AN ATTEMPT TO RECONCILE THE DYNAMICAL AND RADAR DETERMINATIONS OF THE ASTRONOMICAL UNIT

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RÉSUMÉ. — L'auteur a pu obtenir une solution qui satisfait aussi bien les observations optiques que celles par radar, sans augmenter considérablement les résidus. Il s'ensuit que l'unité astronomique est déterminée de façon plus satisfaisante à l'aide des échos-radar sur Vénus que par la méthode dynamique seule. L'auteur ajoute quelques remarques sur les valeurs des masses de Mercure, Vénus et Mars.

ABSTRACT. — It has been found possible to obtain a solution which satisfies both the dynamical and radar investigations without doing any great injustice to any of the observations. It is deduced that the astronomical unit is determined more satisfactorily from the radar bounces off Venus rather than by dynamical methods alone. Some comments are also given concerning the masses of Mercury, Venus and Mars.

ZUSAMMENFASSUNG. — Es wurde dis Möglichkeit einer Lösung gefunden, die sowohl die dynamischen als auch die Radar-Untersuchungen befriedigt, ohne dass man dabei in grossen Widerspruch mit irgend einer Beobachtungen gerät. Man kann zeigen, dass die Astronomische Einheit befriedigender aus den Radio-Echos von Venus bestimmt wird als lediglich durch dynamische Methoden. Es werden auch einige Bemerkungen zu den Massen von Merkur, Venus und Erde gemacht.

Резюме. — Автор смог получить решение удовлетворяющие оптические наблюдения также как и наблюдения радаром, не увеличивая значительно остатки. Из этого следует, что астрономическая единица определена более удовлетворительным образом пользуясь радиоэхом Венеры нежели динамическим методом. Автор делает несколько замечаний о массах Меркурий, Венеры и Марса.

Several independent determinations of the astronomical unit from radar bounces off Venus in 1961 agree to within about 1000 km of 149 599 000 km. On the other hand, the best dynamical derivation — that by Rabe [1] from the motion of (433) *Eros* between 1926 and 1945 — leads to the grossly different value of 149 527 000 ± 10 000 (m. e.) km. There exist, however, strong correlations between some

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of the orbital elements of the Earth and Eros, and Eckstein [2] has resolved this difficulty by substituting into Rabe's equations the corrections to the Earth's orbital elements provided by Duncombe's [3] discussion of the observations of Venus; and he also assumed that no corrections were to be applied to Newcomb's value of the masses of Mercury, Venus and Mars. His results lead to a value for the astronomical unit of 149 546 000 ± 2 000 km, and a very similar figure was obtained by assuming that Newcomb's values for the elements of the Earth's orbit were correct. Thus Eckstein found that it was easily possible to diminish the discrepancy between the dynamical and the radar values by some 26 %, and it was this discovery that motivated the greater part of this present investigation.

The discussion of the Lincoln Laboratory measurements by Pettengill *et al.* [4] shows that if Newcomb's orbits for Venus and the Earth are employed, there appears to be an increase in the value of the astronomical unit over the period covered by the range measurements.

If Duncombe's corrections to the orbital elements of Venus and the Earth are applied, this trend is diminished but is nevertheless still present. There are in addition small periodic variations with a period of about one month. It would seem desirable to attempt to remove the systematic trend, and to examine the periodic variation to ensure that it is not of a gravitational character, such as might be due to the omission of significant terms in Newcomb's tables. To eliminate any possible gravitational effect numerical integrations of the orbits of the two planets have been performed at the Jet Propulsion Laboratory, and these were adjusted so that the agreement with Newcomb's tables was as close as possible. In order to attend to the systematic variation a solution was made of Duncombe's normal equations groups 6 to 9, together with some afforded by the radar measurements. These groups pertain to optical observations of Venus made between 1886 and 1949 and Duncombe made solutions for the following 14 quantities :

$$A = \frac{1}{2}(\Delta M_0 + \Delta r - \Delta l''); \qquad H = \Delta\alpha + \Delta l'';$$

$$B = \frac{1}{2}\Delta p; \qquad I = \Delta\varepsilon;$$

$$C = \frac{1}{2}\Delta q; \qquad J = \Delta d_\delta;$$

$$D = e\Delta r + 0.8\,\Delta e'' - 1.4\,e''\,\Delta\varpi''; \qquad K = \Delta d_\alpha;$$

