). Therefore,

$$\begin{aligned}\beta_0^{L[H]}(M) &= \dim_{\mathcal{Q}(L[B_i])}(\mathcal{Q}(L[B_i] \otimes_{L[B_i]} M_i) \\ &\overset{\text{by (2)}}{=} |G:H| \dim_{\mathcal{Q}(L_\tau[A_i])}(\mathcal{Q}(L_\tau[A_i]) \otimes_{L_\tau[G]} M).\end{aligned}$$

This proves that $|G:H|$ divides $\beta_0^{L[H]}(M)$. Therefore, the proposition holds in the case $k=0$ and M is finitely presented. In particular, the following Sylvester module rank function on $L_\tau[G]$ (see [Jai19b] for definitions)

$$\dim M := \frac{\dim_{\mathcal{R}_{L[H]}}(\mathcal{R}_{L[H]} \otimes_{L[H]} M)}{|G:H|} = \frac{\beta_0^{L[H]}(M)}{|G:H|}$$

is integer-valued. This Sylvester function is induced by the canonical embedding of $L_\tau[G]$ into $\mathrm{Mat}_{|G:H|}(\mathcal{R}_{L[H]})$ (here the endomorphisms act on the right-hand side):

$$L_\tau[G] \hookrightarrow \mathrm{End}_{L[H]}(L_\tau[G])) \hookrightarrow \mathrm{End}_{\mathcal{R}_{L[H]}}(\mathcal{R}_{L[H]} \otimes_{L[H]} L_\tau[G]) \cong \mathrm{Mat}_{|G:H|}(\mathcal{R}_{L[H]}).$$

By an argument of Linnell (see [Lin93, Lemma 3.7]), the division closure $\mathcal{D}_G$ of $L_\tau[G]$ in $\mathrm{Mat}_{|G:H|}(\mathcal{R}_{L[H]})$ is a division ring and

$$\dim M = \dim_{\mathcal{D}_G}(\mathcal{D}_G \otimes_{L_\tau[G]} M).$$

The division closure of $L[H]$ in $\mathrm{Mat}_{|G:H|}(\mathcal{R}_{L[H]})$, and so in $\mathcal{D}_G$, is isomorphic to $\mathcal{R}_{L[H]}$ as $L[H]$-ring. By [Jai20, Proposition 2.7], the canonical map of $(\mathcal{R}_{L[H]}, L_\tau[G])$-bimodules

$$\alpha : \mathcal{R}_{L[H]} \otimes_{L[H]} L_\tau[G] \to \mathcal{D}_G \text{ is bijective.} \tag{3}$$

This is an analog of the isomorphism (1). In particular, $\dim_{\mathcal{R}_{L[H]}} \mathcal{D}_G = |G:H|$.

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Note that $L_\tau[G]$ is a free $L[H]$-module. Thus, every free resolution of an $L_\tau[G]$-module is also a free resolution of it viewed as an $L[H]$-module. Thus, (3) implies that for every $L_\tau[G]$-module M we have that

$$\operatorname{Tor}_k^{L[H]}(\mathcal{R}_{L[H]}, M) \cong \operatorname{Tor}_k^{L_\tau[G]}(D_G, M).$$

Therefore,

$$\beta_k^{L[H]}(M) = \dim_{R_{L[H]}} \operatorname{Tor}_k^{L[H]}(\mathcal{R}_{L[H]}, M) = |G:H| \dim_{\mathcal{D}_G} \operatorname{Tor}_k^{L_\tau[G]}(D_G, M),$$

and, thus, $|G:H|$ divides $\beta_k^{L[H]}(M)$ if it is finite. □

We say that an $L_\tau[G]$-module M is *acceptable* if there exists an $L_\tau[G]$-submodule M_0 of M such that:

(1) $\dim_L(M/M_0) < \infty$;
(2) $H = \ker(\tau) \leq C_G(M/M_0)$;
(3) $\beta_k^{L[H]}(M_0) = 0$ for every $k \geq 1$.

In this paper, acceptable $L_\tau[G]$-modules appear using the construction presented in the following lemma.

Lemma 6.3. *Let G be a group. Let M be a $\mathbb{Q}[G]$-module, and let M_0 be a submodule of M such that:*

(i) $\dim_{\mathbb{Q}} M/M_0 < \infty$;
(ii) $H = C_G(M/M_0)$ *is of finite index in G;*
(iii) $\beta_k^{\mathbb{Q}[G]}(M_0) = 0$ *for $k \geq 1$.*

Put $F = G/H = \{f_1, \dots, f_t\}$ and let $L = \mathbb{Q}(x_f | f \in F)$ be the field of rational functions on t variables over $\mathbb{Q}$. Define $\tau : G \to \operatorname{Aut}(L)$ via the formula

$$\tau(g)(p(x_{f_1}, \dots, x_{f_t})) = p(x_{gf_1}, \dots, x_{gf_t}), \quad p(x_{f_1}, \dots, x_{f_t}) \in L.$$

Put $\widetilde{M} = L_\tau[G] \otimes_{\mathbb{Q}[G]} M$. Then $\widetilde{M}$ is an acceptable $L_\tau[G]$-module. Moreover, if $\widetilde{M_0} = L_\tau[G] \otimes_{\mathbb{Q}[G]} M_0$, then:

(1) $\dim_L(\widetilde{M}/\widetilde{M_0})$ *is finite;*
(2) $H \leq C_G(\widetilde{M}/\widetilde{M_0})$;
(3) $\beta_k^{L[H]}(\widetilde{M_0}) = 0$ *for every $k \geq 1$.*

Proof. (1) As $L_\tau[G]$ is a flat $\mathbb{Q}[G]$-module, we obtain that $\widetilde{M_0} \cong L_\tau[G] \otimes_{\mathbb{Q}[G]} M_0$ and $\widetilde{M}/\widetilde{M_0} \cong L_\tau[G] \otimes_{\mathbb{Q}[G]} (M/M_0)$. In particular, $\dim_L(\widetilde{M}/\widetilde{M_0}) = \dim_{\mathbb{Q}}(M/M_0) < \infty$.

(2) As $H = \ker \tau$, $\ker \tau$ acts trivially on $\widetilde{M}/\widetilde{M_0}$.

