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ON A QUESTION OF HARTWIG AND LUH

SAMUEL J. DITTMER, DINESH KHURANA and PACE P. NIELSEN✉

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Abstract

In 1977 Hartwig and Luh asked whether an element a in a Dedekind-finite ring R satisfying $aR = a^2R$ also satisfies $Ra = Ra^2$. In this paper, we answer this question in the negative. We also prove that if a is an element of a Dedekind-finite exchange ring R and $aR = a^2R$, then $Ra = Ra^2$. This gives an easier proof of Dischinger's theorem that left strongly π-regular rings are right strongly π-regular, when it is already known that R is an exchange ring.

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1. Introduction

As a common generalisation of right artinian rings and algebraic algebras, Arens and Kaplansky in 1948 introduced a class of rings R such that for any $a \in R$ there exist an element $x \in R$ and a positive integer $n \geq 1$ (both depending on a) satisfying $a^{n+1}x = a^n$ (see [2, Theorem 3.1]). If an element $a \in R$ satisfies $a^nR = a^{n+1}R$ for some $n \geq 1$, then we say that a is *right strongly π-regular* (of index n), and define the left version analogously. It is clear that a is right strongly π-regular in R precisely when the chain of right ideals $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ stabilises. If every element of a ring R is right (or left) strongly π-regular, we say R is *right (or left) strongly π-regular*. Strongly π-regular rings have been widely studied; see, for example, [1, 3, 9, 13, 15].

In [10, page 74] Kaplansky asked:

> *Since it is customary in modern algebra to use chain conditions as a fundamental hypothesis, it is natural to ask what can be deduced from just the assumption $a^{n+1}x = a^n$. For example, does it enable one to construct idempotents?*

For more than 25 years the answer to the question raised by Kaplansky was not known and the classes of left and right strongly π-regular rings were studied separately. In 1976 Dischinger [7] proved the amazing result (with a very intriguing proof) that a ring is left strongly π-regular if and only if it is right strongly π-regular. Now such rings are simply called *strongly π-regular rings*. Note that for every element a in a strongly π-regular ring R, there exist $x \in R$ and $n \geq 1$ such that $a^n x$ is a nonzero idempotent. Thus, Kaplansky's question has a positive answer when interpreted globally.

Unfortunately, Dischinger's result does not hold true locally. Recall that a ring R is *Dedekind-finite* if $ab = 1$ implies $ba = 1$. Fix a ring R which is not Dedekind-finite, with elements a, b such that $ab = 1$ but $ba \neq 1$. For example, we can take $R = F\langle a, b : ab = 1\rangle$, where F is a field. It is clear that $aR = a^2R = \cdots = R$, but the chain of left ideals $Ra \supseteq Ra^2 \supseteq \cdots$ does not stabilise.

On the other hand, Azumaya proved in [3, Theorem 1] that if a is an element of a ring R with bounded index of nilpotence and $aR = a^2R$, then $Ra = Ra^2$. Equivalently, Dischinger's result is true locally for rings with bounded index of nilpotence. As a ring R with bounded index of nilpotence is Dedekind-finite (see [11, pages 7–8] for two different proofs), Hartwig and Luh [8, page 94] asked if the result of Azumaya holds in a Dedekind-finite ring. In other words, if a is an element in a Dedekind-finite ring R and $aR = a^2R$, then is it the case that $Ra = Ra^2$?

We construct a ring R with only *trivial idempotents* which has an element a such that $aR = a^2R$ but $Ra \neq Ra^2$. As non-Dedekind-finite rings have infinitely many idempotents, this answers the question of Hartwig and Luh in the negative. This also shows that Kaplansky's condition $a^{n+1}x = a^n$, which in view of Dischinger's result enables one to construct idempotents globally, does not enable one to construct idempotents locally.

On the positive side, we prove the local left–right symmetry of strong π-regularity in Dedekind-finite exchange rings. We end with a number of open problems.