$$E = \Delta e - 1.4\,\Delta e'' - 0.8\,e''\,\Delta\varpi''; \qquad L = \frac{1}{2}\Delta e''$$

$$F = \frac{\Delta m}{5\,m}; \qquad M = \frac{1}{2}e''\,\Delta\varpi'';$$

$$G = \Delta\delta; \qquad N = \frac{1}{4}\Delta l''.$$

The doubly primed quantities refer to the usual orbital elements of the Earth and the unprimed ones to those of Venus. Thus A to E are principally connected with the orbit of Venus and L to N with that of the Earth.

No information about F (one-fifth of the fractional correction to be applied to the mass of Mercury), G to I (concerned with corrections to the fundamental co-ordinate system), or J and K (corrections due to the phase of Venus) can be provided by the radar measurements, so Duncombe's equations were modified by substituting his values of these quantities, the secular variations being applied where relevant to bring them up to 1961.3, the epoch of the radar observations. The 25 Lincoln Laboratory measurements furnish values of τ, the one-way radar time to Venus, and we have

$$\tau = \Lambda\, \Delta,$$

where Δ is the distance to Venus, and if this is measured in astronomical units and τ is measured in milliseconds, Λ is the astronomical unit in light-milliseconds. Now,

$$\begin{aligned} d\tau &= \Lambda\, d\Delta + \Delta d\Lambda \\ &= \Lambda\, d\Delta + \Lambda Q \Delta \end{aligned}$$

where

$$Q = \frac{d\Lambda}{\Lambda};$$

and if Λ is assumed to be 499 000 and $d\Delta$ and Q are measured in seconds of arc, we have

$$d\tau = 2.41922\,(d\Delta + Q\Delta).$$

From the rectangular co-ordinates of the two planets

$$\Delta^2 = (x - x'')^2 + (y - y'')^2 + (z - z'')^2;$$

so that

$$d\Delta = \frac{x - x''}{\Delta}\, d(x - x'') + \frac{y - y''}{\Delta}\, d(y - y'') + \frac{z - z''}{\Delta}\, d(z - z'').$$

The differentials on the right-hand side of this equation may readily be expressed in terms of the usual elements and hence of the eight unknowns A to E, L to N, so that we have 25 equations of the form

$$d\tau = a\mathrm{A} + b\mathrm{B} + c\mathrm{C} + d\mathrm{D} + e\mathrm{E} + l\mathrm{L} + m\mathrm{M} + n\mathrm{N} + q\mathrm{Q},$$

with

$$q = 2.41922\,\Delta.$$

The radar observations may be divided into four groups of different accuracies, and two assumptions were made about their relative weightings : that $\sqrt{w}$ for the four groups are (*a*) 2, 4, 25, 11, and (*b*) 1, 2, 5, 3. The relative weighting of the optical and radar measurements provides a more difficult problem, for whilst the former cover a period of 64 years and the latter only 3 months, the optical measurements do not give any information about the unknown Q in which we are interested. Consequently, several trials were made, the results of some of them being given in table I. We have

$$d\mathrm{A} = 2.41922\,\mathrm{Q},$$

and taking

$$c = 299.7925\ \mathrm{km/ms},$$
$$d\mathrm{A} = 725.264\,\mathrm{Q\ km}.$$

TABLE I.

Solutions of optical and radar data combined.

$\sqrt{w}$ (radar)	(a)	(b)	(a)	Optical
$\sqrt{w}$ (radar v. optical)	25 : 1.	25 : 1.	1 : 1.	only.
	$''$	$''$	$''$	$''$
A	+0.301	+0.275	+0.248	+0.244
B	+0.160	+0.161	+0.161	+0.161
C	+0.142	+0.139	+0.135	+0.135
D	+0.076	+0.080	+0.085	+0.085
E	+0.085	+0.073	+0.058	+0.056
L	−0.012	−0.017	−0.031	−0.033
M	−0.057	−0.067	−0.073	−0.074
N	−0.045	−0.045	−0.045	−0.045
Q	+2.025	+2.002	+2.131	–

TABLE II.

Residuals of the Lincoln Laboratory radar measurements from the first solution in table I.