(3) Observe that $\widetilde{M_0}$ as an $L[H]$ module is isomorphic to $L[H] \otimes_{\mathbb{Q}[H]} M_0$. Hence,

$$\beta_k^{L[H]}(\widetilde{M_0}) = \beta_k^{L[H]}(L[H] \otimes_{\mathbb{Q}[H]} M_0) = \beta_k^{\mathbb{Q}[H]}(M_0) = |G:H| \beta_k^{\mathbb{Q}[G]}(M_0) = 0. \quad \square$$

Proposition 6.4. *Let G be a torsion-free virtually limit group and let L be a subfield of $\mathbb{C}$. Let $\tau : G \to \operatorname{Aut}(L)$ be a homomorphism with finite image. Put $H = \ker \tau$. Let M be an acceptable $L_\tau[G]$-module. Then there exists an $L[H]$-submodule M' of M such that*

$$\beta_1^{L[H]}(M') = 0, \quad \dim_L M/M' \leq \frac{\beta_1^{L[H]}(M)}{|G:H|} \quad \text{and} \quad H \leq C_G(M/M').$$

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Proof. Let M_0 be from the definition of an acceptable $L_\tau[G]$-module. By induction on $\dim_L N/M_0$ we prove that for every $L_\tau[G]$-submodule N of M, satisfying $M_0 \le N$, there exists an $L[H]$-submodule $M_0 \le N' \le N$, such that

$$\beta_1^{L[H]}(N') = 0 \quad \text{and} \quad \dim_L N/N' \le \frac{\beta_1^{L[H]}(N)}{|G:H|}.$$

The base of induction, when $N = M_0$, is clear, because $\beta_1^{L[H]}(M_0) = 0$. Assume now that the proposition holds if $\dim_L N/M_0 < n$ and let us prove it in the case where $\dim_L N/M_0 = n$.

Let N_1 be a maximal $L_\tau[G]$-submodule of N that contains M_0. By Lemma 6.1, $N/N_1 \cong L$. Then, because $\dim_L N_1/M_0 < n$, there exists an $L[H]$-submodule N_1' of N_1, containing M_0, such that $\beta_1^{L[H]}(N_1') = 0$ and

$$\dim_L(N_1/N_1') \le \frac{\beta_1^{L[H]}(N_1)}{|G:H|}.$$

As G is a virtually limit group, by Proposition 3.3, $\beta_2^{L[H]}(L) = 0$. Therefore (see Proposition 8.1(3) for a more general statement),

$$\beta_1^{L[H]}(N_1) \le \beta_1^{L[H]}(N).$$

By Proposition 6.2, $\beta_1^{L[H]}(N)$ and $\beta_1^{L[H]}(N_1)$ are divisible by $|G:H|$. Hence, $\beta_1^{L[H]}(N) \ge \beta_1^{L[H]}(N_1) + |G:H|$ or $\beta_1^{L[H]}(N) = \beta_1^{L[H]}(N_1)$. In the first case, we simply take $N' = N_1'$ and we are done. Thus, let us assume that $\beta_1^{L[H]}(N) = \beta_1^{L[H]}(N_1)$.

Take $a \in N \backslash N_1$ and let N' be the $L[H]$-submodule generated by a and N_1'. As H acts trivially on N/M_0, $\dim_L N'/N_1' = 1$. Therefore, we have that

$$N_1 + N' = N \quad \text{and} \quad N_1 \cap N' = N_1'.$$

This leads to the following exact sequence of $L[H]$-modules:

$$0 \to N_1' \to N_1 \oplus N' \to N \to 0.$$

Using the long exact sequence for Tor, we obtain that

$$\beta_1^{L[H]}(N_1) + \beta_1^{L[H]}(N') = \beta_1^{L[H]}(N_1 \oplus N') \le \beta_1^{L[H]}(N) + \beta_1^{L[H]}(N_1') = \beta_1^{L[H]}(N_1).$$

Thus, $\beta_1^{L[H]}(N') = 0$. The construction of N' implies also that

$$\dim_L(N/N') = \dim_L(N_1/N_1') \le \frac{\beta_1^{L[H]}(N_1)}{|G:H|} = \frac{\beta_1^{L[H]}(N)}{|G:H|}. \qquad \square$$

7. The proofs of Theorems 1.2 and 1.1

In this section, we finish the proof of Theorem 1.2 and deduce from it Theorem 1.1.

Proof of Theorem 1.2. Let G be a surface group and K a subfield of $\mathbb{C}$. As

$$K[G/W] \cong \bigoplus_{x \in U\backslash G/W} K[U/U \cap xWx^{-1}]$$

as $K[U]$-modules we have

$$\beta_1^{K[U]}(K[G/W]) = \sum_{x \in U\backslash G/W} \overline{\chi}(U \cap xWx^{-1}).$$

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Let $M=\mathbb{Q}[G/W]$. Using Theorem 4.4, we obtain that there exists a normal subgroup H of G of finite index such that if M_0 denotes the kernel of the map $\mathbb{Q}[G/W]\to\mathbb{Q}[G/WH]$, then

$$\beta_1^{\mathbb{Q}[G]}(M_0)=0.$$

Define the ring $L_\tau[G]$ as in Lemma 6.3 and put

$$\widetilde{M}=L_\tau[G]\otimes_{\mathbb{Q}[G]}M.$$

Then, by Lemma 6.3, $\widetilde{M}$ is an acceptable $L_\tau[G]$-module. Thus, by Proposition 6.4 there exists an $L[H]$-submodule $\widetilde{M}'$ of $\widetilde{M}$ such that

$$\beta_1^{L[H]}(\widetilde{M}')=0,\quad \dim_L(\widetilde{M}/\widetilde{M}')\le\frac{\beta_1^{L[H]}(\widetilde{M})}{|G:H|}\quad\text{and}\quad H\le C_G(\widetilde{M}/\widetilde{M}').$$

Let us show that $\beta_1^{L[U\cap H]}(\widetilde{M}')=0$. If W is of finite index in G, then $\widetilde{M}'=\{0\}$, thus, we assume that W is of infinite index in G. Then $\mathbb{Q}[G/W]$ is of projective dimension 1 as a $\mathbb{Q}[G]$-module, and so $\widetilde{M}$ is of projective dimension 1 as an $L[H]$-module. By Proposition 5.1, $\beta_1^{L[U\cap H]}(\widetilde{M}')=0$. Therefore, we obtain

$$\begin{aligned}\beta_1^{\mathbb{Q}[U]}(\mathbb{Q}[G/W])&=\frac{\beta_1^{\mathbb{Q}[H\cap U]}(M)}{|U:H\cap U|}=\frac{\beta_1^{L[H\cap U]}(\widetilde{M})}{|U:H\cap U|}\\&\le\frac{\beta_1^{L[H\cap U]}(\widetilde{M}')+\beta_1^{L[H\cap U]}(\widetilde{M}/\widetilde{M}')}{|U:H\cap U|}\\&\le\frac{\beta_1^{L[H\cap U]}(L)\beta_1^{L[H]}(\widetilde{M})}{|U:H\cap U||G:H|}=\frac{\beta_1^{\mathbb{Q}[H\cap U]}(\mathbb{Q})\beta_1^{\mathbb{Q}[H]}(M)}{|U:H\cap U||G:H|}\\&=\beta_1^{\mathbb{Q}[U]}(\mathbb{Q})\beta_1^{\mathbb{Q}[G]}(M)=\overline{\chi}(U)\overline{\chi}(W).\end{aligned}$$

