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Arithmetic $\mathcal{D}$-modules on the unit disk

Richard Crew

With an appendix by Shigeki Matsuda

Abstract

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristic. We classify arithmetic $\mathcal{D}$-modules on $\mathrm{Spf}(\mathcal{V}[\![t]\!])$ up to certain kind of 'analytic isomorphism'. This result is used to construct canonical extensions (in the sense of Katz and Gabber) for objects of this category.

Introduction

Let k be a perfect field of characteristic $p > 0$, and $\mathcal{V}$ a complete discrete valuation ring of mixed characteristic with residue field k and fraction field K. This paper has two related objectives: on one hand, to classify holonomic $F\mathcal{D}^\dagger$-modules on $\mathrm{Spf}(\mathcal{V}[\![t]\!])$ up to isomorphism; on the other hand, to construct a theory of 'canonical extensions' in the sense of Katz and Gabber for these objects.

Neither problem is tractable as posed, since there are far too many arithmetic $F\mathcal{D}^\dagger$-modules on $\mathrm{Spf}(\mathcal{V}[\![t]\!])$. For example, if $M = \mathcal{V}[\![t]\!] \otimes_{\mathcal{V}} K$ has the $\mathcal{D}^\dagger$-module structure given by the constant connection, then there are infinitely many nonisomorphic extensions of M by itself (the isomorphism classes of extensions are given by the de Rham cohomology of M) and infinitely many of these have a Frobenius structure. This shows the difficulty of finding a classification, and also indicates that there can be no such thing as a canonical extension in this situation.

On the other hand, Matsuda [Mat02] gave a classification of quasi-unipotent isocrystals on the Robba ring $\mathcal{R}$, and constructed a theory of canonical extensions for them. Roughly speaking, what makes Matsuda's result possible is that $\mathcal{R}$ has finite-dimensional de Rham cohomology. This suggests that the natural object of study is a suitable localization of the category of holonomic $F\mathcal{D}^\dagger$-modules; roughly speaking, we should want to extend scalars from $\mathcal{O} = \mathcal{V}[\![t]\!]$ to the subring $\mathcal{O}^{\mathrm{an}} \subset K[\![t]\!]$ of power series convergent for $|t| < 1$.

This localization is constructed in § 5. Denote by $\mathcal{D}^\dagger$ Berthelot's ring of arithmetic differential operators on $\mathcal{V}[\![t]\!]$. We will construct a flat $\mathcal{D}^\dagger$-algebra $\mathcal{D}^{\mathrm{an}}$, the ring of *analytic differential differential operators* on $\mathcal{V}[\![t]\!]$. The category of coherent $\mathcal{D}^\dagger$-modules up to analytic isomorphism is the category whose objects are coherent $\mathcal{D}^\dagger$-modules, but morphisms are given by $\mathrm{Hom}(M, N) = \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}}(\mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{D}^\dagger} M, \mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{D}^\dagger} N)$. If M is an F-isocrystal on $\mathcal{R}^b$, then $\mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{D}^\dagger} M$ is isomorphic to $\mathcal{R} \otimes_{\mathcal{R}^b} M$, an F-isocrystal on $\mathcal{R}$; thus a classification of holonomic $F\mathcal{D}^\dagger$-modules up to analytic isomorphism will include a classification of F-isocrystals on $\mathcal{R}$.

Our classification is modeled on Malgrange's classification [Mal91] of germs of holonomic $\mathcal{D}$-modules at a point of a complex curve. In the regular singular case, they are classified by

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pairs of finite-dimensional $\mathbb{C}$-vector spaces (V, W) endowed with a pair of maps $c: V \to W$, $v: W \to V$ (the 'canonical' and 'variation' maps) such that $1 + uv$ is an automorphism. The spaces V, W are respectively the spaces of hyperfunction and microfunction solutions of the $\mathcal{D}$-module, or, alternatively, the spaces of nearby cycles and vanishing cycles of the dual module. In the irregular case, the classification requires supplementary data (Stokes structure) on the space of hyperfunction solutions. We will find a similar picture for holonomic $F\mathcal{D}^\dagger$-modules, except that the Stokes structure is replaced by a Galois action, and there is a Frobenius structure in addition. In the case of F-isocrystals on $\mathcal{R}^b$ (which are quasiunipotent by the p-adic monodromy theorem) we recover Matsuda's classification. Finally, the techniques behind the classification allow us to construct a theory of canonical extensions for the holonomic $F\mathcal{D}^\dagger$-modules up to analytic isomorphism.

The first two sections of this paper describe the classification of F-isocrystals on $\mathcal{R}$, following methods of Fontaine rather than Matsuda, since Fontaine's procedure is more closely related to our classification of holonomic $F\mathcal{D}^\dagger$-modules. The next section reviews briefly Berthelot's theory of arithmetic $\mathcal{D}$-modules, and gives some explanation of how his theory is to be adapted to the formal scheme $\mathrm{Spf}(\mathcal{V}[\![t]\!])$, which falls slightly outside his framework. It also reviews some results of [Cre06] on the preservation of holonomy by cohomological operations in the one-dimensional case. Section 5 is devoted to the construction properties of the $\mathcal{D}^\dagger$-algebra $\mathcal{D}^{\mathrm{an}}$ and the category of coherent $\mathcal{D}^\dagger$-modules up to analytic isomorphism. As an application we give a construction of i^+M when M is a holonomic $F\mathcal{D}^\dagger$-module on $\mathrm{Spf}(\mathcal{V}[\![t]\!])$ and i is the closed immersion defined by the divisor $t = 0$. We show, finally, that certain cohomological operations such as the holonomic dual, and the inverse image functors i^+, $i^!$ extend to the analytic category.

The classification itself is carried out in §§ 6 and 7. In § 6 we associate, to any holonomic $F\mathcal{D}^\dagger$-module M, a pair of finite-dimensional vector spaces $\mathbb{V}(M)$, $\mathbb{W}(M)$ over the maximal unramified extension K^{nr} of K, endowed with various supplementary data: a 'canonical' map $\mathbb{V}(M) \to \mathbb{W}(M)$, a 'variation' map $\mathbb{W}(M) \to \mathbb{V}(M)$, a Frobenius, and a Galois action, all satisfying various compatibilities. We call the category of such objects the category of *solution data* since the definition of the functors $\mathbb{V}$ and $\mathbb{W}$ is modeled on the spaces of hyperfunction and microfunction solutions in Malgrange [Mal91]. The main result of § 6 is that the functor $M \mapsto \mathbb{S}(M) = (\mathbb{V}(M), \mathbb{W}(M))$ is exact. In § 7, we show how to construct a holonomic $F\mathcal{D}^\dagger$-module $\mathbb{M}(S)$ from a solution datum S. For any solution datum S we have $S \simeq \mathbb{S}(\mathbb{M}(S))$, but a holonomic $F\mathcal{D}^\dagger$-module M is not in general isomorphic to $\mathbb{M}(\mathbb{S}(M))$. It is here that we need to use the analytic category: M and $\mathbb{M}(\mathbb{S}(M))$ are analytically isomorphic (i.e. their extensions to $\mathcal{D}^{\mathrm{an}}$ are isomorphic).

The last section of this paper is a theory of canonical extensions for holonomic $F\mathcal{D}^\dagger$-modules up to analytic isomorphism. First, we show how to associate a holonomic $F\mathcal{D}^\dagger$-module on $\mathbb{P}^1/K$ to any solution datum S, whose restriction to $\mathrm{Spf}(\mathcal{V}[\![t]\!])$ (viewed as the completion of the local ring at $0 \in \mathbb{P}^1/\mathcal{V}$) is isomorphic to $\mathbb{M}(S)$; this is basically an extension of the methods of § 7. We then characterize the holonomic $F\mathcal{D}^\dagger$-modules on $\mathbb{P}^1$ that arise in this way, and then show that the restriction functor is an equivalence. As an application, we construct a local Fourier transforms in the manner of Malgrange [Mal91] and Laumon [Lau87].

0.1 Notation and terminology

Throughout this paper, $\mathcal{V}$ is a complete mixed characteristic discrete valuation ring of characteristic $(0, p)$ with maximal ideal $\mathfrak{m}$, perfect residue field k, and fraction field K. Choose once and for all a power q of p such that k contains the field with q elements, and a lifting σ of the qth-power Frobenius of k to $\mathcal{V}$ and K. We also fix a local field $F \simeq k((t))$ of characteristic p, with integer ring $A \simeq k[[t]]$ and residue field k. Fix a lifting $\mathcal{O} \simeq \mathcal{V}[[t]]$ of A and an isomorphism $\mathcal{O} \otimes_{\mathcal{V}} k \simeq A$. Note that t denotes both a local parameter of A and an element of $\mathcal{O}$, and to increase the confusion we will sometimes use t to denote $\mathrm{Spec}(A)$; in context the meaning will be clear.

As always, the *Robba ring* $\mathcal{R}$ relative to $\mathcal{O}$ is the ring of formal Laurent series in t with coefficients in K, convergent in some annulus $r < |t| < 1$. The *bounded Robba ring* $\mathcal{R}^b \subset \mathcal{R}$ is the subring of bounded elements (i.e. represented by formal Laurent series with bounded coefficients, or equivalently the elements of $\mathcal{R}$ representing functions that remain bounded as $|t| \to 1$; some authors denote this by $\mathcal{E}_K^\dagger$), while the *integral Robba ring* $\mathcal{R}^0$ is the set of elements of $\mathcal{R}$ with integral coefficients. It is known to be a Henselian discrete valuation ring, having as a uniformizer any uniformizer of $\mathcal{V}$, and with fraction field $\mathcal{R}^b$ (see [Mat95] and the appendix to this paper for the Henselian property). Finally, we denote by $\mathcal{O}^{\mathrm{an}} \subset \mathcal{R}$ the subring of power series over K convergent for $|t| < 1$.

We denote by $\mathcal{A}^0$ the p-adic completion of $\mathcal{R}^0$ or of $\mathcal{O}[t^{-1}]$; this is the ring of formal Laurent series $\sum_{n\in\mathbb{Z}} a_n t^n$ with $a_n \in \mathcal{V}$ and $a_n \to 0$ as $n \to -\infty$, and is a Cohen ring for $k((t))$. The fraction field of $\mathcal{A}^0$ will be denoted by $\mathcal{A}$; it is sometimes called the *Amice ring*. Note that $\mathcal{R}^b$ is naturally a subring of $\mathcal{A}$, but there is no containment relation between $\mathcal{A}$ and $\mathcal{R}$.

We denote by $G = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ the absolute Galois group of F, and by $I \subset G$ the inertia subgroup; thus G acts on the maximal unramified extension K^{nr} of K via its quotient $G/I \simeq \mathrm{Gal}(k^{\mathrm{sep}}/k)$. We will say that a G-module M is *discrete* if the G-action is continuous for the discrete topology on M, or, equivalently, if $M = \varinjlim_H M^H$ where H runs throughout the directed system of open normal subgroups of G. If V is a K^{nr}-vector space of *finite dimension* and a discrete G-module, there is an open normal subgroup $H \subseteq G$ such that $V \simeq V^H \otimes_{(K^{\mathrm{nr}})^H} K^{\mathrm{nr}}$ as K^{nr}-vector spaces. To see this, it suffices to choose a K^{nr}-basis of V, and an open normal H that fixes every element of the basis.

Certain common tensor products will be denoted by subscripting, e.g. $M_{\mathbb{Q}}$ for $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and M_K for $M \otimes_{\mathcal{V}} K$.

In any category, the morphisms $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W')$ and $\mathrm{Hom}(V', W) \to \mathrm{Hom}(V, W)$ induced by $f : W \to W'$ and $g : V \to V'$ will be written f_* and g^* respectively.

Modules over a noncommutative ring are left modules, unless otherwise indicated.

1. p-adic hyperfunctions

We begin by observing that once the $\mathcal{V}$-algebra $\mathcal{O}$ has been fixed, all of the constructions in the last paragraph (relative to F) are determined up to canonical isomorphism. For example, $\mathcal{O}^{\mathrm{an}}$ is the function algebra of the rigid analytic space X canonically attached to the adic formal scheme $\mathrm{Spf}(\mathcal{O})$ (for the topology defined by the maximal ideal; see [Ber96b, 0.2.6]), while $\mathcal{R}$ is the direct limit $\varinjlim_U \Gamma(X - U, \mathcal{O}_X)$ where $U \subset X$ runs through affinoid subspaces. Finally the bounded Robba ring $\mathcal{R}^b \subset \mathcal{R}$ is the subset of functions bounded on some $X - U$, U affinoid, and the integral Robba ring $\mathcal{R}^0 \subset \mathcal{R}^b$ is the set of power-bounded elements of $\mathcal{R}^b$ for the natural topology induced by $\mathcal{R}$ (for this latter, and for the characterization of the bounded elements, see [Cre98, § 5, especially Lemma 5.2]).

We will consider pairs $u = (F(u), \mathcal{O}(u))$ where $F(u)/F$ is a finite separable extension, and $\mathcal{O}(u)$ is a formally smooth lifting of the integer ring $A(u)$ of the discretely valued field $F(u)$, i.e. a

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power series ring with coefficients in some finite unramified integral extension $\mathcal{V}(u)$ of $\mathcal{V}$. In this situation we define $\mathcal{R}(u)$, $\mathcal{R}^b(u)$, $K(u)$, and so forth to be the analogues relative to the pair u of the various rings constructed in § 0.1. The letter t will denote the 'base pair' $(F, \mathcal{O})$. A morphism of pairs $u \to v$ will be a pair consisting of an F-algebra homomorphism $F(u) \to F(v)$, and an $\mathcal{R}^0$-algebra homomorphism $\mathcal{R}^0(u) \to \mathcal{R}^0(v)$ reducing modulo the maximal ideal of $\mathcal{V}$ to the given map $F(u) \to F(v)$. Note that a morphism $u \to v$ induces ring homomorphisms $\mathcal{R}^b(u) \to \mathcal{R}^b(v)$, $\mathcal{R}(u) \to \mathcal{R}(v)$, but not necessarily $\mathcal{O}(u) \to \mathcal{O}(v)$ or $\mathcal{O}^{\mathrm{an}}(u) \to \mathcal{O}^{\mathrm{an}}(v)$.

If $F(u)/F$ is a finite extension we denote by $H(u) \subseteq G$ the open subgroup corresponding to $F(u)$; if in addition $F(u)/F$ is normal, we denote by $G(u)$ the Galois group of $F(u)/F$, so that $G(u) \simeq G/H(u)$. When $F(u)/F$ is Galois the action of $G(u)$ on $F(u)$ and $A(u)$ lifts uniquely to $\mathcal{R}^0(u)$, $\mathcal{R}^b(u)$, and $\mathcal{R}(u)$; this follows from the Henselian property of $\mathcal{R}^0$. The action does not always lift to $\mathcal{O}(u)$ or $\mathcal{O}^{\mathrm{an}}(u)$; see [Cre87, 3.5] for an analogous case.

Lemma 1.0.1. *If $F(u)/F$ is Galois with group $G(u)$, and $\mathcal{R}(u)/\mathcal{R}$, $\mathcal{R}^b(u)/\mathcal{R}^b$ are the corresponding ring extensions, then $\mathcal{R}(u)^G = \mathcal{R}$ and $(\mathcal{R}^b(u))^G = \mathcal{R}^b$.*

Proof. For the case of $\mathcal{R}^b(u)/\mathcal{R}^b$, which is an unramified extension of discretely valued fields with residue field extension L/K, this is clear; alternatively, one can deduce it from the case of $\mathcal{R}(u)$, using the equality $\mathcal{R}^b(u) \cap \mathcal{R}(u)^G = (\mathcal{R}^b(u))^G$.

Denote by X, Y the open unit disks $|t| < 1$, $|u| < 1$, so that $\mathcal{R} \to \mathcal{R}(u)$ corresponds to a finite map $\pi : Y \to X$. For any interval $I \subseteq [0, 1)$ with rational endpoints we denote by X_I the rigid-analytic subspace of X defined by $|t| \in I$, and we set $Y_I = \pi^{-1}(X_I)$. Finally, denote by $\mathcal{R}_I$ (respectively $\mathcal{R}(u)_I$) the function algebras of X_I (respectively Y_I). We then have

$$\mathcal{R} = \varinjlim_{r \to 1^-} \mathcal{R}_{[r,1)}, \quad \mathcal{R}(u) = \varinjlim_{r \to 1^-} \mathcal{R}(u)_{[r,1)}$$

and for $r < 1$ sufficiently close to 1, G acts on $Y_{[r,1)}$ with quotient $X_{[r,1)}$ (cf. [Cre87, 3.5]). Since the quotient map is finite étale, we have $\mathcal{R}(u)^G_{[r,1)} = \mathcal{R}_{[r,1)}$ by standard descent theory, and since the functor of G-invariants is exact, we find $\mathcal{R}(u)^G = \mathcal{R}$. □

1.1 Stable Robba rings

We now formally adjoin a logarithm of t to $\mathcal{R}$, following a method of Fontaine [Fon94]. Denote by $\mathcal{R}^1 \subset (\mathcal{R}^0)^\times$ the subgroup of units congruent modulo the maximal ideal of $\mathcal{V}$ to an element of $k[\![t]\!]$ with constant term 1. For $f \in \mathcal{R}^1$, the usual power series for the logarithm defines a homomorphism $\log : \mathcal{R}^1 \to \mathcal{R}^b$, where

$$\log(f) = \sum_{n>0} (-1)^{n+1} \frac{(f-1)^n}{n}. \tag{1.1.1}$$

Since k is perfect, there is a canonical embedding of the Witt vector ring $W(k) \hookrightarrow \mathcal{V}$ inducing the identity on the quotient rings. Thus if we set $\mathcal{R}^\sharp = K^\times \mathcal{R}^1$, we can extend the log map (1.1.1) to a homomorphism $\log : R^\sharp \to \mathcal{R}^b$ by requiring that $\log p = 0$ and $\log x = 0$ whenever x is the Teichmuller lifting of an element of $k^\times$.

We now consider the category $\mathcal{C}$ whose objects are pairs (S, λ) where S is an $\mathcal{R}^b$-algebra, and $\lambda : (\mathcal{R}^b)^\times \to S$ is a homomorphism extending $\log : \mathcal{R}^\sharp \to \mathcal{R}^b$. Morphisms $(S, \lambda) \to (S', \lambda')$ in $\mathcal{C}$ are $\mathcal{R}^b$-algebra homomorphisms $S \to S'$ sitting in the obvious commutative diagram. Since $(\mathcal{R}^b)^\times = t^{\mathbb{Z}} \cdot \mathcal{R}^\sharp$, it is clear that this category has an initial object $(\mathcal{R}^{b,\mathrm{st}}, \log)$, and that $\mathcal{R}^{b,\mathrm{st}}$ is a polynomial ring over $\mathcal{R}^b$ in one variable, which can be taken to be $\log t$ for any choice of local parameter t of $\mathcal{O}$. The universal property asserts the existence and uniqueness, for any (S,λ) in $\mathcal{C}$, of the dotted arrow in the diagram

$$\begin{tikzcd}
\mathcal{R}^\sharp \arrow[r, "\log"] \arrow[d] & \mathcal{R}^b \arrow[d] \arrow[ddr, bend left] & \\
(\mathcal{R}^b)^\times \arrow[r, "\log"] \arrow[drr, bend right, "\lambda"'] & \mathcal{R}^{b,\mathrm{st}} \arrow[dr, dotted] & \\
 & & S
\end{tikzcd} \tag{1.1.2}$$

making it commutative. Finally we set $\mathcal{R}^{\mathrm{st}} = \mathcal{R}\otimes_{\mathcal{R}^b}\mathcal{R}^{b,\mathrm{st}}$; since $\mathcal{R}^\times = \mathcal{R}^{b,\times}$ we can set $\log:\mathcal{R}^\times\to\mathcal{R}$ equal to the composite of the universal $\log:\mathcal{R}^{b,\times}\to\mathcal{R}^b$ with the identification $\mathcal{R}^\times=\mathcal{R}^{b,\times}$. As before we have $\mathcal{R}^{\mathrm{st}}\simeq\mathcal{R}[\log t]$ non-canonically, and we could of course have defined $\mathcal{R}^{\mathrm{st}}$ directly in this manner, but this construction makes it clear that $\mathcal{R}^{\mathrm{st}}$ is determined canonically by the original choice of $\mathcal{O}$. The usefulness of this will be evident shortly.

The module of relative one-forms $\Omega^1_{\mathcal{R}^{b,\mathrm{st}}/\mathcal{R}^b}$ is free of rank one, generated by $d\log t$. If t' is any other choice of local parameter, we have $t=ut'$ with $u\in\mathcal{R}^\sharp$, and thus $\log t$ and $\log t'$ differ by an element of $\mathcal{R}^b$ which is killed by the exterior derivative $d:\mathcal{R}^{b,\mathrm{st}}\to\Omega^1_{\mathcal{R}^{b,\mathrm{st}}/\mathcal{R}^b}$, so that $d\log t = d\log t'$. It follows that there is a canonical $\mathcal{R}^b$-derivation $N_{\mathcal{R}}:\mathcal{R}^{b,\mathrm{st}}\to\mathcal{R}^{b,\mathrm{st}}$ of $\mathcal{R}^{\mathrm{st}}$ defined by $N_{\mathcal{R}}(\log t)=1$. By extension of scalars we get a canonical $\mathcal{R}$-derivation $N_{\mathcal{R}}:\mathcal{R}^{\mathrm{st}}\to\mathcal{R}^{\mathrm{st}}$. In either case $N_{\mathcal{R}}$ will be called the *canonical monodromy operator;* it will be important later that it is surjective.

Set $\partial = d/dt$ and denote by $\mathcal{D}$ the ring $\mathcal{R}[\partial]$ of (usual) differential operators with coefficients in $\mathcal{R}$. The canonical $\mathcal{D}$-module structure of $\mathcal{R}$ extends to $\mathcal{R}^{\mathrm{st}}$ via the formula

$$\partial\log t = \frac{1}{t} \tag{1.1.3}$$

(note that this definition is independent of the choice of parameter). It is clear that $N_{\mathcal{R}}$ commutes with ∂, and so respects the $\mathcal{D}$-module structure. Similarly, if we set $\mathcal{D}^b=\mathcal{R}^b[\partial]$, then the $\mathcal{D}^b$-module structure of $\mathcal{R}^b$ extends to $\mathcal{R}^{b,\mathrm{st}}$ as before, and $N_{\mathcal{R}}$ is a $\mathcal{D}^b$-module endomorphism.

Suppose finally that $\varphi:\mathcal{R}^0\to\mathcal{R}^0$ is a lifting of the qth-power Frobenius map of F, and let φ also denote its (unique) extensions to $\mathcal{R}^b$ and $\mathcal{R}$. Since φ preserves the set of Teichmuller liftings of elements of $k^\times$, it commutes with the logarithm $\log:\mathcal{R}^\sharp\to\mathcal{R}^b$. Denote by $\varphi_*\mathcal{R}^{b,\mathrm{st}}$ the ring $\mathcal{R}^{b,\mathrm{st}}$ viewed as an algebra over itself via φ; since φ commutes with the inclusion $\mathcal{R}^\sharp\to(\mathcal{R}^b)^\times$, the pair $(\varphi_*\mathcal{R}^{b,\mathrm{st}},\log\circ\varphi)$ belongs to the category $\mathcal{C}$. Denoting by $i:\mathcal{R}^b\to\mathcal{R}^{b,\mathrm{st}}$ the structure morphism and applying the universal property (1.1.2) yields a commutative diagram

$$\begin{tikzcd}
\mathcal{R}^\sharp \arrow[r, "\log"] \arrow[d] & \mathcal{R}^b \arrow[d, "i"'] \arrow[ddr, bend left, "i\circ\varphi"] & \\
(\mathcal{R}^b)^\times \arrow[r, "\log"] \arrow[drr, bend right, "\log\circ\varphi"'] & \mathcal{R}^{b,\mathrm{st}} \arrow[dr, dotted, "\varphi"] & \\
 & & \mathcal{R}^{b,\mathrm{st}}
\end{tikzcd} \tag{1.1.4}$$

in which the commutative triangle on the right shows that the dotted arrow marked φ is indeed an extension of $\varphi:\mathcal{R}^b\to\mathcal{R}^b$, while the commutative triangle on the bottom shows that this

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extension commutes with $\log\colon (\mathcal{R}^b)^\times \to \mathcal{R}^{b,\mathrm{st}}$. By extension of scalars we get an extension of φ to $\mathcal{R}^{\mathrm{st}}$ commuting with the logarithm. Explicitly, if t is a lifting of a local parameter, then $\varphi(t) = t^q u$ for some $u \in \mathcal{R}^1$, and the extension is determined by

$$\varphi(\log t) = \log(\varphi(t)) = q \log t + \log u.$$

Since $N_{\mathcal{R}}(\log u) = 0$ we find that $N_{\mathcal{R}}\varphi(\log t) = qN_{\mathcal{R}}(\log t) = q$, and the $\mathcal{R}^b$-linearity of $N_{\mathcal{R}}$ yields

$$N_{\mathcal{R}}\varphi = q\varphi N_{\mathcal{R}} \tag{1.1.5}$$

on $\mathcal{R}^{b,\mathrm{st}}$ and $\mathcal{R}^{\mathrm{st}}$.