1961.	Light distance to Venus (ms).	a. u. (km).	1961.	Light distance to Venus (ms).	a. u. (km).
Mar. 6...	−0.49	−340	Apr. 10...	−0.03	− 40
7...	−0.93	−670	12...	+0.09	+100
14...	−0.16	−130	12...	+0.19	+200
16...	+0.04	+ 30	18...	+0.08	+ 80
22...	−0.30	−270	20...	−0.03	− 30
23...	−0.49	−450	21...	−0.03	− 30
24...	−0.13	−120	24...	−0.01	− 10
27...	−0.66	−630	26...	0.00	0
31...	−0.37	−370	28...	0.00	0
Apr. 3...	−0.40	−410	May 3...	+0.03	+ 30
3...	−0.36	−370	16...	+0.40	+280
5...	−0.42	−440	18...	+0.44	+290
8...	−0.13	−130			

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Thus $Q = + 2.025$ corresponds to $dA = + 1469$ km, and added to

$$A = 299.7925 \times 499\,000 = 149\,596\,458 \text{ km}$$

this gives 149 597 927 km for the astronomical unit. The values of Q determined with other weighting-systems lead to values of the astronomical unit within 100 km of this, but if the uncertainty in c is 0.5 km/s that in A is 250 km. The (O—C) residuals of the radar measurements from the first solution are given in table II and it may be noted that the monthly variation remains : no completely satisfactory explanation for this has been offered, the use of numerical integrations ruling out the possibility of a gravitational cause. It is quite possibly connected with variations in solar activity. A slight systematic effect also remains, and this may perhaps be due to solar variations of longer period.

In the work by Rabe on the motion of Eros the following unknowns were adopted :

1	ΔM_0	Orbit of Eros
2	$100\,\Delta n$	
3	$\Delta\varphi$	
4	Δs	
5	Δp	
6	Δq	
7	$10^{-4}\,\theta_{\oplus + ☾}$	Mass corrections
8	$10^{-4}\,\theta_{♀}$	
9	$10^{-4}\,\theta_{♂}$	
10	$10^{-4}\,\theta_{☿}$	
11	$\Delta l''$	Orbit of the Earth
12	$\Delta\varepsilon$	
13	$\Delta e''$	
14	$e''\,\Delta\varpi''$	
15	$\Delta\alpha_0$	Negatives of the equinox and equator point corrections
16	$\Delta\delta_0$	

The mass corrections are applied by means of the relations

$$\theta = 20.6265\,\boldsymbol{\theta}$$

and

$$\frac{m_0^{-1}}{m^{-1}} = 1 + \boldsymbol{\theta},$$

where the m_0^{-1} are

⊕ + ☾	328 390
♀	408 000
♂	3 093 500
☿	6 000 000

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It was not possible to reconcile Rabe's observational equations with his normal equations, but it is understood that the former are correct. The resulting reciprocal masses of the four inner planets are as follows, and Rabe's solutions are given for comparison :

⊕ + ☾	328 434 ± 81	328 452 ± 64
♀	408 658 ± 310	408 645 ± 309
♂	3 113 000 ± 16 000	3 110 000 ± 11 000
☿	6 107 000 ± 70 000	6 120 000 ± 64 000
[vv]	7.60	7.55

It was assumed that

$$\pi_{\odot}^{3} = \frac{2.236933 \times 10^{8}}{1 + m^{-1}_{\oplus + ☾}}$$

the constant being consistent with that given by Brouwer [5] when $\mu^{-1} = 81.364$, though a rather smaller value is probable. An increase of 0.1 in μ^{-1} causes an increase of 33 in the last place of the constant. Thus the solution above corresponds to $\pi_{\odot} = 8''.7983 \pm 0''.0007$, and taking the equatorial radius of the Earth as 6 378 166 m [6], the astronomical unit becomes 149 528 000 ± 12 000 km.

