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EXISTENCE OF SOLUTIONS OF PLANE TRACTION PROBLEMS FOR INEXTENSIBLE TRANSVERSELY ISOTROPIC ELASTIC SOLIDS

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Abstract

A plane strain or plane stress configuration of an inextensible transversely isotropic linear elastic solid with the axis of symmetry in the plane, leads to a harmonic lateral displacement field in stretched coordinates. Various displacement and mixed displacement-traction boundary conditions yield standard boundary-value problems of potential theory for which uniqueness and existence of solutions are well established. However, the important case of prescribed tractions at each boundary point gives a non-standard potential problem involving linking of boundary values at opposite ends of chords parallel to the axis of material symmetry. Uniqueness and existence of solutions, within arbitrary rigid motions, are now established for the traction problem for general domains.

1. Introduction

The common practice of reinforcing an elastic matrix by strong fibres produces, in the simplest case of approximately homogeneous and parallel embedding, a relatively inextensible elastic material transversely isotropic about the fibre direction. For infinitesimal deformations in plane strain or plane stress configurations containing the fibre direction, a stress function satisfying a generalised biharmonic equation is obtained. However, if the inextensible limit is taken, the inextensible theory (IT) described by Morland [1] and England et al. [2], leads to the lower order potential theory in which the lateral displacement is harmonic in stretched coordinates. This simplification is helpful for both analytic and numerical purposes. Typical displacement and mixed displacement — traction boundary conditions for domains with no finite boundary section parallel to the fibre direction lead to standard boundary-value

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problems of potential theory [1] with one exception, a mixed displacement—traction problem. Also the important traction problem, in which the traction is prescribed at each point of the boundary, leads to non-standard boundary conditions.

In the traction problem boundary values at the ends of chords parallel to the fibre direction (fibre chords) where they intersect the domain boundary are linked. When the boundary is not convex to a given fibre line, more than one fibre chord exists on that line, and boundary values are linked in pairs from the ends of each chord. Iterative or other approximate numerical methods will be required for general domains, so it is important to establish uniqueness and existence of solutions for these non-standard boundary conditions to show that the potential theory is a compatible approximation to the full equations allowing low extensibility.

A proof of uniqueness, within an arbitrary rigid motion for the traction problem, is now presented for the typical boundary conditions listed in [1]. It is assumed that prescribed boundary displacements are continuous and prescribed boundary tractions are piecewise-continuous and bounded. Then the traction problem is discussed and under a weak restriction on the form of boundary existence is established for a domain which has a Green's function of the second kind (Neumann function); that is, for which the standard Neumann problem has a solution. A simply connected bounded domain convex to all fibre lines, and with no finite boundary section parallel to the fibre direction, is treated first, and the problem is reduced to a Fredholm integral equation of the second kind with a weakly singular symmetric kernel. The homogeneous equation has a single eigenfunction corresponding to a rigid motion and orthogonal to the right-hand function of the non-homogeneous equation, so existence of a solution in L_2 is established. It then follows that the solution corresponds to continuous shear and transverse stress in the interior of the domain.

The same integral equation is obtained if finite boundary sections parallel to the fibre direction are included, provided that the prescribed tangential traction is not met continuously on such sections. That is, there exists a unique stress field in the potential theory if shear stress discontinuity is allowed at such parallel boundary sections. This boundary layer effect has been explained by Everstine and Pipkin [3] for the idealised theory (IDT) which assumes inextensibility in two orthogonal directions in the plane. The shear stress discontinuity arises in general on boundaries parallel to either direction when both normal and tangential tractions are prescribed, but some simple exact solutions illustrate how the shear stress discontinuity is the limit (as extensibility approaches zero) of a thin boundary layer of high shear stress gradient. A singular perturbation scheme for low extensibility based on IDT has been applied to a cantilever beam problem by Everstine and Pipkin [4], and recently

Spencer [5] has developed boundary layer approximations based on IT. Uniqueness and existence for the traction problem in IDT has been established by Pipkin and Sanchez [6] for a simply connected bounded domain convex to both sets of fibre lines.

For the bounded domain not convex to all fibre lines, simply or multiply connected, the traction problem is reduced to a system of Fredholm equations which in turn can be transformed into a single equation of the previous form. Again there is an eigenfunction corresponding to rigid motion and orthogonal to the right-hand function; but now the interior stress may have shear stress discontinuities (interior boundary layers) along limit fibre lines intersecting the boundary at turning points. When the domain extends to infinity, appropriate limit behaviour leads to a standard boundary value problem.

