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# The Metric Dimension of Circulant Graphs

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*Abstract.* A subset $W$ of the vertex set of a graph $G$ is called a *resolving set* of $G$ if for every pair of distinct vertices $u, v$ of $G$, there is $w \in W$ such that the distance of $w$ and $u$ is different from the distance of $w$ and $v$. The cardinality of a smallest resolving set is called the *metric dimension* of $G$, denoted by $\dim(G)$. The circulant graph $C_n(1, 2, \ldots, t)$ consists of the vertices $v_0, v_1, \ldots, v_{n-1}$ and the edges $v_i v_{i+j}$, where $0 \le i \le n-1$, $1 \le j \le t$ $(2 \le t \le \lfloor \frac{n}{2} \rfloor)$, the indices are taken modulo $n$. Grigorious, Manuel, Miller, Rajan, and Stephen proved that $\dim(C_n(1, 2, \ldots, t)) \ge t + 1$ for $t < \lfloor \frac{n}{2} \rfloor$, $n \ge 3$, and they presented a conjecture saying that $\dim(C_n(1, 2, \ldots, t)) = t + p - 1$ for $n = 2tk + t + p$, where $3 \le p \le t + 1$. We disprove both statements. We show that if $t \ge 4$ is even, there exists an infinite set of values of $n$ such that $\dim(C_n(1, 2, \ldots, t)) = t$. We also prove that $\dim(C_n(1, 2, \ldots, t)) \le t + \frac{p}{2}$ for $n = 2tk + t + p$, where $t$ and $p$ are even, $t \ge 4$, $2 \le p \le t$, and $k \ge 1$.

## 1 Introduction

The concept of metric dimension was introduced by Slater [11], who referred to a metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of a minimum number of loran/sonar detecting devices in a network so that the position of every vertex in the network can be uniquely represented in terms of its distances to the devices in the set. Applications of the study of metric dimension to the problem of pattern recognition and image processing are given in [9]. We study the metric dimension of circulant graphs, which are Cayley graphs of cyclic groups.

Let $G$ be a connected graph with vertex set $V(G)$. The distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path between them. A vertex $w$ *resolves* a pair of vertices $u, v$ if $d(u, w) \ne d(v, w)$. For an ordered set of vertices $W = \{w_1, w_2, \ldots, w_z\}$, the *representation* of distances of a vertex $v$ with respect to $W$ is the ordered $z$-tuple

$$r(v|W) = \big(d(v, w_1), d(v, w_2), \ldots, d(v, w_z)\big).$$

A set of vertices $W \subset V(G)$ is a *resolving set* of $G$ if every two vertices of $G$ have distinct representations (if every pair of vertices of $G$ is resolved by some vertex of $W$). The cardinality of a smallest resolving set is called the *metric dimension*, and it is denoted by $\dim(G)$. Note that the $i$-th coordinate in $r(v|W)$ is 0 if and only if $v = w_i$. This means that in order to show that $W$ is a resolving set of $G$, it suffices to verify $r(u|W) \ne r(v|W)$ for every pair of distinct vertices $u, v \in V(G) \setminus W$.

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The metric dimension of various classes of graphs has been investigated for four decades. For example, the metric dimension of regular graphs was studied in [12]; products of graphs were considered in [6], metric manifolds in [3], the strong metric dimension in [8], and the fractional metric dimension in [13].

We define a circulant graph. Let $n, m$ and $a_1, a_2, \ldots, a_m$ be positive integers such that $1 \le a_1 < a_2 < \cdots < a_m \le \lfloor \frac{n}{2} \rfloor$. The circulant graph $C_n(a_1, a_2, \ldots, a_m)$ consists of vertices $v_0, v_1, \ldots, v_{n-1}$ and edges $v_i v_{i+a_j}$, where $0 \le i \le n-1$, $1 \le j \le m$; the indices are taken modulo $n$. The numbers $a_1, a_2, \ldots, a_m$ are called *generators*. The graph $C_n(a_1, a_2, \ldots, a_m)$ is a regular graph either of degree $2m$ if all generators are smaller than $\frac{n}{2}$, or of degree $2m-1$ if $\frac{n}{2}$ is one of the generators. Vertices with consecutive indices are called *consecutive vertices.* The distance between two vertices $v_i$ and $v_j$ in $C_n(1, 2, \ldots, t)$, where $0 \le i < j < n$, is

$$d(v_i, v_j) = \min\left\{ \left\lceil \frac{j-i}{t} \right\rceil, \left\lceil \frac{n-(j-i)}{t} \right\rceil \right\}. \tag{1.1}$$

This equation can be simplified as

$$d(v_i, v_j) = \left\lceil \frac{j-i}{t} \right\rceil \qquad \text{if } 0 \le j-i \le \frac{n}{2}, \tag{1.2}$$

$$d(v_i, v_j) = \left\lceil \frac{n-(j-i)}{t} \right\rceil \quad \text{if } \frac{n}{2} < j-i < n. \tag{1.3}$$

The metric dimension of circulant graphs has been extensively studied. Javaid, Rahim, and Ali [7] showed that $\dim(C_n(1,2)) = 3$ if $n \equiv 0, 2, 3 \pmod 4$. Imran et al. [4] showed that $\dim(C_n(1,2,3)) = 4$ if $n \equiv 2, 3, 4, 5 \pmod 6$, $n \ge 14$. Borchert and Gosselin [1] found the values of $\dim(C_n(1,2))$ and $\dim(C_n(1,2,3))$ for any $n$. They proved that $\dim(C_n(1,2)) = 4$ if $n \equiv 1 \pmod 4$, and for $n \ge 8$ we have $\dim(C_n(1,2,3)) = 5$ if $n \equiv 1 \pmod 6$ and $\dim(C_n(1,2,3)) = 4$ otherwise. The metric dimension of the circulant graphs $C_n(1,3)$ was studied in [5] and the circulant graphs $C_n(1, \frac{n}{2})$ for even $n$ were considered in [10].