Rabe found corrections to the elements of the Earth's orbit which differ somewhat from those found from other sources. A comparison of the work by Duncombe on Venus, by Clemence in 1943 [7] on Mercury and by Morgan and Scott in 1939 [8] on the Sun gives the following values for the epoch 1932.7 (the mean date of the Eros observations) :

$\Delta l''$	$-0''.124 \pm 0''.047$
$\Delta \varepsilon$	-0.028 ± 0.012
$\Delta e''$	-0.120 ± 0.009
$e'' \Delta \varpi''$	-0.090 ± 0.018

Similarly, a comparison of various determinations of the masses of Venus, Mars and Mercury gives the following values of their reciprocals :

♀	408 000 ± 700
♂	3 088 000 ± 4 000
☿	6 140 000 ± 180 000

The seven equations provided by these data were then solved together with the 74 equations considered by Rabe. The reciprocal masses came out as follows :

⊕ + ☾	328 580	($\pi_{\odot} = 8.''7971$)
♀	408 457	
♂	3 089 000	
☿	6 142 000	

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The $[vv]$ of Rabe's equations is increased only to 8.57, but the resulting $\pi_{\odot}$ is forced 0.3 of the way from the original solution to that determined from the radar measurements. Eckstein's value of 8″.79726, obtained by direct substitution into Rabe's equations of Duncombe's values for the corrections to the Earth's orbital elements, is in very close accordance.

It is instructive to combine both the Eros and the Venus material into one solution. The number of independent unknowns is 23, and it is convenient to take the 16 used by Rabe and to add to them the following from Duncombe's work on Venus :

17 : J; 18 : K; 19 : A; 20 : B; 21 : C; 22 : D; 23 : E.

Eight unknowns are common to the two investigations, and the relations between them are as follows :

7.	$10^{-4}\,\theta_{\oplus+☾}$	$= -0.0314548 - 0.000299858\,Q$
10.	$10^{-4}\,\theta_{☿}$	$= 103.132\,F$
11.	$\Delta l''$	$= 4\,N$
12.	$\Delta\varepsilon$	$= I$
13.	$\Delta e''$	$= 2\,L$
14.	$e''\,\Delta\varpi''$	$= 2\,M$
15.	$\Delta\alpha_0$	$= H - 4\,N$
16.	$\Delta\delta_0$	$= G$

For this purpose Duncombe's groups 8 and 9 only were utilized, the mean epoch being so close to that of the Eros observations that any secular changes in the orbital elements of the Earth could be ignored. Since the p. e. of unit weight for the Eros equations is ± 0″.243 and that of Duncombe's Venus equations is ± 0″.67, the latter were multiplied by 0.36 prior to combining them with the former. The equation concerning the mass of Mars was also included. The resulting solution is given as solution A in table III and the residuals of the Eros equations are given in table IV. The corresponding values of the reciprocal masses of the four inner planets agree very closely with the previous solution :

⊕ + ☾	328 579 ±	96
♀	408 479 ±	946
♂	3 088 900 ±	15 300
☿	6 130 000 ±	179 000

The orbital elements of Venus and the Earth are almost exactly what they would be if the Eros data were omitted, and since the relations between those elements of Eros prone to correlation with some of the Earth's are virtually eliminated, one would expect that the masses of

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the planets are determined rather more reliably. The resulting mass of the Earth-Moon system leads to :

$$\pi_{\odot} = 8.''7971 \pm 0,''0009, \qquad A = 149\,549\,000 \pm 15\,000 \text{ km}.$$

The purely dynamical determination of the solar parallax could undoubtedly be improved by including observations made much earlier than 1926. Perhaps a better value could be provided by (1566) *Icarus* or (1620) *Geographos*, which can approach the Earth much more closely than can Eros. It may be of interest to note that the only other recent dynamical determination (that from *Pioneer V* by the Space Technology Laboratory in 1960) is remarkably accordant with the above value :

$$\pi_{\odot} = 8.''7974 \pm 0.''0012.$$

But these results are still very much at variance with the radar measurements. At this point it might be worth while to consider whether there could be any tremendous systematic error in these measurements. Smith [9] has noted that a reexamination of the 1959 records revealed

TABLE III.

Solutions of Eros and Venus equations combined.