For domains and boundary tractions which do not lead to boundary layers at external boundaries or at limit lines in the interior, the potential solution satisfies the full boundary conditions and is the zero order approximation to a regular perturbation for low extensibility.

2. Elasticity Equations and Potential Theory

Plane strain or plane stress in the Oxy plane of rectangular Cartesian coordinates are considered, and infinitesimal strain is assumed. For a linear elastic material inextensible in the Ox direction and transversely isotropic about Ox, the in-plane stress-displacement relations [1] are

$$\sigma_{yy} = \frac{\mu_L}{c^2}\frac{\partial v}{\partial y}, \qquad \sigma_{xy} = \mu_L\left(\frac{\partial v}{\partial x} + \frac{du}{dy}\right), \tag{2.1}$$

where μ_L is the longitudinal shear modulus and

$$c^2 = \frac{\mu_L}{k_T + \mu_T} \qquad \text{or} \qquad \frac{\mu_L(k_T + \mu_T)}{4\mu_T k_T} \tag{2.2}$$

for plane strain or plane stress respectively, where μ_T, k_T are the transverse shear and bulk moduli respectively. It is supposed that c^2 is of order unity. The displacement field has components

$$u_x = u(y), \qquad u_y = v(x, y), \tag{2.3}$$

where the dependence $u(y)$ reflects the inextensibility. The stress σ_{xx} is arbitrary, not determined by the deformation, until determined by boundary conditions.

In the absence of body force equilibrium in the x and y directions gives repectively

$$\sigma_{xx} = \mu_L\left\{t(y) - xu''(y) - \frac{\partial v}{\partial y}\right\}, \tag{2.4}$$

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$$c^2 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{2.5}$$

The arbitrary fibre direction tension is now represented by the function $t(y)$. However, at boundaries $y = \text{constant}$, $t(y)$, $u(y)$ are constants and in general there is no solution of (2.5) compatible with prescribed shear and normal tractions σ_{xy}, σ_{yy} [1]. Some simple exact solutions for small finite extensibility [3] demonstrate the existence of boundary layers of high shear stress gradient and high tension in the fibre direction, approximated in the inextensible case by shear stress discontinuity and infinite fibre tension. Thus, across an infinitesimal layer of thickness δ at a boundary $y = \text{constant}$, equilibrium in the x-direction requires

$$\frac{\partial \Sigma_{xx}}{\partial x} + [\sigma_{xy}] = 0, \tag{2.6}$$

where $\Sigma_{xx} = \lim(\delta\sigma_{xx})$ as $\delta \to 0$ is the total finite tension in the layer. The shear stress jump allows the prescribed shear traction to differ from the interior solution given by satisfying the prescribed normal traction condition. Such shear stress discontinuities can also arise at interior limit lines in the x-direction starting at turning points of internal boundaries.

Introducing stretched coordinates

$$X = x, \qquad Y = cy, \tag{2.7}$$

and writing

$$u = U(Y), \qquad v = V(X, Y), \qquad t = T(Y), \tag{2.8}$$

the basic equations become

$$\begin{gathered} \sigma_{yy} = \frac{\mu_L}{c}\frac{\partial V}{\partial Y}, \qquad \sigma_{xy} = \mu_L\left\{cU'(Y) + \frac{\partial V}{\partial X}\right\}, \\ \sigma_{xx} = \mu_L\left\{T(Y) - c^2 XU''(Y) - c\frac{\partial V}{\partial Y}\right\}; \end{gathered} \tag{2.9}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} = 0. \tag{2.10}$$

Thus the lateral displacement function $V(X, Y)$ is harmonic and the complete field description contains two further functions of a single variable: $U(Y)$, $T(Y)$.

Figures 1 and 2 show typical domains in the stretched coordinate plane OXY of simply connected bounded cross-sections of a body. In Fig. 1 the domain D is convex to all fibre lines and the boundary C is the union of two simple arcs C_L and C_R (left and right sections) each spanning an interval