□

Proof of Theorem 1.1. First consider the case when U and W are of finite index. Let r be the number of the double (U,W)-cosets in G. Observe that $r\le|G:U|\le\overline{\chi}(U)$. Therefore, we have

$$\begin{aligned}\sum_{x\in U\backslash G/W}\overline{d}(U\cap xWx^{-1})&=r+\sum_{x\in U\backslash G/W}\overline{\chi}(U\cap xWx^{-1})\overset{\text{Theorem 1.2}}{\le}r+\overline{\chi}(U)\overline{\chi}(W)\\&=\overline{d}(U)\overline{d}(W)+r-1-\overline{\chi}(U)-\overline{\chi}(W)\le\overline{d}(U)\overline{d}(W).\end{aligned}$$

If U or W is of infinite index, then $U\cap xWx^{-1}$ is free. Thus, we obtain

$$\begin{aligned}\sum_{x\in U\backslash G/W}\overline{d}(U\cap xWx^{-1})&=\sum_{x\in U\backslash G/W}\overline{\chi}(U\cap xWx^{-1})\overset{\text{Theorem 1.2}}{\le}\overline{\chi}(U)\overline{\chi}(W)\\&\le\overline{d}(U)\overline{d}(W).\end{aligned}$$

□

8. A module theoretic reformulation of the geometric Hanna Neumann conjecture for limit groups

Let G be a group and let K be a field. Let N and M be two left $K[G]$-modules. Consider the tensor product $N\otimes_K M$. The diagonal action on $N\otimes_K M$,

$$g(n\otimes m)=(gn)\otimes(gm)\quad(g\in G,n\in N,m\in M),$$

defines on $N\otimes_K M$ a structure of a left $K[G]$-module.

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For every $k \geq 1$, we put

$$\beta_k^{K[G]}(N, M) = \beta_k^{K[G]}(N \otimes_K M).$$

This definition is different from that used in [Jai17]. In light of [Bro82, Proposition III.2.2] and the Lück approximation (Proposition 3.2) one sees that two definitions are closely related. However, we do not claim that these two definitions always define the same invariant. We are very grateful to Mark Shusterman who suggested this new definition to us.

In the following proposition, we collect the main properties of $\beta_k^{K[G]}(N, M)$.

Proposition 8.1. *Let G be a group and let K be a subfield of $\mathbb{C}$.*

(1) *Let N and M be left $K[G]$-modules. Then*

$$\beta_1^{K[G]}(N, M) = \beta_1^{K[G]}(M, N) \quad \text{and} \quad \beta_1^{K[G]}(N, K) = \beta_1^{K[G]}(N).$$

(2) *Let H be a subgroup of finite index in G. Let N and M be left $K[G]$-modules. Then*

$$\beta_k^{K[G]}(N, M) = \frac{1}{|G:H|}\beta_k^{K[H]}(N, M).$$

(3) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of left $K[G]$-modules. Assume that $\dim_K M_3 < \infty$ and $H = C_G(M_3)$ is of finite index in G. Then for any left $K[G]$-module N and any $k \geq 1$, we have*

$$\beta_k^{K[G]}(N, M_1) - (\dim_K M_3)\beta_{k+1}^{K[G]}(N) \leq \beta_k^{K[G]}(N, M_2) \leq \beta_k^{K[G]}(N, M_1) + (\dim_K M_3)\beta_k^{K[G]}(N).$$

Proof. (1) This follows directly from the definitions.

(2) Observe that $\mathcal{R}_{K[H]} \otimes_{K[H]} K[G]$ is isomorphic to $\mathcal{R}_{K[G]}$ as a right $K[G]$-module and $\dim_{\mathcal{R}_{K[G]}} = |G:H| \dim_{\mathcal{R}_{K[H]}}$. Let $L = N \otimes_K M$. Then we obtain that

$$\begin{aligned}\beta_k^{K[G]}(N, M) &= \beta_k^{K[G]}(L) = \dim_{\mathcal{R}_{K[G]}} \operatorname{Tor}_k^{K[G]}(\mathcal{R}_{K[G]}, L) \\ &= \dim_{\mathcal{R}_{K[G]}} \operatorname{Tor}_k^{K[H]}(\mathcal{R}_{K[H]}, L) = \frac{1}{|G:H|}\dim_{\mathcal{R}_{K[H]}} \operatorname{Tor}_k^{K[H]}(\mathcal{R}_{K[H]}, L) \\ &= \frac{1}{|G:H|}\beta_k^{K[H]}(L) = \frac{1}{|G:H|}\beta_k^{K[H]}(N, M).\end{aligned}$$

(3) From the long exact sequence for the Tor functor, corresponding to the exact sequence

$$0 \to N \otimes_K M_1 \to N \otimes_K M_2 \to N \otimes_K M_3 \to 0,$$

it follows that

$$\beta_k^{K[G]}(N, M_1) - \beta_{k+1}^{K[G]}(N, M_3)) \leq \beta_k^{K[G]}(N, M_2) \leq \beta_k^{K[G]}(N, M_1) + \beta_k^{K[G]}(N, M_3).$$

Observe that

$$\begin{aligned}\beta_k^{K[G]}(N, M_3) &= \frac{1}{|G:H|}\beta_k^{K[H]}(N, M_3) \\ &= \frac{\dim_K M_3}{|G:H|}\beta_k^{K[H]}(N, K) = (\dim_K M_3)\beta_k^{K[G]}(N).\end{aligned}$$

This finishes the proof of part (3). ∎

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In the following proposition, we give an algebraic reinterpretation of the sum which appears in Conjecture 1.