Grigorious et al. [2] showed that $\dim(C_n(1, 2, \ldots, t)) \le t+1$ if $n \equiv r \pmod{2t}$, where $2 \le r \le t+2$ (the graph is resolved by the vertices $v_0, v_1, \ldots, v_t$) and $\dim(C_n(1, 2, \ldots, t)) \le r-1$ if $n \equiv r \pmod{2t}$, where $r \in \{t+3, t+4, \ldots, 2t+1\}$.

## 2 Results

We study upper and lower bounds on the metric dimensions of the circulant graphs $C_n(1, 2, \ldots, t)$. Theorem 2.8 of Grigorious et al. [2] says that if $n \ge 3$, then $\dim(C_n(1, 2, \ldots, t)) \ge t+1$. However, the proof of this theorem is not correct. The authors tried to show by contradiction that there is no resolving set $W$ of $C_n(1, 2, \ldots, t)$ consisting of $t$ vertices. They considered three cases.

Case 1: W consists of $t$ consecutive vertices.

Case 2: W consists of two sets of consecutive vertices.

Case 3: W consists of a set of consecutive vertices $W_1$ and all the other vertices of $W$ belong to a set of (at most) $t$ consecutive vertices. (Note that $N^L_{r-1}(W_1) \cup N^L_r(W_1)'$ used in their Case 3.3 is a set of $t$ vertices.)

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These three cases cover only a small part of possible choices of $t$ vertices (of $W$) from the set $V(C_n(1,2,\dots,t))$, thus the proof is incomplete.

An easy example that contradicts [2, Theorem 2.8] is the graph $C_{20}(1,2,3,4)$. This graph is resolved by the set $W=\{v_0,v_2,v_8,v_{10}\}$, hence $\dim(C_{20}(1,2,3,4))\le 4$.

Let us prove that if $t\ge 4$ is even, then there exists an infinite set of values of $n$ with $n\equiv t \pmod{2t}$, such that $\dim(C_n(1,2,\dots,t))\le t$.

***Theorem 2.1*** *Let $n=2tk+t$ where $t\ge 4$ is even and $k\ge 2$. Then*

$$\dim\big(C_n(1,2,\dots,t)\big)\le t.$$

**Proof** Let $n=2tk+t$, where $t\ge 4$ is even and $k\ge 2$. Let $W_1=\{v_0,v_2,\dots,v_{t-2}\}$ and $W_2=\{v_{tk},v_{tk+2},\dots,v_{tk+t-2}\}$. Note that $|W_1|=|W_2|=\frac{t}{2}$. We show that $W=W_1\cup W_2$ is a resolving set of $C_n(1,2,\dots,t)$. Let us divide the vertex set of $C_n(1,2,\dots,t)$ into three disjoint sets: $V_1=\{v_0,v_1,\dots,v_t\}$, $V_2=\{v_{t+1},v_{t+2},\dots,v_{tk+t-1}\}$, $V_3=\{v_{tk+t},v_{tk+t+1},\dots,v_{n-1}\}$.

First we show that no two vertices in $V_2$ have the same representations of distances with respect to $W$. For $x=1,2,\dots,k-1$; $j=1,2,\dots,t$; $i=0,2,\dots,t-2$, we have $v_i\in W_1$, and by (1.2),

$$d(v_{tx+j},v_i)=x+\Big\lceil\frac{j-i}{t}\Big\rceil=\begin{cases}x+1 & \text{if } i<j,\\ x & \text{if } i\ge j,\end{cases}$$

and if $x=k$ and $j=1,2,\dots,t-1$, by (1.1) we obtain

$$\begin{aligned}d(v_{tk+j},v_i)&=\min\Big\{\Big\lceil\frac{(tk+j)-i}{t}\Big\rceil,\Big\lceil\frac{n-[(tk+j)-i]}{t}\Big\rceil\Big\}\\&=\min\Big\{k+\Big\lceil\frac{j-i}{t}\Big\rceil,k+1+\Big\lceil\frac{i-j}{t}\Big\rceil\Big\}=\begin{cases}k+1 & \text{if } i<j,\\ k & \text{if } i\ge j.\end{cases}\end{aligned}$$

Since $j$ (where $1\le j\le t$) is greater than $\lceil\frac{j}{2}\rceil$ elements from the set $\{0,2,\dots,t-2\}$, the first $\lceil\frac{j}{2}\rceil$ entries of $r(v_{tx+j}\,|\,W_1)$ for $x=1,2,\dots,k$ are $x+1$ and the other $\frac{t}{2}-\lceil\frac{j}{2}\rceil$ entries are equal to $x$; $r(v_{tx+j}\,|\,W_1)=(x+1,\dots,x+1,x,\dots,x)$. The only vertices with the same representations of distances with respect to $W_1$ are the pairs $(v_{tx+j-1},v_{tx+j})$, where $j=2,4,\dots,t$ (if $x=k$, then $j=2,4,\dots,t-2$, because $v_{tk+t}\notin V_2$). But since for $x=1,2,\dots,k$ and $j=2,4,\dots,t-2$, we have $v_{tk+j}\in W_2$ and by (1.2),

$$d(v_{tx+j-1},v_{tk+j})=\Big\lceil\frac{tk-tx+1}{t}\Big\rceil=k-x+1,\, d(v_{tx+j},v_{tk+j})=k-x$$

and if $j=t$ and $x=1,2,\dots,k-1$, then by (1.2) for $v_{tk}\in W_2$, $d(v_{tx+t-1},v_{tk})=k-x$, $d(v_{tx+t},v_{tk})=k-x-1$, vertices of $W_2$ resolve the pairs $(v_{tx+j-1},v_{tx+j})$ for $x=1,2,\dots,k$ and $j=2,4,\dots,t$ ($j\le t-2$ if $x=k$). Thus, any two vertices of $V_2$ have different representations of distances with respect to $W$.