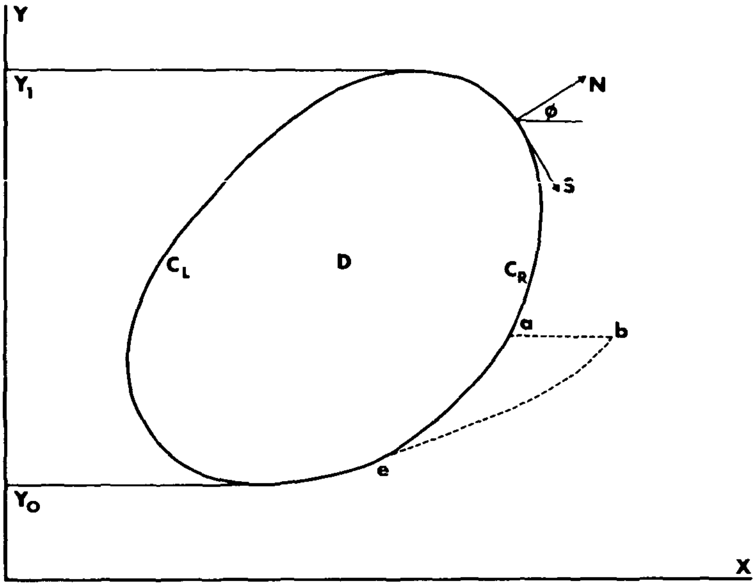

Fig. 1. Domain convex to fibre lines.

$I(Y_0, Y_1)$. At each point on C (S, N) denote local tangential and outward normal coordinates and ϕ is the inclination of the outward normal to OX. If D is extended by taking C_R along the dashed contour abe, then the case of a boundary section ab parallel to the fibre direction is included.

Figure 2 illustrates a domain not convex to all fibre lines. Constructing interior limit lines Y = constant through the re-entrant turning points A, B, E in the Y-direction subdivides D into a set of domains $D_1 - D_7$ each convex to fibre lines. A similar partition can be made for multiply connected domains. Each subdomain $D_r(r = 1, \cdots, R$ in general) has a boundary formed by subsections of C_L and C_R, defined as boundary sections with $\pi/_2 < \phi < 3\pi/_2$ and $-\pi/_2 < \phi < \pi/_2$ respectively, and a limit line Y = constant, and spans a Y-interval I_r. Each fibre chord meets C_L and C_R at its end points. However, a given fibre line Y = constant may pass through several of the convex subdomains, in Fig. 2 for example there are fibre lines common to D_1 and D_2, D_4 and D_5, D_4, D_6 and D_7, D_6 and D_7. The functions $U(Y)$, $T(Y)$ need not be identical on disjoint fibre chords of a common fibre line, so it is convenient to introduce single-valued functions $U_r(Y)$ $T_r(Y)$ on the interval I_r of each subdomain D_r. The displacement $U(Y)$ must be continuous across a limit line, but the tension $T(Y)$ must may be discontinuous there. Thus

$$D_r\colon U = U_r(Y), \qquad T = T_r(Y), \qquad Y \in I_r. \tag{2.11}$$

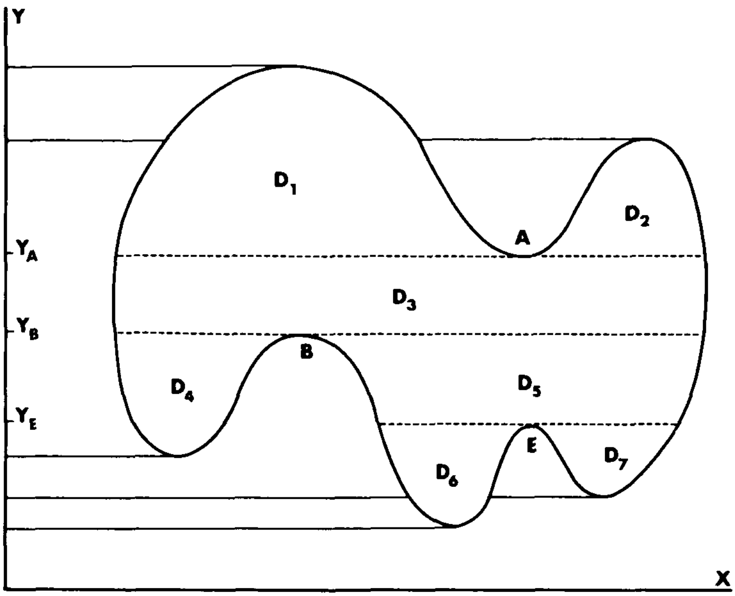

Fig. 2. Non-convex domain.