Proposition 8.2. *Let G be a limit group and let K a subfield of $\mathbb{C}$. Let U and W be two finitely generated subgroups of G. Then*

$$\beta_1^{K[G]}(K[G/U], K[G/W]) = \sum_{x \in U\backslash G/W} \overline{\chi}(U \cap xWx^{-1}).$$

Proof. First let us show that

$$\beta_1^{K[G]}(K[G/U]) = \overline{\chi}(U). \tag{4}$$

Indeed, Proposition 3.4 implies that $\beta_1^{K[G]}(K[G/U]) = \beta_1^{K[U]}(K)$. Now, from Proposition 3.3, it follows that $\beta_1^{K[U]}(K) = \overline{\chi}(U)$.

Observe that

$$K[G/U] \otimes_K K[G/W] \cong \bigoplus_{x \in U\backslash G/W} K[G/(U \cap xWx^{-1})]$$

as $K[G]$-modules. Therefore, we obtain

$$\begin{aligned} \beta_1^{K[G]}(K[G/U], K[G/W]) &= \sum_{x \in U\backslash G/W} \beta_1^{K[G]}(K[G/(U \cap xWx^{-1})]) \\ &\overset{\text{by (4)}}{=} \sum_{x \in U\backslash G/W} \overline{\chi}(U \cap xWx^{-1}). \end{aligned}$$ □

Corollary 8.3. *Conjecture 1 for a limit group G is equivalent to the following statement: for any finitely generated subgroups U and W of G,*

$$\beta_1^{\mathbb{Q}[G]}(\mathbb{Q}[G/U], \mathbb{Q}[G/W]) \le \beta_1^{\mathbb{Q}[G]}(\mathbb{Q}[G/U]) \cdot \beta_1^{\mathbb{Q}[G]}(\mathbb{Q}[G/W]).$$

9. The strengthened Howson property for hyperbolic limit groups

The Howson property for limit groups was proved by Dahmani [Dah03]. In the case of hyperbolic limit group, we can prove the strengthened Howson property (see the statement of Theorem 9.1). In fact, the strengthened Howson property holds for the family of stable subgroups of a given group.

Let $f\colon \mathbb{R}_{\ge 1} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a function. Let $H \le G$ be finitely generated groups, and fix some word metrics.

A quasi-geodesic γ in G is f-stable if for any (λ, ϵ)-quasi-geodesic η with endpoints on γ, we have η is contained in the $f(\lambda, \epsilon)$-neighborhood of γ. The *subgroup H is f-stable* in G if the inclusion of H is a quasi-isometrically embedding (with respect to the word metrics) and the image of any geodesic in H is an f-stable quasi-geodesic in G. A subgroup H is *stable* if it is f-stable for some f as previously.

Examples of stable subgroups include quasi-convex subgroups of hyperbolic groups, subgroups quasi-isometrically embedded in the cone-off graph of relatively hyperbolic groups and convex cocompact subgroups of mapping class groups or $\mathrm{Out}(F_n)$. Note that any stable subgroup must be word hyperbolic and that being a stable subgroup is a property preserved under conjugation. See [AMST19] and references therein for details.

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Theorem 9.1. *Let U and W be two stable subgroups of a finitely generated group G. Then for almost all $x \in U\backslash G/W$, the subgroup $U \cap xWx^{-1}$ is finite. In particular, if U is torsion-free, the sum $\sum_{x\in U\backslash G/W} d(U \cap xWx^{-1})$ is finite.*

Proof. The theorem follows from [AMST19, Lemma 4.2] which states that, under the hypothesis of our theorem, there is a constant $D \geq 0$ such that whenever $|U^{g_1} \cap W^{g_2}| = \infty$ for some $g_1, g_2 \in G$ then the cosets g_1U and g_2W have intersecting D-neighborhoods.

Suppose that $U \cap W^g$ is infinite. Then gW intersects the D-neighborhood of U. By multiplying g by an element of U on the left, we can assume that gW is a distance at most D from the identity. Thus, by multiplying g by an element of W on the right, we can assume that the length of g is less than D. Therefore, for all $UxW \in U\backslash G/W$, having no representative in the ball of radius D and center the identity, the subgroup $U \cap xWx^{-1}$ is finite.

By [AMST19, Lemma 3.1], the intersection of stable subgroups is stable and, hence, finitely generated. Therefore, the 'in particular' claim follows. □

As hyperbolic limit groups are torsion-free and every finitely generated subgroup is quasi-convex (and, hence, stable), we obtain the following corollary.

Corollary 9.2. *Let G be a hyperbolic limit group and let U and W be two finitely generated subgroups of G. Then the sum $\sum_{x\in U\backslash G/W} d(U \cap xWx^{-1})$ is finite.*

The strengthened Howson property is not true for limit groups that are not hyperbolic. A simple example can be constructed on abelian groups, because all conjugates of a subgroup are equal, regardless of conjugating by representatives of different cosets. This is essentially the only reason for which the strengthened Howson property fails for limit groups. For limit groups, or more generally relatively hyperbolic groups, one has a similar statement to Theorem 9.1 if one restricts to non-parabolic intersections.

Let G be a group and $\mathbb{H} = \{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups. Let $\mathcal{H}$ be the disjoint union $\sqcup_{\lambda\in\Lambda} H_\lambda$. A group G is *hyperbolic relative to a family of subgroups* $\mathbb{H}$ if it admits a finite relative presentation with linear relative isoperimetric inequality. The group G has a *finite relative presentation with respect to* $\mathbb{H}$ if G is generated by a finite set X together with the collection of subgroups in $\mathbb{H}$ and it is subject to a finite number of relations involving elements of X and elements of $\mathbb{H}$, formally

$$G = (\langle X \mid \rangle * (*_{\lambda\in\Lambda} H_\lambda))/\langle\langle R\rangle\rangle,$$

with X and R finite. Here $\langle\langle R\rangle\rangle$ denotes the normal closure of R in $\langle X \mid \rangle * (*_{\lambda\in\Lambda} H_\lambda)$. Let $\mathcal{R}$ be all the words over $\mathcal{H}$ that represent trivial elements. The relative presentation has *linear isoperimetric inequality*, if there is a constant C such that for every $w \in (X \cup \mathcal{H})^*$ representing 1 in G, then w is equal in $\langle X \mid \rangle * (*_{\lambda\in\Lambda} H_\lambda)$ to a product of conjugates of elements of $R \cup \mathcal{R}$ using at most $C\ell(w) + C$ conjugates of R. Here $\ell(w)$ denotes the length of w.

An important property that will be used is that the Cayley graph of G with respect to $X \cup \mathcal{H}$, denoted $\Gamma(G, X \cup \mathcal{H})$, is hyperbolic.