Unknown.	Solution A.		Solution B.		Solution C.
	″	″	″	″	″
1.	—0.275	±0.317	—0.607	±0.300	—0.683
2.	+0.0436	±0.0298	+0.0001	±0.0268	+0.0242
3.	—0.056	±0.085	—0.074	±0.084	—0.074
4.	—0.431	±0.451	+0.125	±0.418	+0.223
5.	—0.084	±0.136	—0.124	±0.135	—0.111
6.	—0.111	±0.088	—0.118	±0.088	—0.126
7.	—0.011851	±0.006063	—0.032017	±0.000061	—0.032019
8.	—0.0242	±0.0476	+0.0509	±0.0419	—0.0335
9.	+0.031	±0.103	+0.255	±0.079	+0.490
10.	—0.436	±0.573	—1.328	±0.505	—1.373
11.	—0.111	±0.088	—0.127	±0.088	—0.118
12.	—0.045	±0.037	—0.045	±0.037	—0.045
13.	—0.082	±0.052	—0.069	±0.052	—0.074
14.	—0.107	±0.053	—0.093	±0.052	—0.098
15.	+0.043	±0.092	+0.060	±0.092	+0.051
16.	—0.099	±0.027	—0.094	±0.027	—0.094
17.	+0.359	±0.026	+0.357	±0.026	+0.357
18.	—0.553	±0.019	—0.554	±0.019	—0.554
19.	+0.250	±0.018	+0.256	±0.018	+0.256
20.	+0.059	±0.021	+0.060	±0.021	+0.060
21.	+0.017	±0.029	+0.019	±0.028	+0.019
22.	+0.050	±0.023	+0.049	±0.023	+0.049
23.	+0.120	±0.024	+0.123	±0.024	+0.124

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TABLE IV.

Residuals of Eros observations.

Date 0^h E. T.	Solution A.		Solution B.		Solution C.	
	$\Delta\alpha \cos\delta$.	$\Delta\delta$.	$\Delta\alpha \cos\delta$.	$\Delta\delta$.	$\Delta\alpha \cos\delta$.	$\Delta\delta$.
	$''$	$''$	$''$	$''$	$''$	$''$
1926. July 4.....	−0.43	−0.04	+0.20	+0.72	+0.35	+0.84
1928. Sept. 10.....	−0.22	−0.57	−0.69	−0.87	−1.02	−1.09
1930. Oct. 10.....	+0.14	−0.20	−0.04	−0.14	+0.05	−0.14
20.....	+0.39	−0.02	+0.21	+0.08	+0.32	+0.08
30.....	+0.26	+0.03	+0.09	+0.18	+0.22	+0.15
Nov. 9.....	+0.15	+0.20	+0.01	+0.38	+0.15	+0.33
19.....	+0.19	−0.25	+0.09	−0.05	+0.24	−0.13
29.....	+0.18	+0.12	+0.11	+0.34	+0.27	+0.23
Dec. 9.....	+0.29	−0.02	+0.25	+0.18	+0.40	+0.05
19.....	−0.33	+0.23	−0.35	+0.41	−0.21	+0.26
29.....	+0.17	+0.31	+0.17	+0.43	+0.29	+0.27
1931. Jan. 8.....	+0.23	−0.04	+0.21	+0.02	+0.32	−0.12
18.....	−0.18	+0.31	−0.22	+0.30	−0.14	+0.19
28.....	−0.30	+0.24	−0.36	+0.18	−0.32	+0.11
Feb. 7.....	+0.13	−0.25	+0.08	−0.33	+0.08	−0.34
17.....	−0.53	−0.31	−0.54	−0.37	−0.57	−0.34
27.....	−0.05	−0.30	−0.01	−0.32	−0.07	−0.27
Mar. 9.....	+0.12	−0.18	+0.19	−0.16	+0.12	−0.09
19.....	0.00	−0.05	+0.08	+0.10	0.00	+0.18
29.....	+0.28	−0.10	+0.34	−0.01	+0.27	+0.07
Apr. 8.....	+0.16	−0.26	+0.19	−0.13	+0.12	−0.05
18.....	+0.20	−0.17	+0.22	−0.02	+0.15	+0.06
28.....	+0.47	+0.03	+0.47	+0.18	+0.41	+0.25
1933. Mar. 26.....	−1.05	+0.75	−0.97	+0.84	−0.91	+0.83
May 31.....	−0.16	−0.18	+0.08	+0.01	+0.28	+0.04
Aug. 14.....	+0.24	−0.10	+0.44	−0.03	+0.49	−0.05
1935. July 12.....	−0.31	−0.10	−0.14	−0.01	−0.46	−0.21
Aug. 25.....	+0.85	+0.63	+1.47	+0.99	+0.88	+0.70
Nov. 9.....	+0.70	+0.27	+0.91	+0.44	+0.72	+0.30
1937. Nov. 3.....	+0.06	−0.43	+2.20	−0.39	+0.77	−0.27
1938. Jan. 14.....	−0.29	−0.52	−0.07	−0.44	−0.40	−0.49
Feb. 23.....	+0.50	+0.54	+0.54	+0.47	+0.33	+0.61
1940. July 30.....	−0.59	−0.34	−1.16	−0.33	−1.38	−0.37
1942. Aug. 13.....	−0.28	−0.06	−0.89	−0.47	−0.03	−0.01
1944. Sept. 15.....	+0.43	+0.26	−1.40	−0.33	+0.20	+0.12
Nov. 30.....	+0.12	+0.17	−0.08	+0.08	+0.16	+0.34
1945. Feb. 2.....	−0.58	−0.03	−0.32	−0.07	−0.78	−0.10