If $\mu_L t_x$, $\mu_L t_y$ are the boundary tractions, then writing

$$\int_0^s t_x\, ds' = \frac{1}{c}\int_0^S (\cos^2\phi + c^2\sin^2\phi)^{\frac{1}{2}} t_x\, dS' = g(S), \tag{2.12}$$

$$(\cos^2\phi + c^2\sin^2\phi)^{\frac{1}{2}} t_y = f(S), \tag{2.13}$$

where s denotes arc length on the boundary in Oxy coordinates, it follows that on C [1],

$$V + cXU'(Y) = g(S) + J(Y), \tag{2.14}$$

$$\frac{\partial V}{\partial N} + c\cos\phi\, U'(Y) = f(S), \tag{2.15}$$

where

$$J(Y) = -\frac{1}{c}\int^S \cos\phi\, T(Y')dS' = \frac{1}{c}\int^Y T(Y')dY'. \tag{2.16}$$

For the non-convex domain the first integral defines distinct functions $J_r(Y)$ on each D_r, continuous at the common end points $Y_A \cdots$ between subdomains; it is supposed that T is bounded.

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3. Boundary Conditions and Uniqueness

Since $U'(Y)$, $J(Y)$ in (2.14), (2.15) are constant along fibre chords it is helpful [1] to describe boundary conditions with explicit reference to C_L and C_R, specifically at opposite ends of fibre chords in both convex and non-convex domains. Prescribed displacement component U on C_L and C_R must be compatible with inextensibility — constant on a fibre chord. If different conditions are applied on C_L and C_R there is a corresponding problem with C_L and C_R interchanged, and pairing of conditions on C_L and C_R applies also to subsections of C_L and C_R over a common Y-interval. A list of typical boundary conditions is given in [1].

Let $\hat{V}(S), \hat{g}(S) \cdots$ denotes boundary values of $V, g, \cdots$ respectively at the opposite end of the fibre chord through the arc point S. $J(Y)$ may be elimated from (2.14) to give

$$c(X-\hat{X})U'(Y) = g - \hat{g} - (V - \hat{V}) \tag{3.1}$$

in terms of values at both ends of the fibre chord, and substituting in (2.15) gives

$$\frac{\partial V}{\partial N} - \frac{\cos\phi}{X-\hat{X}}(V-\hat{V}) = f - \frac{\cos\phi}{X-\hat{X}}(g-\hat{g}). \tag{3.2}$$

In the traction problem t_x, t_y, and hence g, f, are prescribed on each point of C, and (3.2) is a non-standard boundary condition for harmonic V, involving the values V and $\hat{V}$ at each S.

In the mixed problem

$$t_x, V \quad \text{on} \quad C_L, \quad t_x, t_y \quad \text{on} \quad C_R, \tag{3.3}$$

g is given everywhere on C, together with

$$V = V_L \quad \text{on} \quad C_L, \qquad f = f_R \quad \text{on} \quad C_R. \tag{3.4}$$

Now (3.1) relates $U'(Y)$ to the unknown V on C_R, and taking $\hat{V}$ in (3.2) to signify V_L, etc., (3.2) gives

$$C_R\colon \frac{\partial V}{\partial N} - \frac{\cos\phi}{X-\hat{X}} V = f_R - \frac{\cos\phi}{X-\hat{X}}(g-\hat{g}+\hat{V}), \tag{3.5}$$

which, with the first of (3.4), defines a mixed boundary value problem for V, but the coefficient $\cos\phi/(X-\hat{X})$ is negative on C_R in contrast to the standard Robin condition. The other sets of boundary conditions listed in [1] prescribe standard boundary value problems for V.

Before investigating existence of solutions V for the traction conditions (3.2) it is important to establish uniqueness properties. If U, V are the displacement difference of two possible solutions for given boundary conditions, corresponding to a stress difference equilibrating with zero body force, it follows that

$$\oint (t_x U + t_y V) ds = \oint_{\bar{D}} \left\{ \left[\frac{\partial V}{\partial X} + cU'(Y) \right]^2 + \left(\frac{\partial V}{\partial Y} \right)^2 \right\} dA \tag{3.6}$$

where $\bar{D}$ is the body cross-section in Oxy. Thus for all boundary conditions which make the difference integrand of the LHS vanish everywhere, which include the traction problem and (3.3) and the list in [1], the RHS integrand must vanish on $\bar{D}$. Hence the difference displacement has the linear form

$$V = V_0 + V_1 X, \qquad U = U_0 - \frac{1}{c} V_1 Y, \tag{3.7}$$

representing a rigid translation and rotation, arbitrary for the traction problem. For the conditions (3.3) the difference reduces to

$$U = U_0, \tag{3.8}$$

a rigid translation in the x-direction.