A subgroup $U \leqslant G$ is *relatively quasi-convex* if U is a quasi-convex set in $\Gamma(G, X \cup \mathcal{H})$. Being relatively quasi-convex is independent of the generating set X.

Conjugates of elements of $\mathbb{H}$ are called *parabolic*. Non-parabolic infinite-order elements are called *loxodromic* and, indeed, they act as a loxodromic isometry of $\Gamma(G, X \cup \mathcal{H})$. Note that parabolic subgroups are bounded subsets of $\Gamma(G, X \cup \mathcal{H})$ and, therefore, they are relatively quasi-convex.

Connecting with the previous notion of stability, if a subgroup of a relatively hyperbolic group is quasi-convex and has no non-trivial parabolic elements, then it is stable.

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Lemma 9.3. *Let U and W be two relatively quasi-convex subgroups of a finitely generated, relatively hyperbolic group G. Then for almost all $x \in U\backslash G/W$, the subgroup $U \cap xWx^{-1}$ does not contain a loxodromic element.*

Proof. The key arguments of this proof are contained in [HW09, Lemma 8.4] whose proof we follow closely. We assume that X is a finite generating set of G.

Suppose that $U \cap gWg^{-1}$ does contain a loxodromic element f. As f is a loxodromic isometry of $\Gamma = \Gamma(G, X \cup \mathcal{H})$, the subgroup $\langle f \rangle$ has two different accumulation points $\{f^{\infty}, f^{-\infty}\} \in \partial\Gamma$, the Gromov boundary of Γ. As $f \in gWg^{-1}$, we have that $\langle f \rangle g \in gW$ and note that $\langle f \rangle g$ has also $\{f^{\infty}, f^{-\infty}\}$ as accumulation points because it is at finite Hausdorff X-distance from $\langle f \rangle$. Thus, the accumulation points of $\langle g^{-1}fg \rangle \leqslant W$ in $\partial\Gamma$ are $\{g^{-1}f^{\infty}, g^{-1}f^{-\infty}\}$.

By [HW09, Lemma 8.3] there are bi-infinite geodesics γ_U and γ_W in Γ from $f^{-\infty}$ to f^{∞} and $g^{-1}f^{-\infty}$ to $g^{-1}f^{\infty}$, respectively, and they are at a finite Hausdorff X-distance from $\langle f \rangle$ and $\langle g^{-1}fg \rangle$, respectively. Finally, the vertices of γ_U lie in the σ-neighborhood of U and the vertices of γ_W lie on the σ-neighborhood of W, where σ is the quasi-convexity constant of U and W.

Note that $g\gamma_W$ has the same end points at infinite as γ_U. Now, by [HW09, Lemma 8.2], there is a constant L, only depending on Γ, such that the vertices of the geodesics γ_U and $g\gamma_W$ are at most L Hausdorff X-distance of each other. Thus, $\langle f \rangle$ and $\langle f \rangle g$ are at Hausdorff X-distance at most $L + 2\sigma$. This implies that U and gW have intersecting $D = L + 2\sigma$ neighborhoods. By multiplying g by an element of U on the left, we can assume that gW is a distance at most D from the identity. Thus, by multiplying g by an element of W on the right, we can assume that the length of g is less than D. Therefore, for all $UxW \in U\backslash G/W$, having no representative in the X-ball of radius D and center the identity, the subgroup $U \cap xWx^{-1}$ does not contain loxodromic elements. □

Theorem 9.4. *Let G be a limit group and let U and W be two finitely generated subgroups of G. Then for almost all $x \in U\backslash G/W$, the subgroup $U \cap xWx^{-1}$ is abelian. In particular, the sum $\sum_{x \in U\backslash G/W} \overline{\chi}(U \cap xWx^{-1})$ is finite.*

Proof. The case where G is hyperbolic follows from Theorem 9.2.

If G is a non-hyperbolic limit group, then G is finitely generated and hyperbolic relative to the family $\mathcal{H}$ of maximal abelian non-cyclic subgroups (see [Dah03, Theorem 4.5]).

Recall that by [Dah03, Proposition 4.6], finitely generated subgroups of limit groups are relatively quasi-convex. In particular, U and W are relatively quasi-convex. By Lemma 9.3, for almost all $x \in U\backslash G/W$ the subgroup $U \cap xWx^{-1}$ does not contain a loxodromic element. As limit groups are torsion-free, this implies that for almost all $x \in U\backslash G/W$, the subgroup $U \cap xWx^{-1}$ is contained in a parabolic subgroup and, hence, it is abelian. Moreover, because limit groups have the Howson property [Dah03, Theorem 4.7], each $U \cap xWx^{-1}$ is finitely generated and, hence, a limit group. Hence, $\overline{\chi}(U \cap xWx^{-1})$ is well-defined and zero when $U \cap xWx^{-1}$ is abelian. □

Theorem 9.1 implies also that a quasi-convex subgroup of a hyperbolic virtually compact special group is virtually malnormal.

Corollary 9.5. *Let G be a hyperbolic virtually compact special group and H a quasi-convex subgroup of G. Then H is virtually malnormal and a virtual retract.*

Proof. By Theorem 2.1, H is a virtual retract.

As G is virtually a subgroup of a RAAG, G is virtually torsion-free and residually finite. By passing to a finite index subgroup of G, we can assume that H is torsion-free. By Theorem 9.1, there is only a finite number of double cosets HxH such that $H^x \cap H \neq \{1\}$. By Theorem 2.2, each of these double cosets is separable. As a finite collection of disjoint separable sets is separable,

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there exists a normal subgroup N of G of finite index that separates these double cosets. Hence, H is malnormal in HN. □

10. The Wilson–Zalesskii property in virtually compact special hyperbolic groups

Let G be a residually finite group and let U and W two subgroups of G. We say that U and W satisfy the *Wilson–Zalesskii property* if

$$\overline{U} \cap \overline{W} = \overline{U \cap W}.$$

Here $\overline{U}$ denotes the closure of U in the profinite completion $\widehat{G}$ of G. When G is virtually free, the Wilson–Zalesskii property for G was proved by Wilson and Zalesskii in [WZ98, Proposition 2.4] for every pair of finitely generated subgroups (see also [RZ96, Lemma 3.6] for the case when U and W are cyclic). In this section, we show that a pair of quasi-convex subgroups of a virtually compact special hyperbolic group satisfies the Wilson–Zalesskii property. Our argument essentially follows the original argument of Wilson and Zalesskii. It uses a beautiful idea of double trick that goes back to the work of Long and Niblo [LN91]. Let us start with the following useful lemma.

