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PERTURBATIONS OF AF-ALGEBRAS

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Introduction. Let A and B be C^*-algebras acting on a Hilbert space H, and let

$$\|A - B\| = \sup \{d(a, B_1), d(b, A_1): a \in A_1, b \in B_1\},$$

where A_1 is the unit ball in A and $d(a, B_1)$ denotes the distance of a from B_1. We shall consider the following problem: if $\|A - B\|$ is sufficiently small, does it follow that there is a unitary operator u such that $uAu^* = B$?

Such questions were first considered by Kadison and Kastler in [**9**], and have received considerable attention. In particular in the case where A is an approximately finite-dimensional (or hyperfinite) von Neumann algebra, the question has an affirmative answer (cf [**3**], [**8**], [**12**]). We shall show that in the case where A and B are approximately finite-dimensional C^*-algebras (AF-algebras) the problem also has a positive answer.

Our approach is as follows. First we shall prove that the local semi-groups (see Section 1.1) of close C^*-algebras are isomorphic, and then use the result of Elliott [**6**] that this semigroup is a complete invariant for AF-algebras to deduce that close AF-algebras are isomorphic. We then apply the result for hyperfinite von Neumann algebras mentioned above to reduce to the case where $\bar{A} = \bar{B}$, and finally we follow Bratteli's modifications [**1**] of Powers' techniques (see [**11**]) to deduce that A and B are unitarily equivalent. We follow this programme in Section 2; our first section contains the versions of Elliott's and Bratteli's work which we shall need.

We wish to acknowledge the help of Ed Effros, who suggested that we try to use K_0 instead of Ext (see [**10**]); the connection between $K_0(A)$ and Elliott's invariant is discussed in [**5**]. We have recently learned that Erik Christensen has also obtained our main result; his methods are completely different and will appear elsewhere [**4**].

1. The theorems of Elliott and Bratteli.

1.1. Let A be a C^*-algebra, and let $P(A)$ denote the set of projections (self-adjoint idempotents) in A. We define an equivalence relation on $P(A)$ by $e \backsim f$ if there is a partial isometry $v \in A$ with $vv^* = e$ and $v^*v = f$; we denote by $[e]_A$, or just $[e]$, the equivalence class containing e. If $e, f \in P(A)$ and there exist e', f' in $P(A)$ with $e'f' = 0$ (e' and f' are orthogonal) and $e \backsim e'$, $f \backsim f'$, then we define $[e] + [f] = [e' + f']$. We call the set of equivalence

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classes in $P(A)$ equipped with this partially defined addition the *local semigroup of A*, and denote it by $S(A)$.

We recall that a unital C^*-algebra A is an AF-algebra if there is an increasing sequence of finite-dimensional C^*-subalgebras $\{A_n\}$ such that $\bigcup_{n=1}^{\infty} A_n$ is dense in A. Elliott [**6**] has proved that the local semigroup $S(A)$ is a complete invariant for AF-algebras; the specific version of his result which we shall need is the following.

THEOREM 1.2. *Let A and B be* AF-*algebras and suppose that $\rho : S(A) \rightarrow S(B)$ is an isomorphism of local semigroups. Then there is a *-isomorphism ϕ of A onto B such that*

$$[\phi(e)]_B = \rho([e]_A) \text{ for } e \in P(A).$$

Proof. The algebraic analogue (where A and B are inductive limits of sequences of finite-dimensional semi-simple algebras) of this is Theorem 4.3 of [**6**]; the statement $[\phi(\cdot)]_B = \rho([\cdot]_A)$ follows from the construction in the proof of that theorem [**6**]. That the same result holds for AF-algebras is a consequence of the remarks in Section 4.4 of [**6**].

1.3. We shall also need the following theorem [**1**], where if A is a C^*-algebra acting on a Hilbert space H, then $\bar{A}$ denotes its weak closure.