If $u = (F(u), \mathcal{O}(u))$ is a pair of the sort defined above, then we can define $\mathcal{R}^{\mathrm{st}}(u)$ relative to $\mathcal{R}(u)$ just as before. Since $\mathcal{R}(u)^0$ is a finite étale $\mathcal{R}^0$-algebra, the lift φ of Frobenius extends uniquely to $\mathcal{R}^0(u)$, $\mathcal{R}^b(u)$, and $\mathcal{R}(u)$ (though not necessarily to $\mathcal{O}(u)$), and for the same reason the $\mathcal{D}^b$-module structure of $\mathcal{R}^b$ (respectively $\mathcal{R}$) extends uniquely to $\mathcal{R}^b(u)$ (respectively $\mathcal{R}(u)$).

The universal property of $\mathcal{R}^b \to \mathcal{R}^{b,\mathrm{st}}$ shows that the composite

$$\mathcal{R}^\sharp \longrightarrow \mathcal{R}^\sharp(u) \xrightarrow{\log} \mathcal{R}(u)^{b,\mathrm{st}}$$

factors through an $\mathcal{R}$-algebra homomorphism

$$\mathcal{R}^{b,\mathrm{st}} \longrightarrow \mathcal{R}^{b,\mathrm{st}}(u) \tag{1.1.6}$$

and we have the following proposition.

Proposition 1.1.1. *The $\mathcal{R}^b$-algebra homomorphisms*

$$\mathcal{R}^b(u) \otimes_{\mathcal{R}^b} \mathcal{R}^{b,\mathrm{st}} \longrightarrow \mathcal{R}^{b,\mathrm{st}}(u),$$
$$\mathcal{R}(u) \otimes_{\mathcal{R}} \mathcal{R}^{\mathrm{st}} \longrightarrow \mathcal{R}^{\mathrm{st}}(u)$$

induced by (1.1.6) are isomorphisms, and if N_t and N_u denote the canonical monodromy operators of $\mathcal{R}^{\mathrm{st}}$ and $\mathcal{R}^{\mathrm{st}}(u)$, then $N_u|\mathcal{R} = (\deg(F(u)/F))N_t$.

Proof. In $\mathcal{R}^0(u)$, we have

$$t = au^n g(u)$$

with $a \in K(u)$, $g \in \mathcal{R}^1(u)$, and $n = \deg(F(u)/F)$. Therefore

$$\log t = n \log u + (\text{element of } \mathcal{R}^b(u)),$$

and the assertion follows easily. □

It follows, in particular, that $\mathcal{R}^{\mathrm{st}}$ is canonically identified with a (φ, N)-stable subring of $\mathcal{R}^{\mathrm{st}}(u)$. From Proposition 1.1.1 and Lemma 1.0.1, we see that if $G(u)$ is the Galois group of $F(u)/F$, then

$$(\mathcal{R}^{b,\mathrm{st}}(u))^{G(u)} = \mathcal{R}^{b,\mathrm{st}} \quad \text{and} \quad (\mathcal{R}^{\mathrm{st}}(u))^{G(u)} = \mathcal{R}^{\mathrm{st}}. \tag{1.1.7}$$

We note, finally, that since $\mathcal{R}^b(u)$ is an étale $\mathcal{R}^b$-algebra, the $\mathcal{D}^b$-module structure on $\mathcal{R}^b$ extends uniquely to a $\mathcal{D}^b$-module structure on $\mathcal{R}^b(u)$, and by (1.1.3) it extends to $\mathcal{R}^{b,\mathrm{st}}(u)$ as well; similar considerations show that there are natural $\mathcal{D}$-module structures on $\mathcal{R}(u)$ and $\mathcal{R}^{\mathrm{st}}(u)$. Explicitly, the inclusion $\mathcal{R}^b \to \mathcal{R}^b(u)$ corresponds to an expression for $t = t(u)$ as a power series in u, such that $t'(u)$ is invertible in $\mathcal{R}^b(u)$; then the $\mathcal{D}^b$-module structure on $\mathcal{R}^b(u)$ is determined by $\partial f(u) = f'(u)t'(u)^{-1}$, where the prime denotes differentiation with respect to u.

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1.2 Hyperfunctions

If $u \to v$ is a morphism of pairs, then the induced map $\mathcal{R}(u) \to \mathcal{R}(v)$ extends by Proposition 1.1.1 to an $\mathcal{R}$-algebra homomorphism $\mathcal{R}^{\mathrm{st}}(u) \to \mathcal{R}^{\mathrm{st}}(v)$. Since the category of pairs u has a small cofinal subcategory, we can define the rings of *hyperfunctions* $\mathcal{B}$ and *bounded hyperfunctions* $\mathcal{B}^b$ by

$$\mathcal{B} = \varinjlim_u \mathcal{R}^{\mathrm{st}}(u) \quad \text{and} \quad \mathcal{B}^b = \varinjlim_u \mathcal{R}^{b,\mathrm{st}}(u). \tag{1.2.1}$$

From the preceding discussion, the following properties are clear.

- The ring $\mathcal{B}^b$ (respectively $\mathcal{B}$) has a natural $\mathcal{D}^b$-module (respectively $\mathcal{D}$-module) structure, and the natural maps $\mathcal{R}^{b,\mathrm{st}}(u) \to \mathcal{B}^b$, $\mathcal{R}^{\mathrm{st}}(u) \to \mathcal{B}$ are morphisms of differential modules.
- The canonical monodromy operators of $\mathcal{R}^{\mathrm{st},b}$ and $\mathcal{R}^{\mathrm{st}}$ induce derivations $N_{\mathcal{B}} : \mathcal{B}^b \to \mathcal{B}^b$, $N_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ which are surjective morphisms of differential modules.
- The map φ extends to $\varphi : \mathcal{B}^b \to \mathcal{B}^b$, $\varphi : \mathcal{B} \to \mathcal{B}$, and the extension satisfies $N_{\mathcal{B}}\varphi = q\varphi N_{\mathcal{B}}$.
- The absolute Galois group G of F acts on $\mathcal{B}^b$ and $\mathcal{B}$, and we have $(\mathcal{B}^b)^{H(u)} = \mathcal{R}^{b,\mathrm{st}}(u)$ and $\mathcal{B}^{H(u)} = \mathcal{R}^{\mathrm{st}}(u)$ (recall that $H(u) \subseteq G$ is the open subgroup of G corresponding to $F(u)/F$). It follows that the action of G on $\mathcal{B}^b$ and $\mathcal{B}$ is discrete in the sense of § 0.1.

LEMMA 1.2.1. *There are canonical isomorphisms* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{R}, \mathcal{B}) = K^{\mathrm{nr}}$ *and* $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{R}, \mathcal{B}) = 0$ *for* $i > 0$.

Proof. Since $\mathcal{R}$ has the free resolution $\mathcal{D} \xrightarrow{\partial} \mathcal{D}$ we have $\mathrm{RHom}(\mathcal{R}, \mathcal{B}) = [\mathcal{B} \xrightarrow{\partial} \mathcal{B}]$, so we must show that ∂ is surjective, with kernel K^{nr}. Since $\mathcal{B}$ is the inductive limit of the $\mathcal{R}^{\mathrm{st}}(u)$, it suffices to prove that ∂ is surjective on $\mathcal{R}^{\mathrm{st}}(u)$, with kernel $K(u)$. As above we have $\partial f(u) = f'(u)t'(u)^{-1}$. This makes it clear that $\mathrm{Ker}\,\partial = K(u)$, and that ∂ will be surjective if the endomorphism $f \mapsto f'$ of $\mathcal{R}(u)[\log u]$ is so. In fact, $u^{-1}(\log u)^n$ is obviously integrable, and for any formal Laurent series $f(u)$ with no term u^{-1}, $f(u)(\log u)^n$ can be integrated by repeated integration by parts. □

If we put

$$\mathcal{R}^{b,\mathrm{st}}_n(u) = \mathrm{Ker}\ N^n_{\mathcal{R}} | \mathcal{R}^{\mathrm{st},b}(u), \quad \mathcal{R}^{\mathrm{st}}_n(u) = \mathrm{Ker}\ N^n_{\mathcal{R}} | \mathcal{R}^{\mathrm{st}}(u) \tag{1.2.2}$$

then $\mathcal{R}^{b,\mathrm{st}}_n(u)$ is a finite free $\mathcal{R}^b(u)$-module, and in fact is a finite free $\mathcal{R}^b$-module since $\mathcal{R}^b(u)$ is itself finite free over $\mathcal{R}^b$; the same goes for $\mathcal{R}^{\mathrm{st}}_n(u)$. They are stable under the action of G, φ, and $N_{\mathcal{R}}$. Note that $\mathcal{R}^{\mathrm{st}}_0 = \mathcal{R}$ and $\mathcal{R}^{\mathrm{st},b}_0 = \mathcal{R}^b$. We can also define $\mathcal{B}_n = \mathrm{Ker}\ N^{n+1}_{\mathcal{B}} | \mathcal{B}$, and then $\mathcal{B}_0 = \varinjlim_u \mathcal{R}(u)$. Finally, since $\mathcal{B} = \mathcal{B}_0[\log t]$, we have

$$\mathcal{B}^b = \varinjlim_n \mathcal{B}^b_n = \varinjlim_{u,n} \mathcal{R}^{b,\mathrm{st}}_n(u) \quad \text{and} \quad \mathcal{B} = \varinjlim_n \mathcal{B}_n = \varinjlim_{u,n} \mathcal{R}^{\mathrm{st}}_n(u). \tag{1.2.3}$$

2. *F*-isocrystals on $\mathcal{R}$

2.1 An application of the monodromy theorem

As always, an *isocrystal on* $\mathcal{R}$ is a pair (M, ∇) consisting of a finite free $\mathcal{R}$-module M and a connection $\nabla : M \to M \otimes \Omega^1_{\mathcal{R}/K}$ (or equivalently a $\mathcal{D}$-module structure on M). An F-*isocrystal on* $\mathcal{R}$ is a triple (M, ∇, F) where (M, ∇) is an isocrystal on $\mathcal{R}$ and F is a Frobenius structure, i.e. a φ-linear isomorphism $F : \varphi^* M \to M$ commuting with ∇. Morphisms of isocrystals (respectively F-isocrystals) are $\mathcal{R}$-linear maps horizontal for the connections (respectively and commuting with the Frobenius structure). The category of isocrystals on $\mathcal{R}$ (respectively F-isocrystals on $\mathcal{R}$) will be denoted $\mathrm{Isoc}(\mathcal{R})$ (respectively $F\text{-}\mathrm{Isoc}(\mathcal{R})$). We will often omit explicit mention of ∇

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and F, and speak, for example, of 'the F-isocrystal M'. Later on we will need the notion of an *F-isocrystal on* $\mathcal{R}^b$ (respectively on the Amice ring $\mathcal{A}$), which is a finite-dimensional $\mathcal{R}^b$-vector space (respectively $\mathcal{A}$-vector space) endowed with a connection and a Frobenius structure as above.

The structure of quasi-unipotent isocrystals on $\mathcal{R}$ was determined by Matsuda [Mat02], and his results can be combined with the p-adic monodromy theorem of André, Kedlaya and Mebkhout [And02, Ked04, Meb02] to obtain a classification of F-isocrystals on $\mathcal{R}$. Here we will use a more direct approach based on ideas of Fontaine; the essential ideas are doubtless well known by now but it will nonetheless be useful to go through it in some detail, if only to establish notation.

One form of the monodromy theorem asserts that for any F-isocrystal M there is a finite separable extension $F(u)/F$ such that the $\mathcal{D}$-module $M \otimes_{\mathcal{R}} R^{\mathrm{st}}(u)$ has a full set of horizontal sections. Equivalently, there is a finite-dimensional K^{nr}-vector space V and an isomorphism $M \otimes_{\mathcal{R}} \mathcal{B} \simeq V \otimes_{K^{\mathrm{nr}}} \mathcal{B}$ (in the terminology of Fontaine [Fon94], M is $\mathcal{B}$-admissible) and we can recover V as the K^{nr}-space of horizontal sections $V \simeq (M \otimes_{\mathcal{R}} \mathcal{B})^{\nabla}$, which is then endowed with a discrete Galois action, a Frobenius action, and a nilpotent operator, all coming from the corresponding structures on $\mathcal{B}$. However, for the purposes of § 6, it is more convenient to work with a contravariant formulation than a covariant one, so we proceed as follows: for any F-isocrystal M on $\mathcal{R}$, we set

$$\mathbb{V}(M) = \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{B}) \tag{2.1.1}$$

and observe that the mononodromy theorem implies that

$$\mathrm{Hom}_{\mathcal{R}}(M, \mathcal{B}) \simeq V \otimes_{K^{\mathrm{nr}}} \mathcal{B}$$

for some finite-dimensional K^{nr}-vector space V whose dimension is the rank of M. Taking horizontal sections yields an isomorphism $V \simeq \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{B})$, so we may rewrite the above as

$$\mathrm{Hom}_{\mathcal{R}}(M, \mathcal{B}) \simeq \mathbb{V}(M) \otimes_{K^{\mathrm{nr}}} \mathcal{B} \tag{2.1.2}$$

which is a canonical isomorphism of $\mathcal{D}$-modules. The left-hand side is endowed with a φ-linear isomorphism, a linear endomorphism arising from the canonical monodromy, and an action of the Galois group G arising from the action of G on $\mathcal{B}$. The right-hand side is endowed with corresponding maps and actions, and as these are compatible with the $\mathcal{D}$-module structure, we see that $V = \mathbb{V}(M)$ is naturally endowed with the following structures:

(i) a σ-linear isomorphism $F : V \to V$;

(ii) a linear endomorphism $N : V \to V$ satisfying $NF = qFN$ (because of the $N_{\mathcal{B}}\varphi = q\varphi N_{\mathcal{B}}$ in $\mathcal{B}$); and

(iii) an action ρ of the Galois group G that is semilinear with respect to the natural action of G on K^{nr}, and discrete in the sense of § 0.1.

The relation $NF = qFN$ implies that N decreases the slopes of F, and since V has finite dimension this means that N is nilpotent. The Galois action is discrete because $Hom_{\mathcal{D}}(M, \mathcal{R}^{\mathrm{st}}(u)) \simeq V \otimes_{K^{\mathrm{nr}}} \mathcal{R}^{\mathrm{st}}(u)$ for some finite extension $t \to u$.

We denote by $\mathrm{Mod}_K(G, F, N)$ the category of objects (V, F, N, ρ), V being a K^{nr}-vector space of finite dimension endowed with F, N, ρ satisfying conditions (i)–(iii). Thus (2.1.1) defines a functor

$$\mathbb{V} : F - \mathrm{Isoc}(\mathcal{R}) \longrightarrow \mathrm{Mod}_K(G, F, N). \tag{2.1.3}$$

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Later on we need the category $\mathrm{Mod}_K(G, F)$ consisting of triples (V, F, ρ) with V, F, ρ as above; it can be considered a full subcategory of $\mathrm{Mod}_K(G, F, N)$ consisting of objects with $N = 0$.

Suppose now that $V = (V, F, n, \rho)$ is an object of $\mathrm{Mod}_K(G, F, N)$, and set

$$\mathbb{M}(V) = \mathrm{Hom}_{K^{\mathrm{nr}}}(V, \mathcal{B})^{G,N} \tag{2.1.4}$$

where the $(\ )^{G,N}$ denotes the intersection of the G-invariants with the kernel of $N^* - N_{\mathcal{B}*}$, where $N_{\mathcal{B}*}$ and N^* are the induced actions of $N_{\mathcal{B}}$ and N. This is evidently an $\mathcal{R}$-module, endowed with a connection and Frobenius structure. For later use we define the $\mathcal{R}^b$-module

$$\mathbb{M}^b(V) = \mathrm{Hom}_{K^{\mathrm{nr}}}(V, \mathcal{B}^b)^{G,N} \tag{2.1.5}$$

which likewise is an $\mathcal{R}^b$-module with connection and Frobenius.

LEMMA 2.1.1. *For any V in $\mathrm{Mod}_K(G, F, N)$, $\mathbb{M}(V)$ (respectively $\mathbb{M}^b(V)$) is an F-isocrystal on $\mathcal{R}$ (respectively $\mathcal{R}^b$).*

Proof. It suffices to show that $\mathbb{M}^b(M)$ is a finite-dimensional $\mathcal{R}^b$-vector space, since $\mathbb{M}(V) = \mathbb{M}^b(V) \otimes_{\mathcal{R}^b} \mathcal{R}$. We first observe that in (2.1.4) and (2.1.5), the G-action factors through some finite quotient $G(u)$ attached to some finite extension $F(u)/F$. By (1.1.7), we have

$$\mathbb{M}^b(V) = \mathrm{Hom}_{K^{\mathrm{nr}}}(V, \mathcal{R}^{\mathrm{st},b}(u))^{G(u),N}$$

for this u. Furthermore $N^n = 0$ on V for some n, so we have

$$\mathbb{M}^b(V) = \mathrm{Hom}_{K^{\mathrm{nr}}}(V, \mathcal{R}_n^{\mathrm{st},b}(u))^{G(u),N}$$

where $\mathcal{R}_n^{\mathrm{st},b}(u)$ is defined by (1.2.2). Since $\mathcal{R}_n^{\mathrm{st},b}(u)$ is a finite-dimensional $\mathcal{R}^b$-vector space, we see that the same is true for $\mathbb{M}^b(V)$. □

It is clear that the construction $V \mapsto \mathbb{M}(V)$ defines a functor

$$\mathbb{M} : \mathrm{Mod}_K(G, F, N) \to F - \mathrm{Isoc}(\mathcal{R}) \tag{2.1.6}$$

and our goal is to show that the functors (2.1.3) and (2.1.6) are inverse equivalences.

LEMMA 2.1.2. *Suppose V and W are K^{nr}-vector spaces with a discrete G-action. If V has finite dimension, $\mathrm{Hom}_{K^{\mathrm{nr}}}(V, W)$ is discrete.*

Proof. Since V has finite dimension there is an open normal subgroup $H \subseteq G$ such that $V \simeq V^H \otimes_{(K^{\mathrm{nr}})^H} K^{\mathrm{nr}}$. Via the adjunction isomorphism we can identify an $f \in \mathrm{Hom}_{K^{\mathrm{nr}}}(V, W)$ with a $(K^{\mathrm{nr}})^H$-linear map $f : V^H \to W$; since H is normal in G, V^H is G-stable and the action of G on f is $(\sigma f)(v) = \sigma(f(\sigma^{-1}(v)))$. If $v_1, \ldots, v_r$ is a basis of V^H, there is an open normal subgroup $H' \subseteq G$ fixing each of the $f(v_i)$, and then that f is fixed by the open normal subgroup $H \cap H'$. Since f was arbitrary, the action of G on $\mathrm{Hom}_{K^{\mathrm{nr}}}(V, W)$ is discrete. □

The category of K^{nr}-vector spaces with a discrete G-action is abelian, and the functor of G-invariants is exact. This is actually a general fact about the category of K-vector spaces with a discrete G-action, where G is any profinite group and K is a field of characteristic 0. The exactness of the functor of G-invariants is of course well known if G is finite, and in general it follows from the isomorphisms

$$H^i(G, V) = \varinjlim_H H^i(G/H, V^H) = 0 \quad \text{for } i > 0.$$

We will need the following elementary observation.

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Lemma 2.1.3. *If N and A are elements of an associative algebra such that N is nilpotent and A has a right inverse commuting with N, then $A-N$ has a right inverse.*

Proof. If $AB=I$ and $BN=NB$, then NB is nilpotent, $I-NB$ is invertible, and the right inverse is $B(I-NB)^{-1}$. □

Lemma 2.1.4. *The functors $\mathbb{V}$, $\mathbb{M}$ are exact.*

Proof. We first consider $\mathbb{V}$. Since M is a free $\mathcal{R}$-module, there are isomorphisms

$$\mathrm{Ext}^1_{\mathcal{D}}(M,\mathcal{B})=\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{R},\mathrm{Hom}_{\mathcal{R}}(M,\mathcal{B}))=\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{R},\mathbb{V}(M)\otimes_{K^{\mathrm{nr}}}\mathcal{B}).$$

Since $\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{R},\mathcal{B})=0$ by Lemma 1.2.1, $\mathrm{Ext}^1_{\mathcal{D}}(M,\mathcal{B})=0$ for any M in F-Isoc$(\mathcal{R})$, and the exactness of (2.1.6) follows.

As for the functor $\mathbb{M}$, we first observe that the functor $V\mapsto\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})$ on K-vector spaces is exact. We claim that $V\mapsto\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})^N$ is exact; recall that $\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})^N$ denotes the kernel of $N^*-N_{\mathcal{B}*}$ on $\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})$. As N_B has a right inverse on $\mathcal{B}$, $N_{\mathcal{B}*}$ has a right inverse on $\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})$ which commutes with N^*. By Lemma 2.1.3, $N^*-N_{\mathcal{B}*}$ is surjective on $\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})$ for any V, and a simple argument using the snake lemma shows that $V\mapsto\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})^N$ is exact. By Lemma 2.1.2 the G-action on $\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})$ is discrete; then $\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})^N$ is a discrete G-module, and consequently $V\mapsto\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})^{N,G}=\mathbb{V}(M)$ is exact.

Theorem 2.1.1. *The functors (2.1.3), (2.1.6) are inverse equivalences of categories.*

Proof. The canonical isomorphism (2.1.2) can be rewritten

$$M^{\vee}\otimes_{\mathcal{R}}\mathcal{B}=\mathbb{V}(M)\otimes_{K^{\mathrm{nr}}}\mathcal{B}.$$

Since this has been shown for all M, and $\mathbb{V}(M^{\vee})\simeq\mathbb{V}(M)^{\vee}$ canonically, we get

$$M\otimes_{\mathcal{R}}\mathcal{B}=\mathrm{Hom}_{K^{\mathrm{nr}}}(\mathbb{V}(M),\mathcal{B})$$

and thus

$$M=\mathrm{Hom}_{K^{\mathrm{nr}}}(\mathbb{V}(M),\mathcal{B})^{G,N}=\mathbb{M}(\mathbb{V}(M))$$

since $\mathcal{B}^{G,N}=\mathcal{R}$.