Lemma 10.1. *Let G be a residually finite group and let U and W be two finitely generated subgroups of G. Let H be a subgroup of $\widehat{G}$ of finite index. Assume that*

$$\overline{(U \cap H)(W \cap H)} \cap G = (U \cap H)(W \cap H) \quad \text{and} \quad \overline{U \cap H} \cap \overline{W \cap H} = \overline{U \cap W \cap H}.$$

Then $\overline{U} \cap \overline{W} = \overline{U \cap W}$.

Proof. Note that one always has that $\overline{U \cap W} \subseteq \overline{U} \cap \overline{W}$. Let $v \in \overline{U} \cap \overline{W}$. Then we can write $v = u_1 u_2 = w_1 w_2$, where $u_1 \in U$, $u_2 \in \overline{U \cap H}$, $w_1 \in W$ and $w_2 \in \overline{W \cap H}$. Thus,

$$k = u_1^{-1} w_1 = u_2 w_2^{-1} \in \overline{(U \cap H)(W \cap H)} \cap G = (U \cap H)(W \cap H).$$

Therefore, there are $u_3 \in U \cap H$ and $w_3 \in W \cap H$ such that $k = u_3 w_3^{-1}$. Hence,

$$u_1 u_3 = w_1 w_3 \in U \cap W \quad \text{and} \quad u_3^{-1} u_2 = w_3^{-1} w_2 \in \overline{U \cap H} \cap \overline{W \cap H} = \overline{U \cap W \cap H}.$$

Thus,

$$v = u_1 u_2 = (u_1 u_3)(u_3^{-1} u_2) \in \overline{U \cap W}.$$

□

Lemma 10.2. *Let G be a hyperbolic virtually compact special group, U a malnormal retract of G and W a quasi-convex subgroup of G. Put $K = W \cap U$. Then $G *_U G$ is hyperbolic virtually compact special and $W *_K W$ is quasi-convex in $G *_U G$.*

Proof. As U is a retract in G, U is quasi-isometrically embedded in G and, hence, quasi-convex. By [Git96, Lemma 5.2], $G *_U G$ is hyperbolic and by [HW15, Theorem A], it is virtually compact special.

For the sake of the proof, let G' denote a copy of G and U', W' and K' the corresponding copies of U, W and K in G'. Let $B = G *_{U=U'} G'$ and $A = \langle W, W' \rangle \leqslant B$. From Bass–Serre theory (or normal forms on amalgamated free products), it follows easily that the natural map $W *_{K=K'} W' \to \langle W, W' \rangle \leqslant G *_{U=U'} G'$ is injective and, hence, an isomorphism.

Fix finite generating sets Y and Y' of G and G', respectively, and let $X = Y \cup Y'$ a finite generating set of B. In [Git96] terminology, a path p on the Cayley graph of $\Gamma(B, X)$ is in normal

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form if p is the concatenation of subpaths

$$p \equiv p_1 p_2 \dots p_n,$$

such that the label of each p_i is the label of a geodesic word either on $\Gamma(G, Y)$ or on $\Gamma(G', Y')$, no two labels of consecutive subpaths p_i and p_{i+1} lie in the same set Y or Y' and, finally, no label represents an element of $U = U'$ except maybe the label of p_1. Now, [Git96, Lemma 4.1] in view of [Git96, Lemma 5.2] claims that there is a constant C such that any geodesic path in $\Gamma(B, X)$ is in the C-neighborhood of any path in normal form with the same endpoints. From this, it easily follows that A is quasi-convex in B. Indeed, let q be any geodesic path in $\Gamma(B, X)$ with endpoints in A. By the equivariance of the action, we can assume that q goes from 1 to $a \in A$. As $A \cong W *_K W'$, there is a path p in normal form $p \equiv p_1 \dots p_n$ from 1 to a where the labels of each p_i represents an element of W or W'. By the mentioned result of Gitik, q is in the C-neighborhood of p. Let σ denote the quasi-convexity constant of W and W' as subspaces of $\Gamma(G, Y)$ and $\Gamma(G', Y')$, respectively. We claim that each p_i is in the σ-neighborhood of A. Indeed, let $a_0 = 1$ and $a_i \in A$ be the element represented by the label of $p_1 p_2 \dots p_{i-1}$. Suppose that the label of p_i is a word in Y. The case where the label of p_i is a word in Y' is analogous. Then $a_i^{-1} p_i$ is a geodesic path in $\Gamma(G, Y)$ with endpoints in W, and therefore lies in the σ-neighborhood of W. Thus, p_i lies in the σ-neighborhood of $a_i W \subseteq A$ as claimed. Therefore, p is in the σ-neighborhood of A. □

Theorem 10.3. *Let G be a virtually compact special hyperbolic group and U and W quasi-convex subgroups. Then U and W satisfy the Wilson–Zalesskii property.*

Proof. Put $K = U \cap W$. We want to show that $\overline{U} \cap \overline{W} = \overline{K}$. By Corollary 9.5, there exists a subgroup H of G of finite index, containing U, such that U is retract and malnormal in H. Moreover, by Lemma 10.1, it is enough to prove that $\overline{(U \cap H)(W \cap H)} \cap G = (U \cap H)(W \cap H)$ and $\overline{U} \cap \overline{W \cap H} = \overline{K \cap H}$. The first condition follows from Theorem 2.2. Thus, without loss of generality, we may assume that $H = G$.

Let $G' = \{g' : g \in G\}$ be a group isomorphic to G such that the map $g \mapsto g'$ is an isomorphism between G and G'. Put $F = G *_U G'$. In the following, we identify U and U' in F. Let P be a subgroup of F generated by W and W'. As $G \cap G' = U$, we obtain that $W \cap W' = W \cap W' \cap U = K$, and so $P \cong W *_K W'$.

Observe that G is a retract of F. Hence, the closure $\overline{G}$ of G in $\widehat{F}$ is isomorphic to $\widehat{G}$. Thus, the closures of U, W and K in $\widehat{F}$ are isomorphic to the closures of U, W and K in $\widehat{G}$, respectively. In particular, $\widehat{F}$ is isomorphic to the profinite amalgamated product $\overline{G} \widehat{*}_{\overline{U}} \overline{G'}$, and so $\overline{G} \cap \overline{G'} = \overline{U}$.

As W is quasi-convex in G, it is virtually a retract (Theorem 2.1). Hence, $\overline{W} = \widehat{W}$. Thus, the closures of K in $\widehat{W}$ and $\widehat{G}$ are isomorphic. Therefore, $\widehat{P}$ is isomorphic to the profinite amalgamated product $\overline{W} \widehat{*}_{\overline{K}} \overline{W'}$.

