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A Note on the Height of the Formal Brauer Group of a $K3$ Surface

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Abstract. Using weighted Delsarte surfaces, we give examples of $K3$ surfaces in positive characteristic whose formal Brauer groups have height equal to 5, 8 or 9. These are among the four values of the height left open in the article of Yui [11].

1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. Let X be a $K3$ surface over k (*i.e.* a smooth projective surface over k with a trivial canonical sheaf and irregularity 0). In [2], Artin and Mazur defined the formal Brauer group $\widehat{\mathrm{Br}}_X$ of X as the one-dimensional formal group representing the following functor on the category of finite local k-algebras A with residue field k:

$$\widehat{\mathrm{Br}}_X(A) = \ker\bigl(H^2_{\mathrm{et}}(X_A, \mathbb{G}_m) \to H^2_{\mathrm{et}}(X, \mathbb{G}_m)\bigr)$$

where $X_A = X \times \operatorname{Spec} A$ and $\mathbb{G}_m$ is the sheaf of multiplicative groups. By the p-rank of the kernel of the multiplication-by-p map on $\widehat{\mathrm{Br}}_X$, one defines the height $h := \operatorname{ht} \widehat{\mathrm{Br}}_X$ of the formal Brauer group of X:

$$p^h = \# \ker([p]\colon \widehat{\mathrm{Br}}_X \to \widehat{\mathrm{Br}}_X).$$

Since X is a $K3$ surface, its second Betti number is 22. Hence $\rho(X) \leq 22 - 2h$ if h is finite (*cf.* [2]), where $\rho(X)$ is the Picard number of X (*i.e.* the $\mathbb{Z}$-rank of the Néron-Severi group of X). As $\rho(X)$ and h are positive integers, the above inequality implies $h \leq 10$; in fact, it is proved in [1] that h takes all the integer values between 1 and 10 if $h < \infty$.

In [11], Yui gave concrete examples of $K3$ surfaces with $h = 1, 2, 3, 4, 6$ or 10. These $K3$ surfaces are obtained by using weighted diagonal or quasi-diagonal $K3$ surfaces. Such surfaces are quotients of Fermat surfaces by finite group actions and are special classes of weighted Delsarte $K3$ surfaces [6].

In this paper, we generalize the results of [11] to weighted Delsarte $K3$ surfaces and realize three (out of four) values of h missing in [11]. First we describe a general

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algorithm for computing the height of the formal Brauer groups of weighted Delsarte $K3$ surfaces. Then with this algorithm, we carry out calculations numerically to obtain $K3$ surfaces with $h = 5, 8$ and 9.

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2 Weighted Delsarte Surfaces

We summarize some geometric properties of weighted Delsarte surfaces. All the facts given in this section are proved in [6].

Let $Q = (q_0, q_1, q_2, q_3)$ be a quadruple of positive integers such that $p \nmid q_i$ $(0 \leq i \leq 3)$ and $\gcd(q_\alpha, q_\beta, q_\gamma) = 1$ for every triple $\{\alpha, \beta, \gamma\} \subset \{0, 1, 2, 3\}$. The weighted projective 3-space over k of type Q, denoted by $\mathbb{P}^3(Q)$, is the projective variety $\mathbb{P}^3(Q) := \operatorname{Proj} k[x_0, x_1, x_2, x_3]$, where the polynomial algebra is graded by $\deg(x_i) = q_i$ for $0 \leq i \leq 3$ (*cf.* [5]).

Let m be a positive integer such that $p \nmid m$. Let $A = (a_{ij})$ be a 4×4 matrix of integer entries satisfying the conditions

(i) $a_{ij} \geq 0$ and $p \nmid a_{ij}$ for every (i, j),
(ii) $p \nmid \det A$,
(iii) $\sum_{j=0}^{3} q_j a_{ij} = m$ for $0 \leq i \leq 3$,
(iv) given j, $a_{ij} = 0$ for some i.