2. The main example

Let $R = F\langle x, y : x^2y = x\rangle$, where F is any commutative ring with only trivial idempotents. Clearly $x^2R = xR$. We will prove that R has only trivial idempotents.

Monomials in the variables x and y can be put into *reduced* form, simply by repeatedly replacing instances of x^2y by x. This is because the given relation $x^2y = x$ yields a reduction system in the sense of [4]. Monomials in their reduced form provide a free F-module basis for R, which we denote by $\mathfrak{B}$. Given $m \in \mathfrak{B}$ and an element $r \in R$, we say that m is in the support of r, denoted $m \in \mathrm{supp}(r)$, if the coefficient of m in the reduced representation of r is nonzero. Notice that we have $Rx^2 \neq Rx$ since any monomial ending in x^2 will continue to end in x^2 after reductions. Given any monomial $m \in R$, we write $\mathrm{red}(m) \in \mathfrak{B}$ for the reduced form of m.

Any monomial $m \in \mathfrak{B}$ has the form

$$m = y^{n_k} x y^{n_{k-1}} \cdots x y^{n_1} x^{n_0}$$

for some unique integers $k \geq 1$, $n_1, n_2, \ldots, n_{k-1} \geq 1$ and $n_0, n_k \geq 0$. We will call k the *depth* of m, and denote it by $\mathrm{depth}(m)$. We say that $m' \in \mathfrak{B}$ is a *right subword* of $m \in \mathfrak{B}$ if firstly $m = m''m'$ for some $m'' \in \mathfrak{B}$ and secondly there are no reductions to perform

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when we concatenate m'' and m'. If $m = y^{n_k} x y^{n_{k-1}} \cdots x y^{n_1} x^{n_0}$, then for any $j \le k$, we define

$$m(j) = y^{n_j} x y^{n_{j-1}} \cdots x y^{n_1} x^{n_0},$$

which is a right subword of m.

If $m_1 = y^{s_j} x y^{s_{j-1}} \cdots x y^{s_1} x^{s_0}$ and $m_2 = y^{t_j} x y^{t_{j-1}} \cdots x y^{t_1} x^{t_0}$ are two elements of $\mathfrak{B}$ with the same depth, we say $m_1 \prec_j m_2$ if either $s_0 > t_0$ or there exists some j' such that $1 \le j' \le j$, $s_i = t_i$ for $i < j'$, and $s_{j'} < t_{j'}$.

Let $m_1, m_2 \in \mathfrak{B}$ have depths d_1 and d_2 respectively and set $d = \min\{d_1, d_2\}$. We say that $m_1 \prec m_2$ if either $m_1(d) \prec_d m_2(d)$, or $m_1(d) = m_2(d)$ and $d = d_1 < d_2$. Given these definitions, the following result is straightforward.

Lemma 2.1. *The relation $\prec$ is a total ordering on $\mathfrak{B}$.*

Lemma 2.2. *Let $m \in \mathfrak{B}$.*

(1) *If m starts on the left with y, then* $\mathrm{red}(x^2 m)(\ell) \prec_\ell m(\ell)$*, where*

$$\ell = \mathrm{depth}(\mathrm{red}(x^2 m)) - 1.$$

(2) *If m starts on the left with x, then* $\mathrm{red}(xm)(\ell) \prec_\ell m(\ell)$*, where*

$$\ell = \mathrm{depth}(\mathrm{red}(xm)) - 1.$$

In particular, if $m_1, m_2 \in \mathfrak{B}$, then either $m_1 m_2$ is already reduced or $\mathrm{red}(m_1 m_2) \prec m_2$.