We consider representations of distances of the vertices in $V_3$. For $x=1,2,\dots,k-1$; $j=0,1,\dots,t-1$; $i=0,2,\dots,t-2$, we have $v_i\in W_1$ and by (1.3),

$$d(v_{n-tx+j},v_i)=\Big\lceil\frac{n-[(n-tx+j)-i]}{t}\Big\rceil=x+\Big\lceil\frac{i-j}{t}\Big\rceil=\begin{cases}x & \text{if } i\le j,\\ x+1 & \text{if } i>j,\end{cases}$$

and if $x = k$, by (1.1) we obtain

$$d(v_{tk+t+j}, v_i) = \min\left\{\left\lceil \frac{(tk+t+j)-i}{t}\right\rceil, \left\lceil \frac{n-[(tk+t+j)-i]}{t}\right\rceil\right\}$$
$$= \min\left\{k+1+\left\lceil \frac{j-i}{t}\right\rceil, k+\left\lceil \frac{i-j}{t}\right\rceil\right\} = \begin{cases} k & \text{if } i \le j, \\ k+1 & \text{if } i > j. \end{cases}$$

Since $j$ (where $0 \le j \le t-1$) is greater than or equal to $\lfloor \frac{j}{2} \rfloor + 1$ elements from the set $\{0, 2, \dots, t-2\}$, the first $\lfloor \frac{j}{2} \rfloor + 1$ entries of $r(v_{n-tx+j} \mid W_1)$ (for $x = 1, 2, \dots, k$) are $x$ and the other entries are equal to $x+1$. The only vertices with the same representations of distances with respect to $W_1$ are the pairs $(v_{n-tx+j}, v_{n-tx+j+1})$, where $j = 0, 2, \dots, t-2$. Since for $v_{tk+j} \in W_2$, by (1.2),

$$d(v_{n-tx+j}, v_{tk+j}) = \left\lceil \frac{n-tx+j-(tk+j)}{t}\right\rceil = k-x+1,$$
$$d(v_{n-tx+j+1}, v_{tk+j}) = k-x+1+\left\lceil \frac{1}{t}\right\rceil = k-x+2,$$

vertices of $W_2$ resolve the pairs $(v_{n-tx+j}, v_{n-tx+j+1})$. Thus, any two vertices of $V_3$ are resolved by $W$.

Note that a vertex $v \in V_2$ and a vertex in $V_3$ can have the same representation of distances with respect to $W_1$ only if all entries of $r(v \mid W_1)$ are the same numbers. For $x = 1, 2, \dots, k-1$, we have $v_{tx+t-1}, v_{tx+t} \in V_2$ and $r(v_{tx+t-1} \mid W_1) = r(v_{tx+t} \mid W_1) = (x+1, \dots, x+1)$, and for $v_{tk+t-1} \in V_2$, we have $r(v_{tk+t-1} \mid W_1) = (k+1, \dots, k+1)$. For $x = 1, 2, \dots, k$, we have $v_{n-tx+t-2}, v_{n-tx+t-1} \in V_3$ and $r(v_{n-tx+t-2} \mid W_1) = r(v_{n-tx+t-1} \mid W_1) = (x, \dots, x)$, which implies that for $x = 1, 2, \dots, k-1$, we have

$$r(v_{tx+t-1} \mid W_1) = r(v_{tx+t} \mid W_1) = r(v_{n-tx-2} \mid W_1) = r(v_{n-tx-1} \mid W_1).$$

Since for $v_{tk} \in W_2$, by (1.2),

$$d(v_{tx+t-1}, v_{tk}) = \left\lceil \frac{tk-(tx+t-1)}{t}\right\rceil = k-x, \qquad d(v_{tx+t}, v_{tk}) = k-x-1,$$
$$d(v_{n-tx-2}, v_{tk}) = \left\lceil \frac{(n-tx-2)-tk}{t}\right\rceil = k-x+1, \quad d(v_{n-tx-1}, v_{tk}) = k-x+1,$$

the vertices $v_{tx+t-1}, v_{tx+t} \in V_2$ are of distance at most $k-x$ from $v_{tk}$ and the vertices $v_{n-tx+t-2}, v_{n-tx+t-1} \in V_3$ are of distance $k-x+1$ from $v_{tk} \in W_2$. Therefore, any vertex in $V_2$ and any vertex in $V_3$ have different representations of distances with respect to $W$.