an echo from Venus corresponding to that value of the astronomical unit determined from the 1961 bounces, and Peabody [10] has reported that the preliminary results from the 1962 inferior conjunction also confirm

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those of 1961, the recent measurements yielding a value perhaps some 800 km greater. Kotelnikov [11] has succeeded in contacting Mercury by radar and the resulting measurements are in accord with his value of 149 599 300 km for the astronomical unit determined from Venus in 1961.

But the most definite argument that the radar information is essentially correct (in particular, that the assumption that propagation takes place with the speed of light in vacuo is satisfactory) is that the value thus provided for the distance of the Moon is in excellent agreement with that derived by other methods; and that even if it is in error by 2 km it would be unlikely to leave an uncertainty exceeding 500 km in the case of Mercury near inferior conjunction.

It is now therefore necessary to see how well the radar parallax satisfies the Eros equations. Rabe [12] has solved his equations forcing $m^{-1}_{\oplus+☾}$ to be 328 906, and finds that $[vv]$ is increased to 22.29, that many residuals are greater than $1''$ and that there are conspicuous systematic tendencies. We have solved the radar equations in conjunction with those provided by Duncombe and Rabe, the difference in epoch being allowed for, and the equations multiplied by $\sqrt{w} = 2.78$. Tables III and IV, solution B, give the results and residuals, but the residuals can scarcely be described as satisfactory, $[vv]$ being 20.72. Unknowns 7 to 10 lead to the following values of the reciprocals of the planetary masses :

⊕ + ☾	328 900.5 ± 1.0	♂..........	3 055 700 ± 11 500
♀..........	406 996 ± 823	☿	6 413 000 ± 164 000

and we should have

$$\pi_{\odot} = 8.''794188 \pm 0.''000009, \qquad A = 149\,597\,800 \pm 150 \text{ km},$$

though these mean errors are those actually resulting from the equations as solved and are consequently extremely conservative. This value of the mass of Venus is in very good agreement with that determined by Morgan and Scott [8] from their discussion of observations of the Sun. This value of the mass of Mercury is in remarkable agreement with those determined from the secular variations of the orbits of Mercury and the Earth by Clemence [13] and Brouwer [14], namely, $m^{-1}_{☿} = 6\,400\,000$ and $6\,430\,000$ respectively, where the latter's value has been adjusted on account of the change indicated in the mass of the Earth-Moon system. Duncombe's [3] value of 5 970 000 has a greater uncertainty, particularly if one compares the values furnished by his groups of equations individually. The close agreement between Duncombe's value and that determined from Encke's comet by Makover and Bokhan [15] is perphaps fortuitous, the latter being the weighted mean of 5 885 000 ± 300 000 (from observations between 1898 and 1911) and 6 280 000 ± 500 000 (from observations between 1937 and 1954), but apart from the obvious difficulties associated with observations of a diffuse object this latter method of determination depends heavily on the secular changes in the mean motion of the comet, some doubt having been expressed by Roemer [16] as to whether they actually exist. The value for Mars from solution B does not seem very reasonable, but it might be argued that this quantity is not well determined from these observations of Eros. Consequently, it is of interest to transfer it to the right-hand sides of the equations and then to see if a value can be assigned which leaves a satisfactory set of residuals. Thus there results solution C (*see* tables III and IV), in which $[vv]$ is only 13.73. For the reciprocal masses :

⊕ + ☾	328 900.5	♂	3 021 700
♀	408 664	☿	6 430 000

that of Venus having returned to its former value and that of Mars being understandably quite wild.