For an unbounded domain with finite stress at infinity, a corresponding homogeneous stress field can first be subtracted. However, t_x traction on finite boundary sections may penetrate to infinity — finite $T(Y)$ — within the influence strip (Y-interval) of the finite boundary section. An illustration is seen in [1]. By (2.9), bounded σ_{xx} as $X \to \pm\infty$ implies $U''(Y) \equiv 0$, and continuity of U across the limit lines of the strip, supposed zero at infinity outside the strip, implies $U \equiv 0$ except in subdomains not extending to infinity in the X-direction. Here rigid motion at infinity is assumed absent. Thus there is no contribution to the boundary integral of (3.6) from finite $T(Y)$ at infinity, and the uniqueness result follows. Note that $U \equiv 0$ on a boundary section reduces a t_y prescription (2.15) to a standard normal derivative condition.

Summarizing, for displacement and traction conditions which make the boundary integral in (3.6) of a possible difference solution vanish, and on a domain for which the divergence theorem holds, there can be only one interior stress field with bounded gradient. For the pure traction problem the possible displacement difference represents a rigid motion.

4. Traction Problem for Convex Domain

First consider a bounded simply connected domain D convex to all fibre lines, and with no boundary section parallel to the fibre direction, as illustrated by the solid boundary in Fig. 1. $U(Y)$, $J(Y)$ are defined on an interval $I(Y_0 < Y < Y_1)$, and the left and right-hand boundary arcs can be represented as

$$C_L: S = S_L(Y), \qquad C_R: S = S_R(Y), \qquad Y \in I. \tag{4.1}$$

Each arc C_L, C_R may have a finite number of points at which the tangent turns through an angle $< \pi$ (no cusps), but a smoothness restriction at the turning points at Y_0, Y_1 is found to be necessary. It is assumed that the prescribed tractions t_x, t_y are bounded and piecewise continuous, so that $g(s)$ is bounded, continuous, and piecewise differentiable, and $f(s)$ is bounded and piecewise continuous. In fact, from self-equilibration of the tractions, $g(s)$ is periodic over the contour length and hence single-valued at each point on C, and

$$0 = \oint t_y \, ds = \frac{1}{c} \oint_C f(S) \, dS. \tag{4.2}$$

Since $\cos\phi \, U'(Y) = -d/dS[U(Y)]$, and U is single-valued, (2.15), (4,2) show that

$$\oint_C \frac{\partial V}{\partial N} \, dS = 0, \tag{4.3}$$

as required for harmonic V.

The zero moment condition is

$$\oint (x t_y - y t_x) ds = 0. \tag{4.4}$$

Now

$$\oint y t_x \, ds = -\oint g \, dy = \frac{1}{c} \oint_C g(S) \cos\phi \, dS, \tag{4.5}$$

since g is single-valued, and

$$\oint x t_y \, ds = \frac{1}{c} \oint_C X f(S) dS, \tag{4.6}$$

so (4.4) becomes

$$\oint_C (Xf - \cos\phi \, g) dS = 0. \tag{4.7}$$

For a simply connected bounded domain D in the plane there exists a symmetric Green's function of the second kind, see for example Sternberg and Smith [7],

$$G(P, Q) = G(Q, P) = -\frac{1}{2\pi} \log r + W(P, Q) \tag{4.8}$$

where P, Q are distinct points in D, r = length PQ, and $W(P, Q)$ is analytic in P

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and Q. Then any harmonic V in D is expressed in terms of its normal derivative on C by

$$V(P) = V_0 + \oint_C G(P, Q) \frac{\partial V(Q)}{\partial N_Q} dS_Q, \tag{4.9}$$

where V_0 is an arbitrary constant and the representation (4.9) is continuous onto C. Applying (2.15) and writing $G(S, \xi)$ for P at arc position S and Q at position ξ on C,

$$V(S) = V_0 + \oint_C G(S,\xi)f(\xi)d\xi + c\oint_C G(S,\xi)U'(\eta)\frac{d\eta}{d\xi}d\xi. \tag{4.10}$$