Proof. (1) Write $m = y^{n_k} x y^{n_{k-1}} \cdots x y^{n_1} x^{n_0}$ with $n_k \ge 1$. Then

$$x^2 m = x y^{n_k - 1} x y^{n_{k-1}} \cdots x y^{n_1} x^{n_0}.$$

Note that the monomial on the right-hand side is in reduced form if $n_k > 1$ and in that case it is clear that $\mathrm{red}(x^2 m)(\ell) \prec_\ell m(\ell)$. If $n_k = 1$, then

$$x^2 m = x y^{n_{k-1} - 1} \cdots x y^{n_1} x^{n_0}.$$

The right-hand side is reduced if $n_{k-1} > 1$ and in that case it is again clear that $\mathrm{red}(x^2 m)(\ell) \prec_\ell m(\ell)$. Proceeding in this way we see that if $n_i > 1$ for any $1 \le i \le k$, then $\mathrm{red}(x^2 m)(\ell) \prec_\ell m(\ell)$. Otherwise $\mathrm{red}(x^2 m) = x^{n_0+1} \prec_0 m(0)$ and in this case $\mathrm{depth}(x^{n_0+1}) = 1$ so $\ell = 0$ still works.

The proof of (2) runs along the same lines as that of (1). Also the last assertion follows quickly from (1) and (2). □

Theorem 2.3. *The ring $R = F\langle x, y : x^2 y = x\rangle$, where F is a commutative ring with trivial idempotents, contains only trivial idempotents.*

Proof. Let $e^2 = e \in R$. The only monomials which multiply to a constant are constants (as there are no reductions which remove all instances of x and y). Thus, we see that the constant term of e must be an idempotent. As F contains only trivial idempotents we may assume $1 \notin \operatorname{supp}(e)$, replacing e by $1 - e$ if necessary. Suppose, by way of contradiction, that $e \neq 0$. Let M be the maximal monomial (with respect to $\prec$) in the support of e, and let m_0 be the shortest right subword of M in the support of e (that is, a right subword of M for which no proper right subword is also in the support). As $e^2 = e$ we must have $m_0 = m_1 m_2$ for some $m_1, m_2 \in \operatorname{supp}(e)$.

Case 1. Assume $m_1 m_2$ is already reduced. Then m_2 is a right subword of m_0. From the minimality of m_0, we have $m_2 = m_0$. Thus $m_1 = 1 \in \operatorname{supp}(e)$, a contradiction.

Case 2. If $m_1 m_2$ is not reduced, then by Lemma 2.2 we have $m_0 = \operatorname{red}(m_1 m_2) \prec m_2$. For any $j < \operatorname{depth}(m_0) = k$ we have $m_0(j) = m_2(j)$, because otherwise $M(j) = m_0(j) \prec_j m_2(j)$ which is not possible in view of the maximality of M. Thus the only way to have $m_0 \prec m_2$ is if $m_0 = m_0(k) \prec_k m_2(k)$ or $m_0 = m_0(k) = m_2(k)$ and $\operatorname{depth}(m_2) > k$. In either case, this implies that m_0 is a (proper) right subword of m_2.

Writing $m_2 = m_3 m_0$ we have $m_0 = m_1 m_3 m_0 = m_4 m_0$, where m_4 is the reduced form of $m_1 m_3$. If $m_4 m_0$ requires reduction to put it into its reduced form, then $m_0 = \operatorname{red}(m_4 m_0) \prec m_0$ by Lemma 2.2, a contradiction. So $m_4 m_0$ is already in reduced form and hence $m_4 = 1$. Thus $m_1 m_3 = 1$, which implies that $m_1 = 1$, another contradiction. □

3. Dedekind-finite exchange rings

In this section we will prove that if a is an exchange element in a Dedekind-finite ring R and $aR = a^2R$, then $Ra = Ra^2$. Towards that end we first introduce a new definition.

Definition 3.1. Given an element $a \in R$ we say that a is a *right exchange element* if for any right ideal $I \subseteq R$ with $aR + I = R$, there exists an idempotent $e \in aR$ such that $1 - e \in I$. Left exchange elements are defined similarly. Throughout this paper, we will only work with right exchange elements.