Let us study the vertices in $V_1$. For $j = 1, 2, \dots, t$ and $i = 0, 2, \dots, t-2$, where $i \ne j$, we have $v_i \in W_1$ and $d(v_j, v_i) = \lceil \frac{|j-i|}{t} \rceil = 1$. Thus, $r(v_j \mid W_1) = (1, \dots, 1)$ for $v_j \in V_1 \setminus W_1$. From the previous part of this proof it follows that the only vertices in $V_2 \cup V_3$ with the representation of distances with respect to $W_1$ equal to $(1, \dots, 1)$ are $v_{n-2}$ and $v_{n-1}$. For $j = 1, 3, \dots, t-1$ and $i = 0, 2, \dots, t-2$, we have $v_{tk+i} \in W_2$, and by

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(1.1),

$$d(v_j, v_{tk+i}) = \min\left\{\left\lceil \frac{(tk+i)-j}{t}\right\rceil, \left\lceil \frac{n-[(tk+i)-j]}{t}\right\rceil\right\}$$

$$= \min\left\{k + \left\lceil \frac{i-j}{t}\right\rceil, k+1+\left\lceil \frac{j-i}{t}\right\rceil\right\} = \begin{cases} k & \text{if } i<j,\\ k+1 & \text{if } i>j.\end{cases}$$

Since $j$ is greater than $\frac{j+1}{2}$ elements from the set $\{0,2,\ldots,t-2\}$, the first $\frac{j+1}{2}$ entries of $r(v_j \mid W_2)$ are $k$ and the other entries are equal to $k+1$. This means that the vertices $v_1, v_3, \ldots, v_{t-1}$ have different representations of distances with respect to $W_2$. Since for $v_{tk} \in W_2$, we have $d(v_t, v_{tk}) = k-1$, $d(v_{n-2}, v_{tk}) = d(v_{n-1}, v_{tk}) = k+1$ and for $j = 1, 3, \ldots, t-1$, we have $d(v_j, v_{tk}) = k$, all vertices of $C_n(1,2,\ldots,t)$ are resolved by $W$. Hence, $\dim(C_n(1,2,\ldots,t)) \le |W| = t$. ∎

In [2] the authors proposed a conjecture saying that $\dim(C_n(1,2,\ldots,t)) = t+p-1$ for $n = 2tk+t+p$, where $3 \le p \le t+1$. We disprove this conjecture if $t$ and $p$ are even. Let us present a new upper bound on the metric dimension of $C_n(1,2,\ldots,t)$ for $n \equiv r \pmod{2t}$, where $r = 0$ and $r = t+2, t+4, \ldots, 2t-2$.

***Theorem 2.2*** *Let $n = 2tk+t+p$ where $t$ and $p$ are even, $t \ge 4$, $2 \le p \le t$, and $k \ge 1$. Then*

$$\dim\big(C_n(1,2,\ldots,t)\big) \le t + \frac{p}{2}.$$

**Proof** Let $n = 2tk+t+p$ where $k \ge 1$, $t \ge 4$ is even and $p = 2,4,\ldots,t$. Let

$$W_1 = \{v_0, v_2, \ldots, v_{t-2}\}, \quad W_2 = \{v_{t-1}, v_{t+1}, \ldots, v_{t+p-3}\},$$
$$W_3 = \{v_{tk+p-2}, v_{tk+p}, \ldots, v_{tk+p+t-4}\}.$$

Note that $|W_1| = |W_3| = \frac{t}{2}$ and $|W_2| = \frac{p}{2}$. We show that $W = W_1 \cup W_2 \cup W_3$ is a resolving set of $C_n(1,2,\ldots,t)$.

Let us divide the vertex set of $C_n(1,2,\ldots,t)$ into four disjoint sets:

$$V_1 = \{v_0, v_1, \ldots, v_t\}, \qquad V_2 = \{v_{t+1}, v_{t+2}, \ldots, v_{tk+t}\},$$
$$V_3 = \{v_{tk+t+1}, v_{tk+t+2}, \ldots, v_{tk+t+p-1}\}, \qquad V_4 = \{v_{tk+t+p}, v_{tk+t+p+1}, \ldots, v_{n-1}\}.$$

First we show that no two vertices in $V_2$ have the same representation of distances with respect to $W$. For $x = 1,2,\ldots,k-1$; $j = 1,2,\ldots,t$; $i = 0,2,\ldots,t-2$, we have $v_i \in W_1$, and by (1.2),

$$d(v_{tx+j}, v_i) = x + \left\lceil \frac{j-i}{t}\right\rceil = \begin{cases} x+1 & \text{if } i<j,\\ x & \text{if } i \ge j,\end{cases}$$

and if $x = k$; $j = 1,2,\ldots,t$, by (1.1), we obtain

$$d(v_{tk+j}, v_i) = \min\left\{\left\lceil \frac{(tk+j)-i}{t}\right\rceil, \left\lceil \frac{n-[(tk+j)-i]}{t}\right\rceil\right\}$$

$$= \min\left\{k + \left\lceil \frac{j-i}{t}\right\rceil, k+1+\left\lceil \frac{p+i-j}{t}\right\rceil\right\} = \begin{cases} k+1 & \text{if } i<j,\\ k & \text{if } i \ge j.\end{cases}$$

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Since $j$ (where $1 \le j \le t$) is greater than $\lceil \frac{j}{2} \rceil$ elements from the set $\{0, 2, \dots, t-2\}$, the first $\lceil \frac{j}{2} \rceil$ entries of $r(v_{tx+j} \mid W_1)$ for $x = 1, 2, \dots, k$ are $x+1$ and the other $\frac{t}{2} - \lceil \frac{j}{2} \rceil$ entries are equal to $x$; $r(v_{tx+j} \mid W_1) = (x+1, \dots, x+1, x, \dots, x)$. Thus, the only vertices in $V_2$ with the same representations of distances with respect to $W_1$ are the pairs $(v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \dots, (v_{tk+t-1}, v_{tk+t})$. Let us show that most of these pairs are resolved by vertices in $W_3$.

