Note on a Property of Circulating Decimals with an even number of Repeating Figures equivalent to a Vulgar Fraction with a Prime Number as Denominator.

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1. The property is, that if such a decimal has $2k$ repeating figures, the first k are a_1, a_2, &c., in order, and the second k are b_1, b_2, &c., in order, and the a's and b's are such that

$$a_1 + b_1 = a_2 + b_2 = \&c. = 9\ ;$$

for example, $\frac{42}{73} = \cdot\dot{5}753424\dot{6}$.

2. The proof of this is as follows. Let n be the prime number, then by Fermat's Theorem $10^{n-1} - 1$ is a multiple of n ;

$\therefore$ any number x/n will, at most, give a repeating decimal with $\overline{n-1}$ repeating figures.

Now $10^{n-1} - 1 = (10^m - 1)(10^m + 1)$, n being of course odd, where $m = \frac{1}{2}(n-1)$, and n is prime, $\therefore$ n is a factor of $10^m - 1$ or of $10^m + 1$, not of both. Take the case that it is not a factor of $10^m - 1$, then the decimal must have $(n-1)$ repeating figures.

The first m figures are got by dividing $x.10^m$ by n. Then if $10^m + 1 = yn$, where y is an integer,

$$x.10^m/n = x(y - 1/n) = xy - x/n = xy - 1 + (n-x)/n\ ;$$

$\therefore$ the remainder is $(n-x)$, and the first m figures arranged in order form the number $xy - 1$.

The second set of m figures is got by dividing $(n-x)10^m$ by n, and the result is $10^m - x.10^m/n$,

$$i.e.,\ 10^m - xy + x/n\ ;$$

$\therefore$ the remainder is x, and we have $\overline{n-1}$ repeating figures, as we expected.

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The second set of m figures makes up the number $10^m - xy$, *i.e.*, a number with m figures, each 9, $-(xy-1)$.
But the first set of m figures is the number $xy - 1$;
$\therefore$ the figures are

$$a_1,\ a_2,\ \text{\&c., and } b_1,\ b_2,\ \text{\&c., where } 9 = a_1 + a_2 = b_1 + b_2 = \text{\&c.}$$

If n is not a factor of $10^m + 1$, but of $10^m - 1$, then x/n becomes a repeating decimal of m figures at most. If m is odd, the case is not one under discussion; if m is even, then

$$10^m - 1 = (10^l - 1)(10^l + 1), \text{ where } 2l = m,$$

and n is a factor of $10^2 - 1$ or $10^2 + 1$, and the proof is exactly as before, if n is a factor of the latter; if of the former, proceed to the next factorization in the same way, and continue till we have $10^p + 1$, and not $10^p - 1$, a multiple of n.

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