Note that this generalises the notion of a *suitable* element defined in [12]. Every (von Neumann) regular element is a right (and left) exchange element, which is seen by modifying [12, Lemma 2.8]. Furthermore, a ring R is an exchange ring if and only if every element is a right (and left) exchange element [12, Proposition 1.11].

Example 3.2. Note that if $a \in R$ has the property that ar is suitable for every $r \in R$, then a is an exchange element. The converse is not true. Take $a = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{Z})$. This is an idempotent, hence regular and an exchange element. However, $ae_{2,1} = 3e_{1,1}$ is not suitable.

Lemma 3.3. *If a is a regular element in a Dedekind-finite ring R, then $aR = a^2R$ implies that $Ra = Ra^2$.*

Proof. First note that if $a^2x = a$ and $aya = a$, then $a^2xya^2 = a^2$, so a^2 is also regular. The equality $aR = a^2R$ implies $Ra \cong Ra^2$. But as $Ra^2 \subseteq^{\oplus} Ra \subseteq^{\oplus} {}_RR$ and R is Dedekind-finite, it follows that $Ra^2 = Ra$. □

The Dedekind-finite hypothesis in the lemma cannot be dropped. Indeed, if R is not Dedekind-finite, then there exist $a, b \in R$ with $ab = 1 \neq ba$. So a is regular and $aR = a^2R$ but $Ra \neq Ra^2$.

With this lemma in place, we are now able to obtain a local version of Dischinger's theorem, for exchange elements. This generalises [14, Proposition 3], where R is assumed to be a semiperfect ring with nil Jacobson radical.

Theorem 3.4. *Let a be a right exchange element in a Dedekind-finite ring R. If $aR = a^2R$, then $Ra = Ra^2$.*

Proof. In view of Lemma 3.3 it is enough to prove that a is regular. As $aR = a^2R$,

$$aR + r_R(a) = R,$$

where $r_R(a) = \{r \in R : ar = 0\}$. Since a is an exchange element, there exists an idempotent $1 - e \in aR$ such that $e \in r_R(a)$. So

$$aR + eR = R \quad \text{and} \quad r_R(a) + (1 - e)R = R.$$

In particular, $e \cdot r_R(a) = eR$ and so $-e = ek$ for some $k \in r_R(a)$. We will show that $a - e$ is a unit.

Note that $(a - e) \cdot r_R(a) = e \cdot r_R(a) = eR$. Also $(a - e)(1 + k) = a - e - ek = a$. Thus both aR and eR are in $(a - e)R$. As $aR + eR = R$, we have that $(a - e)R = R$. This implies that $a - e$ is right invertible. But as R is Dedekind-finite, $a - e$ is a unit.

Write $a - e = u$. Then $e = (a - e)k = uk$ implies $u^{-1}e = k$. Thus $au^{-1}(a - u) = au^{-1}e = ak = 0$ implies that $au^{-1}a = a$. □

Corollary 3.5. *Dischinger's result holds locally in Dedekind-finite exchange rings.*

In Dischinger's proof of the left–right symmetry of strongly π-regular rings, the most difficult portion is showing that such a ring is π-regular, and hence an exchange ring. It would be nice if a more conceptual proof of this fact could be found.

As another application of the ideas in Theorem 3.4, we next prove that we can lift the left–right symmetry of strongly π-regular elements *locally*, if idempotents lift modulo the radical.

Corollary 3.6. *Let a be an element in a Dedekind-finite ring R with $a^nR = a^{n+1}R$ for some $n \geq 1$. Assume that idempotents lift modulo the Jacobson radical J. If $\overline{a} \in R/J$ is strongly π-regular, then so is $a \in R$.*

Proof. It suffices to treat the case where $n = 1$ and $\overline{a}$ is strongly regular. By Theorem 3.4, it is also enough to show that there exists an idempotent $e \in aR$ such that $1 - e \in r_R(a)$.