Since for $x = 0, 1, \dots, k$ and $j = 0, 2, \dots, t-2$, we have $v_{tk+p+j-2} \in W_3$ and by (1.2), $d(v_{tx+p+j-2}, v_{tk+p+j-2}) = k - x$ and $d(v_{tx+p+j-3}, v_{tk+p+j-2}) = k - x + \lceil \frac{1}{t} \rceil = k - x + 1$, vertices in $W_3$ resolve the pairs $(v_{p-3}, v_{p-2})$, $(v_{p-1}, v_p), \dots, (v_{tk+p+t-5}, v_{tk+p+t-4})$. Note that

$$\{(v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \dots, (v_{tk+t-3}, v_{tk+t-2})\} \subset \{(v_{p-3}, v_{p-2}), (v_{p-1}, v_p), \dots, (v_{tk+p+t-5}, v_{tk+p+t-4})\},$$

which implies that all pairs of vertices in $V_2$ except for the pair $(v_{tk+t-1}, v_{tk+t})$ are resolved. Since $v_{t-1} \in W_2$ resolves the pair $(v_{tk+t-1}, v_{tk+t})$, any two vertices of $V_2$ have different representations of distances with respect to $W$.

We consider representations of distances of the vertices in $V_4$. For $x = 1, 2, \dots, k-1$; $j = 0, 1, \dots, t-1$; $i = 0, 2, \dots, t-2$; we have $v_i \in W_1$, and by (1.3),

$$d(v_{n-tx+j}, v_i) = \left\lceil \frac{n - [(n - tx + j) - i]}{t} \right\rceil = x + \left\lceil \frac{i-j}{t} \right\rceil = \begin{cases} x & \text{if } i \le j, \\ x+1 & \text{if } i > j, \end{cases}$$

and if $x = k$, we obtain

$$\begin{aligned} d(v_{n-tk+j}, v_i) &= \min\left\{ \left\lceil \frac{(n-tk+j)-i}{t} \right\rceil, \left\lceil \frac{n - [(n-tk+j)-i]}{t} \right\rceil \right\} \\ &= \min\left\{ k+1+\left\lceil \frac{p+j-i}{t} \right\rceil, k + \left\lceil \frac{i-j}{t} \right\rceil \right\} = \begin{cases} k & \text{if } i \le j, \\ k+1 & \text{if } i > j. \end{cases} \end{aligned}$$

Since $j$ (where $0 \le j \le t-1$) is greater than or equal to $\lfloor \frac{j}{2} \rfloor + 1$ elements from the set $\{0, 2, \dots, t-2\}$, the first $\lfloor \frac{j}{2} \rfloor + 1$ entries of $r(v_{n-tx+j} \mid W_1)$ (for $x = 1, 2, \dots, k$) are $x$ and the other entries are equal to $x+1$. The only vertices with the same representations of distances with respect to $W_1$ are the pairs

$$(v_{tk+t+p}, v_{tk+t+p+1}), (v_{tk+t+p+2}, v_{tk+t+p+3}), \dots, (v_{n-2}, v_{n-1}).$$

But since for $x = 1, 2, \dots, k$ and $j = 0, 2, \dots, t-2$, we have $v_{tk+p+j-2} \in W_3$, and by (1.2),

$$d(v_{n-tx+j-2}, v_{tk+p+j-2}) = k - x + 1,$$
$$d(v_{n-tx+j-1}, v_{tk+p+j-2}) = k - x + 1 + \left\lceil \frac{1}{t} \right\rceil = k - x + 2,$$

vertices of $W_3$ resolve all pairs except for the pair $(v_{n-2}, v_{n-1})$, which is resolved by $v_{t-1} \in W_2$. Thus, any pair of vertices in $V_4$ is resolved by $W$.

Note that a vertex $v \in V_2$ and a vertex in $V_4$ can have the same representation of distances with respect to $W_1$ only if all entries of $r(v \mid W_1)$ are the same numbers.

For $x = 1,2,\dots,k$, we have $v_{tx+t-1}, v_{tx+t} \in V_2$ and $r(v_{tx+t-1}|W_1) = r(v_{tx+t}|W_1) = (x+1,\dots,x+1)$. For $v_{n-tx+t-2}, v_{n-tx+t-1} \in V_4$, we have

$$r(v_{n-tx+t-2}|W_1) = r(v_{n-tx+t-1}|W_1) = (x,\dots,x),$$

which implies that for $x = 1,2,\dots,k-1$, we have $r(v_{tx+t-1}|W_1) = r(v_{tx+t}|W_1) = r(v_{n-tx-2}|W_1) = r(v_{n-tx-1}|W_1)$. For $v_{tk+p-2} \in W_3$, by (1.2),

$$\begin{aligned}
d(v_{tx+t-1}, v_{tk+p-2}) &= k-x-1+\left\lceil \frac{p-1}{t} \right\rceil = k-x,\\
d(v_{tx+t}, v_{tk+p-2}) &= k-x-1+\left\lceil \frac{p-2}{t} \right\rceil \le k-x,\\
d(v_{n-tx-2}, v_{tk+p-2}) &= k-x+1,\\
d(v_{n-tx-1}, v_{tk+p-2}) &= k-x+1+\left\lceil \frac{1}{t} \right\rceil = k-x+2,
\end{aligned}$$

so the vertices $v_{tx+t-1}, v_{tx+t} \in V_2$ are of distance at most $k-x$ from $v_{tk+p-2}$, and the vertices $v_{n-tx-2}, v_{n-tx-1} \in V_4$ are of distance at least $k-x+1$ from $v_{tk+p-2} \in W_3$. Therefore, any vertex in $V_2$ and any vertex in $V_4$ have different representations of distances with respect to $W$.