THEOREM. *Let A and B be* AF-*algebras acting on a separable Hilbert space H, and suppose that $\bar{A} = \bar{B} = M$. If there is an isomorphism $\phi : A \rightarrow B$ such that $\phi(e) \backsim e$ in M for each $e \in P(A)$, then there is a unitary operator $u \in M$ such that $uAu^* = B$.*

Proof. We first observe that this result is Lemma 4.11 of [**1**] with the condition $A \mathrel{\overline{\subseteq}} M$ ("A is permanently locally unitarily equivalently embedded in M") replaced by the condition

$$\alpha(e) \backsim e \text{ for every } e \in P(A). \tag{*}$$

This condition is weaker than Bratteli's; $A \mathrel{\overline{\subseteq}} M$ means every isomorphism $\beta : A \rightarrow B$ where $\bar{B} = M$ should satisfy (*). To see that condition (*) suffices, we first note that Lemma 4.8 of [**1**] remains true under this weaker condition. All subsequent constructions in the proofs of 4.9 and 4.10 of [**1**] involve building new isomorphisms from an isomorphism $\beta : A \rightarrow B$ in two equivalent ways:

$$\delta(x) = u\beta(x)u^*, \; \gamma(x) = \beta(uxu^*) \;\; (x \in A),$$

where u is a unitary element of B and A respectively. It is easy to see that such δ's and γ's also satisfy (*); hence Lemma 4.10 is valid in our setting, and the isomorphism constructed also satisfies (*). It follows in the same way that the inductive step in Lemma 4.11 of [**1**] is valid with the additional inductive hypothesis

(v) for all $r \geqq 0$ $\alpha_r(e) \smile e$ for every $e \in P(A)$,

and the rest of the proof goes through unchanged.

2. The main result. We begin with a sequence of lemmas which are modifications of results from [**2**], [**7**] and [**9**]. We provide proofs for the sake of completeness.

LEMMA 2.1. *If A is a C^*-algebra on H and e is a projection on H such that there is an x in A, $\|x\| \leqq 1$ with $\|x - e\| < a(\leqq \frac{1}{2})$ then there is a projection f in A with $\|f - e\| < 2a$.*

Proof. Since $\|(x + x^*)/2 - e\| < a$ we can assume that x is self-adjoint. Since the spectrum of x must lie in $(-a, a) \cup (1 - a, 1]$ we see that there is a spectral projection f of x such that $\|f - x\| < a$ and so $\|f - e\| < 2a$. Since $a \leqq \frac{1}{2}$, the spectrum of x is disconnected so that f is a continuous function of x. Hence, f is in A.

LEMMA 2.2. *If e and f are projections in a C^*-algebra A and $\|e - f\| < a(\leqq 1)$ then there is a v in A such that $e = vv^*$ and $f = v^*v$.*

Proof. Let $x = ef$. Then $x^*x = fef$ is in fAf. Moreover,

$$\|x^*x - f\| = \|fef - f\| \leqq \|e - f\| < a \leqq 1$$

so that x^*x is invertible in the C^*-algebra fAf. In particular, $(x^*x)^{1/2}$ has initial projection f, and there is an element $y(= (x^*x)^{-1/2})$ in fAf such that $(x^*x)^{1/2}y = f$. Hence, if $x = v(x^*x)^{1/2}$ is the polar decomposition of x, then v has initial projection f and

$$xy = v(x^*x)^{1/2}y = vf = v$$

is in A. Similarly, the range projection of v is e.