The equality $aR = a^2R$ is equivalent to $aR + r_R(a) = R$. So write $ar + k = 1$, where $r \in R$ and $k \in r_R(a)$. As $\overline{a} \in \overline{R} = R/J$ is strongly regular we have $\overline{a}\overline{R} \cap r_{\overline{R}}(\overline{a}) = (0)$. Thus, $\overline{ar}$ and $\overline{k}$ are complementary idempotents. In particular, $\overline{ar}$ is clean (the sum of a unit and an idempotent) in $\overline{R}$. As idempotents lift modulo J by assumption (and units always lift modulo J) this implies ar is clean, and hence suitable by the proof of [12, Proposition 1.8(1)]. Therefore, there exists an idempotent $e \in arR$ with $1 - e \in (1 - ar)R = kR \subseteq r_R(a)$ as desired. □

We do not know whether the condition that idempotents lift modulo the Jacobson radical can be dropped or not.

4. Open questions

What other properties, like Dedekind-finiteness, are easily provable using only a one-sided strongly π-regular assumption, and may lead to a local proof of Dischinger's theorem? Note that the Jacobson radical in a strongly π-regular ring (or more generally, a π-regular ring) is always nil. The example we constructed in Section 2 already has zero Jacobson radical, so that condition does not suffice to yield a local form of Dischinger's theorem. Another obvious fact is that factor rings of strongly π-regular rings are still strongly π-regular. In particular, factor rings must stay Dedekind-finite. In this case, the example we constructed does fail to have the necessary property. Indeed, if $R = F\langle x, y : x^2y = x\rangle$ with F a field, then $R/(1 - xy)$ is not Dedekind-finite.

Question 4.1. If R is a ring for which every factor ring is Dedekind-finite, are right strongly π-regular elements also left strongly π-regular?

There are other conditions that a strongly π-regular ring possesses, which are not quite as easy to obtain. For example, unifying numerous results in the literature Ara [1] proved that strongly π-regular rings have stable range one. Thus, we might also ask whether the ring we constructed in Section 2 has stable range one. The answer is no. Recall that if R has stable range one, so does every factor ring, and we just found a factor of R which is not Dedekind-finite, hence does not have stable range one.

Can we modify our example so that it has stable range one? We begin with the following result of Canfell [6, Corollary 4.5].

Lemma 4.2. *Let R be a ring with stable range one. If $aR = bR$, then $a = bu$ for some $u \in U(R)$.*

Proof. For completeness we include a proof here. As $aR = bR$ we have $a = br$ and $b = as$ for some $r, s \in R$. Then $rR + (1 - rs)R = R$ and so from the stable range condition there exists some $z \in R$ with $r + (1 - rs)z = u \in U(R)$. Multiplying on the left by b we have $a = br = b(r + (1 - rs)z) = bu$ as desired. □

With this lemma in mind, we see that if R has stable range one and $xR = x^2R$, we may as well assume that $x^2y = x$, with y a unit. Consider the ring $R' = F\langle x, y, z : x^2y = x,\ xz = x^2,\ yz = zy = 1\rangle$. An analysis similar to the one done before shows that if F is a commutative ring with only trivial idempotents, then R' has only trivial idempotents. (We think of $z = y^{-1}$. The only major change is that while we consider right subwords in the previous sense, we now also allow multiplication on the left by y^{-1}. So, for example, $y^{-3}xyx^2$ is a right subword of xy^3xyx^2.) This shows that right strongly π-regular elements, with *unit right quasi-inverses* are not left π-regular, even in a Dedekind-finite ring. It is not clear whether R' has stable range one. Thus, we ask the following question which is strictly weaker than Question 4.1.

QUESTION 4.3. If R has stable range one, are right strongly π-regular elements also left strongly π-regular?

Modifying [8, Proposition 7] we have the following lemma.