Suppose, on the other hand, that V is an object of $\mathrm{Mod}_K(G,F,N)$. We want to show that the canonical map $\mathrm{ev}:V\mapsto\mathbb{V}(\mathbb{M}(V))$ given by evaluation is an isomorphism for all V. By Lemma 2.1.4 the functor $V\mapsto\mathbb{V}(\mathbb{M}(V))$ is exact, so it is enough to check that $V\mapsto\mathbb{V}(\mathbb{M}(V))$ is an isomorphism for irreducible objects V. When (V,F,N,ρ) is irreducible, $N=0$ and ρ is an irreducible representation of G. For such V we have

$$M=\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B})^{G,N}=\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B}_0)^{G}=\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{R}(u))^{G(u)}$$

for some $F(u)/F$. As in the proof of Lemma 2.1.1, this M arises from

$$M^b=\mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{R}^b(u))^{G(u)} \tag{2.1.7}$$

by extension of scalars from $\mathcal{R}^b$ to $\mathcal{R}$. Since $\mathcal{R}^b(u)$ is an étale extension of $\mathcal{R}^b$, there is an isomorphism

$$\mathcal{R}^b(u)\otimes_{\mathcal{R}^b}\mathcal{R}^b\simeq\mathcal{R}^b[G(u)]$$

of Galois bimodules; here the Galois action in (2.1.7) corresponds to the standard action on the group ring, while the other is the natural action of $G(u)$ on the scalars. Extending scalars to $\mathcal{B}$,

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we see that there are isomorphisms

$$\mathcal{B} \otimes_{\mathcal{R}} M \simeq \mathrm{Hom}_{K^{\mathrm{nr}}}(V, \mathcal{B}[G(u)])^{G(u)} \simeq \mathrm{Hom}_{K^{\mathrm{nr}}}(V, \mathcal{B})$$

which imply that $\mathbb{V}(M) \simeq V$.

COROLLARY 2.1.1 (Matsuda [Mat02]). *An indecomposable F-isocrystal on $\mathcal{R}$ is the tensor product of a twist of a unit-root isocrystal corresponding to an irreducible representation of G, and an indecomposable unipotent F-isocrystal.*

Remark. Actually, Matsuda in [Mat02] treats the slightly more general case of isocrystals on $\mathcal{R}$ which are not assumed to have a Frobenius structure, but which are assumed to be quasi-unipotent.

3. $\mathcal{D}^\dagger$-modules on $\mathrm{Spf}(\mathcal{V}[[t]])$

Berthelot's theory of arithmetic $\mathcal{D}$-modules is set in the context of a scheme X smooth over a base S on which p is nilpotent, or of a smooth morphism $\mathfrak{X} \to \mathfrak{S}$ of p-adic formal schemes. As such it does not apply to $\mathrm{Spf}(\mathcal{V}[[t]])$ with the adic topology defined by the maximal ideal of $\mathcal{V}[[t]]$; nonetheless Berthelot's theory extends virtually without modification to this case. The purpose of this section is to review some aspects of this theory, and explain when needed the modifications for the case of $\mathrm{Spf}(\mathcal{V}[[t]])$. A more complete summary of these ideas can be found in [Ber02b], and the full story is in [Ber96a, Ber02a] and work of Berthelot currently in preparation.

3.1 Arithmetic differential operators

If S is a $\mathbb{Z}_{(p)}$-scheme and X/S is smooth, then module $\mathcal{D}^{(m)}$ of partially divided power differential operators of level m is the direct limit

$$\mathcal{D}^{(m)} = \varinjlim_{n} \mathrm{Hom}_{\mathcal{O}_X}(P^n_{(m)}, \mathcal{O}_X) \tag{3.1.1}$$

where $P^n_{(m)}$ is the divided power neighborhood of level m and order n of the diagonal $X \subset X \times_S X$, and the *Hom* is taken with respect to the left $\mathcal{O}_X$-algebra structure (cf. [Ber96a, 2.2.1] for this). A $\mathcal{D}^{(m)}$-module structure on an $\mathcal{O}_X$-module M is the same as an m-PD-stratification, i.e. a compatible collection of isomorphisms

$$P^n_{(m)} \otimes_{\mathcal{O}_X} M \xrightarrow{\sim} M \otimes_{\mathcal{O}_X} P^n_{(m)} \tag{3.1.2}$$

which restrict to the identity on the diagonal and satisfy a cocycle condition (see [Ber96a, 2.3.1–2]). For example, $M = \mathcal{O}_X$ has an natural $\mathcal{D}^{(m)}$-module structure for all m, corresponding to the identity in (3.1.2). It is by means of this description that one proves that the structure sheaf of certain blowups of X have a natural $\mathcal{D}^{(m)}$-module structure.

LEMMA 3.1.1. *Suppose $f \in \Gamma(X, \mathcal{O}_X)$ and*

$$B(f, r) = \mathcal{O}_X[T]/(f^r T - p), \quad C(f, r) = \mathcal{O}_X[T]/(pT - f^r). \tag{3.1.3}$$

If p^{m+1} divides r, then the natural $\mathcal{D}^{(m)}$-module structure on $\mathcal{O}_X$ extends to $B(f, r)$ and $C(f, r)$ compatibly with the $\mathcal{O}_X$-module structure. If $m' \geqslant m$ and $p^{m'+1}$ divides r, then the natural $\mathcal{D}^{(m)}$-module structure on $B(f, r)$ and $C(f, r)$ coincides with the restriction of the natural $\mathcal{D}^{(m')}$-module structure. Finally, for any multiple $r' = ar$ of r, the morphisms $B(f, r) \to B(f, r')$, $C(f, r') \to C(f, r)$ induced by $T \mapsto f^a T$ are $\mathcal{D}^{(m)}$-linear.

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Proof. For $B(f,r)$ this is [Ber96a, 4.2.1], so we shall just sketch the proof. We can reduce to the universal case $X = \mathrm{Spec}(\mathbb{Z}_{(p)}\langle t\rangle)$ and $f = t$. The condition on m guarantees that there is a level m divided power polynomial $\varphi^{(m)}(s,t)$ such that $t^r - s^r = p\varphi^{(m)}(s,t)$ and $\varphi^{(m)}(t,t) = 0$. If we set $P = P^n_{(m)}$, then we can identify $P \otimes_{\mathcal{O}_X} B(t,r)$ and $B(t,r) \otimes_{\mathcal{O}_X} P$ with $\mathcal{O}_P[T]/((1 \otimes t^r)T - p)$ and $\mathcal{O}_P[T]/((t^r \otimes 1)T - p)$ respectively, and similarly for $C(t,r)$. With these identifications, the stratification is given by

$$T \mapsto T(1 + T\varphi^{(m)}(t \otimes 1, 1 \otimes t))^{-1}$$

in the case of $B(t,r)$, and

$$T \mapsto T + \varphi^{(m)}(t \otimes 1, 1 \otimes t)$$

in the case of $C(f,r)$; for $B(f,r)$ the map is a well-defined isomorphism since $\varphi^{(m)}(s,t)$ is nilpotent in $P^n_{(m)}$, and for $C(f,r)$ this is obvious. The cocycle condition follows from the identity $\varphi^{(m)}(s,t) + \varphi^{(m)}(t,u) = \varphi^{(m)}(s,u)$, the above maps restrict to the identity on the diagonal since $\varphi^{(m)}(t,t) = 0$. The remaining assertions are just as in [Ber96a]; we will just remind the reader that the linearity of the maps $B(f,r) \to B(f,r')$, $C(f,r') \to C(f,r)$ follow from the homogeneity of $\varphi^{(m)}$. □

We are mainly interested in the case $S = \mathrm{Spf}(\mathcal{V})$, $X = \mathrm{Spf}(\mathcal{V}[[t]])$; here the $B(t,r)$ are used in defining the rings of differential operators with overconvergent coefficients, and the $C(t,r)$ will be used to construct the analytification functor and the ring $\mathcal{D}^{\mathrm{an}}$ in § 5.

If $\mathfrak{S}$ is a p-adic formal $\mathbb{Z}_p$-scheme and $\mathfrak{X}/\mathfrak{S}$ is smooth, we set $S_i = \mathbb{Z}/p^{i+1}\mathbb{Z} \otimes \mathfrak{S}$, $X_i = \mathbb{Z}/p^{i+1}\mathbb{Z} \otimes \mathfrak{X}$, and

$$\hat{\mathcal{D}}^{(m)}_{\mathfrak{X}} = \varprojlim_i \hat{\mathcal{D}}^{(m)}_{X_i/S_i}, \quad \hat{\mathcal{D}}^{(m)}_{\mathfrak{X}\mathbb{Q}} = \hat{\mathcal{D}}^{(m)}_{\mathfrak{X}} \otimes \mathbb{Q}.$$

A $\hat{\mathcal{D}}^{(m)}$-module structure on a p-adically complete and separated $\mathcal{O}_{\mathfrak{X}}$-module M is the same as a formal level m stratification

$$P^n_{(m)} \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} M \xrightarrow{\sim} M \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} P^n_{(m)}, \tag{3.1.4}$$

just as before.

Both $\hat{\mathcal{D}}^{(m)}_{\mathfrak{X}}$ and $\hat{\mathcal{D}}^{(m)}_{\mathfrak{X}\mathbb{Q}}$ are noetherian. The ring $\mathcal{D}^{\dagger}_{\mathfrak{X}\mathbb{Q}}$ of overconvergent differential operators is the direct limit

$$\mathcal{D}^{\dagger}_{\mathfrak{X}\mathbb{Q}} = \varinjlim_m \hat{\mathcal{D}}^{(m)}_{\mathfrak{X}\mathbb{Q}}$$

and is coherent, but not noetherian.

When $X = \mathrm{Spf}((\mathcal{V}/p^n\mathcal{V})[[t]])$, we modify $P^n_{(m)}$ in the definition (3.1.1) by taking $X \times_S X$ to be the product in the category of (t)-adic formal schemes. The remaining constructions are unchanged. In the case of $\mathfrak{X} = \mathrm{Spf}(\mathcal{V}[[t]])$ we will drop the subscripts, at least until § 8, and write $\hat{\mathcal{D}}^{(m)}$, $\hat{\mathcal{D}}^{(m)}_{\mathbb{Q}}$, and $\mathcal{D}^{\dagger}$ for $\hat{\mathcal{D}}^{(m)}_{\mathfrak{X}}$, $\hat{\mathcal{D}}^{(m)}_{\mathfrak{X}\mathbb{Q}}$, and $\mathcal{D}^{\dagger}_{\mathfrak{X}\mathbb{Q}}$ (this differs from the notation of [Ber96a], where $\mathcal{D}^{\dagger}_{\mathfrak{X}}$ is a proper subring of $\mathcal{D}^{\dagger}_{\mathfrak{X}\mathbb{Q}}$). Since $\mathcal{V}[[t]]$ is noetherian, the proof that $\hat{\mathcal{D}}^{(m)}$ is noetherian works in the present case without modification. If ∂ is the element of $D^{(m)}$ dual to dt and $\partial^{[k]}$ is the usual divided-power differential operator $(1/k!)\partial^k$, then these rings are

$$\begin{aligned}\hat{\mathcal{D}}^{(m)} &= \left\{ \sum_{k\geqslant 0} a_k \partial^{[k]} \;\middle|\; a_k \in \mathcal{O}, \text{ and } a_k \text{ is divisible by } [k/p^m]! \right\}\\ \mathcal{D}^\dagger &= \left\{ \sum_{k\geqslant 0} a_k \partial^{[k]} \;\middle|\; \begin{array}{l} a_k \in \mathcal{O}_K, \text{and there exist positive constants } C,\\ \eta < 1 \text{ such that } |a_k| \leqslant C\eta^k \text{ for all } k \end{array} \right\}\end{aligned} \tag{3.1.5}$$

where $|\ |$ is the p-adic valuation. Note, in the first equality, that the condition $[k/p^m]!|a_k$ implies $a_k \to 0$ as $k \to \infty$.

The same construction can be made for $\mathfrak{X} = \mathrm{Spf}(\mathcal{A}^0)$ where $\mathcal{A}^0$ has the p-adic topology. The resulting rings analogous to those in (3.1.5) will be denoted by $\hat{\mathcal{D}}^{(m)}_{\mathcal{A}^0}$ and $\mathcal{D}^\dagger_{\mathcal{A}}$ respectively; they are given explicitly by formulas like (3.1.5), but now a_k is element of $\mathcal{A}^0$ or $\mathcal{A}$ respectively. There are natural $\mathcal{O}$-algebra homomorphisms $\hat{\mathcal{D}}^{(m)}_{\mathcal{A}^0} \to \hat{\mathcal{D}}^{(m)}$, $\mathcal{D}^\dagger \to \mathcal{D}^\dagger_{\mathcal{A}}$.

Suppose $\mathfrak{X}$ is a formally smooth p-adic formal scheme and $Z \subset \mathfrak{X}$ is a closed subset defined by a section f of $\mathcal{O}_{\mathfrak{X}}$. Let us briefly recall the construction of the ring $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}(Z)$ of differential operators with overconvergent singularities around Z (in [Ber96a] this is denoted $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}({}^\dagger Z)$). By (3.1.3), the ring $B(f, p^{m+1})$ has a natural $\mathcal{D}^{(m)}$-module structure, so that $B(f, p^{m+1}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{D}^{(m)}$ has a natural $\mathcal{O}_{\mathfrak{X}}$-algebra structure. Denote by $\hat{\mathcal{D}}^{(m)}(Z)_{\mathbb{Q}}$ its p-adic completion, tensored with $\mathbb{Q}$. For variable m, the $\hat{\mathcal{D}}^{(m)}(Z)_{\mathbb{Q}}$ naturally form an inductive system, and one defines

$$\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}(Z) = \varinjlim_m \hat{\mathcal{D}}^{(m)}(Z)_{\mathbb{Q}}$$

which like $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}$ is a coherent, but not noetherian, $\mathcal{O}_{\mathfrak{X}\mathbb{Q}}$-algebra. Up to canonical isomorphism, it depends only on the reduction of Z modulo the uniformizer of $\mathcal{V}$; in particular, this construction globalizes (though we shall not need this). When $\mathfrak{X} = \mathrm{Spf}(\mathcal{V}[\![t]\!])$, $f = t$, we modify the construction as before, and write $\mathcal{D}^\dagger(0)$ for $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}(Z)$. An argument parallel to [Ber96a, 4.3.11] shows that $\mathcal{D}^\dagger(0)$ is a left and right flat $\mathcal{D}^\dagger$-algebra. Finally, the ring $\mathcal{D}^\dagger_{\mathcal{A}}$ is naturally a $\mathcal{D}^\dagger(0)$-algebra, thanks to the inclusion homomorphism $\mathcal{R}^b \to \mathcal{A}$.

3.2 Coherent $\mathcal{D}^\dagger$-modules

A coherent $\mathcal{D}^\dagger$-module arises by extension of scalars $\hat{\mathcal{D}}^{(m_0)} \to \mathcal{D}^\dagger$ from some coherent $\hat{\mathcal{D}}^{(m_0)}$-module $M^{(m_0)}$. If for $m \geqslant m_0$ we set $M^{(m)} = \hat{\mathcal{D}}^{(m)} \otimes_{\hat{\mathcal{D}}^{(m_0)}} M^{(m_0)}$, then $M \simeq \varinjlim_m M^{(m)}_{\mathbb{Q}}$, and this observation would suffice for the construction, in the next section, of the analytification of a coherent $\mathcal{D}^\dagger$-module. It is technically more convenient, however, to start with a construction on the level of derived categories, and for this we need Berthelot's description in [Ber02b, § 4.2] of $\mathcal{D}^b_{\mathrm{coh}}(\mathcal{D}^\dagger)$ as a localized inductive limit. There is no point in going into the details of this rather elaborate construction, but a few points should be mentioned.

We denote by $D^b(\hat{\mathcal{D}}^{(\cdot)})$ the (bounded) derived category of the category of inductive systems $M^{(\cdot)}$ where each $M^{(m)}$ is a $\hat{\mathcal{D}}^{(\cdot)}$-module and the transition maps $M^{(m)} \to M^{(m+1)}$ are semilinear with respect to $\hat{\mathcal{D}}^{(m)} \to \hat{\mathcal{D}}^{(m+1)}$. Berthelot applies two successive localizations to $D^b(\hat{\mathcal{D}}^{(\cdot)})$ to obtain a category $\underrightarrow{LD}^b_{\mathbb{Q}}(\hat{\mathcal{D}}^{(\cdot)})$ and a functor $\varinjlim : \underrightarrow{LD}^b_{\mathbb{Q}}(\hat{\mathcal{D}}^{(\cdot)}) \to D^b(\mathcal{D}^\dagger)$ extending the usual direct limit. The full subcategory $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)})$ of $\underrightarrow{LD}^b_{\mathbb{Q}}(\hat{\mathcal{D}}^{(\cdot)})$ consists of objects isomorphic in $\underrightarrow{LD}^b_{\mathbb{Q}}(\hat{\mathcal{D}}^{(\cdot)})$ to an $M^{(m)}$ for which there exists an increasing $\lambda : \mathbb{N} \to \mathbb{N}$ with $\lambda(m) \geqslant m$ such

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that the following properties hold.

(i) The $\hat{\mathcal{D}}^{(\cdot)}$-module $M^{(m)}$ is an object of $D^b_{\mathrm{coh}}(\hat{\mathcal{D}}^{(\lambda(m))})$.

(ii) For every $m' \geqslant m$, the canonical map

$$\hat{\mathcal{D}}^{(\lambda(m'))} \overset{L}{\otimes}_{\hat{\mathcal{D}}^{(\lambda(m))}} M^{(m)} \to M^{(m')}$$

is an isomorphism.

Finally, Berthelot shows [Ber02b, 4.2.4] that the functor $(\otimes_{\mathbb{Z}}\mathbb{Q}) \circ \varinjlim$ induces an equivalence of categories (by an abuse of notation, denoted here simply by $\varinjlim$)

$$\varinjlim : \underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)}) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{D}^\dagger). \tag{3.2.1}$$

Of course, if M is a coherent $\mathcal{D}^\dagger$-module and $M^{(m_0)}$ is a coherent $\hat{\mathcal{D}}^{(m_0)}$-module such that $M \simeq \mathcal{D}^\dagger \otimes_{\hat{\mathcal{D}}^{(m_0)}} M^{(m_0)}_{\mathbb{Q}}$, then the inductive system $\{\hat{\mathcal{D}}^{(m)} \otimes_{\hat{\mathcal{D}}^{(m_0)}} M^{(m_0)}\}$ (considered as a complex whose only nonvanishing term is in degree zero) is an object of $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)})$ whose image under $\varinjlim : \underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)}) \to D^b_{\mathrm{coh}}(\mathcal{D}^\dagger)$ is M.

3.3 F-Isocrystals on $\mathcal{R}^b$

The differential module structure of an isocrystal M on $\mathcal{R}^b$ does not, in general, extend to a $\hat{\mathcal{D}}^{(m)}$-module structure for any m. However, if M is *solvable* in the sense of Robba (i.e. the generic radius of convergence of the connection on the circle $|t| = r$ approaches one as r tends to 1, cf. [CM02, §§ 4.1 and 8.3]), then the differential module structure extends to a $\mathcal{D}^\dagger(0)$-module structure, and makes M into a coherent $\mathcal{D}^\dagger(0)$-module. The argument is the same as that of [Ber90, § 3.1] and [Ber96a, § 4.4] with no particular modifications. Nonetheless some of the constructions of [Ber96a, § 4.4] will be needed later, so we will briefly recall them.

With $B(f, r)$ as in (3.1.3), we set

$$B^{(m)} = \hat{B}(t, p^{m+1}), \quad B^{(m)}_{\mathbb{Q}} = B^{(m)} \otimes \mathbb{Q}, \tag{3.3.1}$$

where the hat denotes the p-adic completion, and, for any $n \geqslant m$,

$$\hat{\mathcal{D}}^{(n,m)} = B^{(n)} \hat{\otimes}_{\mathcal{O}_{\mathfrak{X}}} \hat{\mathcal{D}}^{(m)}, \quad \hat{\mathcal{D}}^{(n,m)}_{\mathbb{Q}} = \hat{\mathcal{D}}^{(n,m)} \otimes \mathbb{Q} \tag{3.3.2}$$

(the condition on n is necessary to get an action of $\hat{\mathcal{D}}^{(m)}$ on $B^{(n)}$). It will also be useful to set

$$\hat{\mathcal{D}}^{(m)}(0) = \hat{\mathcal{D}}^{(m,m)} \tag{3.3.3}$$

so that

$$\mathcal{D}^\dagger(0) \simeq \varinjlim_m \hat{\mathcal{D}}^{(m)}(0)_{\mathbb{Q}}. \tag{3.3.4}$$

For $n \geqslant m$ the maps $B^{(m)} \to B^{(n)}$, $\hat{\mathcal{D}}^{(m)} \to \hat{\mathcal{D}}^{(n)}$ induce maps

$$\hat{\mathcal{D}}^{(m)}(0) = \hat{\mathcal{D}}^{(m,m)} \to \hat{\mathcal{D}}^{(n,m)} \to \hat{\mathcal{D}}^{(n,n)} = \hat{\mathcal{D}}^{(n)}(0),$$

and thus for any increasing $\lambda : \mathbb{N} \to \mathbb{N}$ such that $\lambda(m) \geqslant m$ there is a natural isomorphism

$$\mathcal{D}^\dagger(0) \simeq \varinjlim_m \hat{\mathcal{D}}^{(\lambda(m),m)}_{\mathbb{Q}}. \tag{3.3.5}$$

If M is a solvable isocrystal on $\mathcal{R}^b$, the argument of [Ber96a, Theorem 4.4.5] with the solvability condition in place of the overconvergence condition of [Ber96a, § 4.4] shows that we can find the following:

- a $n_0 \in \mathbb{N}$ and a $B^{(n_0)}$-module M_0 of finite type;
- an isomorphism
$$\varinjlim_{n \geqslant n_0} B^{(n)}_{\mathbb{Q}} \otimes M_0 \xrightarrow{\sim} M; \tag{3.3.6}$$
- an increasing function $\lambda : \mathbb{N} \to \mathbb{N}$ such that $\lambda(0) = n_0$ and $\lambda(m) \geqslant m$; and
- a $\hat{\mathcal{D}}^{(\lambda(m),m)}$-structure on $B^{(\lambda(m))} \otimes M_0$ such that the natural homomorphisms
$$B^{(\lambda(m))} \hat{\otimes} M_0 \to B^{(\lambda(m+1))} \otimes M_0$$
and the isomorphism (3.3.6) are $\hat{\mathcal{D}}^{(\lambda(m),m)}$-linear.

Set $M^{(m)} = B^{(\lambda(m))} \otimes M_0$, so that $M^{(0)} \simeq M_0$. The argument of [Ber96a, 4.4.6–4.4.11] shows that the canonical morphism
$$M^{(m)}_{\mathbb{Q}} \longrightarrow \hat{\mathcal{D}}^{(\lambda(m),m)}_{\mathbb{Q}} \otimes_{\hat{\mathcal{D}}^{(\lambda(0),0)}} M^{(0)} \tag{3.3.7}$$
is an isomorphism. Combining the isomorphisms (3.3.5) and (3.3.7), we get an isomorphism
$$M \xrightarrow{\sim} \mathcal{D}^{\dagger}(0) \otimes_{\hat{\mathcal{D}}^{(\lambda(0),0)}} M^{(0)} \tag{3.3.8}$$
by passing to the limit in m. This is the essential step in showing that M is a coherent $\mathcal{D}^{\dagger}(0)$-module [Ber96a, Theorem 4.4.12]. If we set $\hat{\mathcal{D}}^{(\infty,0)} = \varinjlim_m \hat{\mathcal{D}}^{(m,0)}$, then (3.3.8) yields an isomorphism
$$M \xrightarrow{\sim} \mathcal{D}^{\dagger}(0) \otimes_{\hat{\mathcal{D}}^{(\infty,0)}} M. \tag{3.3.9}$$
On the other hand, an argument parallel to [Ber96a, 4.4.9] and [Ber90, 3.1.3] shows that the canonical map
$$M \longrightarrow \hat{\mathcal{D}}^{(\infty,0)} \otimes_{\mathcal{D}^b} M$$
is an isomorphism, where, as before, $\mathcal{D}^b = \mathcal{R}^b[\partial]$. From (3.3.9) we then obtain a natural isomorphism
$$M \xrightarrow{\sim} \mathcal{D}^{\dagger}(0) \otimes_{\mathcal{D}^b} M, \tag{3.3.10}$$
for any F-isocrystal M on $\mathcal{R}^b$.