By Lemma 10.2, P is quasi-convex in F and F is virtually compact special. Therefore, by Theorem 2.1, P is virtually retract in F, and so the closure $\overline{P}$ of P in $\widehat{F}$ is isomorphic its profinite completion $\overline{W} \widehat{*}_{\overline{K}} \overline{W'}$. Hence, $\overline{W} \cap \overline{W'} = \overline{K}$. On the other hand,

$$\overline{W} \cap \overline{W'} = \overline{U} \cap \overline{W} \cap \overline{W'} = \overline{U} \cap \overline{W}.$$

We conclude that $\overline{U} \cap \overline{W} = \overline{K}$. □

Corollary 10.4. *Let G be a hyperbolic limit group and let U and W two finitely generated subgroups of G. Then for every normal subgroup T of G of finite index, there exists a finite-index*

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normal subgroup H of G such that

$$UH \cap WH \leq (U \cap W)T.$$

Proof. Assume that for every normal subgroup H of G of finite index there exists $x_H \in (UH \cap WH) \setminus (U \cap W)T$. Let $G > H_1 > H_2 > \cdots$ be a chain of normal subgroups of G of finite index that form a base of neighbors of 1 in the profinite topology of G. Without loss of generality, we may assume that there exists

$$x = \lim_{i \to \infty} x_{H_i} \in \widehat{G}.$$

Clearly, $x \in \overline{U} \cap \overline{W}$. By Theorem 10.3, U and W satisfy the Wilson–Zalesskii property. Hence, $x \in \overline{U \cap W}$. Therefore, there exists n such that if $i \geq n$, $x_{H_i} \in (U \cap W)T$. We have arrived at a contradiction. □

11. Constructions of submodules with trivial $\beta_1^{K[G]}$

In this section, we assume that G is an L^2-Hall hyperbolic limit group. For example, by Theorem 4.4, G can be the fundamental group of a closed orientable surface. Let K be a subfield of $\mathbb{C}$. Let W be a finitely generated subgroup of G. Then, because G is L^2-Hall, there exists a normal subgroup H of G of finite index such that W is L^2-independent in WH. Let N_0 be the kernel of the map $K[G/W] \to K[G/HW]$, then by Proposition 4.2, $\beta_1^{K[G]}(N_0) = 0$.

Now, let U be another finitely generated subgroup of G. The main result of this section is the proposition which generalizes the result described in the previous paragraph (it is a particular case corresponding to $U = G$).

PROPOSITION 11.1. *Let G be an L^2-Hall hyperbolic limit group and U, W two finitely generated subgroups of G. There exists a normal subgroup H of G of finite index such that if L_0 denotes the kernel of the map*

$$K[G/U] \otimes_K K[G/W] \to K[G/U] \otimes_K K[G/WH],$$

then $\beta_1^{K[G]}(L_0) = 0$.

Remark. Let N_0 denote the kernel of the map $K[G/W] \to K[G/WH]$. Then

$$L_0 \cong K[G/U] \otimes_K N_0.$$

Proof. By Corollary 9.2, there are only finitely many double cosets UxW such that $U \cap xWx^{-1}$ is non-trivial. By a result of Minasyan (Theorem 2.2), each of these double cosets is separable. It is easy to see that a finite family of disjoint separable sets is separable. Therefore, there exists a normal subgroup H_0 of G of finite index that separates these cosets.

Let UsW be a double coset with $U \cap sWs^{-1}$ non-trivial. Recall that

$$K[G](1U \otimes sW) \cong K[G/(U \cap sWs^{-1})]$$

as $K[G]$-modules. Let T_s be a normal subgroup of G of finite index such that $U \cap sWs^{-1}$ is L^2-independent in $(U \cap sWs^{-1})T_s$. By Corollary 10.4, there exists a normal subgroup H_s of G of finite index such that $U \cap sWH_s s^{-1} \leq (U \cap sWs^{-1})T_s$.

Now we put $H = H_0 \cap (\cap_s H_s)$, where the intersection is over double cosets UsW with $U \cap sWs^{-1}$ non-trivial.

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Let S be a set of representatives of the (U, HW)-double cosets and extend it to $\widetilde{S}$, $S \subset \widetilde{S}$, a set of representatives of the (U, W)-cosets. Observe that if $x \in \widetilde{S} \setminus S$, then $U \cap xWx^{-1}$ is trivial. Define the map $\pi \colon \widetilde{S} \to S$ in such way that $U\pi(x)WH = UxWH$. Then we obtain the following decomposition of $K[G/U] \otimes_K K[G/W]$:

$$K[G/U] \otimes_K K[G/W] = (\oplus_{s\in S} K[G](1U \otimes sW)) \oplus (\oplus_{x\in\widetilde{S}\setminus S} K[G](1U \otimes (xW - \pi(x)W))). \quad (5)$$

Moreover, if $s \in S$ and $x \in \widetilde{S} \setminus S$, then

$$K[G](1U \otimes sW) \cong K[G/(U \cap sWs^{-1})] \quad \text{and} \quad K[G](1U \otimes (xW - \pi(x)W)) \cong K[G].$$

Observe also that

$$K[G/U] \otimes K[G/WH] = \oplus_{s\in S} K[G](1U \otimes sWH).$$

Moreover, $K[G](1U \otimes sWH) \cong K[G/U \cap sWHs^{-1}]$.

For each $s \in S$ we denote by I_s the kernel of the map

$$K[G/U \cap sWs^{-1}] \to K[G/U \cap sWHs^{-1}].$$

Then, from the decomposition (5), we obtain that

$$L_0 \cong (\oplus_{s\in S} I_s) \oplus (\oplus_{x\in\widetilde{S}\setminus S} K[G](1U \otimes (xW - \pi(x)W))).$$

If $x \in \widetilde{S} \setminus S$, then $K[G](1U \otimes (xW - \pi(x)W)) \cong K[G]$. Hence,

$$\beta_1^{K[G]}(K[G](1U \otimes (xW - \pi(x)W))) = 0.$$

If $s \in S$, then I_s is a submodule of the kernel J_s of the map

$$K[G/U \cap sWs^{-1}] \to K[G/(U \cap sWs^{-1})T_s].$$

As $U \cap sWs^{-1}$ is L^2-independent in $(U \cap sWs^{-1})T_s$, $\beta_1^{K[G]}(J_s) = 0$. Hence, by Proposition 8.1(3) (applied with $N = K$, $M_1 = I_s$ and $M_2 = J_s$), $\beta_1^{K[G]}(I_s) = 0$. Thus, $\beta_1^{K[G]}(L_0) = 0$. □

12. The geometric Hanna Neumann conjecture

12.1 The proof of Theorem 1.3

Let G be a hyperbolic limit group, and assume that G is L^2-Hall.