LEMMA 4.4. *Let $R \subseteq S$ be rings. If $a \in R$ satisfies $aR = a^2R$ and $Sa = Sa^2$, then $Ra = Ra^2$. In particular, any subring of a ring satisfying Dischinger's theorem locally, also satisfies Dischinger's theorem locally.*

PROOF. Write $a = a^2x = ya^2$ with $x \in R$ and $y \in S$. We have $ya = ya^2x = ax \in R$. So $y^2a = yax = ax^2 \in R$. Now we compute

$$(y^2a)a^2 = y(ya^2)a = ya^2 = a.$$

Hence $Ra^2 = Ra$, as desired. □

According to a result of Burgess and Raphael [5, Theorem 2.1], every ring can be embedded in a clean (and hence exchange) ring. Those rings which can be embedded in Dedekind-finite exchange rings satisfy Dischinger's theorem locally. Thus, the ring we constructed cannot be embedded in a Dedekind-finite exchange ring.

QUESTION 4.5. Can a ring with stable range one be embedded in a Dedekind-finite exchange ring?

We do not know the answer to this question, even for semilocal rings. A positive answer to this question would lead to a positive answer to Question 4.3.

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References

[1] P. Ara, 'Strongly π-regular rings have stable range one', *Proc. Amer. Math. Soc.* **124**(11) (1996), 3293–3298.

[2] R. F. Arens and I. Kaplansky, 'Topological representation of algebras', *Trans. Amer. Math. Soc.* **63** (1948), 457–481.

[3] G. Azumaya, 'Strongly π-regular rings', *J. Fac. Sci. Hokkaido Univ. Ser.* I **13** (1954), 34–39.

[4] G. M. Bergman, 'The diamond lemma for ring theory', *Adv. Math.* **29**(2) (1978), 178–218.

[5] W. D. Burgess and R. Raphael, 'On embedding rings in clean rings', *Comm. Algebra* **41** (2013), 552–564.

[6] M. J. Canfell, 'Completion of diagrams by automorphisms and Bass' first stable range condition', *J. Algebra* **176**(2) (1995), 480–503.

https://doi.org/10.1017/S0004972713000373 Published online by Cambridge University Press

[7] F. Dischinger, ‘Sur les anneaux fortement π-réguliers’, *C. R. Acad. Sci. Paris Sér.* A–B **283**(8) (1976), Aii, A571–A573 (in French).
[8] R. E. Hartwig and J. Luh, ‘On finite regular rings’, *Pacific J. Math.* **69**(1) (1977), 73–95.
[9] L. S. Huang and W. Xue, ‘An internal characterisation of strongly regular rings’, *Bull. Aust. Math. Soc.* **46**(3) (1992), 525–528.
[10] I. Kaplansky, ‘Topological representation of algebras. II’, *Trans. Amer. Math. Soc.* **68** (1950), 62–75.
[11] T. Y. Lam, ‘Exercises in modules and rings’, in: *Problem Books in Mathematics* (Springer, New York, NY, 2007).
[12] W. K. Nicholson, ‘Lifting idempotents and exchange rings’, *Trans. Amer. Math. Soc.* **229** (1977), 269–278.
[13] K. M. Rangaswamy and N. Vanaja, ‘A note on modules over regular rings’, *Bull. Aust. Math. Soc.* **4** (1971), 57–62.
[14] L. H. Rowen, ‘Finitely presented modules over semiperfect rings’, *Proc. Amer. Math. Soc.* **97**(1) (1986), 1–7.
[15] H.-P. Yu, ‘On strongly pi-regular rings of stable range one’, *Bull. Aust. Math. Soc.* **51**(3) (1995), 433–437.

SAMUEL J. DITTMER, Department of Mathematics,
Brigham Young University, Provo, UT 84602, USA
e-mail: samuel.dittmer@gmail.com

DINESH KHURANA, Department of Mathematics, Panjab University,
Chandigarh 160 014, India
e-mail: dkhurana@pu.ac.in

PACE P. NIELSEN, Department of Mathematics, Brigham Young University,
Provo, UT 84602, USA
e-mail: pace@math.byu.edu

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