If (M, F) is an F-isocrystal on $\mathcal{R}^b$, then M is known to be solvable in the above sense ([CM02, Proposition 8.16]; this is the analogue for $\mathcal{R}^b$ of the well-known 'Dwork trick', but is considerably more difficult), so M is naturally a coherent $\mathcal{D}^{\dagger}(0)$-module. In fact, in this case the restriction of scalars of M to $\mathcal{D}^{\dagger}$ is coherent as a $\mathcal{D}^{\dagger}$-module, as was shown in [Cre06].

There is no analogue of this last result for F-isocrystals on $\mathcal{A}$, and we just *define* a convergent F-isocrystal on $\mathcal{A}$ to be an F-isocrystal whose differential module structure is induced by a $\mathcal{D}^{\dagger}_{\mathcal{A}}$-module structure.

3.4 Cohomological operations on $F\mathcal{D}^{\dagger}$-modules

Let φ be a lifting to $\mathfrak{X}$ of the q-power Frobenius of the reduction of $\mathfrak{X}$, where q is a fixed power of p, and recall that a *Frobenius structure* relative to φ on a $\mathcal{D}^{\dagger}$-module M is an isomorphism $M \xrightarrow{\sim} \varphi^* M$ of $\mathcal{D}^{\dagger}$-modules. An $F\mathcal{D}^{\dagger}$-module on $\mathfrak{X}$ is a $\mathcal{D}^{\dagger}$-module equipped with a Frobenius structure, and morphisms of $F\mathcal{D}^{\dagger}$-modules are defined in the obvious way; the category of $F\mathcal{D}^{\dagger}$-modules on $\mathfrak{X}$ is independent of the choice of φ. The presence of a Frobenius structure allows one to define the *characteristic variety* of an $F\mathcal{D}^{\dagger}$-module [**?**, § 5.2], and Berthelot has proven an analogue of Bernstein's inequality, enabling him to define the notion of a *holonomic* $F\mathcal{D}^\dagger$-module. The definition is a bit involved and will not be given here. It is unknown whether there is a reasonable definition of characteristic variety in the case of $\mathcal{D}^\dagger$-modules without Frobenius structure. As in the classical case, the holonomy condition is essential for the preservation of coherence under the standard cohomological operations f_+, f^+, $f_!$, $f^!$ on F-complexes of $\mathcal{D}^\dagger$-modules, although it is still a matter of conjecture whether these operations actually preserve holonomy.

The cohomological operations mentioned above are constructed in [?] and described in [Ber02b, §§ 2–4]. Relative to $\mathfrak{X} = \mathrm{Spf}(\mathcal{V}[\![t]\!])$ they have fairly simple explicit descriptions, similar to the classical case.

Consider first the closed immersion $i : 0 \to \mathfrak{X}$ defined by the divisor $t = 0$ in the formal scheme $\mathfrak{X} = \mathrm{Spf}(\mathcal{V}[\![t]\!])$. A holonomic module on 0 is just a finite-dimensional K-vector space V endowed with a Frobenius action, and $i_+V \simeq i_!V$ can be identified simply with $V \otimes_K \delta$, where δ is the 'Dirac' $\mathcal{D}^\dagger$-module $\mathcal{R}^b/\mathcal{O}_K$. On the other hand, if M is a holonomic $F\mathcal{D}^\dagger$-module on $\mathfrak{X}$, $i^!M$ is given by

$$i^!M \xrightarrow{\sim} [M \xrightarrow{t\cdot} M] \tag{3.4.1}$$

where the complex is supported in degrees $[0, 1]$. In general, the holonomicity of extraordinary inverse images is still an open problem, but for formally smooth formal curves over $\mathcal{V}$, such as $\mathrm{Spf}(\mathcal{V}[\![t]\!])$, this was proven in [Cre06] (the case of a proper, formally smooth curve was treated by Caro [Car06b]). It was shown in [Cre06, Theorem 2.2] that the functor i^+ on holonomic $F\mathcal{D}^\dagger$-modules can be computed by

$$i^+M \simeq \mathrm{RHom}_{\mathcal{D}^\dagger}(M^*, \mathcal{O}^{\mathrm{an}}) \tag{3.4.2}$$

where M^* is the holonomic dual of M (see below). We will give another, simpler version of this formula later, in § 5. Finally, it was shown in [Cre06, 2.2] that for any holonomic $F\mathcal{D}^\dagger$-module M, the spaces $\mathrm{Ext}^i(M, \mathcal{O}^{\mathrm{an}})$ have finite dimension, so that i^+ preserves holonomy as well.

There are several operations relative to the inclusion $j : \eta = \mathrm{Spec}(k((t))) \to \mathrm{Spec}(k[\![t]\!])$ of the generic point, even though we are regarding $\mathrm{Spf}(\mathcal{O})$ as a formal scheme. Two of these operations could be called restriction to the generic point, namely the extensions of scalars $j^*M = \mathcal{D}^\dagger_{\mathcal{A}} \otimes_{\mathcal{D}^\dagger} M$ and $j^+M = \mathcal{D}^\dagger(0) \otimes_{\mathcal{D}^\dagger} M$ of a $\mathcal{D}^\dagger$-module M, which, when M is a coherent $\mathcal{D}^\dagger$-module, are coherent modules over $\mathcal{D}^\dagger_{\mathcal{A}}$ and $\mathcal{D}^\dagger(0)$ respectively. Of these two constructions, j^+ will be the more useful one.

When (M, F) is a holonomic $F\mathcal{D}^\dagger$-module, one can show without too much difficulty that j^*M is a convergent F-isocrystal on $\mathcal{A}$ (see [Ber10]); this is a variant of the argument showing that a holonomic $F\mathcal{D}^\dagger$-module whose characteristic variety is the 0-section of the cotangent bundle is a convergent F-isocrystal, cf. [Ber02b, § 5.2, Remark]. A more difficult result states that when (M, F) is holonomic, j^+M is an F-isocrystal on $\mathcal{R}^b$. For this one needs a result of Berthelot [Ber07] which in the present situation says that when M is a coherent $\mathcal{D}^\dagger(0)$-module such that $\mathcal{D}^\dagger_{\mathcal{A}} \otimes_{\mathcal{D}^\dagger(0)} M$ is a convergent isocrystal on $\mathcal{A}$, M is an isocrystal on $\mathcal{R}^b$. If now (M, F) is a holonomic $F\mathcal{D}^\dagger$-module, $j^+M = \mathcal{D}^\dagger(0) \otimes_{\mathcal{D}^\dagger} M$ is a coherent $\mathcal{D}^\dagger(0)$-module, and $\mathcal{D}^\dagger_{\mathcal{A}} \otimes_{\mathcal{D}^\dagger(0)} j^+M$ can be identified with j^*M, which as before is a convergent F-isocrystal on $\mathcal{A}$, so it follows that j^+M is an isocrystal on $\mathcal{R}^b$.

On the other hand an F-isocrystal on $\mathcal{R}^b$ can be viewed (since it is Robba-solvable, cf. 3.3) as a $\mathcal{D}^\dagger(0)$-module with Frobenius structure, and the extension of scalars $\mathcal{D}^\dagger_{\mathcal{A}} \otimes_{\mathcal{D}^\dagger(0)} M$ is holonomic, as it can be shown to be a convergent F-isocrystal on $\mathcal{A}$. We are therefore justified in simply *defining* the category of holonomic $F\mathcal{D}^\dagger$-modules on η to be the category of F-isocrystals on $\mathcal{R}^b$, and regarding the functor j^+ as a functor from the category of holonomic $F\mathcal{D}^\dagger$-modules on $\mathfrak{X}$ to the category of holonomic $F\mathcal{D}^\dagger$-modules on η.

Finally, if M is a holonomic $F\mathcal{D}^\dagger$-module on η, i.e. an F-isocrystal on $\mathcal{R}^b$ viewed as $\mathcal{D}^\dagger(0)$-module, the direct image j_+M is defined to be the restriction of scalars of M by $\mathcal{D}^\dagger \to \mathcal{D}^\dagger(0)$; it is known to be a holonomic $F\mathcal{D}^\dagger$-module on $\mathfrak{X}$ (see [Cre06, Theorem 3.1]). With these definitions, (j_+, j^+) is an adjoint pair.

We will say that a holonomic $F\mathcal{D}^\dagger$-module M on $\mathfrak{X} = \mathrm{Spf}(\mathcal{V}[\![t]\!])$ is *punctual* if it has the form $M = i_! V \simeq V \otimes_K \delta$ for some F-isocrystal V on K; conversely, it is easily checked that any such module is holonomic. Using the localization triangle [Ber02b, 5.3.6]

$$i_! i^! M \longrightarrow M \longrightarrow j_+ j^+ M \xrightarrow{+1}, \tag{3.4.3}$$

we see that M is punctual if and only if $j^+M = 0$ (this is a special case of the arithmetic version [Ber02b, 5.3.3] of Kashiwara's theorem).

We will say that a holonomic $F\mathcal{D}^\dagger$-module M on $\mathfrak{X} = \mathrm{Spf}(\mathcal{V}[\![t]\!])$ is of *connection type* if the natural map $M \to j_+j^+M$ is an isomorphism, or in other words the natural map

$$M \longrightarrow \mathcal{D}^\dagger(0) \otimes_{\mathcal{D}^\dagger} M$$

is an isomorphism. The localization triangle (3.4.3) shows that $i^!M = 0$ is an equivalent condition. Here we are following (roughly) the terminology of [Mal91]; note, for example that the $\mathcal{D}^\dagger$-module $\mathcal{O}_K$, regarded as a free $\mathcal{O}_{\mathfrak{X}}$-module with connection, is not of 'connection type'.

From the localization triangle (3.4.3) we see that, for any holonomic $F\mathcal{D}^\dagger$-module M, there are exact sequences

$$\begin{gathered} 0 \longrightarrow N_1 \longrightarrow M \longrightarrow \tilde{M} \longrightarrow 0 \\ 0 \longrightarrow \tilde{M} \longrightarrow j_+j^+M \longrightarrow N_2 \longrightarrow 0 \end{gathered} \tag{3.4.4}$$

with N_1, N_2 punctual; we will call this the *standard devissage* of M (this is a very special case of a more general devissage constructed by Caro, cf. [Car06a]).

For any coherent $\mathcal{D}^\dagger$-module M (or more generally, perfect complex of $\mathcal{D}^\dagger$-modules), the dualizing functor is given by

$$\mathbb{D}(M) = \mathrm{RHom}_{\mathcal{D}^\dagger}(M, \mathcal{D}^\dagger \otimes \omega_{\mathfrak{X}}^{-1})[1]$$

(normally one translates by the dimension of $\mathfrak{X}$; cf. [Ber02b, 4.3.10]; in the formal case we should translate by the rank of $\Omega^1_{\mathfrak{X}/\mathcal{V}}$ rather than by the dimension of $\mathfrak{X}$ as a formal scheme). Now any holonomic $F\mathcal{D}^\dagger$-module on $\mathrm{Spf}(\mathcal{V}[\![t]\!])$ has cohomological dimension one (again, the rank of $\Omega^1_{\mathfrak{X}/\mathcal{V}}$; cf. Virrion [Vir00]) so for M a holonomic $F\mathcal{D}^\dagger$-module on $\mathrm{Spf}(\mathcal{V}[\![t]\!])$ we have $\mathbb{D}(M) \simeq M^*$ for some holonomic $F\mathcal{D}^\dagger$-module M^*, the so-called holonomic dual of M. The functor $M \mapsto M^*$ defines an autoequivalence of the category of $F\mathcal{D}^\dagger$-modules (cf. [Vir00]).

4. The ring of analytic differential operators

In this section we construct the analytification of an object of $D^b_{\mathrm{coh}}(\mathcal{D}^\dagger)$ and the ring of analytic differential operators $\mathcal{D}^{\mathrm{an}}$. We show furthermore that $\mathcal{D}^{\mathrm{an}}$ is a left and right flat $\mathcal{D}^\dagger$-algebra, and that the analytification functor is the derived tensor product with $\mathcal{D}^{\mathrm{an}}$.

4.1 Analytification

Recalling that $\mathcal{O} = \mathcal{V}[[t]]$, we set for any positive integers r, n

$$\mathcal{O}_{r,n} = \mathcal{O}[T]/(p^n, pT - t^r) \quad \text{and} \quad \mathcal{O}_r = \varprojlim_n \mathcal{O}_{r,n} \simeq \mathcal{O}\langle T\rangle/(pT - t^r).$$

By (3.1.3), $\mathcal{O}_{r,n}$ has a natural $\mathcal{D}^{(m)}$-module structure if r is divisible by p^{m+1}; it is determined by the condition that the $\partial^{[k]}$ annihilate $pT - t^r$:

$$\partial^{[k]}T = p^{-1}\binom{r}{k}t^{r-k} \quad \text{for } k \leqslant p^m. \tag{4.1.1}$$

(the condition on r implies that these binomial coefficients are divisible by p). It follows that if M is a $\hat{\mathcal{D}}^{(m)}$-module, then $\mathcal{O}_{r,n} \otimes_{\mathcal{O}} M$ has a natural $\hat{\mathcal{D}}^{(m)}$-module structure.

Since $\hat{\mathcal{D}}^{(m)}$ is left and right flat over $\mathcal{O}$, a projective $\hat{\mathcal{D}}^{(m)}$-module is $\mathcal{O}$-flat. Thus if $M^{\cdot}$ is an object of $D^b(\hat{\mathcal{D}}^{(m)})$, a choice of quasi-isomorphism $P^{\cdot} \to M^{\cdot}$ with $P^{\cdot}$ a complex of projective $\hat{\mathcal{D}}^{(m)}$-modules yields an isomorphism

$$R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{\cdot} \simeq R\varprojlim_n \mathcal{O}_{r,n} \otimes_{\mathcal{O}} P^{\cdot} \tag{4.1.2}$$

in $D(\mathcal{O})$. Since $\mathcal{O}$ and $R\varprojlim$ have finite homological dimension, the above complex is actually in $D^b(\mathcal{O})$. On the other hand, the previous paragraph shows that $R\varprojlim_n \mathcal{O}_{r,n} \otimes_{\mathcal{O}} P^{\cdot}$ is naturally a complex of $\hat{\mathcal{D}}^{(m)}$-modules, so in the end we see that $R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{\cdot}$ is naturally an object of $D^b(\hat{\mathcal{D}}^{(m)})$.

Lemma 4.1.1. *If M is a $\hat{\mathcal{D}}^{(m)}$-module of finite type, then*

$$H^i\left(R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M\right)_{\mathbb{Q}} = \begin{cases} (\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M)_{\mathbb{Q}} & i = 0, \\ 0 & i \neq 0. \end{cases} \tag{4.1.3}$$

Proof. The p-torsion submodule of M is also of finite type, so we can reduce to the cases where M is either p-torsion or p-torsion free. In the first case M is annihilated by a fixed power of p, the same is true for the $H^i(R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M)$, and everything in (4.1.3) is zero. If M is p-torsion-free, then

$$\mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M \simeq [\mathcal{O}\langle T\rangle/p^n \otimes_{\mathcal{O}} M \xrightarrow{pT-t^r} \mathcal{O}\langle T\rangle/p^n \otimes_{\mathcal{O}} M],$$

and consequently

$$R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M \simeq [\mathcal{O}\langle T\rangle \hat{\otimes}_{\mathcal{O}} M \xrightarrow{pT-t^r} \mathcal{O}\langle T\rangle \hat{\otimes}_{\mathcal{O}} M]. \tag{4.1.4}$$

The right-hand side of (4.1.4) is the p-adic completion of

$$[\mathcal{O}[T] \otimes_{\mathcal{O}} M \xrightarrow{pT-t^r} \mathcal{O}[T] \otimes_{\mathcal{O}} M],$$

and, since M has no p-torsion, multiplication by $pT - t^r$ is injective on $\mathcal{O}[T] \otimes_{\mathcal{O}} M$. It is then injective on $\mathcal{O}\langle T\rangle \hat{\otimes}_{\mathcal{O}} M$, and we find

$$H^i\left(R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M\right) = \begin{cases} \mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M & i = 0, \\ 0 & i \neq 0. \end{cases} \qquad \square$$

Remark. If M is a $\hat{\mathcal{D}}^{(m)}$-module of finite type, the $\mathcal{O}$-submodule $M \subset M_K$ defines a Banach norm on M_K, and the equivalence class of the norm (and the resulting topology on M_K) is

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independent of M in the following sense: if $M' \subset M_K$ is a $\hat{\mathcal{D}}^{(m)}$-submodule of finite type such that $M_K = M'_K$, then M and M' define equivalent norms. We can assume that M and M' are p-torsion free and identify them with $\hat{\mathcal{D}}^{(m)}$-submodules of M_K. Since M and M' are finitely generated $\hat{\mathcal{D}}^{(m)}$-modules, there are positive integers a, b such that $p^a M \subseteq M'$ and $p^b M' \subset M$, and the equivalence of norms follows. Now, in the terminology of [Bos84, ch. 2], $\mathcal{O}_K$ is a Banach K-algebra and $(\mathcal{O}_r)_K$ is a normed $\mathcal{O}_K$-module, and we can identify $(\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M)_{\mathbb{Q}}$ with the completed tensor product $(\mathcal{O}_r)_K \hat{\otimes}_{\mathcal{O}_K} M_K$ of normed $\mathcal{O}_K$-modules. It follows that $(\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M)_{\mathbb{Q}}$ is functorial in M_K, not just in M; this point will be important in the proof of Proposition 4.1.2.

The proof of Lemma 4.1.1 shows that there is a second way of making $R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{\cdot}$ into an object of $D^b(\hat{\mathcal{D}}^{(m)})$ when $M^{\cdot}$ is an object of $D^b(\hat{\mathcal{D}}^{(m)})$. In fact, equation (4.1.1) defines a $\mathcal{D}^{(m)}$-module structure on $\mathcal{O}[T]$ and $\mathcal{O}\langle T\rangle$, and since $(p^n, pT - t^r)$ is a regular sequence in $\mathcal{O}[T]$, the associated Koszul complex $K^{\cdot}_{r,n}$ is a resolution of $\mathcal{O}_{r,n}$ by flat $\mathcal{O}$-modules. By (4.1.1) it is also a resolution by $\mathcal{D}^{(m)}$-modules, so

$$R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{\cdot} \simeq R\varprojlim_n K^{\cdot}_{r,n} \otimes M^{\cdot} \tag{4.1.5}$$

is naturally an object of $D^b(\hat{\mathcal{D}}^{(m)})$, and is in fact is isomorphic to the construction (4.1.2).

Suppose now $M^{\cdot}$ is an object of $D^b(\hat{\mathcal{D}}^{(n,m)})$. Since $\hat{\mathcal{D}}^{(n,m)}$ is not a flat $\mathcal{O}$-algebra (neither is $B^{(n)}$), we cannot use (4.1.2) to construct $R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{\cdot}$ as an object of $D^b(\hat{\mathcal{D}}^{(n,m)})$. We can, however, use a combination of the two constructions (4.1.2) and (4.1.5). In fact, if $P^{\cdot} \to M^{\cdot}$ is a quasi-isomorphism with $P^{\cdot}$ a complex of projective $\hat{\mathcal{D}}^{(n,m)}$-modules (bounded above), then

$$R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{\cdot} \simeq R\varprojlim_n K^{\cdot}_{r,n} \otimes P^{\cdot} \tag{4.1.6}$$

by (4.1.5). Since a projective $\hat{\mathcal{D}}^{(n,m)}$-module is p-torsion free, the same argument that led to (4.1.4) shows that

$$R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{\cdot} \simeq [\mathcal{O}\langle T\rangle \xrightarrow{pT - t^r} \mathcal{O}\langle T\rangle] \hat{\otimes}_{\mathcal{O}} P^{\cdot}. \tag{4.1.7}$$

We now observe that if P (respectively Q) is a $\hat{\mathcal{D}}^{(m)}$-module (respectively $\hat{\mathcal{D}}^{(n,m)}$-module) then $P \hat{\otimes}_{\mathcal{O}} Q$ has a natural $\hat{\mathcal{D}}^{(n,m)}$-module structure. It follows that the right-hand side of (4.1.7) is naturally an object of $D^-(\hat{\mathcal{D}}^{(n,m)})$.

In any case the argument of Lemma 4.1.1 is valid when M is a $\hat{\mathcal{D}}^{(n,m)}$-module of finite type; in fact for such M it is still true that if M is p-torsion, it is annihilated by a power of p. The rest of the argument does not appeal to the $\hat{\mathcal{D}}^{(n,m)}$-module structure of M.

Let M be an object of $D^b_{\mathrm{coh}}(\mathcal{D}^\dagger)$. As in § 3, $M \simeq \varinjlim_m M^{(m)}_{\mathbb{Q}}$ for some object $M^{(\cdot)}$ of $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)})$. We define

$$M^{\mathrm{an}} = \varinjlim_m \left(R\varprojlim_r \left(R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{(m)} \right) \otimes \mathbb{Q} \right) \tag{4.1.8}$$

where it is understood that $r \to \infty$ in such a way that $v_p(r) \to \infty$ as well, so that $(R\varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M^{(m)})$ is naturally an object of $D^b(\hat{\mathcal{D}}^{(m)})$ for $r \gg 0$. It follows that the right-hand side of (4.1.8) is naturally an object of $D^b(\mathcal{D}^\dagger)$. Note that the right-hand side of (4.1.8) is independent of the choice of $M^{(\cdot)}$. In fact, an ind-isogeny $M^{(\cdot)} \to N^{(\cdot)}$ of $\hat{\mathcal{D}}^{(\cdot)}$-modules (cf. [Ber02b, 4.2.1]) induces for all $m \geqslant 0$ isomorphisms

$$\Big(R \varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} M\Big)_{\mathbb{Q}} \xrightarrow{\sim} \Big(R \varprojlim_n \mathcal{O}_{r,n} \overset{L}{\otimes}_{\mathcal{O}} N\Big)_{\mathbb{Q}}$$

in $D^b(\hat{\mathcal{D}}^{(m)})$, and these are compatible with the localization [Ber02b, 4.2.2]. It follows that if $M^{(\cdot)} \to N^{(\cdot)}$ is an isomorphism in $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)})$, the induced map $M^{\mathrm{an}} \to N^{\mathrm{an}}$ is an isomorphism in $D^b(\mathcal{D}^\dagger)$.

Although the construction (4.1.8) looks like a hopeless tangle of interwoven direct and inverse limits, we will show that $M \mapsto H^0(M^{\mathrm{an}})$ induces an exact functor on the category of coherent $\mathcal{D}^\dagger$-modules. The following variant of the standard Mittag–Leffler criterion ought to be well-known; it is in any case an immediate consequence of [Gro61, Remark 13.2.4].

Lemma 4.1.2. *Suppose (A_α) is an inverse system of complete metrizable topological groups whose transition maps $f_{\alpha\beta} : A_\beta \to A_\alpha$ satisfy the following condition: for every α there is a $\beta \geqslant \alpha$ such that for all $\gamma \geqslant \beta$, $f_{\alpha\gamma}(A_\gamma)$ is dense in $f_{\alpha\beta}(A_\beta)$. Then $R^1 \varprojlim_\alpha A_\alpha = 0$.*

(Note that the hypothesis in [Gro61, Remark 13.2.4] of uniform continuity is automatic here.)