Let $N = \mathbb{Q}[G/U]$ and $M = \mathbb{Q}[G/W]$. Using Proposition 11.1, we obtain that there exists a normal subgroup H of G of finite index such that:

(1) if N_0 denotes the kernel of the map $\mathbb{Q}[G/U] \to \mathbb{Q}[G/UH]$, then

$$\beta_1^{\mathbb{Q}[G]}(N_0) = 0;$$

(2) if M_0 denotes the kernel of the map $\mathbb{Q}[G/W] \to \mathbb{Q}[G/WH]$, then

$$\beta_1^{\mathbb{Q}[G]}(N, M_0) = 0.$$

Observe that $H \le C_G(M/M_0) \cap C_G(N/N_0)$.

Define the ring $L_\tau[G]$ as in Lemma 6.3 and put

$$\widetilde{M} = L_\tau[G] \otimes_{\mathbb{Q}[G]} M \quad \text{and} \quad \widetilde{N} = L_\tau[G] \otimes_{\mathbb{Q}[G]} N.$$

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Then, by Lemma 6.3, $\widetilde{N}$ is an acceptable $L_\tau[G]$-module. Thus, by Proposition 6.4 there exists an $L[H]$-submodule $\widetilde{N}'$ of $\widetilde{N}$ such that

$$\beta_1^{L[H]}(\widetilde{N}')=0,\quad \dim_L(\widetilde{N}/\widetilde{N}')\le \frac{\beta_1^{L[H]}(\widetilde{N})}{|G:H|}\quad\text{and}\quad H\le C_G(\widetilde{N}/\widetilde{N}').$$

Let us show first that $\beta_1^{L[H]}(\widetilde{N}',\widetilde{M})=0$:

$$\begin{aligned}\beta_1^{L[H]}(\widetilde{N}',\widetilde{M}) &\overset{\text{Proposition 8.1(3)}}{\le} \beta_1^{L[H]}(\widetilde{N}',\widetilde{M_0})\\ &\overset{\text{Proposition 8.1(3)}}{\le} \beta_1^{L[H]}(\widetilde{N},\widetilde{M_0})=\beta_1^{\mathbb{Q}[H]}(N,M_0)=\frac{\beta_1^{\mathbb{Q}[G]}(N,M_0)}{|G:H|}=0.\end{aligned}$$

Thus,

$$\begin{aligned}\beta_1^{L[H]}(\widetilde{N},\widetilde{M}) &\overset{\text{Proposition 8.1(3)}}{\le} \dim_L(\widetilde{N}/\widetilde{N}')\beta_1^{L[H]}(\widetilde{M})+\beta_1^{L[H]}(\widetilde{N}',\widetilde{M})\\ &= \dim_L(\widetilde{N}/\widetilde{N}')\beta_1^{L[H]}(\widetilde{M})\le \frac{\beta_1^{L[H]}(\widetilde{N})\beta_1^{L[H]}(\widetilde{M})}{|G:H|}.\end{aligned}$$

Therefore,

$$\begin{aligned}\beta_1^{\mathbb{Q}[G]}(N,M) &\overset{\text{Proposition 8.1(2)}}{=} \frac{\beta_1^{\mathbb{Q}[H]}(N,M)}{|G:H|}=\frac{\beta_1^{L[H]}(\widetilde{N},\widetilde{M})}{|G:H|}\\ &\le \frac{\beta_1^{L[H]}(\widetilde{N})\beta_1^{L[H]}(\widetilde{M})}{|G:H|^2}=\frac{\beta_1^{\mathbb{Q}[H]}(N)\beta_1^{\mathbb{Q}[H]}(M)}{|G:H|^2}=\beta_1^{\mathbb{Q}[G]}(N)\beta_1^{\mathbb{Q}[G]}(M).\end{aligned}$$

By Corollary 8.3, we are done.

12.2 The geometric Hanna Neumann conjecture beyond the surface groups

As we have shown in order to settle the case of hyperbolic limit groups of Conjecture 1, it is enough to prove the L^2-Hall property for these groups. We strongly believe that L^2-Hall property holds for arbitrary limit groups.

In the case of limit groups, the generalized Howson property holds if one replaces d by $\overline{\chi}$ (Theorem 9.4) because the reduced Euler characteristic for finitely generated abelian groups is zero. However, if G is a limit group, we do not know whether the double cosets with respect to two finitely generated subgroups are separated (in the hyperbolic case this follows from [Mina06]) and we do not know whether the Wilson–Zalesskii property holds for every pair of finitely generated subgroups of a limit group[1].

As quasi-convex subgroups of hyperbolic virtually compact special groups satisfy the Howson property we may ask whether they satisfy also the conclusion of the geometric Hanna Neumann conjecture. This is not true. Fix a natural number d. In a 2-generated free group F, we can find a finitely generated malnormal subgroup H with $d(H)=d$. Then, by [Git96, Corollary 5.3] and [HW15, Corollary B], $G=F*_H F$ is hyperbolic and virtually special compact. The two copies of F are quasi-convex in G, and their intersection has rank d.

In addition, we want to mention that a hyperbolic virtually special compact group is not always L^2-Hall with respect to a quasi-convex subgroup. For example, take a free non-abelian

[1] After this paper was accepted, Minayan [Mina22] proved that the Wilson–Zalesskii property holds for limit groups in general.

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retract H in the fundamental group of a compact hyperbolic 3-manifold group G. Then the first L^2-Betti number of G and, thus, of all its subgroups of finite index are equal to zero. However, $\beta_1^{(2)}(H) > 0$.

Acknowledgements

This paper is partially supported by the Spanish MINECO through the grants MTM2017-82690-P and PID2020-114032GB-I00, and the 'Severo Ochoa' programs for Centres of Excellence SEV-2015-0554 and CEX2019-000904-S4.

We would like to thank Mark Shusterman for a very useful comment about the definition of $\beta_1^{K[G]}(N, M)$, Pavel Zalesskii for bringing the paper [WZ98] to our attention and Enric Ventura for explaining to us the proof of Corollary 1.5. We are grateful to Henrique Souza and an anonymous referee for many useful comments.

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