LEMMA 2.3. *Let A and B be C^*-algebras on H with $\|A - B\| < a(\leqq \frac{1}{4})$ and let e, f (respectively, e', f') be projections in A (respectively, B) with $\|e - e'\| < a$ and $\|f - f'\| < a$. Then $e \underset{A}{\sim} f$ if and only if $e' \underset{B}{\sim} f'$.*

Proof. Let v in A be such that $v^*v = f$ and $vv^* = e$. Let y in the unit ball of B be such that $\|v - y\| < a$. Let $x = e'yf'$, so that $x^*x = f'y^*e'yf'$ is in $f'Bf'$ and

$$\begin{aligned}\|x^*x - f'\| &\leqq \|y^*e'y - f'\| \leqq \|y^*e'y - y^*e'v\| + \|y^*e'v - y^*ev\| \\ &\quad + \|y^*ev - v^*ev\| + \|v^*ev - f'\| \leqq \|y - v\| + \|e' - e\| \\ &\quad + \|y^* - v^*\| + \|f - f'\| < 4a < 1\end{aligned}$$

so that x^*x is invertible in the C^*-algebra $f'Bf'$. Proceeding as in the proof of Lemma 2.2, we see that if $x = w(x^*x)^{1/2}$ then w is in B and $w^*w = f'$, $ww^* = e'$.

LEMMA 2.4. *Let A and B be unital C^*-algebras on H such that $\|A - B\| < a(\leqq 1/6)$. Then if e is a projection in A there is a projection $e' \in B$ with*

$\|e - e'\| < a$. *Further, if* e' *is such a projection and* f *is another projection in* A *orthogonal to* e, *then there is a projection* $f' \in B$ *orthogonal to* e' *such that* $\|f - f'\| < 6a$.

Proof. Let $u = 1 - 2e$; then u is unitary in A and so there is an operator $u' \in B$ with $\|u'\| \leqq 1$ and $\|u - u'\| < a$. Letting $x = \frac{1}{2}(1 - u')$ we have $\|x\| \leqq 1$ and $\|x - e\| < \frac{1}{2}a$. If we apply Lemma 2.1, we get an $e' \in P(B)$ with $\|e - e'\| < a$. Similarly, we can find an $f_1 \in P(B)$ with $\|f - f_1\| < a$. Let $y = (1 - e')f_1(1 - e')$; then y is self-adjoint and a computation yields that

$$\|y - f\| = \|(1 - e')f_1(1 - e') - (1 - e)f(1 - e)\| < 3a.$$

By Lemma 2.1 there is a projection $f' \in (1 - e')B(1 - e')$ with $\|f - f'\| < 6a$, and the result is proved.

Definition 2.5. Suppose that A and B are unital C^*-algebras acting on the Hilbert space H, and suppose that $\|A - B\| < a(\leqq 1/6)$. Then for $[e] \in S(A)$ we define $\rho([e]) \in S(B)$ by

$$\rho([e]) = [f] \text{ where } f \in P(B) \text{ satisfies } \|e - f\| < a.$$

We observe that such an f always exists by Lemma 2.4 and that ρ is well-defined by Lemma 2.3.

THEOREM 2.6. *Let* A *and* B *be unital* C^*-*algebras on* H *and suppose that* $\|A - B\| < a(\leqq \frac{1}{8})$. *Then* ρ *is an isomorphism of* $S(A)$ *onto* $S(B)$.

Proof. Interchanging A and B in the definition gives an inverse for ρ, so that ρ is one-to-one and onto; we must show that ρ and ρ^{-1} are local semigroup homomorphisms. Suppose that $[e] + [f]$ is defined in $S(A)$ so that we may assume e is orthogonal to f. Then $\rho([e]) = [e']$ where $e' \in P(B)$ and $\|e - e'\| < a$ and $\rho([f]) = [f']$ for $f' \in P(B)$ and $\|f - f'\| < a$. By Lemma 2.4 we can find a projection $f'' \in P(B)$ such that $\|f - f''\| < 6a$ and f'' is orthogonal to e'. Hence $\|f' - f''\| < 7a < 1$ and so by Lemma 2.2 $[f'] = [f'']$; thus $[e'] + [f']$ is defined and equals $[e' + f'']$. Now $\rho([e + f]) = [g']$ where $g' \in P(B)$ and $\|e + f - g'\| < a$; then

$$\|g' - (e' + f'')\| < 8a \leqq 1$$

and so $[g'] = [e' + f'']$ by Lemma 2.2. It follows that ρ is a homomorphism of local semigroups; the same argument implies that ρ^{-1} is also, and the result is proved.