Lemma 4.1.3. *If M is a coherent $\mathcal{D}^\dagger$-module and $M = \varinjlim_m M^{(m)}$, then*

$$H^i(M^{\mathrm{an}}) = \begin{cases} \varinjlim_m \Big(\varprojlim_r \Big(\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M_{\mathbb{Q}}{}^{(m)}\Big)\Big) & i = 0, \\ 0 & i \neq 0. \end{cases} \tag{4.1.9}$$

Proof. If $M = \varinjlim_m M^{(m)}$, then Lemma 4.1.1 shows that

$$M^{\mathrm{an}} = \varinjlim_m \Big(R \varprojlim_r (\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M^{(m)})_{\mathbb{Q}}\Big).$$

Each $(\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M^{(m)})_{\mathbb{Q}}$ has the structure of a Banach space, and, for $r < r'$, the image of $(\mathcal{O}^{r'} \hat{\otimes}_{\mathcal{O}} M^{(m)})_{\mathbb{Q}} \to (\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M^{(m)})_{\mathbb{Q}}$ is dense. Therefore $R^1 \varprojlim_r ((\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M^{(m)})_{\mathbb{Q}}) = 0$ by Lemma 4.1.2, and by 'Roos's theorem' the $R^i \varprojlim_r$ vanish for $i > 1$. The result follows from the exactness of inductive limits. $\square$

The last lemma justifies writing M^{an} for $H^0(M^{\mathrm{an}})$ when M is a coherent $\mathcal{D}^\dagger$-module, which we shall do from now on. In the particular case $M = \mathcal{D}^\dagger$ we write

$$\begin{aligned} \mathcal{D}^{\mathrm{an}} &= H^0((\mathcal{D}^\dagger)^{\mathrm{an}}) \\ &= \varinjlim_m \Big(\varprojlim_r (\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} \hat{\mathcal{D}}^{(m)})_{\mathbb{Q}}\Big) \end{aligned} \tag{4.1.10}$$

and observe that it has a natural structure as a $\mathcal{D}^\dagger$-algebra; we will call it the ring of *analytic differential operators*. Explicitly, it is

$$\mathcal{D}^{\mathrm{an}} = \left\{ \sum_k a_k \partial^{[k]} \;\middle|\; \begin{array}{l} a_k \in \mathcal{O}^{\mathrm{an}} \text{ and there is an } \eta < 1 \text{ such that for all positive } r < 1, \\ \text{there exists a } C_r > 0 \text{ such that } |a_k|_r \leqslant C_r \eta^k \end{array} \right\} \tag{4.1.11}$$

where $|a|_r$ denotes the supremum norm on the disk $|t| \leqslant r$.

Note that the $\mathcal{D}^\dagger$-module structure of M^{an} extends naturally to a $\mathcal{D}^{\mathrm{an}}$-module structure, and accordingly we will view $M \mapsto M^{\mathrm{an}}$ as a functor from the category of coherent $\mathcal{D}^\dagger$-modules to the category of (left) $\mathcal{D}^{\mathrm{an}}$-modules. It should be clear that similar constructions hold for right $\mathcal{D}^\dagger$-modules.

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We can now prove the main result of this section.

THEOREM 4.1.1. *The $\mathcal{D}^\dagger$-algebra $\mathcal{D}^{\mathrm{an}}$ is left and right flat, and for any $\mathcal{D}^\dagger$-module M of finite presentation there are natural isomorphisms*

$$\mathcal{D}^{\mathrm{an}} \overset{L}{\otimes}_{\mathcal{D}^\dagger} M \xrightarrow{\sim} \mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{D}^\dagger} M \xrightarrow{\sim} M^{\mathrm{an}}. \tag{4.1.12}$$

Proof. Choose a presentation

$$(\mathcal{D}^\dagger)^r \longrightarrow (\mathcal{D}^\dagger)^s \longrightarrow M \longrightarrow 0$$

of M. By Lemma 4.1.3, the functor $M \mapsto M^{\mathrm{an}}$ is exact on the category of coherent $\mathcal{D}^\dagger$-modules, so we have

$$(\mathcal{D}^{\mathrm{an}})^r \longrightarrow (\mathcal{D}^{\mathrm{an}})^s \longrightarrow M^{\mathrm{an}} \longrightarrow 0$$

and this implies the last isomorphism of (4.1.12). However, this means that the functor $M \mapsto \mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{D}^\dagger} M$ is exact on the category of $\mathcal{D}^\dagger$-modules of finite presentation, which implies that $\mathcal{D}^{\mathrm{an}}$ is flat as a right module. The proof that it is flat as a left module is similar.

The corresponding results for coherent $\mathcal{D}^\dagger(0)$-modules are proven similarly and will just sketch the results. A construction parallel to that of § 3.2 produces a category $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)}(0))$ and an equivalence of categories

$$\varinjlim : \underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)}(0)) \xrightarrow{\sim} D^b_{\mathrm{coh}}(\mathcal{D}^\dagger(0)) \tag{4.1.13}$$

parallel to (3.2.1). If $M^{(\cdot)}$ is an object of $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)}(0))$ corresponding via (4.1.13) to an object of $D^b_{\mathrm{coh}}(\mathcal{D}^\dagger(0))$, the right-hand side of (4.1.8) is naturally an object of $D(\mathcal{D}^\dagger(0))$, which we write $M^{\mathrm{an}(0)}$. We set $\mathcal{D}^{\mathrm{an}}(0) = (\mathcal{D}^\dagger(0))^{\mathrm{an}}$, and leave to the reader the exercise of working out the analogue of (4.1.11); then, for M as above, $M^{\mathrm{an}(0)}$ is naturally an object of $D(\mathcal{D}^{\mathrm{an}}(0))$. Lemma 4.1.3 holds as stated for any coherent $\mathcal{D}^\dagger(0)$-module, so $M^{\mathrm{an}(0)}$ is in fact an object of $D^b(\mathcal{D}^{\mathrm{an}}(0))$, and the same argument as before leads to the following theorem.

THEOREM 4.1.2. *The $\mathcal{D}^\dagger(0)$-algebra $\mathcal{D}^{\mathrm{an}}(0)$ is left and right flat, and for any $\mathcal{D}^\dagger(0)$-module of finite presentation M there are natural isomorphisms*

$$\mathcal{D}^{\mathrm{an}}(0) \overset{L}{\otimes}_{\mathcal{D}^\dagger(0)} M \xrightarrow{\sim} \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^\dagger(0)} M \xrightarrow{\sim} M^{\mathrm{an}(0)}. \tag{4.1.14}$$

It follows from the constructions that $\mathcal{D}^{\mathrm{an}}(0)$ is a $\mathcal{D}^{\mathrm{an}}$-algebra, and that

$$\begin{tikzcd} \mathcal{D}^\dagger \arrow[r] \arrow[d] & \mathcal{D}^\dagger(0) \arrow[d] \\ \mathcal{D}^{\mathrm{an}} \arrow[r] & \mathcal{D}^{\mathrm{an}}(0) \end{tikzcd}$$

commutes.

In § 3.3 we explained why a solvable isocrystal M on $\mathcal{R}^b$ has a natural structure as a coherent $\mathcal{D}^\dagger(0)$-module. By construction $\mathcal{D}^{\mathrm{an}}(0)$ is an $\mathcal{R}$-module, so it follows that $M^{\mathrm{an}} = \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^\dagger(0)} M$ has a natural $\mathcal{R}$-module structure.

PROPOSITION 4.1.1. *If M is a solvable isocrystal on $\mathcal{R}^b$, the natural map*

$$\mathcal{R} \otimes_{\mathcal{R}^b} M \longrightarrow M^{\mathrm{an}(0)}$$

is an isomorphism.

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Proof. We resume the notation of § 3.3: $M^{(m)} = B^{(\lambda(m))} \otimes_{\mathcal{O}} M_0$ for some finitely generated $B^{(n_0)}$-module M_0 and some λ. The image of $M^{(\cdot)}$ in $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)}(0))$ corresponds via the equivalence (4.1.13) to the $\mathcal{D}^\dagger(0)$-module M, so

$$M^{\mathrm{an}(0)} \simeq \varinjlim_m \varprojlim_r (\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M^{(m)}_{\mathbb{Q}})$$

by the analogue of Lemma 4.1.3 for $\mathcal{D}^\dagger(0)$. Then

$$\begin{aligned}\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M^{(m)}_{\mathbb{Q}} &\simeq \mathcal{O}_r \hat{\otimes}_{\mathcal{O}} (B^{(\lambda(m))} \otimes_{\mathcal{O}} M_0)_{\mathbb{Q}} \\ &\simeq (\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} B^{(\lambda(m))})_{\mathbb{Q}} \otimes_{B^{(\lambda(m))}} (B^{(\lambda(m))} \otimes_{B^{(n_0)}} M_0)_{\mathbb{Q}}. \\ &\simeq (\mathcal{O}_r \hat{\otimes}_{\mathcal{O}} B^{(\lambda(m))})_{\mathbb{Q}} \otimes_{B^{(\lambda(m))}} M^{(m)}_{\mathbb{Q}}.\end{aligned}$$

Since M_0 is a finitely generated $B^{(n_0)}$-module, Lemma 4.1.2 yields an isomorphism

$$\varprojlim_r \mathcal{O}_r \hat{\otimes}_{\mathcal{O}} M^{(m)}_{\mathbb{Q}} \simeq (\mathcal{O}^{\mathrm{an}} \hat{\otimes}_{\mathcal{O}} B^{(\lambda(m))}_{\mathbb{Q}}) \otimes_{B^{(\lambda(m))}_{\mathbb{Q}}} M^{(m)}_{\mathbb{Q}}.$$

Since

$$\mathcal{R} \simeq \varinjlim_m \varprojlim_r \mathcal{O}_r \hat{\otimes}_{\mathcal{O}} B^{(\lambda(m))} \quad \text{and} \quad \mathcal{R}^b \simeq \varinjlim_m \varprojlim_r B^{(\lambda(m))}_{\mathbb{Q}}$$

the proposition follows by taking the direct limit in m. □

For any $\mathcal{D}^{\mathrm{an}}(0)$-module M we denote by j_+M the restriction of scalars of M to the subring $\mathcal{D}^{\mathrm{an}}$ (compare with the definition of j_+ for coherent $\mathcal{D}^\dagger(0)$-modules in § 3.4). If M is a coherent $\mathcal{D}^\dagger(0)$-module that is also coherent as a $\mathcal{D}^\dagger$-module, it make sense to compare $(j_+M)^{\mathrm{an}}$ and $j_+(M^{\mathrm{an}(0)})$. We claim that there is a natural morphism

$$(j_+M)^{\mathrm{an}} \longrightarrow j_+(M^{\mathrm{an}(0)}) \tag{4.1.15}$$

of $\mathcal{D}^{\mathrm{an}}$-modules. In fact, we have $M = \varinjlim_m M^{(m)}_{\mathbb{Q}}$ for some $\hat{\mathcal{D}}^{(\cdot)}$-module $M^{(\cdot)}$, and then

$$M(0)^{(m)} = B^{(m)} \hat{\otimes}_{\mathcal{O}} M^{(m)} \tag{4.1.16}$$

is a coherent $\hat{\mathcal{D}}^{(m,m)}$-module for which $M \simeq \varinjlim_m M(0)^{(m)}_{\mathbb{Q}}$ as well. The map (4.1.15) is induced by the maps $M^{(m)}_{\mathbb{Q}} \to M(0)^{(m)}_{\mathbb{Q}}$.

In order to understand when (4.1.15) is an isomorphism, we first make the following observation. If M is a coherent $\mathcal{D}^\dagger(0)$-module, say $M = \varinjlim_m M^{(m)}_{\mathbb{Q}}$ as above, then each $M^{(m)}_{\mathbb{Q}}$ has a canonical Banach space topology, and M inherits an inductive limit topology which is independent of the choice of $M^{(m)}$, since any two choices are cofinal. We will call this the natural topology of M; when separated, it is an LF-space topology.

Proposition 4.1.2. *If M is a coherent $\mathcal{D}^\dagger(0)$-module that is coherent as a $\mathcal{D}^\dagger$-module, and separated in its natural topology, then the natural map (4.1.15) is an isomorphism.*

Proof. We start with $M^{(m)}$, $M(0)^{(m)}$ as in (4.1.16), which are coherent modules over $\hat{\mathcal{D}}^{(m)}$ an $\hat{\mathcal{D}}^{(m,m)}$ respectively. Since these are noetherian rings, we can replace $M^{(m)}$ and $M(0)^{(m)}$ by their images in M, to obtain coherent modules (still called $M^{(m)}$, $M(0)^{(m)}$), injective maps

$$M^{(m)}_{\mathbb{Q}} \hookrightarrow M(0)^{(m)}_{\mathbb{Q}} \hookrightarrow M$$

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and isomorphisms

$$\varinjlim_m M_{\mathbb{Q}}^{(m)} \xrightarrow{\sim} \varinjlim_m M(0)_{\mathbb{Q}}^{(m)} \xrightarrow{\sim} M \tag{4.1.17}$$

although (4.1.16) no longer holds. The natural topology of M is the inductive limit topology induced by the right-hand isomorphism in (4.1.17), and is by hypothesis separated. It follows that the inductive limit on the far left is separated; furthermore, both inductive limits are LF-space topologies, so the isomorphisms in (4.1.16) are all topological isomorphisms by the open mapping theorem for LF-spaces.

In particular, for any m the map

$$M(0)_{\mathbb{Q}}^{(m)} \longrightarrow \varinjlim_n M_{\mathbb{Q}}^{(n)}$$

is injective and continuous, so it follows from Grothendieck's factorization theorem (e.g. [Sch02, Corollary 8.9]) that there is an $\lambda(m) \geqslant m$ such that the above map factors through a continuous map $M(0)_{\mathbb{Q}}^{(m)} \longrightarrow M_{\mathbb{Q}}^{(\lambda(m))}$. In other words, the two inductive systems $\{M_{\mathbb{Q}}^{(m)}\}_m$, $\{M(0)_{\mathbb{Q}}^{(m)}\}_m$ of $\mathcal{O}$-submodules of M are cofinal. We can choose $\lambda(m)$ to be an increasing function of m, in which case $M^{(\lambda(\cdot))}$ defines an object of $\underrightarrow{LD}^b_{\mathbb{Q},\mathrm{coh}}(\hat{\mathcal{D}}^{(\cdot)})$ isomorphic to $M^{(\cdot)}$.

By the remarks after Lemma 4.1.1 and the functorial properties of the completed tensor product, the maps

$$M_{\mathbb{Q}}^{(m)} \longrightarrow M(0)_{\mathbb{Q}}^{(m)}, \quad M(0)_{\mathbb{Q}}^{(m)} \longrightarrow M_{\mathbb{Q}}^{(\lambda(m))}$$

extend uniquely to continuous maps

$$(\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M^{(m)})_{\mathbb{Q}} \longrightarrow (\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M(0)^{(m)})_{\mathbb{Q}},$$
$$(\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M(0)^{(m)})_{\mathbb{Q}} \longrightarrow (\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M^{(\lambda(m))})_{\mathbb{Q}}$$

for any $r < 1$ (existence follows from [Bos84, § 2.1, Proposition 5] and uniqueness is clear since $M_{\mathbb{Q}}^{(m)}$ is dense in $(\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M^{(m)})_{\mathbb{Q}}$). By uniqueness, the composite

$$(\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M^{(m)})_{\mathbb{Q}} \longrightarrow (\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M(0)^{(m)})_{\mathbb{Q}} \longrightarrow (\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M^{(\lambda(m))})_{\mathbb{Q}} \tag{4.1.18}$$

is the morphism induced by $M^{(m)} \to M^{(\lambda(m))}$, while the composite

$$(\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M(0)^{(m)})_{\mathbb{Q}} \longrightarrow (\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M^{(\lambda(m))})_{\mathbb{Q}} \longrightarrow (\mathcal{O}_r \mathbin{\hat{\otimes}_{\mathcal{O}}} M(0)^{(\lambda(m))})_{\mathbb{Q}} \tag{4.1.19}$$

is induced by $M(0)^{(m)} \to M(0)^{(\lambda(m))}$. Passing to the inverse limit in r and the direct limit in m in (4.1.18) yields the map (4.1.15), while passing to the limits in (4.1.19) yields an inverse to (4.1.15). □

Corollary 4.1.1. *If M is an F-isocrystal on $\mathcal{R}^b$, the natural map*

$$(j_+ M)^{\mathrm{an}} \longrightarrow j_+(M^{\mathrm{an}(0)}) \tag{4.1.20}$$

is an isomorphism.

Proof. It suffices to show that the natural topology of $M^{\mathrm{an}(0)}$ is separated. In fact, M is a finite free $\mathcal{R}$-module, and the natural topology is induced by the topology of $\mathcal{R}$, so this is clear. □

In particular, if M is an F-isocrystal on $\mathcal{R}^b$, there is a functorial isomorphism

$$\mathcal{R} \otimes_{\mathcal{R}^b} M \xrightarrow{\sim} M^{\mathrm{an}} \tag{4.1.21}$$

arising from Proposition 4.1.2 and Corollary 4.1.1.

The analytification of the adjunction $M \to j_+j^+M$ is $M^{\mathrm{an}} \to (j_+j^+M)^{\mathrm{an}}$, and since $(j_+j^+M)^{\mathrm{an}}$ has a $\mathcal{D}^{\mathrm{an}}(0)$-module structure, we obtain a natural map

$$\mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^{\mathrm{an}}} M^{\mathrm{an}} \longrightarrow (j_+j^+M)^{\mathrm{an}}. \tag{4.1.22}$$

Lemma 4.1.4. *For any holonomic $F\mathcal{D}^\dagger$-module M, the natural map (4.1.22) is an isomorphism.*

Proof. Since $j^+M \simeq \mathcal{D}^\dagger(0) \otimes_{\mathcal{D}^\dagger} M$ is an F-isocrystal on $\mathcal{R}^b$ we may apply Corollary 4.1.1, obtaining an isomorphism

$$(j_+j^+M)^{\mathrm{an}} \longrightarrow j_+(j^+M)^{\mathrm{an}(0)}.$$

It is then easily checked that (4.1.22) is the composite of the isomorphisms

$$\begin{aligned}
\mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^{\mathrm{an}}} M^{\mathrm{an}} &\simeq \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^{\mathrm{an}}} (\mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{D}^\dagger} M)\\
&\simeq \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^\dagger} M.\\
&\simeq \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^\dagger(0)} (\mathcal{D}^\dagger(0) \otimes_{\mathcal{D}^\dagger} M)\\
&\simeq \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^\dagger(0)} (j^+M)\\
&\simeq (j^+M)^{\mathrm{an}(0)}
\end{aligned}$$

where the first and last isomorphisms are from Theorems 4.1.1 and 4.1.2 respectively. □

Remark. If, as in § 3.4, we define $j^+M = \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^{\mathrm{an}}} M$ when M is a $\mathcal{D}^{\mathrm{an}}$-module, the natural map (4.1.22) takes the form

$$j_+j^+(M^{\mathrm{an}}) \longrightarrow (j_+j^+M)^{\mathrm{an}}.$$

I do not know if (4.1.22) is an isomorphism for coherent $\mathcal{D}^\dagger$-modules M. One can show, using Lemma 4.1.9 and its analogue for $\mathcal{D}^\dagger(0)$, that (4.1.22) becomes an isomorphism if the tensor product is replaced by a suitable completion.

5. Coherent $\mathcal{D}^\dagger$-modules up to analytic isomorphism

5.1 Basic calculations

We begin with some elementary computations. The resolution

$$\mathcal{O}_K \simeq [\mathcal{D}^\dagger \xrightarrow{\cdot\partial} \mathcal{D}^\dagger]$$

shows that

$$(\mathcal{O}_K)^{\mathrm{an}} = \mathcal{O}^{\mathrm{an}} \tag{5.1.1}$$

while on the other hand Corollary 4.1.1 yields an isomorphisms

$$(\mathcal{R}^b)^{\mathrm{an}} = (\mathcal{R}^b)^{\mathrm{an}(0)} = \mathcal{R} \tag{5.1.2}$$

(use the F-isocrystal structure $\nabla(1) = 0$, $F(1) = 1$ on $\mathcal{R}^b$). The isomorphism

$$\delta \simeq [\mathcal{O}_K \longrightarrow \mathcal{R}^b]$$

together with (5.1.1) and (5.1.2) then yields

$$\delta^{\mathrm{an}} \simeq \mathcal{R}/\mathcal{O}^{\mathrm{an}} \simeq \delta$$

and, as any punctual holonomic $F\mathcal{D}^\dagger$-module is a finite sum of copies of δ, we obtain the following proposition.

Proposition 5.1.1. *If N is a punctual holonomic $F\mathcal{D}^\dagger$-module, then $N \simeq N^{\mathrm{an}}$.*

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This elementary fact allows us to complete Theorem 4.1.1.

Corollary 5.1.1. *The analytification functor is exact and faithful on the category of holonomic $F\mathcal{D}^\dagger$-modules.*

Proof. Exactness follows from Theorem 4.1.1. To show faithfulness it suffices, by an elementary argument, to show that $M^{\mathrm{an}} = 0$ only if $M = 0$. Suppose that $M^{\mathrm{an}} = 0$ and let N_1, N_2 be defined by the exactness of

$$0 \to N_1 \to M \to j_+j^+M \to N_2 \to 0.$$

By Theorem 4.1.1 and Proposition 5.1.1, the analytification of this is the exact sequence.

$$0 \to N_1 \to M^{\mathrm{an}} \to (j_+j^+M)^{\mathrm{an}} \to N_2 \to 0.$$

Then $M^{\mathrm{an}} = 0$ implies $N_1 = 0$; on the other hand Lemma 4.1.4 shows that $(j_+j^+M)^{\mathrm{an}} = 0$, and consequently $N_2 = 0$. It follows that $M \simeq j_+j^+M$, i.e. M is an F-isocrystal on $\mathcal{R}^b$. Finally the formula (4.1.21) shows, since $\mathcal{R}^b$ is a field, that $M = 0$. □

We now consider duality, and first remark that the adjunction formula yields an isomorphism

$$\mathrm{RHom}_{\mathcal{D}^\dagger}(M, N) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, N) \tag{5.1.3}$$

for all coherent $\mathcal{D}^\dagger$-modules M, N.

Proposition 5.1.2. *For any two coherent $\mathcal{D}^\dagger$-modules M, N, there is a functorial isomorphism*

$$\mathrm{RHom}_{\mathcal{D}^\dagger}(M, N^{\mathrm{an}}) \simeq \mathrm{RHom}_{\mathcal{D}^\dagger}(\mathbb{D}(N), (\mathbb{D}(M))^{\mathrm{an}}). \tag{5.1.4}$$

Proof. Since M and N have a finite resolutions by finitely generated projective $\mathcal{D}^\dagger$-modules, and $\mathcal{D}^{\mathrm{an}}$ is flat over $\mathcal{D}^\dagger$, M^{an} and N^{an} likewise have a finite resolutions by finitely generated projective $\mathcal{D}^{\mathrm{an}}$-modules. It follows that the natural biduality homomorphism

$$\mathrm{RHom}_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, N^{\mathrm{an}}) \to \mathrm{RHom}_{\mathcal{D}^{\mathrm{an}}}(\mathrm{RHom}_{\mathcal{D}^{\mathrm{an}}}(N^{\mathrm{an}}, \mathcal{D}^{\mathrm{an}}), \mathrm{RHom}_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{D}^{\mathrm{an}}))$$

is an isomorphism. By (5.1.3) and the flatness of $\mathcal{D}^{\mathrm{an}}$ we have

$$\begin{aligned}\mathrm{RHom}_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{D}^{\mathrm{an}}) &\simeq \mathrm{RHom}_{\mathcal{D}^\dagger}(M, \mathcal{D}^{\mathrm{an}})\\ &\simeq \mathrm{RHom}_{\mathcal{D}^\dagger}(M, \mathcal{D}^\dagger)^{\mathrm{an}}\\ &\simeq \mathbb{D}(M)^{\mathrm{an}} \otimes_{\mathcal{O}} \omega_{\mathfrak{X}}[-1]\end{aligned}$$

for any coherent $\mathcal{D}^\dagger$-module M. It follows that the target of the above biduality isomorphism is

$$\mathrm{RHom}_{\mathcal{D}^{\mathrm{an}}}(\mathbb{D}(N)^{\mathrm{an}}, \mathbb{D}(M)^{\mathrm{an}}) \simeq \mathrm{RHom}_{\mathcal{D}^\dagger}(\mathbb{D}(N), \mathbb{D}(M)^{\mathrm{an}}).$$ □

We can now give the promised description of i^+M, for M a holonomic $F\mathcal{D}^\dagger$-module.