Introduce the fibre chord length

$$d(Y) = X[S_R(Y)] - X[S_L(Y)], \tag{4.11}$$

and write

$$g_R(Y) = g[S_R(Y)], \qquad g_L(Y) = g[S_L(Y)], \tag{4.12}$$

$$F(S) = \oint_C G(S,\xi)f(\xi)d\xi, \tag{4.13}$$

$$F_R(Y) = F[S_R(Y)], \qquad F_L(Y) = F[S_L(Y)], \tag{4.14}$$

$$H(Y) = \frac{1}{c}\{g_R(Y) - g_L(Y) - F_R(Y) + F_L(Y)\}. \tag{4.15}$$

$F(S)$ is the boundary value of a harmonic function on D with normal derivative $f(S)$ on C, and is continuous and single-valued at each point of C. Thus $H(Y)$ is continuous on I and

$$H(Y_0) = H(Y_1) = 0. \tag{4.16}$$

Further, for $Y = Y_0 + \delta$ or $Y_1 - \delta$,

$$H(Y) = O(\delta\log\delta) \quad \text{as} \quad \delta \to 0. \tag{4.17}$$

Eliminating V and $\hat{V}$ in (3.1) by (4.10) gives

$$Y \in I: d(Y)U'(Y) + \int_I K(Y,\eta)U'(\eta)d\eta = H(Y), \tag{4.18}$$

where

$$\begin{aligned} K(Y,\eta) &= G[S_R(Y), \xi_L(\eta)] - G[S_L(Y), \xi_L(\eta)] \\ &\quad - G[S_R(Y), \xi_R(\eta)] + G[S_L(Y), \xi_R(\eta)], \\ &= K(\eta, Y), \end{aligned} \tag{4.19}$$

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using the symmetry of $G(P, Q)$. That is, a Fredholm integral equation of the second kind with logarithmically singular kernel. This is reduced to a standard form, Tricomi [8]:

$$Y \in I: p(Y) + \int_I \bar{K}(Y, \eta) p(\eta) d\eta = \bar{H}(Y), \tag{4.20}$$

by the substitutions

$$p(Y) = d^{\frac{1}{2}}(Y) U'(Y), \qquad H(Y) = d^{\frac{1}{2}}(Y) \bar{H}(Y),$$
$$K(Y, \eta) = d^{\frac{1}{2}}(Y) d^{\frac{1}{2}}(\eta) \bar{K}(Y, \eta), \tag{4.21}$$

and $\bar{K}(Y, \eta)$ is symmetric and logarithmically singular at $Y = \eta \in I$.

5. Integral Equation

While $K(Y, \eta)$ is weakly singular the kernel $\bar{K}(Y, \eta)$ of the standard Fredholm equation (4.20) has additional singular behaviour at the end points Y_0, Y_1 of I due to the vanishing of $d(Y)$ there. To apply the standard Fredholm theorem [8] for L_2-kernels it is required that

$$\int_I \int_I \bar{K}^2(Y, \eta) dY d\eta < \infty. \tag{5.1}$$

For $Y = Y_0 + \delta$ or $Y_1 - \delta$ let

$$d(Y) = O(\delta^\nu) \quad \text{as} \quad \delta \to 0, \quad \nu > 0. \tag{5.2}$$

From the structure (4.19) and form of $G(P, Q)$, (4.8), $K(Y, \eta) \to 0$ linearly as $\eta \to Y_0, Y_1$, fixed $Y \in I$, and as $Y \to Y_0, Y_1$, fixed $\eta \in I$, but is logarithmically singular on $Y = \eta$. For each chord length associated with the points in (4.19), writing $\eta = Y_0 + \varepsilon$ or $Y_1 - \varepsilon$,

$$r^2 > (\delta - \varepsilon)^2, \tag{5.3}$$

and hence as $\delta, \varepsilon \to 0$,

$$|\log r| < |\log(\delta - \varepsilon)|. \tag{5.4}$$

Thus in the corners of the square $I \times I$ where both $d(Y)$, $d(\eta)$ vanish, the dominant contributions to the integral (5.1) have the form

$$\int_0^h \int_0^h \frac{(\log|\delta - \varepsilon|)^2}{\delta^\nu \varepsilon^\nu} d\delta\, d\varepsilon.$$

In terms of polar coordinates (ρ, ω), $-\pi/4 < \omega < \pi/4$, the most significant term is

$$\int_0^1 \frac{(\log \rho)^2}{\rho^{2\nu - 1}} d\rho \int_{-\pi/4}^{\pi/4} \frac{(\log|\sin \omega|)^2}{(\cos 2\omega)^\nu} d\omega,$$

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so the necessary and sufficient condition for (5.1) is

$$\nu < 1.$$

That is, a slope discontinuity ($\nu = 1$) at Y_0, Y_1 is excluded and dY/dX vanishes at the turning points, but the curvature need not be finite ($\nu \leqq \frac{1}{2}$). Thus, with the conditions (5.2), (5.5), $\bar{K}(Y, \eta)$ is a symmetric L_2-kernel. Further, recalling (4.17), (4.21), $\bar{H}(Y)$ vanishes at Y_0 and Y_1, and is continuous and bounded on I.