We consider representations of distances of the vertices in $V_3$. For $j = 1,2,\dots,p-1$ and $i = 0,2,\dots,t-2$, we have $v_i \in W_1$, and by (1.1),

$$d(v_{tk+t+j}, v_i) = \min\left\{ k+1+\left\lceil \frac{j-i}{t} \right\rceil, k+\left\lceil \frac{p+i-j}{t} \right\rceil \right\} = k+1;$$

thus, $r(v_{tk+t+j}|W_1) = (k+1,\dots,k+1)$. The only vertices in $V_2 \cup V_4$ with the same representations with respect to $W_1$ are $v_{tk+t-1}$ and $v_{tk+t}$.

We show that any two vertices in $V_3 \cup \{v_{tk+t-1}, v_{tk+t}\}$ have different representation of distances with respect to $W$. It suffices to consider the vertices in

$$\begin{aligned}
V' &= (V_3 \cup \{v_{tk+t-1}, v_{tk+t}\}) \smallsetminus W_3\\
&= \{v_{tk+t-1}, v_{tk+t+1}, \dots, v_{tk+t+p-3}\} \cup \{v_{tk+t+p-2}, v_{tk+t+p-1}\}.
\end{aligned}$$

For $j = -1,1,\dots,p-1$ and $i = -1,1,\dots,p-3$, we have $v_{t+i} \in W_2$ and

$$d(v_{tk+t+j}, v_{t+i}) = k+\left\lceil \frac{j-i}{t} \right\rceil = \begin{cases} k & \text{if } i \ge j,\\ k+1 & \text{if } i < j. \end{cases}$$

Since $j$ is greater than $\frac{j+1}{2}$ elements from the set $\{-1,1,\dots,p-3\}$, the first $\frac{j+1}{2}$ entries of $r(v_{tk+t+j}|W_2)$ are $k+1$ and the other $\frac{p}{2} - \frac{j+1}{2}$ entries are equal to $k$. For $i = -1,1,\dots,p-3$,

$$d(v_{tk+t+p-2}, v_{t+i}) = k+\left\lceil \frac{p-i-2}{t} \right\rceil = k+1;$$

thus, $r(v_{tk+t+p-2}|W_2) = (k+1,\dots,k+1)$. The only pair of vertices in $V'$ having the same representations with respect to $W_2$ is $(v_{tk+t+p-2}, v_{tk+t+p-1})$, which is resolved by $v_{tk+p-2} \in W_3$, since

$$d(v_{tk+t+p-2}, v_{tk+p-2}) = 1 \quad \text{and} \quad d(v_{tk+t+p-1}, v_{tk+p-2}) = 1+\left\lceil \frac{1}{t} \right\rceil = 2.$$

Let us study the vertices in $V_1$. For $j = 1,3,\dots,t-1$ and $t$; $i = 0,2,\dots,t-2$, we have $v_i \in W_1$ and $d(v_j, v_i) = \lceil \frac{|j-i|}{t} \rceil = 1$, thus $r(v_j \mid W_1) = (1,\dots,1)$ for $v_j \in V_1 \setminus W_1$. From the previous part of the proof it follows that the only vertices in $V_2 \cup V_3 \cup V_4$ with the representation of distances with respect to $W_1$ equal to $(1,\dots,1)$ are $v_{n-2}$ and $v_{n-1}$. So it remains to resolve the vertices $v_1, v_3, \dots, v_{t-1}; v_t, v_{n-2}, v_{n-1}$. First we give representations of $v_1, v_3, \dots, v_{p-3}$ and $v_{n-2}, v_{n-1}$ with respect to $W_3$; then we give representations of $v_{p-1}, v_{p+1}, \dots, v_{t-3}$ with respect to $W_3$, and then we consider the vertex $v_t$.

We show that for $j = 1,3,\dots,p-3$ and $j = n-2, n-1$, we have $r(v_j \mid W_3) = (k+1,\dots,k+1)$. For $j = 1,3,\dots,p-3$ and $i = 0,2,\dots,t-2$, we have $v_{tk+p+i-2} \in W_3$, and by (1.1),

$$d(v_j, v_{tk+p+i-2}) = \min\left\{ k + \left\lceil \frac{p+i-j-2}{t} \right\rceil, k+1+\left\lceil \frac{j+2-i}{t} \right\rceil \right\} = k+1,$$

$$d(v_{n-1}, v_{tk+p+i-2}) = \min\left\{ k+1+\left\lceil \frac{1-i}{t} \right\rceil, k+\left\lceil \frac{p+i-1}{t} \right\rceil \right\} = k+1,$$

$$d(v_{n-2}, v_{tk+p+i-2}) = \min\left\{ k+1+\left\lceil \frac{-i}{t} \right\rceil, k+\left\lceil \frac{p+i}{t} \right\rceil \right\} = k+1,$$

which means that $r(v_j \mid W_3) = r(v_{n-1} \mid W_3) = r(v_{n-2} \mid W_3) = (k+1,\dots,k+1)$.

We give representations of $v_{p-1}, v_{p+1}, \dots, v_{t-3}$ with respect to $W_3$. For $j = 1,3,\dots,t-1-p$ and $i = 0,2,\dots,t-2$, we have $v_{tk+p+i-2} \in W_3$ and

$$d(v_{p+j-2}, v_{tk+p+i-2}) = k + \left\lceil \frac{i-j}{t} \right\rceil = \begin{cases} k & \text{if } i < j, \\ k+1 & \text{if } i > j. \end{cases}$$

Since $j$ is greater than $\frac{j+1}{2}$ elements from the set $\{0,2,\dots,t-2\}$, the first $\frac{j+1}{2}$ entries of $r(v_j \mid W_3)$ are $k$ and the other entries are equal to $k+1$. Note that the first entry of $r(v_j \mid W_3)$ is always $k$.