Proposition 5.1.3. *For any holonomic $F\mathcal{D}^\dagger$-module M, there is a natural isomorphism*

$$i^+M \xrightarrow{\sim} [M^{\mathrm{an}} \xrightarrow{\partial} M^{\mathrm{an}}] \tag{5.1.5}$$

where the complex is in degrees $[-1, 0]$.

Proof. On the one hand, there is a natural isomorphism (3.4.2)

$$i^+M \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}^\dagger}(M^*, \mathcal{O}^{\mathrm{an}})[1]$$

and, on the other hand, (5.1.4) yields

$$\mathrm{RHom}_{\mathcal{D}^\dagger}(M^*, \mathcal{O}^{\mathrm{an}}) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}^\dagger}(\mathcal{O}_K, M^{\mathrm{an}})$$

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since $\mathcal{O}_K$ is self-dual. By the explicit description (3.1.5) of the ring $\mathcal{D}^\dagger$, $\mathcal{O}_K$ has the free resolution $\mathcal{D}^\dagger \xrightarrow{\cdot\partial} \mathcal{D}^\dagger$, and from this we deduce a natural isomorphism

$$\mathrm{RHom}_{\mathcal{D}^\dagger}(\mathcal{O}_K, M^{\mathrm{an}}) \simeq [M^{\mathrm{an}} \xrightarrow{\partial} M^{\mathrm{an}}],$$

and the result follows by composing these isomorphisms. □

Proposition 5.1.4. *A holonomic $F\mathcal{D}^\dagger$-module M is of connection type if and only if* $\mathrm{RHom}_{\mathcal{D}^\dagger}(M, \mathcal{O}^{\mathrm{an}}) = 0$.

Proof. Again using [Cre06, 2.2], there is a natural isomorphism

$$\mathrm{RHom}_{\mathcal{D}^\dagger}(M, \mathcal{O}^{\mathrm{an}})[1] \xrightarrow{\sim} i^+(M^*),$$

while, on the other hand, biduality and the definition of i^+ yield an isomorphism

$$i^+(M^*) \simeq (i^! M)^*$$

so we have $\mathrm{RHom}_{\mathcal{D}^\dagger}(M, \mathcal{O}^{\mathrm{an}}) = 0$ if and only if $i^! M = 0$, i.e. if and only if M has connection type. □

5.2 The analytic category

The category $\mathrm{Coh}^{\mathrm{an}}(\mathcal{D}^\dagger)$ of coherent $\mathcal{D}^\dagger$-modules up to analytic isomorphism is a localization of the category of coherent $\mathcal{D}^\dagger$-modules: objects are just coherent $\mathcal{D}^\dagger$-modules on $\mathrm{Spf}(\mathcal{V}[\![t]\!])$, while morphisms are given by

$$\mathrm{Hom}_{\mathrm{Coh}^{\mathrm{an}}}(M, N) = \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, N^{\mathrm{an}}). \tag{5.2.1}$$

We denote by

$$\mathrm{an} : \mathrm{Coh}(\mathcal{D}^\dagger) \longrightarrow \mathrm{Coh}^{\mathrm{an}}(\mathcal{D}^\dagger) \tag{5.2.2}$$

the localization functor, which by Theorem 4.1.1 is exact.

If (M, Φ) is an $F\mathcal{D}^\dagger$-module, the Frobenius structure Φ induces a Frobenius structure on the $\mathcal{D}^{\mathrm{an}}$-module M^{an}. Thus we can define the category $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ of holonomic $F\mathcal{D}^\dagger$-modules up to isomorphism, by taking as objects the holonomic $F\mathcal{D}^\dagger$-modules, while a morphism $M \to N$ in $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ is a morphism $M^{\mathrm{an}} \to N^{\mathrm{an}}$ of $\mathcal{D}^{\mathrm{an}}$-modules compatible with the induced Frobenius structure. We denote the associated localization, as above, by

$$\mathrm{an} : \mathrm{Hol}(F\mathcal{D}^\dagger) \longrightarrow \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger). \tag{5.2.3}$$

The above results show that certain cohomological functors on the category of holonomic $F\mathcal{D}^\dagger$-modules extend to $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$. First of all, this is the case for the construction $M \mapsto M \otimes_{\mathcal{O}} \omega_{\mathfrak{X}}$ which turns a left $\mathcal{D}^\dagger$-module into a right $\mathcal{D}^\dagger$-module. In fact if we denote by ω^{an} the analytification of the right $\mathcal{D}^\dagger$-module $\omega_{\mathfrak{X}/\mathcal{V}}$, then

$$\omega^{\mathrm{an}} \simeq \mathcal{O}^{\mathrm{an}} \otimes_{\mathcal{O}} \omega_{\mathfrak{X}/\mathcal{V}},$$

and a simple calculation using Lemma 4.1.3 shows that

$$(M \otimes_{\mathcal{O}} \omega_{\mathfrak{X}/\mathcal{V}})^{\mathrm{an}} \simeq M^{\mathrm{an}} \otimes_{\mathcal{O}^{\mathrm{an}}} \omega^{\mathrm{an}} \tag{5.2.4}$$

for any left coherent $\mathcal{D}^\dagger$-module M; there is a similar formula for right coherent $\mathcal{D}^\dagger$-modules. It follows from (5.2.4) that the 'left-to-right' functor extends to $\mathrm{Coh}^{\mathrm{an}}$. Similarly, the duality

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functor $M \mapsto M^*$ satisfies

$$\begin{aligned}(M^*)^{\mathrm{an}} &\simeq \mathrm{Ext}^1_{\mathcal{D}^\dagger}(M, \mathcal{D}^\dagger \otimes_{\mathcal{O}} \omega^{-1}_{\mathfrak{X}/\mathcal{V}})^{\mathrm{an}} \\ &\simeq \mathrm{Ext}^1_{\mathcal{D}^\dagger}(M, \mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{O}^{\mathrm{an}}} (\omega^{\mathrm{an}})^{-1}) \\ &\simeq \mathrm{Ext}^1_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{O}^{\mathrm{an}}} (\omega^{\mathrm{an}})^{-1})\end{aligned}$$

so that $M \mapsto M^*$ is actually functorial in M^{an}, i.e. induces a functor $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger) \to \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ (in fact an autoequivalence). Again, the formulas

$$i^+ M \simeq [M^{\mathrm{an}} \xrightarrow{\partial} M^{\mathrm{an}}]$$

and

$$i^! M \simeq \mathrm{RHom}(M^*, \delta) \simeq \mathrm{RHom}((M^*)^{\mathrm{an}}, \delta)$$

imply that i^+, $i^!$ define cohomological functors on $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$. That the same is true for j_+ and j^+ follows from Proposition 4.1.2 and the isomorphism (4.1.21).

Continuing in the same vein, we note that the property of being a holonomic $F\mathcal{D}^\dagger$-module of connection type is visible in the analytic category.

Proposition 5.2.1. *If M is a holonomic $F\mathcal{D}^\dagger$-module, then M is of connection type if and only if the natural map*

$$M^{\mathrm{an}} \longrightarrow \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^{\mathrm{an}}} M^{\mathrm{an}}$$

is an isomorphism. In particular, if M and M' are holonomic $F\mathcal{D}^\dagger$-modules with isomorphic analytifications, then M is of connection type if and only if the same is true for M'.

Proof. By Theorem 4.1.1 and Proposition 5.1.1, the analytification of the standard devissage of M

$$0 \to N_1 \to M \to j_+ j^+ M \to N_2 \to 0$$

is the exact sequence

$$0 \to N_1 \to M^{\mathrm{an}} \to (j_+ j^+ M)^{\mathrm{an}} \to N_2 \to 0.$$

Using Lemma 4.1.4, we can write this as an exact sequence

$$0 \to N_1 \to M^{\mathrm{an}} \to \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^{\mathrm{an}}} M^{\mathrm{an}} \to N_2 \to 0$$

where the map in the middle is the natural one. As M is of connection type if and only if $N_1 = N_2 = 0$, the proposition follows. □

6. Solution data

6.1 Microfunctions

The first step in extending the classification of § 2 is to show that the differential module structures of $\mathcal{B}^b$ and $\mathcal{B}$ extend to $\mathcal{D}^\dagger$-module structures. By (1.2.3) it suffices to show that the $\mathcal{R}_n^{b,\mathrm{st}}(u)$ have a $\mathcal{D}^\dagger$-module structure; in fact, they are F-isocrystals on $\mathcal{R}^b$, so we can apply the remarks of § 3.3.

It is not hard to construct the $\mathcal{D}^\dagger$-module structure on $\mathcal{R}_n^{b,\mathrm{st}}(u)$ directly. Choose a local parameter t and set

$$\mathcal{R}_n^{0,\mathrm{st}} = \mathrm{Ker}\, N^n | \mathcal{R}^0[\log t]$$

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(unlike $\mathcal{R}^{b,\mathrm{st}}$, this is *not* independent of the choice of t). Since

$$\mathcal{R}_n^{b,\mathrm{st}}(u) \simeq \mathcal{R}^0(u) \otimes_{\mathcal{R}^0} \mathcal{R}_n^{0,\mathrm{st}} \otimes \mathbb{Q}$$

it suffices to construct $\mathcal{D}^\dagger$-module structures on $\mathcal{R}^0(u)_\mathbb{Q}$ and on $\mathcal{R}_{n\mathbb{Q}}^{0,\mathrm{st}}$. Now $\mathcal{R}^0(u)$ is an étale $\mathcal{R}^0$-algebra, so $\mathcal{R}^0(u)$ has a formal m-PD-stratification, and hence a $\hat{\mathcal{D}}^{(m)}$-module structure for all m. On the other hand $\mathcal{R}_n^{0,\mathrm{st}}$ has a PD-stratification (i.e. a stratification of level zero) defined by

$$1 \otimes (\log t) \mapsto (\log t) \otimes 1 + \sum_{k>0} (-1)^{k-1}(k-1)!(t^{-k} \otimes 1)\gamma_k(1 \otimes t - t \otimes 1),$$

which makes $\mathcal{R}_n^{0,\mathrm{st}}$ into a $\hat{\mathcal{D}}^{(0)}$-module; one checks, finally, that the natural morphism

$$\mathcal{R}_n^{0,\mathrm{st}} \to \hat{\mathcal{D}}^{(m)} \otimes_{\hat{\mathcal{D}}^{(0)}} \mathcal{R}_n^{0,\mathrm{st}}$$

is an isogeny for any m; this gives $\mathcal{R}_{n\mathbb{Q}}^{0,\mathrm{st}}$ the structure of a $\hat{\mathcal{D}}_\mathbb{Q}^{(m)}$-module for all m, whence a $\mathcal{D}^\dagger$-module structure.

Since $(\mathcal{R}_n^{b,\mathrm{st}}(u))^{\mathrm{an}} = \mathcal{R}_n^{\mathrm{st}}(u)$, the $\mathcal{D}^\dagger$-module structure of $\mathcal{R}_n^{\mathrm{st}}(u)$ extends to a $\mathcal{D}^{\mathrm{an}}$-module structure. We obtain thereby a $\mathcal{D}^{\mathrm{an}}$-module structure on $\mathcal{B}$. In fact this extends to a $\mathcal{D}^{\mathrm{an}}(0)$-module structure, as one sees from the construction of $\mathcal{D}^{\mathrm{an}}(0)$.

We now set $\mathcal{O}_{K^{\mathrm{nr}}}^{\mathrm{an}} = \mathcal{O}^{\mathrm{an}} \hat{\otimes}_K K^{\mathrm{nr}}$, and define the *microfunction spaces* $\mathcal{C}^b$, $\mathcal{C}$ and the canonical maps $\mathcal{B}^b \to \mathcal{C}^b$, $\mathcal{B} \to \mathcal{C}$ by the exact sequences

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_{K^{\mathrm{nr}}} \longrightarrow \mathcal{B}^b \xrightarrow{\mathrm{can}} \mathcal{C}^b \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_{K^{\mathrm{nr}}}^{\mathrm{an}} \longrightarrow \mathcal{B} \xrightarrow{\mathrm{can}} \mathcal{C} \longrightarrow 0. \end{gathered} \tag{6.1.1}$$

As a quotient of $\mathcal{B}$, $\mathcal{C}$ is a discrete G-module, and it inherits an action of the Frobenius φ and of the canonical monodromy operator, which we denote by $N_\mathcal{C}$. Since $\mathcal{O}_{K^{\mathrm{nr}}}$ has a $\mathcal{D}^{\mathrm{an}}$-module structure, so does $\mathcal{C}$ (but there is no $\mathcal{D}^{\mathrm{an}}(0)$-module structure on $\mathcal{C}$). Since N annihilates $\mathcal{O}^{\mathrm{an}} \subset \mathcal{B}$, the maps $N_\mathcal{B}$, $N_\mathcal{C}$ factor through the *variation*

$$\mathrm{var} : \mathcal{C} \longrightarrow \mathcal{B} \tag{6.1.2}$$

so that

$$N_\mathcal{B} = \mathrm{var} \circ \mathrm{can} \quad \text{and} \quad N_\mathcal{C} = \mathrm{can} \circ \mathrm{var}. \tag{6.1.3}$$

Recall that $I \subset G$ denotes the inertia subgroup of the absolute Galois group of $k((t))$. For $\sigma \in I$, the action of $\sigma - 1$ on $\mathcal{B}$ is zero on $\mathcal{O}_{K^{\mathrm{nr}}}^{\mathrm{an}}$, and so factors through a K^{nr}-linear map

$$v(\sigma) : \mathcal{C} \longrightarrow \mathcal{B} \tag{6.1.4}$$

so that

$$\sigma|\mathcal{B} = 1 + v(\sigma) \cdot \mathrm{can} \quad \text{and} \quad \sigma|\mathcal{C} = 1 + \mathrm{can} \cdot v(\sigma) \tag{6.1.5}$$

for any $\sigma \in I$. These deserve to be called variation maps as well, but it will be convenient to reserve this term for the map induced by N, and refer to the collection of maps $v(\sigma)$ as the 'Galois variation'. We note, finally, that the canonical maps commute with φ and the Galois action, while the variations satisfy

$$\mathrm{var} \cdot \varphi = q\varphi \cdot \mathrm{var}, \quad v(\sigma) \cdot \varphi = q\varphi \cdot v(\sigma), \quad \sigma \in G. \tag{6.1.6}$$

All of the above constructions have versions for $\mathcal{B}^b$ and $\mathcal{C}^b$.

If now M is a holonomic F-$\mathcal{D}^\dagger$-module on $\mathcal{O}$, we define K^{nr}-vector spaces $\mathbb{V}(M)$, $\mathbb{W}(M)$ by

$$\mathbb{V}(M) = \mathrm{Hom}_{\mathcal{D}^\dagger}(M, \mathcal{B}), \quad \mathbb{W}(M) = \mathrm{Hom}_{\mathcal{D}^\dagger}(M, \mathcal{C}) \tag{6.1.7}$$

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and denote by

$$\mathrm{can}_* : \mathbb{V}(M) \to \mathbb{W}(M), \quad \mathrm{var}_*, v(\sigma)_* : \mathbb{W}(M) \to \mathbb{V}(M) \tag{6.1.8}$$

the maps induced by the canonical map (6.1.1) and the variation maps (6.1.2), and (6.1.4). Since $\mathcal{B}$ and $\mathcal{C}$ are actually $\mathcal{D}^{\mathrm{an}}$-modules, there are isomorphisms

$$\mathbb{V}(M) = \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{B}), \quad \mathbb{W}(M) = \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}}, \mathcal{C}) \tag{6.1.9}$$

by Corollary 5.1.3; thus the functors $\mathbb{V}$ and $\mathbb{W}$ extend to the category $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$. They also have a natural Frobenius structure given by $F^* \circ \varphi_*$ which commutes with the canonical map, and satisfies and analogue of (6.1.6) for the variations. We observe, finally, that since $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{D}^{\mathrm{an}}$-modules, $\mathbb{V}(M)$ and $\mathbb{W}(M)$ depend only on the analytic isomorphism class of M.

This should motivate the following definition: the category Soln_K of *solution data* has as objects quintuples $(V, W, c, v, v(\cdot))$ consisting of the following.

- The vector spaces V and W are objects of $\mathrm{Mod}_K(G, F)$, i.e. finite-dimensional K^{nr}-vector spaces endowed with σ-linear isomorphisms $F : V \to V$ and $F : W \to W$ and a discrete semilinear action of G commuting with F.
- The maps $c : V \to W$ and $v : W \to V$ are K^{nr}-linear maps such that c commutes with F, while v satisfies $vF = qFv$.
- For any $\sigma \in I$, the action of σ on V (respectively W) is given by $1 + v(\sigma) \cdot c$ (respectively $1 + c \cdot v(\sigma)$) for some K^{nr}-linear map $v(\sigma) : W \to V$ satisfying (6.1.6).

A morphism $f : (V, W, c, v, v(\sigma)) \to (V', W', c', v', v'(\sigma))$ is a pair of maps $f : V \to V'$, $f : W \to W'$ with all of the above structure, i.e. which are morphisms both of G-representations and of F-isocrystals on K^{nr}, such that the diagrams

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[d, "f"] & W \arrow[d, "f"] \\ V' \arrow[r, "c'"] & W' \end{tikzcd} \quad \begin{tikzcd} W \arrow[r, "v"] \arrow[d, "f"] & V \arrow[d, "f"] \\ W' \arrow[r, "v'"] & V' \end{tikzcd} \quad \begin{tikzcd} W \arrow[r, "v(\sigma)"] \arrow[d, "f"] & V \arrow[d, "f"] \\ W' \arrow[r, "v'(\sigma)"] & V' \end{tikzcd}$$

commute.

It should be clear that Soln_K is a K-linear abelian category in which every object has finite length. A morphism f as above is a monomorphism (respectively epimorphism) if and only if the maps $V \to V'$, $W \to W'$ are injective (respectively surjective).

To lighten the notation, we will occasionally write objects of Soln_K as (V, W, c, v) or (V, W) if the other data do not need to be specified.

The goal of this section is to show that the construction

$$M \mapsto \mathbb{S}(M) = (\mathbb{V}(M), \mathbb{W}(M), \mathrm{can}_*, \mathrm{var}_*, v(\sigma)_*) \tag{6.1.10}$$

described above defines an exact functor $\mathbb{S} : \mathrm{Hol}(M) \longrightarrow \mathrm{Soln}_K$. We first compute $\mathbb{V}(M)$ for some basic types.

Proposition 6.1.1. *For any holonomic $F\mathcal{D}^\dagger$-module there is a functorial isomorphism*

$$\mathbb{V}(M) \xrightarrow{\sim} \mathbb{V}(j_+ j^+ M). \tag{6.1.11}$$

If M is of connection type, there is a functorial isomorphism

$$\mathbb{V}(M) \xrightarrow{\sim} \mathbb{V}(M^{\mathrm{an}}) \tag{6.1.12}$$

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where the $\mathbb{V}$-functor on the right is the functor (2.1.1) applied to the F-isocrystal M^{an} on $\mathcal{R}$. If M is of punctual type, then $\mathrm{RHom}_{\mathcal{D}^\dagger}(M, \mathcal{B}) \simeq 0$, and in particular $\mathbb{V}(M) = 0$.

Proof. Suppose first that M is of connection type, i.e. an F-isocrystal on $\mathcal{R}^b$. Tensoring the isomorphism (3.3.10) with $\mathcal{D}^{\mathrm{an}}(0)$ yields, by (4.1.14), an isomorphism

$$M^{\mathrm{an}} \xrightarrow{\sim} \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}} M^{\mathrm{an}}$$

where $\mathcal{D} = \mathcal{R}[\partial]$ as in § 2. Consequently

$$\mathrm{Hom}_{\mathcal{D}}(M^{\mathrm{an}}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}(0)}(\mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}} M^{\mathrm{an}}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}(0)}(M^{\mathrm{an}}, \mathcal{B})$$

where the first isomorphism is the adjunction. On the other hand, Proposition 4.1.2 yields isomorphisms

$$M^{\mathrm{an}} \simeq \mathcal{D}^{\mathrm{an}} \otimes_{\mathcal{D}^\dagger} M \simeq \mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^\dagger(0)} M,$$

and from this we get isomorphisms

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}(0)}(M^{\mathrm{an}}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}^{\mathrm{an}}(0)}(\mathcal{D}^{\mathrm{an}}(0) \otimes_{\mathcal{D}^\dagger(0)} M, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}^\dagger(0)}(M, \mathcal{B})$$

where the second map is the adjunction. Since M is of connection type we have $M \simeq \mathcal{D}^\dagger(0) \otimes_{\mathcal{D}^\dagger} M$ and isomorphisms

$$\mathrm{Hom}_{\mathcal{D}^\dagger(0)}(M, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}^\dagger(0)}(\mathcal{D}^\dagger(0) \otimes_{\mathcal{D}^\dagger} M, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}^\dagger}(M, \mathcal{B})$$

where again the second map is the adjunction. Putting everything together, we get an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(M^{\mathrm{an}}, \mathcal{B}) \simeq \mathrm{Hom}_{\mathcal{D}^\dagger}(M, \mathcal{B})$$

which is (6.1.12).

Suppose now $M = V \otimes_K \delta$. Since δ has the resolution

$$\delta \simeq [\mathcal{D}^\dagger \xrightarrow{\cdot t} \mathcal{D}^\dagger] \tag{6.1.13}$$

we have

$$\mathrm{RHom}_{\mathcal{D}^\dagger}(V \otimes \delta, \mathcal{B}) \simeq V^* \otimes_K [\mathcal{B} \xrightarrow{t\cdot} \mathcal{B}] \simeq 0,$$

which is the last assertion of Proposition 6.1.1. The isomorphism (6.1.11) then follows using the standard devissage (3.4.4). □

Lemma 6.1.1. *There is a canonical isomorphism*

$$\mathrm{Ext}^1_{\mathcal{D}^\dagger}(\mathcal{R}^b, \mathcal{B}) = 0.$$

Proof. Since $\mathcal{R}^b$ has the resolution

$$\mathcal{R}^b \simeq [\mathcal{D}^\dagger \xrightarrow{\cdot \partial t} \mathcal{D}^\dagger]$$

we have

$$\begin{aligned} \mathrm{RHom}_{\mathcal{D}^\dagger}(\mathcal{R}^b, \mathcal{B}) &\simeq [\mathcal{B} \xrightarrow{\partial t\cdot} \mathcal{B}] \\ &\simeq [\mathcal{B} \xrightarrow{\partial\cdot} \mathcal{B}] \end{aligned}$$

since t is invertible on $\mathcal{B}$. The lemma follows since ∂ is surjective on $\mathcal{B}$, as we saw in the proof of Lemma 1.2.1. □

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Lemma 6.1.2. *For any holonomic F-$\mathcal{D}^\dagger$-module M on $\mathfrak{X}$,*

$$\operatorname{Ext}^i_{\mathcal{D}^\dagger}(M,\mathcal{B}) = \operatorname{Ext}^i_{\mathcal{D}^\dagger}(M,\mathcal{C}) = 0$$

for all $i>0$.