The homogeneous equation associated with (4.20), $\bar{H}(Y) \equiv 0$, corresponds to the zero traction problem $t_x \equiv t_y \equiv 0$. Hence there must be an eigensolution

$$p_0(Y) = kd^{\frac{1}{2}}(Y) \tag{5.6}$$

corresponding to the rigid motion (3.7). That is

$$\int_I K(Y, \eta)d\eta = -d(Y). \tag{5.7}$$

The left hand integral is $\psi[S_R(Y)] - \psi[S_L(Y)]$ where

$$\psi(S) = -\oint_C G(S, \xi)\cos\phi\, d\xi \tag{5.8}$$

is the boundary value of a harmonic function on D with normal derivative $-\cos\phi$. Thus ψ is determined uniquely within an arbitrary contact ψ_0, and is

$$\psi = \psi_0 - X, \tag{5.9}$$

which confirms (5.7). By the uniqueness theorem there is no other non-trivial linearly independent eigensolution corresponding to the zero traction problem. For $t_x \equiv t_y \equiv 0$ the solution (5.6), $U'(Y) = \text{constant}$, implies V is linear in X, independent of Y, by (2.15), and hence $J(Y) = \text{constant}$ and (3.7) follows by (2.14). That is, the eigensolution (5.6) implies a rigid motion (3.7).

It remains to show that $\bar{H}(Y)$ is orthogonal to $p_0(Y)$ to establish the existence of a unique $p(Y)$ within an arbitrary multiple of $p_0(Y)$ by the Fredholm theorem. That is

$$\int_I \bar{H}(Y)d^{\frac{1}{2}}(Y)dY = \int_I H(Y)dY = 0. \tag{5.10}$$

By (4.12)–(4.15),

$$c\int_I H(Y)dY = \oint_C \cos\phi\, g(S)dS + \oint_C F(S)dY. \tag{5.11}$$

$F(S)$ is the boundary value of a harmonic function on D with normal derivative $f(S)$ on C. Let $E(S)$ be the boundary value of the conjugate harmonic function; then

$$C\colon \frac{\partial E}{\partial S} = -f(S), \tag{5.12}$$

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and hence

$$-\oint_C Xf(S)dS = -\oint_C E(S)dX = \oint_C F(S)dY \tag{5.13}$$

since $F + iE$ is analytic on D, and continuous onto C. Hence, combining (5.11), (5.13), and recalling the zero moment condition (4.7), (5.10) is satisfied.

Thus there exists an L_2-solution $p(Y)$ to (4.20). Since the integral and $\bar{H}(Y)$ are continuous and bounded on I, then so is $p(Y)$ and $U'(Y) = O(\delta^{-\frac{1}{2}\nu})$ near Y_0, Y_1 at worst. But direct from (3.1), noting that $g - \hat{g}$ and $V - \hat{V}$ are $O(\delta S) = O(\delta^\nu)$ as $Y \to Y_0, Y_1$, by (2.12) and (4.10), and $|X - \hat{X}| = O(\delta^\nu)$, it follows that $U'(Y)$ is bounded as $Y \to Y_0, Y_1$. Given $U'(Y)$ on I, (2.15) is a standard boundary condition to determine harmonic V on D, then (2.14) gives continuous and bounded $J(Y)$ on I. More specific, $T(Y)$ is determined by the boundary derivative of (2.14) and t_x.

Now consider finite boundary sections parallel to the fibre direction, such as the section ab of the dashed contour in Fig. 1. On such sections, not part of C_L nor C_R, the normal traction is prescribed, hence t_y and $f(S)$ are given and (2.15) applies, but the prescribed tangential traction t_x is not satisfied continuously in the potential approximation, so (2.14) is not applied. Using the representation (4.9) for V, (4.10) follows from (2.15) but now there is no contribution to the second integral from the contour ab where $d\eta/d\xi \equiv 0$. The boundary function g(s) using the known t_x on ab is defined by (2.12), but the functions $g_L(Y)$, $g_R(Y)$ given by (4.12) do not include values from ab. Applying (2.14), (3.1) to points on C_L, C_R gives the integral equation (4.18) again with the kernel K defined by (4.19) in which the functions $G(S, \xi)$ do not take values for S, ξ on ab. Also $d(Y)$ is discontinuous at Y_a.