This last solution appears to be reasonably acceptable and a reconciliation between dynamics and radar has been achieved. One might still have qualms about the mass of Mars. But do we really know the mass of Mars as accurately as is claimed? Newcomb [17] adopted the value $m_{♂}^{-1} = 3\,093\,500$ determined by Hall in 1878 from the satellites, stating that he did not think it could be in error by more than one fiftieth part; the value from solution C differs from Hall's by one forty-third part. Van den Bosch [18] arrived at a value of 3 088 000 ± 7 000 as an average of 27 determinations from the satellites between 1877 and 1909, but drew no distinction between the direct measurements of position angle and distance of a satellite from the estimated center of the Martian disk 12 or 13 magnitudes brighter and the measurements relative to the Martian limbs. The measurements of this second type are much more consistent and yield values of $m_{♂}^{-1}$ quite significantly smaller than those of the first :

	From center.	From limbs.
Phobos	3 107 000	3 078 000
Deimos	3 092 000	3 080 000

Systematic errors are undoubtedly present, and it would seem that the only sure way of eliminating them is to measure the relative positions of the satellites photographically, leaving Mars out of the picture entirely. Photographs taken by Kuiper with the 82-inch reflector in 1956 have recently been measured and reduced by van Biesbroeck [19] A further promising opportunity for determining the mass of Mars is offered by a study of the motion of (1011) *Laodamia* [20].

An extension of the calculation from the secular variations by Brouwer [13] using also the observations of Venus produces the following values for the reciprocals of the planetary masses :

$$♀ : 408\,800 \pm 1\,500 ; \quad ♂ : 3\,092\,000 \pm 39\,000 ; \quad ☿ : 6\,260\,000 \pm 540\,000.$$

but one would have greater confidence in the use of secular variations if a combined discussion could be made of all the observations of the four inner planets, and perhaps the principal asteroids too.

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REFERENCES.

[1] E. Rabe, *Astron. J.*, vol. 55, 1950, p. 112.
[2] M. C. Eckstein, *Astron. J.*, vol. 68, 1963, p. 231.
[3] R. L. Duncombe, *Astron. Papers Amer. Ephemeris*, vol. 16, 1958, part 1.
[4] G. H. Pettengill, H. W. Briscoe, J. V. Evans, E. Gehrels, G. M. Hyde, L. G. Kraft, R. Price and W. B. Smith, *Astron. J.*, vol. 67, 1962, p. 181.
[5] D. Brouwer, *I. A. U. Symposium No.* 21, (*Bull. Astron*.t. XXV, p. 250).
[6] I. Fischer, *J. Geophys. Res.*, vol. 65, 1960, p. 2067.
[7] G. M. Clemence, *Astron. Papers Amer. Ephemeris*, vol. 11, 1943, part 1.
[8] H. R. Morgan and F. P. Scott, *Astron. J.*, vol. 47, 1939, p. 193.
[9] W. B. Smith, *Astron. J.*, vol. 68, 1963, p. 15.
[10] P. R. Peabody, Personal communication, 1963.
[11] V. A. Kotelnikov, *Dokl. U. S. S. R. Acad. Sc.*, vol. 147, 1962, p. 1320.
[12] E. Rabe, *I. A. U. Symposium No.* 21, (*Bull. Astron*.t. XXV, p. 220).
[13] G. M. Clemence, *Proc. Amer. Philosoph. Soc.*, vol. 93, 1949, p. 7.
[14] D. Brouwer, *Bull. Astron.*, vol. 15, 1950, p. 177.
[15] S. G. Makover and N. A. Bokhan, *Trudy Inst. Theor. Astron. Leningrad*, vol. 8, 1961, p. 135.
[16] E. Roemer, *Astron. J.*, vol. 66, 1961, p. 368.
[17] S. Newcomb, *The elements of the four Inner planets and the Fundamental constants of Astronomy*, Washington Government Printing Office, 1895, p. 99.
[18] C. A. van den Bosch, *De Massa's van de Groote planeten*, University of Utrecht Dissertation, 1927, p. 126.
[19] G. van Biesbrook, Personnal communication, 1963.
[20] E. Rabe, *Astron. J.* vol. 64, 1959, p. 344.