Proof. By (6.1.1) and the fact that a holonomic F-$\mathcal{D}^\dagger$-module has cohomological dimension less than or equal to 1, we see that it is enough to prove the vanishing for $\mathcal{B}$ and for $i=1$. By the last assertion of Proposition 6.1.1, it suffices to prove this for holonomic modules of connection type. Now an M of connection type is an F-isocrystal on $\mathcal{R}^b$, so M^{an} is free of finite type over $\mathcal{R}$, and so using (2.1.2) and the flatness of the $\mathcal{D}^\dagger$-algebra $\mathcal{D}^{\mathrm{an}}$ produces isomorphisms

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{D}^\dagger}(M,\mathcal{B}) &\simeq \operatorname{Ext}^1_{\mathcal{D}^{\mathrm{an}}}(M^{\mathrm{an}},\mathcal{B})\\ &\simeq \operatorname{Ext}^1_{\mathcal{D}^{\mathrm{an}}}(\mathcal{R},\operatorname{Hom}_{\mathcal{R}}(M^{\mathrm{an}},\mathcal{B}))\\ &\simeq \operatorname{Ext}^1_{\mathcal{D}^{\mathrm{an}}}(\mathcal{R},\mathbb{V}(M^{\mathrm{an}})\otimes_{K^{\mathrm{nr}}}\mathcal{B})\\ &\simeq \operatorname{Ext}^1_{\mathcal{D}^\dagger}(\mathcal{R}^b,\mathbb{V}(M^{\mathrm{an}})\otimes_{K^{\mathrm{nr}}}\mathcal{B})\end{aligned}$$

and the last Ext group vanishes by Lemma 6.1.1. □

Corollary 6.1.1. *For any holonomic $F\mathcal{D}^\dagger$-module, there is an exact sequence*

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{D}^\dagger}(M,\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}}) \longrightarrow \mathbb{V}(M) \xrightarrow{\mathrm{can}} \mathbb{W}(M) \longrightarrow \operatorname{Ext}^1_{\mathcal{D}^\dagger}(M,\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}}) \longrightarrow 0. \tag{6.1.14}$$

Proof. This follows from Lemma 6.1.2 and the exact sequence (6.1.1). □

Corollary 6.1.2. *A holonomic $F\mathcal{D}^\dagger$-module M is of connection type if and only if the canonical map* $\mathrm{can}:\mathbb{V}(M)\to\mathbb{W}(M)$ *is an isomorphism.*

Proof. By the last corollary, $\mathrm{can}:\mathbb{V}(M)\to\mathbb{W}(M)$ is an isomorphism if and only if $\operatorname{RHom}(M,\mathcal{O}^{\mathrm{an}})\simeq 0$, but we saw in Proposition 5.1.4 that this last isomorphism is equivalent to M being of connection type. □

Remark. The classification that is the aim of this paper was modeled on that of Malgrange [Mal91]. We see, however, that our classification is closer to the 'formal' classification of [Mal91] rather than the holomorphic one. In fact, with $\hat{\mathcal{D}}$ equal to the (classical) ring of differential operators on $K[[t]]$, and $\mathcal{O}=K[[t]]$, we have $\operatorname{RHom}_{\hat{\mathcal{D}}}(M,\mathcal{O})=0$ for any $\hat{\mathcal{D}}$-module M of connection type (if M is totally irregular, this follows from [Mal91, ch. 3, Theorem 2.3]; for M regular it can be checked directly). In the holomorphic case, on the other hand, the dimension of $\operatorname{Ext}^1(M,\mathcal{O})$ for an M of connection type is the irregularity of M (cf. [Mal91, ch. 4, § 4]). This is a little disturbing in view of the well-known analogy between the irregularity of a connection and wild ramification; it would seem that our construction is not capable of seeing the wild vanishing cycles.

We can now compute $\mathbb{W}$ of a punctual module.

Corollary 6.1.3. *If $M=V\otimes_K\delta$, then*

$$\mathbb{W}(M)\simeq V^*_{K^{\mathrm{nr}}}.$$

Proof. Since $\mathbb{V}(M)=0$, (6.1.14) shows there are isomorphisms

$$\mathbb{W}(V\otimes_K\delta) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathcal{D}^\dagger}(V\otimes_K\delta,\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}}) \simeq V^*\otimes_K \operatorname{Ext}^1_{\mathcal{D}^\dagger}(\delta,\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}}),$$

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and, again using the resolution (6.1.13), we find

$$\mathrm{RHom}_{\mathcal{D}^\dagger}(\delta, \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}}) \simeq [\mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}} \xrightarrow{t\cdot} \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}}]$$

so that $\mathrm{Ext}^1_{\mathcal{D}^\dagger}(\delta, \mathcal{O}^{\mathrm{an}}_{K^{\mathrm{nr}}}) \simeq K^{\mathrm{nr}}$. □

Theorem 6.1.1. *The construction (6.1.10) defines an exact functor*

$$\mathbb{S} : \mathrm{Hol}(F\mathcal{D}^\dagger) \longrightarrow \mathrm{Soln}_K.$$

Proof. To show that $\mathbb{S}(M)$ is an object of Soln_K, the only thing that remains to be seen is that $\mathbb{V}(M)$ and $\mathbb{W}(M)$ have finite dimension over K^{nr}. In fact, the extreme terms of (6.1.14) have finite dimension by [Cre06, Theorem 2.2], and by the isomorphism (6.1.11), the dimension of $\mathbb{V}(M)$ is the rank of the F-isocrystal $(j_+j^+M)^{\mathrm{an}}$. It follows that $\mathbb{W}(M)$ has finite dimension.

To show that $\mathbb{S}$ is exact, it suffices to show that the functors $\mathbb{V}$ and $\mathbb{W}$ are exact; this however, follows immediately from Lemma 6.1.2. □

Motivated by Corollary 6.1.2, we will say that an object $(V, W, c, v, v(\cdot))$ of Soln_K is of *connection type* if c is an isomorphism. For example, if (V, N, F, ρ) is an object of $\mathrm{Mod}_K(G, F, N)$, then $(V, V, \mathrm{id}_V, N, \rho(\cdot) - 1)$ is a connection type object of Soln_K. This construction induces a fully faithful functor

$$\mathrm{Mod}_K(G, F, N) \to \mathrm{Soln}_K. \tag{6.1.15}$$

On the other hand, if (V, W, c, v) is a solution datum of connection type, then

$$(V, W, c, v) \simeq (V, V, \mathrm{id}_V, vc),$$

and vc is nilpotent. Thus the full subcategory of connection type solution data is the essential image of the functor (6.1.15).

We will say that an object of Soln_K is *punctual* if it has the form $(0, W, 0, 0)$. Since the Galois variation is zero, the inertia group $I \subset G$ must act trivially on W. By Proposition 6.1.1, punctual F-$\mathcal{D}^\dagger$-modules give rise to solution data of this type, for if $V = 0$ then v, c, and $v(\cdot)$ are zero as well. An object (V, W, c, v) of Soln_K has a 'standard devissage'

$$\begin{aligned} 0 \to (0, \operatorname{Ker} c, 0, 0) \to (V, V, \mathrm{id}_V, vc) \to (V, V/\operatorname{Ker} c, \mathrm{proj}, vc) \to 0,\\ 0 \to (V, V/\operatorname{Ker} c, \mathrm{proj}, vc) \to (V, W, c, v) \to (0, \operatorname{Coker} c, 0, 0) \to 0 \end{aligned} \tag{6.1.16}$$

corresponding to the standard devissage (3.4.4) of an $F\mathcal{D}^\dagger$-module on $\mathcal{O}$.

7. Construction of holonomic $F\mathcal{D}^\dagger$-modules from solution data

7.1

Our aim is to recover a holonomic F-$\mathcal{D}^\dagger$-module up to analytic isomorphism from its solution data. Suppose M is a holonomic $F\mathcal{D}^\dagger$-module and $S = \mathbb{S}(M)$ is the corresponding solution data. The sextuple

$$\mathcal{S} = (\mathcal{B}, \mathcal{C}, \mathrm{can}, \mathrm{var}, v(\cdot), \varphi)$$

can be viewed as an ind-object of the category Soln_K, and there is a natural evaluation map

$$M \longrightarrow \mathrm{Hom}(S, \mathcal{S})$$

where the Hom is in the ind-category of Soln_K. We will show that $\mathrm{Hom}(S, \mathcal{S})$ is isomorphic as a $\mathcal{D}^{\mathrm{an}}$-module to M^{an}, and the induced Frobenius structures coincide. To obtain an $F\mathcal{D}^\dagger$-module we must replace this construction by

$$\mathrm{Hom}(S, \mathcal{S}^b) = \mathbb{M}(S)$$

where $\mathcal{S}^b$ is the ind-object

$$\mathcal{S}^b = (\mathcal{B}^b, \mathcal{C}^b, \mathrm{can}, \mathrm{var}, v(\cdot)).$$

We will see that $\mathbb{M}(S)$ is a holonomic $F\mathcal{D}^\dagger$-module with the same analytification as M, so that we have recovered M as an object of $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$. For reasons explained in the introduction, we cannot hope to do better than this.

It will be convenient to work with a more explicit form of this construction. For $S = (V, W, c, v, v(\cdot))$ we denote by $\mathbb{M}^\bullet(S)$ the two-term complex

$$(\mathrm{Hom}(V, \mathcal{B}^b) \oplus \mathrm{Hom}(W, \mathcal{C}^b))^G \xrightarrow{d} (\mathrm{Hom}(V, \mathcal{C}^b) \oplus \mathrm{Hom}(W, \mathcal{B}^b))^G \tag{7.1.1}$$

in degrees zero and one, where the differential is defined by the matrix

$$d = \begin{pmatrix} \mathrm{can}_* & -c^* \\ -v^* & \mathrm{var}_* \end{pmatrix} \tag{7.1.2}$$

considered as acting on column vectors. Then

$$\mathbb{M}(S) = H^0(\mathbb{M}^\bullet(S)) \tag{7.1.3}$$

clearly coincides with the $\mathbb{M}(S)$ defined above, and the $F\mathcal{D}^\dagger$-module structures on $\mathcal{B}^b$ and $\mathcal{C}^b$ induce an $F\mathcal{D}^\dagger$-module structure on $\mathbb{M}(S)$. We first consider some examples.

The kernel of the variation $\mathrm{var} : \mathcal{C}^b \to \mathcal{B}^b$ is $\overline{\mathcal{R}}^b/\mathcal{O}_{K^{\mathrm{nr}}}$, so by the exactness of Galois-invariants there is an exact sequence

$$0 \longrightarrow \mathcal{R}^b/\mathcal{O} \longrightarrow (\mathcal{C}^b)^G \longrightarrow (\mathcal{B}^b)^G. \tag{7.1.4}$$

Thus if $S = (0, W, 0, 0, F)$ is of punctual type, $W_0 = W^G$ with the induced action of F is an F-isocrystal on K, and

$$\begin{aligned} \mathbb{M}(S) &\simeq \mathrm{Ker}(\mathrm{Hom}(W, \mathcal{C}^b)^G \xrightarrow{\mathrm{var}_*} \mathrm{Hom}(W, \mathcal{B}^b)^G) \\ &\simeq \mathrm{Hom}_K(W_0, \mathcal{R}^b/\mathcal{O}) \\ &\simeq \mathrm{Hom}_K(W_0, \delta) \simeq W_0^* \otimes_K \delta \end{aligned} \tag{7.1.5}$$

is a holonomic $F\mathcal{D}^\dagger$-module of punctual type.

If $S = (V, V, \mathrm{id}_V, N)$ is of connection type, $\mathbb{M}^\bullet(S)$ is

$$(\mathrm{Hom}(V, \mathcal{B}) \oplus \mathrm{Hom}(V, \mathcal{C}))^G \xrightarrow{d} (\mathrm{Hom}(V, \mathcal{C}) \oplus \mathrm{Hom}(V, \mathcal{B}))^G,$$
$$d = \begin{pmatrix} \mathrm{can}_* & -\mathrm{id}^* \\ -N^* & \mathrm{var}_* \end{pmatrix},$$

and the equation

$$\begin{pmatrix} \mathrm{can}_* & -\mathrm{id}^* \\ -N^* & \mathrm{var}_* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

yields $\mathrm{can}_*\, a = b$ and $\mathrm{var}_*\, b = N^* a$, so that $N^* a = (\mathrm{var} \cdot \mathrm{can})_* a = N_{\mathcal{B}*} a$. Thus the kernel of d is the same as the kernel of $N^* - N_{\mathcal{B}*}$, and there is a functorial isomorphism

$$H^0(\mathbb{M}(V, V, \mathrm{id}_V, N)) \simeq \mathrm{Hom}_{K^{\mathrm{nr}}}(V, \mathcal{B}^b)^{G,N} \simeq \mathbb{M}^b(V), \tag{7.1.6}$$

in the notation of (2.1.5). By Lemma 2.1.1, this is an F-isocrystal on $\mathcal{R}^b$, i.e. a holonomic $F\mathcal{D}^\dagger$-module of connection type.

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Proposition 7.1.1. *The construction $S \mapsto \mathbb{M}(S)$ defines an exact functor*

$$\mathrm{Soln}_K \longrightarrow \mathrm{Hol}(F\mathcal{D}^\dagger).$$

Proof. We first show that $S \mapsto \mathbb{M}(S)$ is an exact functor from Soln_K to the category of $F\mathcal{D}^\dagger$-modules. The functoriality of the construction is clear. We first show that

$$\mathrm{Hom}(V, \mathcal{B}^b) \oplus \mathrm{Hom}(W, \mathcal{C}^b) \xrightarrow{d} \mathrm{Hom}(V, \mathcal{C}^b) \oplus \mathrm{Hom}(W, \mathcal{B}^b) \tag{7.1.7}$$

is surjective for any solution data (V, W, c, v), where d is the map (7.1.2). It suffices to show that the endomorphism of $\mathrm{Hom}(V, \mathcal{C}^b) \oplus \mathrm{Hom}(W, \mathcal{B}^b)$ given by the matrix

$$\begin{pmatrix} \mathrm{can}_* & -c^* \\ -v^* & \mathrm{var}_* \end{pmatrix} \begin{pmatrix} \mathrm{var}_* & c^* \\ v^* & \mathrm{can}_* \end{pmatrix} = \begin{pmatrix} N_{\mathcal{C}*} - (vc)^* & 0 \\ 0 & N_{\mathcal{B}*} - (cv)^* \end{pmatrix}$$

is surjective; in computing the matrix product we have used (6.1.3) and the fact that anything of the form f^* commutes with anything of the form g_*. The maps $(vc)^*$, $(cv)^*$ are nilpotent, and the maps $N_{\mathcal{B}*}$, $N_{\mathcal{C}*}$ have right inverses commuting with $(cv)^*$ and $(vc)^*$ respectively, so the surjectivity of this endomorphism follows from Lemma 2.1.3.

Since $\mathcal{B}^b$ and $\mathcal{C}^b$ are discrete G-modules, all four of the Hom spaces in (7.1.7) are discrete G-modules by Lemma 2.1.2. It follows that the map d in (7.1.1) is surjective, and consequently that the functor $\mathbb{M}$ is exact, by the same argument as in Lemma 2.1.4.

Now that we know $\mathbb{M} : \mathrm{Soln}_K \to \mathrm{Mod}(F\mathcal{D}^\dagger)$ is exact, the holonomicity of $\mathbb{M}(S)$ for S in Soln_K reduces, by the standard devissage (6.1.16), to the case where S is either punctual, or of connection type. This was checked in the calculations (7.1.5) and (7.1.6). □

The isomorphisms (6.1.9) show that the functors $\mathbb{V}(M)$ and $\mathbb{W}(M)$ extend naturally to functors on $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$. It follows that $\mathbb{S}$ extends to a functor

$$\mathbb{S}^{\mathrm{an}} : \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger) \to \mathrm{Soln}_K. \tag{7.1.8}$$

We denote by $\mathbb{M}^{\mathrm{an}}$ the composite functor

$$\mathbb{M}^{\mathrm{an}} : \mathrm{Soln}_K \longrightarrow \mathrm{Hol}(F\mathcal{D}^\dagger) \xrightarrow{\mathrm{an}} \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger). \tag{7.1.9}$$

Lemma 7.1.1. *For any S in Soln_K there is a functorial isomorphism*

$$\mathbb{M}(S)^{\mathrm{an}} \simeq \mathrm{Ker} : (\mathrm{Hom}(V, \mathcal{B}) \oplus \mathrm{Hom}(W, \mathcal{C}))^G \xrightarrow{d} (\mathrm{Hom}(V, \mathcal{C}) \oplus \mathrm{Hom}(W, \mathcal{B}))^G. \tag{7.1.10}$$

Proof. Suppose $S = (V, W, c, v)$. Since the action of G on V and W is discrete, and since vc and cv are nilpotent we can replace $\mathcal{B}^b$ and $\mathcal{C}^b$ in (7.1.1) by $\mathcal{R}_n^{b,\mathrm{st}}(u)$ and $\mathcal{R}_n^{b,\mathrm{st}}(u)/\mathcal{O}_K$ for some u and n depending on S. These are coherent $\mathcal{D}^\dagger$-modules whose analytifications are $\mathcal{R}_n^{\mathrm{st}}(u)$ and $\mathcal{R}_n^{\mathrm{st}}(u)/\mathcal{O}^{\mathrm{an}}$ respectively. Taking the limits over u and n results in (7.1.10). □

Theorem 7.1.1. *The functors (7.1.8), (7.1.9) define inverse equivalences of categories between $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ and Soln_K.*

Proof. We first observe that there are functorial morphisms

$$S \longrightarrow \mathbb{S}(\mathbb{M}(S)), \quad M^{\mathrm{an}} \longrightarrow \mathbb{M}(\mathbb{S}(M))^{\mathrm{an}} \tag{7.1.11}$$

for any object S of Soln_K and any object M of $\mathrm{Hol}(F\mathcal{D}^\dagger)$. The first is a straightforward evaluation map, of the sort described at the beginning of this section. The second is more subtle since in general there is no morphism $M \to \mathbb{M}(\mathbb{S}(M))$, let alone an isomorphism, but Lemma 7.1.1 shows that there is a functorial evaluation map $M \to \mathbb{M}(\mathbb{S}(M))^{\mathrm{an}}$ and this induces the second morphism in (7.1.11) by adjunction. Since $\mathbb{S}$ and $\mathbb{M}$ factor through the localization functor $\mathrm{Hol} \to \mathrm{Hol}^{\mathrm{an}}$, we get functorial morphisms

$$1 \longrightarrow \mathbb{S}^{\mathrm{an}}\mathbb{M}^{\mathrm{an}}, \quad 1 \longrightarrow \mathbb{M}^{\mathrm{an}}\mathbb{S}^{\mathrm{an}}. \tag{7.1.12}$$

These will be isomorphisms if the morphisms in (7.1.11) are for all S and M, and since $\mathbb{S}$ and $\mathbb{M}$ are exact, it suffices by devissage to treat the cases where S and M are either punctual or of connection type. For punctual modules this follows from Corollary 6.1.3 and the isomorphism (7.1.5). For connection type modules, this follows from (7.1.6) and Theorem 2.1.1. □

We have already remarked that the category Soln_K is artinian, whence we have the following corollary.

Corollary 7.1.1. *The category* $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ *is artinian.*

The equivalence in Theorem 7.1.1 and the remarks in § 5.2 show that the basic cohomological operations on $\mathrm{Hol}(F\mathcal{D}^\dagger)$ (i.e. j_+, j^+, and so forth) are reflected by operations on Soln_K. Certain of them are easily deduced from the above results. For example, the operations j_+, j^+ correspond to the functors

$$\begin{aligned} j_+ : \mathrm{Mod}_K(G, F, N) &\to \mathrm{Soln}_K & j^+ : \mathrm{Soln}_K &\to \mathrm{Mod}_K(G, F, N), \\ (V, N) &\mapsto (V, V, \mathrm{id}, N) & (V, W, c, v) &\mapsto (V, cv). \end{aligned} \tag{7.1.13}$$

The standard devissage in Soln_K shows that the $H^i(i^!)$ are given by

$$H^0(i^!(V, W, c, v)) = \mathrm{Ker}(c), \quad H^{-1}(i^!(V, W, c, v)) = \mathrm{Coker}(c) \tag{7.1.14}$$

while the functor i_+ corresponds to

$$\begin{aligned} i_+ : \mathrm{Mod}_K &\to \mathrm{Soln}_K \\ V &\mapsto (0, V, 0, 0). \end{aligned} \tag{7.1.15}$$

The remaining operations ($j_!$, i^+, and duality) are more involved and will be treated in another paper.

8. Canonical extensions

8.1 Special étale covers

We now use Theorem 7.1.1 to construct canonical extensions of objects of $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$. We first recall the Katz–Gabber theory of canonical extensions of étale covers.

Set $\mathbb{G}_m = \mathbb{P}^1_k - \{0, \infty\}$. A finite étale morphism $X \to \mathbb{G}_m$ is *special* if its Galois closure $Y \to \mathbb{G}_m$ has the following properties.

(i) The morphism $Y \to \mathbb{G}_m$ is tame at infinity.
(ii) The Galois group G of $Y \to \mathbb{G}_m$ has a single p-Sylow subgroup P, and G/P is cyclic of order prime to p.

Note that these conditions are satisfied by the Galois groups of finite separable extensions of a local field of equicharacteristic p. Fix an identification of the completion of the fraction field of the local ring of $\mathbb{P}^1_k$ at 0 with $F = k((t))$. The theorem of Katz and Gabber is that the restriction functor

$$(\text{special étale covers of } \mathbb{G}_m) \longrightarrow (\text{étale covers of } \mathrm{Spec}(F))$$

is an equivalence of categories. It follows that the natural homomorphism $\pi_1(\mathrm{Spec}(F)) \to \pi_1(\mathbb{G}_m)$ has a canonical section, and any ℓ-adic representation of $\pi_1(\mathrm{Spec}(F))$ has a canonical extension to an ℓ-adic representation of $\pi_1(\mathbb{G}_m)$.

In the p-adic setting, canonical extensions of the this sort were constructed, on one hand by Garnier [Gar95] for $\mathcal{D}^\dagger$-modules of connection type and of rank one (but not necessarily with a Frobenius structure), and on the other hand by Matsuda [Mat02] for quasi-unipotent isocrystals on $\mathcal{R}$ (and in particular, for F-isocrystals on $\mathcal{R}$). Nonetheless it is easy to see that there cannot be any canonical extension of this sort for $F\mathcal{D}^\dagger$-modules on $\mathcal{V}[\![t]\!]$; it is enough to remark that the Ext group $\mathrm{Ext}^1_{\mathcal{D}^\dagger}(\mathcal{O}_K, \mathcal{O}_K)$ is infinite dimensional (it is the de Rham cohomology of $\mathcal{O}_K$), whereas if M is an $F\mathcal{D}^\dagger$-module on $\mathbb{G}_m$ extending $\mathcal{O}_K$, the Ext group $\mathrm{Ext}^1(M, M)$ is of finite dimension; thus the restriction functor cannot be essentially surjective. Matsuda avoids this problem by working over $\mathcal{R}$ instead of $\mathcal{R}^b$, and we shall do essentially the same thing here, by working with the category of holonomic $F\mathcal{D}^\dagger$-modules up to analytic isomorphism.

8.2 Canonical extensions

The first step is to construct a global version of the ring $\mathcal{B}$. We begin by choosing a lift $\mathfrak{P}$ of $\mathbb{P}^1_k$ to a formally smooth p-adic formal scheme over $\mathcal{V}$. Next, choose k-rational points $0, \infty \in \mathbb{P}^1_k(k)$, and denote by $j : \mathbb{A}^1_k \to \mathbb{P}^1_k$ the inclusion of the complement of ∞. Set $\mathcal{O}^\dagger = j^\dagger \mathcal{O}_{\mathfrak{P}}$, and fix a lifting φ of Frobenius to $\mathcal{O}^\dagger$; the ring of global sections of $\mathcal{O}^\dagger$ is a Monsky–Washnitzer weak completion of a polynomial ring over K. It is clear that any two triples of data $(j : \mathbb{A}^1_k \to \mathbb{P}^1_k, \mathfrak{P}, \varphi)$ are isomorphic in an obvious way, and all subsequent constructions will depend canonically on the initial choice of $(j : \mathbb{A}^1_k \to \mathbb{P}^1_k, \mathfrak{P}, \varphi)$.