The eigensolution property (5.6)–(5.9) holds as before since the contour integral definition (5.8) applies with section ab included since $\cos\phi \equiv 0$ there. Similarly, the g_R, g_L contributions in (5.11) may be expressed as the entire contour integral, and the orthogonality condition (5.10) follows as before. Thus the standard equation (4.20) is obtained, with kernel and $\bar{H}(Y)$ discontinuities through $d(Y)$. Thus there exists a unique solution $U'(Y)$ within an arbitrary constant which is bounded and continuous except (generally) at Y_a.

For both types of convex domain, σ_{xy}, σ_{yy} are continuous where $U'(Y)$ is continuous. By (2.14) differentiated along the boundary $U''(Y)$ is piecewise continuous (not defined at points such as Y_a) with discontinuities corresponding to those of t_x and t_y and implying discontinuous $T(Y)$ and σ_{xx}.

6. Non-convex Domain

Now consider a domain not convex to all fibre lines, as illustrated in Fig. 2. The partition by fibre limit lines though the re-entrant turning points such as

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A, B, E subdivides D into a set of domains D_r ($r = 1, \cdots, R$) each convex to fibre lines and on which there are defined single-valued functions $d_r(Y)$, $U_r(Y)$, $J_r(Y)$, $Y \in I_r$, with $U_r(Y)$, $J_r(Y)$, continuous across common limit lines. Any multiply connected domain can be partitioned in similar manner, and where unbounded in the X-direction any limit line extends to infinity. There C_L or C_R are parts of the arc at infinity, and appropriate limit behaviour is prescribed. Extension to boundary sections parallel to the fibre direction follows as before.

With $G(P, Q)$ again the Green's function of the second kind for the total domain D, the representation (4.10) becomes

$$V(S) = V_0 + \oint_C G(S, \xi) f(\xi)\, d\xi$$
$$+ c \sum_{r=1}^{R} \int_{I_r} \{G(S, \xi_R) - G(S, \xi_L)\} U'_r(\eta) d\eta. \tag{6.1}$$

Applying (3.1) separately to each subdomain D_s gives for each interval $I_s (s = 1, \cdots, R)$:

$$d_s(Y) U'_s(Y) + \sum_{r=1}^{R} K_{sr}(Y, \eta) U'_r(\eta) d\eta = H_s(Y). \tag{6.2}$$

Each $K_{sr}(Y, \eta)$ is defined by (4.19) with the S arguments referring to C_L, C_R of D_s and the ξ arguments to C_L, C_R of D_r. The kernels $K_{rr}(Y, \eta)$ have the logarithmic singularities. $H_s(Y)$ is given by (4.15) with the boundary functions for D_s. Substitutions of the form (4.21) reduces the equations (6.2) to a standard system of simultaneous Fredholm integral equations [8] p. 150.

Introducing composite functions $U'(Y), H(Y)$ over the total interval $I = I_1 + \cdots + I_R$ (end to end), equal to $U'_r(Y), H'_r(Y)$ on each subinterval I_r, and a composite kernel $K(Y, \eta)$ over the square $I \times I$ similarly, the system reduces to a single standard Fredholm equation for $U'(Y)$ on I. The properties (5.7) for $K(Y, \eta)$ and (5.10) for $H(Y)$ again follow since the contributions from each K_r, H_r add up to the complete contour integrals (5.8), (5.11) as before. Thus there exists a unique solution $U'(Y)$ within an arbitrary constant. Since $\bar{H}(Y)$ is discontinuous (bounded) at the joining points of adjacent I_r in I, $U'(Y)$ will also be discontinuous at these points, in general, so that shear stress discontinuities arise on the interior limit lines.

Summarizing, the existence of a piecewise continuous stress field for the traction problem has been established on a plane domain for which a Green's function of the second kind exists (including multiply connected domains and domains extending to infinity) under the following conditions:

a) The boundary tangent slope has only a finite number of discontinuities, with no cusps.

b) Approaching any turning point $Y = Y_1$ where dY/dX changes sign, the fibre chord length $d(Y) = O(|Y - Y_1|^{\nu})$, $0 < \nu < 1$, excluding a slope discontinuity there; that is, $dY/dX = 0$.

c) The applied boundary tractions t_x, t_y are bounded and piecewise continuous.

d) On boundary sections $Y =$ constant only the prescribed normal traction t_y is applied, and a shear traction discontinuity is allowed.

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