COROLLARY 2.7. *Let* A *and* B *be* (*unital*) AF-*algebras on* H *such that* $\|A - B\| < a(\leqq \frac{1}{8})$. *Then there is an isomorphism* ϕ *of* A *onto* B *such that* $[\phi(e)] = \rho([e])$ *for* $e \in P(A)$.

Proof. This follows immediately from Theorems 2.6 and 1.2.

THEOREM 2.8. *Let* A *and* B *be* (*unital*) AF-*algebras on* H *such that*

$$\|A - B\| < a(\leqq (1/305)^2).$$

Then there is a unitary operator $u \in (A \cup B)''$ *such that* $uAu^* = B$.

Proof. If $\|A - B\| < a$ then $\|\bar{A} - \bar{B}\| < a$ by Lemma 5 of [**9**]. Since $\bar{A}$ and $\bar{B}$ satisfy the hypotheses of Theorem 4.1 of [**3**], we can conclude that there is a unitary $w \in (A \cup B'')$ such that $w\bar{A}w^* = \bar{B}$ and $\|1 - w\| < 19\sqrt{a}$. Now let $C = wAw^*$ so that

$$\bar{C} = \bar{B} \text{ and } \|C - B\| < a + 38\sqrt{a} < 1/8.$$

By Corollary 2.7 there is an isomorphism $\phi : C \rightarrow B$ such that $[\phi(e)] = \rho([e])$ for $e \in P(C)$. Thus for each $e \in P(C)$ there is a projection $f \in P(B)$ such that $\|e - f\| < a + 38\sqrt{a}$ and $\phi(e) \backsim f$ in B; hence by Lemma 2.2 $\phi(e) \backsim e$ in $\bar{C} = \bar{B}$. Now Theorem 1.3 applies and we can deduce that there is a unitary $v \in \bar{C}$ such that $vCv^* = B$; taking $u = vw$ gives the result.

2.9. *Remarks.* It would be interesting to obtain a bound on $\|1 - u\|$. If we knew that the isomorphism $\phi : A \rightarrow B$ constructed by Elliott was close to the inclusion map $i : A \rightarrow B(H)$, then it would follow from Theorem 2 of [**12**] (or Theorem 5.1 of [**8**]) that we could choose u close to 1.

In his most recent paper E. Christensen has also proven a version of Theorem 2.8 [**4**, cf. 7.1, 7.2, 7.3]. Note that by combining 7.1 and 7.3 of [**4**], one can omit the hypothesis that both A and B are AF-algebras.

References

1. O. Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. *171* (1972), 195–234.

2. E. Christensen, *Perturbations of type* 1 *von Neumann algebras*, J. London Math. Soc. *9* (1975), 395–405.

3. ——— *Perturbations of operator algebras*, Invent. Math. *43* (1977), 1–13.

4. ——— *Near inclusions of C*-algebras*, preprint.

5. E. Effros and J. Rosenberg, *C*-algebras with approximately inner flip*, Pac. J. Math. *77* (1978), 417–443.

6. G. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra *38* (1976), 29–44.

7. J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. *95* (1960), 318–340.

8. B. E. Johnson, *Perturbations of Banach algebras*, Proc. Lond. Math. Soc. (3) *34* (1977), 439–458.

9. R. Kadison and D. Kastler, *Perturbations of von Neumann algebras I. Stability of type*, Amer. J. Math. *94* No. 1 (1972), 38–54.

10. J. Phillips and I. Raeburn, *On extensions of* A.F.*-algebras*, Amer. J. Math., to appear.

11. R. Powers, *Representations of uniformly hyperfinite algebras and their associated von Neumann rings*, Ann. of Math. *86* (1967), 138–171.

12. I. Raeburn and J. Taylor, *Hochschild cohomology and perturbations of Banach algebras*, J. Functional Anal. *25* (1977), 258–266.

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