Let us show that $v_{tk+t-2} \in W_3$ resolves $v_t$ from the other vertices in the set $\{v_1, v_3, \dots, v_{t-1}; v_t, v_{n-2}, v_{n-1}\}$. By (1.2), we have $d(v_t, v_{tk+t-2}) = k + \lceil \frac{-2}{t} \rceil = k$ and $d(v_j, v_{tk+t-2}) = k+1+\lceil \frac{-j-2}{t} \rceil = k+1$ for $j = p-1, p+1, \dots, t-3$, and $d(v_j, v_{tk+t-2}) = k+1$, also for $j = 1,3,\dots,p-3$ and $j = n-2, n-1$. It follows that the vertices $v_{p-1}, v_{p+1}, \dots, v_{t-3}$ and $v_t$ are resolved.

It remains to resolve the vertices $v_1, v_3, \dots, v_{p-3}$ and $v_{n-2}, v_{n-1}$; thus, we study their representations with respect to $W_2$. For $j = 1,3,\dots,p-3$ and $i = -1,1,\dots,p-3$, we have $v_{t+i} \in W_2$, and by (1.2),

$$d(v_j, v_{t+i}) = 1 + \left\lceil \frac{i-j}{t} \right\rceil = \begin{cases} 1 & \text{if } i \le j, \\ 2 & \text{if } i > j. \end{cases}$$

Since $j$ is greater than or equal to $\frac{j+3}{2}$ elements from the set $\{-1,1,\dots,p-3\}$, the first $\frac{j+3}{2}$ entries of $r(v_j \mid W_2)$ are 1 and the other $\frac{p}{2} - \frac{j+3}{2}$ entries are equal to 2. Note that the first two entries of $r(v_j \mid W_3)$ are always 1.

For $i = -1, 1, \dots, p-3$, by (1.3),

$$d(v_{n-1}, v_{t+i}) = 1 + \left\lceil \frac{i+1}{t} \right\rceil = \begin{cases} 1 & \text{if } i = -1, \\ 2 & \text{if } i \geq 1. \end{cases}$$

Thus, $r(v_{n-1} | W_2) = (1, 2, \dots, 2)$. We have $d(v_{n-2}, v_{t+i}) = 1 + \lceil \frac{i+2}{t} \rceil = 2$, so

$$r(v_{n-2} | W_2) = (2, \dots, 2).$$

No two vertices of $C_n(1, 2, \dots, t)$ have the same representations of distances with respect to $W$; hence, $W$ is a resolving set of $C_n(1, 2, \dots, t)$ and $\dim(C_n(1, 2, \dots, t)) \leq |W| = t + \frac{p}{2}$. ∎

Now we focus on lower bounds on the metric dimension of circulant graphs. For any vertex $v_j$ of $C_n(1, 2, \dots, t)$, the vertex $v_{j+\lfloor n/2 \rfloor}$ will be called the *opposite vertex* of $v_j$. Clearly, for any $t$ consecutive vertices $v_i, v_{i+1}, \dots, v_{i+t-1} \in V(C_n(1, 2, \dots, t)) \smallsetminus \{v_j, v_{j+\lfloor n/2 \rfloor}\}$, we have

$$x = d(v_j, v_i) \leq d(v_j, v_{i+1}) \leq \cdots \leq d(v_j, v_{i+t-1}) \leq x + 1 \tag{2.1}$$

or

$$x = d(v_j, v_{i+t-1}) \leq d(v_j, v_{i+t}) \leq \cdots \leq d(v_j, v_i) \leq x + 1 \tag{2.2}$$

for some positive integer $x$. These inequalities will be used in the proofs of Theorems 2.3 and 2.5.

***Theorem 2.3*** *Let $n \geq t^2 + 1$ where $t \geq 2$. Then*

$$\dim(C_n(1, 2, \dots, t)) \geq t.$$

**Proof** We prove the result by contradiction. Suppose that $\dim(C_n(1, 2, \dots, t)) \leq t-1$. Let $W' = \{w_0, w_1, \dots, w_{t-2}\}$ be a resolving set of the graph $C_n(1, 2, \dots, t)$, where the vertices $w_0, w_1, \dots, w_{t-2}$ are not necessarily different. Without loss of generality we can assume that $w_0 = v_0$. For $j = 1, 2, \dots, \lfloor \frac{t}{2} \rfloor$, by (1.2),

$$d(v_0, v_{jt}) = d(v_0, v_{jt-1}) = \cdots = d(v_0, v_{jt-(t-1)}) = j,$$

and by (1.3) we have

$$d(v_0, v_{n-jt}) = d(v_0, v_{n-jt+1}) = \cdots = d(v_0, v_{n-jt+(t-1)}) = j.$$

Let $V_j = \{v_{jt-(t-1)}, v_{jt-(t-2)}, \dots, v_{jt}\}$ and $V_{-j} = \{v_{n-jt}, v_{n-jt+1}, \dots, v_{n-jt+(t-1)}\}$ for $j = 1, 2, \dots, \lfloor \frac{t}{2} \rfloor$. Note that all vertices in $V_j$ (in $V_{-j}$) have the same distance from $v_0$ and

$$\sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} |V_j| = \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} |V_{-j}| = t \left\lfloor \frac{t}{2} \right\rfloor = \begin{cases} \frac{t^2}{2} & \text{if } t \text{ is even}, \\ \frac{t(t-1)}{2} & \text{if } t \text{ is odd}. \end{cases}$$

Thus, $\sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} |V_j| + \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} |V_{-j}| \leq t^2$. Since $n \geq t^2 + 1$, the sets $V_j$ and $V_{-j}$ are disjoint. Since we have $2\lfloor \frac{t}{2} \rfloor$ pairwise disjoint sets and $2\lfloor \frac{t}{2} \rfloor \geq t - 1$, there is at least one set $V' = V_l$, $l \in \{\pm 1, \pm 2, \dots, \pm \lfloor \frac{t}{2} \rfloor\}$, containing no opposite vertices of $w_1, w_2, \dots, w_{t-2}$.