We now consider pairs $(X, i : \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}})$ where $X \to \mathbb{G}_m$ is a special étale cover, $\mathfrak{X}/\mathcal{V}$ is a formally smooth lifting of X/k, and $i : \mathfrak{X} \to \overline{\mathfrak{X}}$ is an open immersion into a proper formally smooth formal $\mathcal{V}$-scheme. For any such pair we denote by $\mathcal{O}^\dagger_X$ the $\mathcal{D}^\dagger_{\overline{\mathfrak{X}}}$-module $j^\dagger \mathcal{O}_{\overline{\mathfrak{X}}}$ on $\overline{\mathfrak{X}}$. If $\pi : Y \to X$ is a morphism over $\mathbb{G}_m$ of special étale covers, and if $(X, i : \mathfrak{X} \to \overline{\mathfrak{X}})$, $(Y, i' : \mathfrak{Y} \to \overline{\mathfrak{Y}})$ are pairs as above, then π does not necessarily lift to a map $\overline{\mathfrak{Y}} \to \overline{\mathfrak{X}}$; nonetheless the theory of $\mathcal{D}^\dagger$-modules allows one to construct a direct image $\pi_+ \mathcal{O}^\dagger_Y$. As the direct image satisfies a canonical transitivity isomorphism, we may define

$$\overline{\mathcal{R}}^{\mathrm{gl}} = \varinjlim_{\pi : X \to \mathbb{G}_m} \pi_+ \mathcal{O}^\dagger_X \tag{8.2.1}$$

where the direct limit is over the category of special étale covers of $\mathbb{G}_m$. It has a natural action of $G = \mathrm{Gal}(F^{\mathrm{sep}}/F)$, and the lifting φ extends uniquely to each of the $\pi_+ \mathcal{O}^\dagger_X$, and thus to the whole of $\overline{\mathcal{R}}^{\mathrm{gl}}$.

Finally, we formally adjoin a logarithm of (a lifting of) an affine parameter of $\mathbb{A}^1_k$ to $\mathcal{O}^\dagger$ and $\mathcal{R}^{\mathrm{gl}}$, following the procedure of § 1.1. Denote by $\mathcal{O}^{\dagger,1} \subset (\mathcal{O}^\dagger)^\times$ the subgroup of integral power series congruent to 1 modulo $\mathfrak{m}$. The logarithm $\log : \mathcal{O}^{\dagger,1} \to \mathcal{O}^\dagger$ is defined as in (1.1.1), and, as before, we extend it to $\mathcal{O}^\sharp = K^\times \cdot \mathcal{O}^{\dagger,1}$ by requiring it to vanish at $p \in \mathcal{V}$ and at the Teichmuller liftings of elements of $k^\times$. Finally, there is a ring $\mathcal{O}^{\mathrm{st}}$ and a homomorphism $\log : \mathcal{O}^\dagger \to \mathcal{O}^{\mathrm{st}}$ solving the same kind of universal problem as in § 1.1; it is isomorphic to a polynomial ring in one variable over $\mathcal{O}^\dagger$, and there is a canonical $\mathcal{O}^\dagger$-derivation $N : \mathcal{O}^{\mathrm{st}} \to \mathcal{O}^{\mathrm{st}}$ such that $N(\log t) = 1$, where t is any lifting of an affine parameter of $\mathbb{A}^1_k$. The map φ lifts uniquely to an endomorphism of $\mathcal{O}^{\mathrm{st}}$ compatible with $\log : \mathcal{O}^{\mathrm{st}} \to \mathcal{O}^\dagger$, and, as before, we have $N\varphi = q\varphi N$.

If we now set

$$\mathcal{B}^{\mathrm{gl}} = \mathcal{O}^{\mathrm{st}} \otimes_{\mathcal{O}^\dagger} \overline{\mathcal{R}}^{\mathrm{gl}} \tag{8.2.2}$$

we see that $\mathcal{B}^{\mathrm{gl}}$ is a ring endowed with the following structures:

- an action of $G=\mathrm{Gal}(F^{\mathrm{sep}}/F)$;
- a lifting φ of Frobenius compatible with the G-action;
- a nilpotent $\mathcal{R}^{\mathrm{gl}}$-derivation N_{gl} commuting with the Galois action, and satisfying $N_{\mathrm{gl}}\varphi = q\varphi N_{\mathrm{gl}}$.

Next, we set

$$\mathcal{C}^{\mathrm{gl}} = \mathcal{B}^{\mathrm{gl}}/\mathcal{O}^{\dagger} \tag{8.2.3}$$

and, as before, there are canonical and variation maps,

$$\mathrm{can}:\mathcal{B}^{\mathrm{gl}} \to \mathcal{C}^{\mathrm{gl}}, \quad \mathrm{var}:\mathcal{C}^{\mathrm{gl}} \to \mathcal{B}^{\mathrm{gl}}, \tag{8.2.4}$$

and for every σ in the inertia group $I \subset G$ a Galois variation

$$v(\sigma):\mathcal{C}^{\mathrm{gl}} \to \mathcal{B}^{\mathrm{gl}} \tag{8.2.5}$$

satisfying analogues of (6.1.5) and (6.1.6).

If $S=(V,W,c,v,v(\cdot))$ is a solution datum in the sense of § 6, we denote by $\mathbb{M}^{\mathrm{gl}\bullet}$ the complex

$$(\mathrm{Hom}_{\mathcal{D}^{\dagger}}(V,\mathcal{B}^{\mathrm{gl}}) \oplus \mathrm{Hom}_{\mathcal{D}^{\dagger}}(W,\mathcal{C}^{\mathrm{gl}}))^G \xrightarrow{d} (\mathrm{Hom}_{\mathcal{D}^{\dagger}}(V,\mathcal{C}^{\mathrm{gl}}) \oplus \mathrm{Hom}_{\mathcal{D}^{\dagger}}(W,\mathcal{B}^{\mathrm{gl}}))^G \tag{8.2.6}$$

supported in degrees zero and one, where, as before,

$$d = \begin{pmatrix} \mathrm{can}_* & -c^* \\ -v^* & \mathrm{var}_* \end{pmatrix}, \tag{8.2.7}$$

and we define

$$\mathbb{M}^{\mathrm{gl}}(S) = H^0(\mathbb{M}^{\mathrm{gl}\bullet}(S)). \tag{8.2.8}$$

The same argument that led to the exact sequence (7.1.4) yields, in the global case, an exact sequence

$$0 \longrightarrow \delta_0 \longrightarrow (\mathcal{C}^{\mathrm{gl}})^G \xrightarrow{\mathrm{var}} (\mathcal{B}^{\mathrm{gl}})^G \tag{8.2.9}$$

where δ_0 is the $\mathcal{D}^{\dagger}_{\mathfrak{P}\mathbb{Q}}$-module $\mathcal{R}^b/\mathcal{O}$ supported at $0 \in \mathbb{P}^1_k$. Then if $S=(0,W,0,0,F)$ is of punctual type, a calculation exactly parallel to (7.1.5) shows that

$$\mathbb{M}^{\mathrm{gl}}(S) \simeq W_0^* \otimes_K \delta_0 \tag{8.2.10}$$

where $W_0 = W^G$; thus $\mathbb{M}^{\mathrm{gl}}$ is a holonomic $F\mathcal{D}^{\dagger}_{\mathfrak{P}\mathbb{Q}}$-module. Similarly, if $S=(V,V,\mathrm{id}_V,N,F)$ is of connection type, a repetition of the argument for (7.1.6) shows that

$$\mathbb{M}^{\mathrm{gl}}(S) \simeq \mathrm{Hom}_{K^{\mathrm{nr}}}(V,\mathcal{B}^{\mathrm{gl}})^{G,N} \tag{8.2.11}$$

which is the direct image onto $\mathfrak{P}$ of an overconvergent F-isocrystal on $\mathbb{G}_m$, in fact of Matsuda's canonical extension of the F-isocrystal $\mathbb{M}^{\mathrm{an}}(S)$. An argument parallel to the proof of Proposition 7.1.1 proves the following proposition.

PROPOSITION 8.2.1. *The construction $S \mapsto \mathbb{M}^{\mathrm{gl}}(S)$ defines an exact functor*

$$\mathrm{Soln}_K \to \mathrm{Hol}(F\mathcal{D}^{\dagger}_{\mathfrak{P}\mathbb{Q}}). \tag{8.2.12}$$

From now on we set $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}) = \mathrm{Spf}(\mathcal{V}[[t]])$ and we fix an identification of $\mathfrak{X}$ with the formal completion of $\mathfrak{P}$ at $0 \in \mathbb{P}^1_k$. This determines a restriction functor

$$R:\mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathfrak{P}\mathbb{Q}}) \longrightarrow \mathrm{Coh}(\mathcal{D}^{\dagger}_{\mathfrak{X}\mathbb{Q}}) \tag{8.2.13}$$

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where $\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}$ is the ring denoted by $\mathcal{D}^\dagger$ in the preceding sections. The choice of lifting φ of Frobenius to $\mathcal{O}^\dagger$ induces a lifting of Frobenius to $\mathcal{O}$, whence another restriction functor

$$R : \mathrm{Hol}(F\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}) \longrightarrow \mathrm{Hol}(F\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}). \tag{8.2.14}$$

Either of these may be composed with the analytification functors (5.2.2), (5.2.3) to get further restrictions:

$$\begin{aligned} R^{\mathrm{an}} &: \mathrm{Coh}(\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}) \longrightarrow \mathrm{Coh}^{\mathrm{an}}(\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}),\\ R^{\mathrm{an}} &: \mathrm{Hol}(F\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}) \longrightarrow \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}). \end{aligned} \tag{8.2.15}$$

All of these functors are exact and faithful; for the functors (8.2.13) and (8.2.14) this follows from the faithful flatness of completions, and for R^{an} it suffices to invoke the faithfulness of the analytification functors (5.2.2), (5.2.3) (Corollary 5.1.1).

Since we have chosen compatible liftings of Frobenius to $\mathfrak{P}$ and $\mathfrak{X}$, the natural restriction morphisms

$$\mathcal{B}^{\mathrm{gl}} \to \mathcal{B}, \quad \mathcal{C}^{\mathrm{gl}} \to \mathcal{C} \tag{8.2.16}$$

are compatible with all of the structures N_{gl}, φ, etc. From this we get a canonical isomorphism

$$R^{\mathrm{an}}(\mathbb{M}^{\mathrm{gl}}(S)) \simeq \mathbb{M}^{\mathrm{an}}(S) \tag{8.2.17}$$

for any solution datum S.

We will say that an $F\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}$-module is *canonical* if it is in the essential image of the functor (8.2.12). We denote by Can_K the full subcategory of the category of holonomic $F\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}$-modules consisting of canonical $F\mathcal{D}^\dagger$-modules, and by

$$\mathbb{M}^{\mathrm{can}} : \mathrm{Soln}_K \longrightarrow \mathrm{Can}_K \tag{8.2.18}$$

the functor induced by $\mathbb{M}^{\mathrm{gl}}$. We can rewrite (8.2.17) as an isomorphism of functors

$$R^{\mathrm{an}} \circ \mathbb{M}^{\mathrm{can}} \simeq \mathbb{M}^{\mathrm{an}}, \tag{8.2.19}$$

and we can now prove the main result of this section.

Theorem 8.2.1. *The restriction functor*

$$R^{\mathrm{an}} : \mathrm{Can}_K \longrightarrow \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}})$$

is an equivalence.

Proof. We know that $\mathbb{M}^{\mathrm{an}}$ is an equivalence of categories, so by the functorial isomorphism (8.2.19) it suffices to show that $\mathbb{M}^{\mathrm{can}}$ is an equivalence. By construction it is essentially surjective. That it is fully faithful is a formal consequence of $\mathbb{M}^{\mathrm{an}}$ being an equivalence and R^{an} being faithful. □

Since $\mathbb{S}^{\mathrm{an}}$ is an inverse to $\mathbb{M}^{\mathrm{an}}$, the inverse of $R^{\mathrm{an}} : \mathrm{Can}_K \to \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}})$ is the *canonical extension*

$$\mathrm{Can} = \mathbb{M}^{\mathrm{gl}} \circ \mathbb{S}^{\mathrm{an}} : \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}\mathbb{Q}}) \longrightarrow \mathrm{Can}_K. \tag{8.2.20}$$

Objects of Can_K have the following properties: (1) they are of 'connection type at infinity', i.e. isomorphic to the direct image of their restriction by $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$; (2) their restriction to $\mathbb{G}_m$ is the direct image by specialization of an overconvergent F-isocrystal on $\mathbb{G}_m/K$; and (3) this restriction to $\mathbb{G}_m$ becomes isomorphic, on some special cover of $\mathbb{G}_m$, to a successive extension of geometrically constant F-isocrystals. It seems probable that these properties characterize Can_K as a full subcategory of $\mathrm{Hol}(F\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}})$, but we will not discuss this question here.

8.3 Fourier transforms

The restriction and extension functors R^{an}, Can in the last subsection were all defined relative to a choice of points $0,\ \infty \in \mathbb{P}^1_k$: in fact we used these points to define the notion of a special cover of $\mathbb{P}^1_k$, and $\mathfrak{X} = \mathrm{Spf}(\mathcal{O})$ was identified with the formal completion of $\mathfrak{P}$ at 0. We could of course reverse the roles of 0 and ∞. Denote by $\mathrm{Can}_K(0)$ the category previously denoted by Can_K, and by

$$R_0 : \mathrm{Can}_K(0) \longrightarrow \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}_0\mathbb{Q}}),$$
$$\mathrm{Can}_0 : \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}_0\mathbb{Q}}) \longrightarrow \mathrm{Can}_K(0)$$

the functors of 'restriction to 0' and 'canonical extension from 0', i.e. the ones considered in the last subsection; here $\mathfrak{X}_0$ is the formal completion of $\mathfrak{P}$ at 0. Reversing 0 and ∞, we get an inverse pair of functors

$$R_\infty : \mathrm{Can}_K(\infty) \longrightarrow \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}_\infty\mathbb{Q}}),$$
$$\mathrm{Can}_\infty : \mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger_{\mathfrak{X}_\infty\mathbb{Q}}) \longrightarrow \mathrm{Can}_K(\infty)$$

where $\mathrm{Can}_K(\infty)$ is defined analogously to $\mathrm{Can}_K(0)$. If $i : \mathbb{P}^1 \to \mathbb{P}^1$ is the map $t \mapsto t^{-1}$, there are functorial isomorphisms

$$R_\infty = R_0 \circ i^+, \quad \mathrm{Can}_\infty = i^+ \circ \mathrm{Can}_0.$$

We denote by $\mathcal{F}$ the one-dimensional Fourier transform of Noot-Huyghe [Noo04], normalized so that $\mathcal{F}$ preserves the category of complexes supported in degree zero (i.e. $\mathcal{F}(M)$ denotes what in [Noo04] would be $\mathcal{F}(M)[1]$). Fix a smooth model $\mathfrak{P}$ of $\mathbb{P}^1_k$, and denote by $\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}(\infty)$ the ring of arithmetic differential operators overconvergent at ∞. It is known [Noo04, Proposition 5.3.5] that if M is a holonomic F-$\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}(\infty)$-module, then so is $\mathcal{F}(M)$. Since we are in dimension one, we can apply [Car06b] or [Cre06] to conclude that $\mathcal{F}(M)$ is actually a holonomic F-$\mathcal{D}^\dagger_{\mathfrak{P}\mathbb{Q}}$-module.

This means that we can use the above constructions to define local Fourier transforms, as in [Lau87, § 2.4]. In fact, if M is a holonomic $F\mathcal{D}^\dagger$-module on $\mathrm{Spf}(\mathcal{O})$, we can define

$$\begin{aligned}\mathcal{F}^{0,\infty'}(M) &= R_\infty(\mathcal{F}(\mathrm{Can}_0(M))),\\ \mathcal{F}^{\infty,0'}(M) &= R_0(\mathcal{F}(\mathrm{Can}_\infty(M))),\\ \mathcal{F}^{\infty,\infty'}(M) &= R_\infty(\mathcal{F}(\mathrm{Can}_\infty(M))),\end{aligned} \tag{8.3.1}$$

and, as in [Lau87], one expects that $\mathcal{F}^{0,\infty'}$ and $\mathcal{F}^{\infty,0'}$ are exact functors giving an equivalence between $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ and the full subcategory of $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ consisting of objects whose Christol–Mebkhout slopes are strictly less than one, while $\mathcal{F}^{\infty,\infty'}$ is an autoequivalence of the subcategory of $\mathrm{Hol}^{\mathrm{an}}(F\mathcal{D}^\dagger)$ of objects whose slopes are strictly greater than one.

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Appendix. Henselian dagger algebras

Shigeki Matsuda

This appendix reproduces a letter from Matsuda to Crew, which generalizes (and corrects some points of) the proof of the Henselian property of the bounded Robba ring given in [Mat95].

Let R be a ring and I an ideal of R. Assume that R is I-adically complete. Let $T_1, \dots, T_n$ be indeterminates. For $i = (i_1, \dots, i_n) \in \mathbb{Z}^n_{\geqslant 0}$, we denote $T_1^{i_1} \cdots T_n^{i_n}$ by T^i and $i_1 + \cdots + i_n$ by $|i|$. For a formal power series $f = \sum_{i \in \mathbb{Z}^n_{\geqslant 0}} a_i T^i \in R[[T_1, \dots, T_n]]$, we consider the following condition:

$$\begin{gathered} \exists \lambda > 0, \exists \mu \in \mathbb{R} \text{ and } \exists f^{(u)} \in I^u R[T_1, \dots, T_n] \text{ s.t.} \\ f = \sum f^{(u)} \quad \text{and} \quad \deg f^{(u)} \leqslant \lambda u + \mu. \end{gathered} \tag{$\dagger$}$$

Here deg means the degree in $T_1, \dots, T_n$. We denote by $R[T_1, \dots, T_n]^\dagger$ the sub R-algebra of $R[[T_1, \dots, T_n]]$ consisting of series which satisfy ($\dagger$).

Lemma A.1. *Let $f \in R[[T_1, \dots, T_n]]$. Then $f = \sum_i a_i T^i \in R[T_1, \dots, T_n]^\dagger$ if and only if*

$$\exists \alpha > 0, \beta \in \mathbb{R} \text{ s.t. } \forall i, \quad a_i \in I^{\max\{0, \alpha|i| - \beta\}}.$$

Lemma A.2. *For $f \in A = R[[T_1, \dots, T_n]]$ we denote by $\deg^{(u)} f$ the degree of the image of f in $R/I^u[[T_1, \dots, T_n]]$. Then f satisfies ($\dagger$) if and only if there exists some $\lambda > 0$ and $\mu \in \mathbb{R}$ such that $\deg^{(u)} f \leqslant \lambda u + \mu$ for any $u \in \mathbb{Z}$, $u \geqslant 0$.*

Proof. If $f = \sum f^{(u)}$ with $f^{(u)}$ satisfying the condition in ($\dagger$), then $f \equiv \sum_{i=0}^{u-1} f^{(i)} \pmod{I^u A}$ for any integer $u \geqslant 0$ and $\deg^{(u)} f \leqslant \lambda(u-1) + \mu$. Conversely, assume that there exist some $\lambda > 0$ and $\mu \in \mathbb{R}$ such that $\deg^{(u)} f \leqslant \lambda u + \mu$ for any $u \in \mathbb{Z}$, $u \geqslant 0$. Then, for any integer $i \geqslant 0$, there exist $f_i \in R[T]$ such that $f \equiv f_i \pmod{I^{i+1} A}$ and $\deg f_i \leqslant \lambda i + \mu$. We put $f^{(0)} = f_0$ and define $f^{(i)}$ to be $f_i - f_{i-1}$ for $i > 0$. Then $\deg f^{(i)} \leqslant \lambda i + \mu$, $f^{(i)} \in I^i A$ and $f \equiv \sum_{i=0}^{u-1} f^{(i)} \pmod{I^u A}$. Thus $\sum_{u=0}^{\infty} f^{(u)}$ converges to f in $R[[T]]$. □

Lemma A.3. *In general, let $a \in A$, and assume that a satisfies ($\dagger$) and that $a \equiv c \pmod{IA}$ for some $c \in R$. Then a satisfies the following condition:*

$$\begin{gathered} \exists \lambda > 0, \text{ and } \exists a^{(u)} \in I^u R[T_1, \dots, T_n] \text{ s.t.} \\ a = \sum a^{(u)} \quad \text{and} \quad \deg a^{(u)} \leqslant \lambda u. \end{gathered} \tag{$\dagger\dagger$}$$

Moreover, if $a, b \in A$ satisfy ($\dagger\dagger$) for a common $\lambda > 0$, then $a + b$ and ab satisfy ($\dagger\dagger$) for the same λ.

Proof. The proof is elementary. □

Proposition A.1. *Let A be an R-algebra such that there exists a surjection φ: $R[T_1, \dots, T_n]^\dagger \to A$ and that $IA \subset \mathrm{rad}(A)$. Then (A, IA) is a Henselian pair.*

Proof. We can assume that $A = R[T_1, \dots, T_n]^\dagger$. By [Ray70, ch. XI, § 2, Proposition 1], it is enough to show that, for any monic polynomial $f = f(X) \in A[X]$, if $\overline{f}(X)$ factors as $\overline{f}(X) = X^d (X-1)^d$, then f factors as $f = \delta P$ with monic polynomials δ, P, such that $\overline{\delta} = X^d$, $\overline{P} = (X-1)^d$. We use the method of Abhyanker [Abh64, ch. II]. Put

$$\begin{aligned} f^\dagger &= f_0 + f_1 X + \cdots + f_{d-1} X^{d-1}, \\ f^* &= f_d + f_{d+1} X + \cdots + f_{2d} X^d. \end{aligned}$$

Then $f = f^\dagger + X^d f^*$, $\overline{f^\dagger} = 0$, and $\overline{f^*} = (X-1)^d$. Since $f^* \equiv 1 \pmod{(I, X)}$, f^* is invertible in $A[\![X]\!]$. Let $F = \sum_{i\geqslant 0} F_i X^i = -f^\dagger/f^*$.

Since $f^* \equiv (X-1)^d$ and $f^\dagger \equiv 0 \pmod{IA[\![X]\!]}$, the F_i satisfy (††) with a common λ by Lemma A.3. Therefore there exist $F^{(u)} = \sum_{i\geqslant 1} F_i^{(u)} X^i$ such that $F_i^{(u)} \in I^u A$ and $\deg F_i^{(u)} \leqslant \lambda u$. We define $Q^{(u)} \in I^u A[\![X]\!]$ and $r^{(u)} \in I^u A[X]$ successively for $u = 0, 1, 2 \ldots$ by

$$Q^{(0)} = 1, \quad r^{(0)} = 0,$$
$$\sum_{\substack{k+l=u\\ k\geqslant 1, l\geqslant 0}} F^{(k)} Q^{(l)} = X^d Q^{(u)} + r^{(u)},$$
$$\deg_X(r^{(u)}) < d.$$

Then we have

$$\deg Q^{(u)} \leqslant \lambda u, \quad \deg r^{(u)} \leqslant \lambda u$$

and $Q = \sum_u Q^{(u)}$ (respectively $r = \sum_u r^{(u)}$) converges in $A[\![X]\!]$ (respectively $A[X]$). By construction, $QF = X^d(Q-1) + r$, and hence

$$X^d - r = (X^d - F)Q = \frac{f^\dagger + X^d f^*}{f^*} Q = \frac{f}{f^*} Q.$$

Since $Q^{(0)} = 1$, we have the facts that $Q \equiv 1 \pmod{IA[\![X]\!]}$ and Q is invertible in $A[\![X]\!]$. Put $\delta = f^*/Q$ and $P = X^d - r$; then $\delta \in A[X]$ and $f = P\delta$ gives the desired factorization. □

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