We show that $V'$ cannot be resolved by $W'$. Let $p$ be the number of vertices of $W'$ in $V'$ $(0 \leq p \leq t-2)$. Without loss of generality we can assume that $w_1, w_2, \dots, w_p \in V'$ and $w_{p+1}, w_{p+2}, \dots, w_{t-2} \notin V'$. Let $v_{a_1}, v_{a_2}, \dots, v_{a_{t-p}}$ be $t - p$ different vertices of

$V' \setminus W'$, where $a_1 \le a_2 \le \cdots \le a_{t-p}$. We know that $d(v_{a_s}, w_r) = 1$ for any $s = 1, 2, \ldots, t-p$ and any $r = 1, 2, \ldots, p$. Thus, $t-p-1$ pairs

$$(v_{a_1}, v_{a_2}), (v_{a_2}, v_{a_3}), \ldots, (v_{a_{t-p-1}} v_{a_{t-p}})$$

have the same representations of distances with respect to the vertices $w_0, w_1, \ldots, w_p$. From (2.1) and (2.2) it follows that any of $t-p-2$ vertices $w_{p+1}, w_{p+2}, \ldots, w_{t-2}$ can resolve at most one pair $(v_{a_s}, v_{a_{s+1}})$, where $s \in \{1, 2, \ldots, t-p-1\}$, which implies that there exists a pair (two vertices of $V'$) that cannot be resolved by $W'$. Hence, $W'$ is not a resolving set of $C_n(1, 2, \ldots, t)$, a contradiction. ∎

From Theorems 2.1 and 2.3 we obtain the following corollary.

***Corollary 2.4*** *Let $n \equiv t \pmod{2t}$, where $n \ge t^2 + 1$ and $t \ge 4$ is even. Then*

$$\dim(C_n(1, 2, \ldots, t)) = t.$$

Finally, we state a lower bound on $C_n(1, 2, \ldots, t)$ for $n \equiv r \pmod{2t}$, where $r \in \{0, 1\} \cup \{t+2, t+3, \ldots, 2t-1\}$.

***Theorem 2.5*** *Let $n = 2tk + r$ where $t \ge 2$, $k \ge 0$ and $t + 2 \le r \le 2t + 1$. Then*

$$\dim(C_n(1, 2, \ldots, t)) \ge t + 1.$$

**Proof** Let $n = 2tk + r$ where $t \ge 2$, $k \ge 0$ and $t + 2 \le r \le 2t + 1$. By Theorem 2.3, we have $\dim(C_n(1, 2, \ldots, t)) \ge t$. We prove that $\dim(C_n(1, 2, \ldots, t)) \ge t + 1$. Suppose to the contrary that it is possible to resolve the graph $C_n(1, 2, \ldots, t)$ by $t$ vertices. Let $W' = \{w_0, w_1, \ldots, w_{t-1}\}$ be a resolving set of $C_n(1, 2, \ldots, t)$. Without loss of generality we can assume that $w_0 = v_0$. Let $V' = \{v_{tk+1}, v_{tk+2}, \ldots, v_{tk+t+1}\}$. Note that for $j = 1, 2, \ldots, t+1$,

$$\begin{aligned} d(v_0, v_{tk+j}) &= \min\left\{ \left\lceil \frac{tk+j}{t} \right\rceil, \left\lceil \frac{n-(tk+j)}{t} \right\rceil \right\} \\ &= \min\left\{ k + \left\lceil \frac{j}{t} \right\rceil, k + \left\lceil \frac{r-j}{t} \right\rceil \right\} = k+1. \end{aligned}$$

We show that $V'$ cannot be resolved by $W'$. Let $p$ be the number of vertices of $W'$ in $V'$ $(0 \le p \le t-1)$. We can assume that $w_1, w_2, \ldots, w_p \in V'$ and $w_{p+1}, w_{p+2}, \ldots, w_{t-1} \notin V'$. The distance between any vertex in $V' \setminus W'$ and $w_i$ is 1 for $i = 1, 2, \ldots, p$, thus all vertices in $V' \setminus W'$ have the same representations of distances with respect to the vertices $w_0, w_1, \ldots, w_p$. Let $v_{a_1}, v_{a_2}, \ldots, v_{a_{t-p+1}}$ be the vertices of $V' \setminus W'$, where $a_1 \le a_2 \le \cdots \le a_{t-p+1}$. By (2.1) and (2.2), any of the $t-p-1$ vertices $w_{p+1}, w_{p+2}, \ldots, w_{t-1}$ can resolve at most one of $t-p$ pairs $(v_{a_1}, v_{a_2}), (v_{a_2}, v_{a_3}), \ldots, (v_{a_{t-p}} v_{a_{t-p+1}})$; therefore, there exists a pair that cannot be resolved by $W'$, a contradiction. ∎

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