We define a *weighted Delsarte surface in* $\mathbb{P}^3(Q)$ *of degree* m *with matrix* A (*cf.* [3], [7], [6]) to be the surface

$$X_A \colon \sum_{i=0}^{3} x_0^{a_{i0}} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}} = 0 \quad \subset \mathbb{P}^3(Q).$$

We say that X_A is *quasi-smooth* (*cf.* [5]) if its affine quasi-cone is smooth outside the origin and that X_A is *in general position relative to* $\mathbb{P}^3(Q)_{\text{sing}}$ if

$$\operatorname{codim}_X \left(X \cap \mathbb{P}^3(Q)_{\text{sing}} \right) \geq 2,$$

where $\mathbb{P}^3(Q)_{\text{sing}}$ denotes the singular locus of $\mathbb{P}^3(Q)$.

Weighted Delsarte surfaces are, in general, singular surfaces; they have cyclic quotient singularities of type A. Throughout the paper, we write $\tilde{X}_A$ for the minimal resolution (of singularities) of X_A.

Lemma 2.1 *Let X_A be a weighted Delsarte surface in $\mathbb{P}^3(Q)$ of degree m with matrix A. Assume that X_A is quasi-smooth and in general position relative to $\mathbb{P}^3(Q)_{\text{sing}}$. Then the minimal resolution $\tilde{X}_A$ of X_A is a $K3$ surface if and only if $m = q_0 + q_1 + q_2 + q_3$.*

Proof See [4] and [5]. ■

Definition 2.1 A weighted Delsarte surface X_A satisfying the assumptions and condition $m = q_0 + q_1 + q_2 + q_3$ of Lemma 2.1 will be called a *weighted Delsarte K3 surface* in $\mathbb{P}^3(Q)$ of degree m with matrix A.

It should be noted that weighted Delsarte surfaces are birational to finite quotients of Fermat surfaces and many properties about their cohomology groups are derived from those of Fermat surfaces (see [6], Section 2). In fact, if $d = |\det A|$, then X_A is covered (rationally) by the Fermat surface of degree d:

$$F_d\colon x_0^d + x_1^d + x_2^d + x_3^d = 0.$$

This covering induces a dominant rational map $F_d \to \tilde{X}_A$. Applying suitable birational transformations on F_d, we obtain a dominant morphism $\tilde{F}_d \to \tilde{X}_A$, where $\tilde{F}_d$ is a smooth surface birational to F_d. Through this morphism, the cohomology groups of $\tilde{X}_A$ can be embedded into those of $\tilde{F}_d$. For instance, the crystalline cohomology $H^2_{\text{cris}}(\tilde{X}_A/W)$ can be described similarly to that of a Fermat surface; specifically

$$H^2_{\text{cris}}(\tilde{X}_A/W) \cong \mathbb{E} \oplus V(0) \oplus \bigoplus_{\boldsymbol{\alpha} \in \mathfrak{A}(\tilde{X}_A)} V(\boldsymbol{\alpha})$$

where the isomorphism is given over the ring W of Witt vectors, $\mathbb{E}$ is a submodule of $H^2_{\text{cris}}(\tilde{X}_A/W)$ corresponding to exceptional divisors, $V(0)$ and $V(\boldsymbol{\alpha})$ are W-modules of rank 1 defined in [6] and

$$\mathfrak{A}(X_A) = \Big\{ \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathfrak{A}(F_d) \;\Big|\; \sum_{i=0}^{3} a_{ij}\alpha_i \equiv 0 \,(\text{mod } d) \text{ for } 0 \le j \le 3 \Big\}$$

$$\mathfrak{A}(F_d) = \Big\{ \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \;\Big|\; \alpha_i \in \mathbb{Z}/d\mathbb{Z}, \alpha_i \ne 0 \,(0 \le i \le 3), \sum_{i=0}^{3} \alpha_i = 0 \Big\}$$

(recall that a_{ij}'s are entries of matrix $A = (a_{ij})$).

For each $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathfrak{A}(X_A)$, define an integer

$$\|\boldsymbol{\alpha}\| = \sum_{i=0}^{3} \Big\langle \frac{\alpha_i}{d} \Big\rangle - 1$$

where $\langle \alpha_i/d \rangle$ denotes the fractional part of α_i/d. Then, since $\tilde{X}_A$ is $K3$, there exists a unique element $\boldsymbol{\alpha}_{ss} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathfrak{A}(X_A)$ such that $\|\boldsymbol{\alpha}_{ss}\| = 0$; equivalently, if we choose every α_i as $1 \le \alpha_i < d$, then $\boldsymbol{\alpha}_{ss}$ is the element satisfying

$$\boldsymbol{\alpha}_{ss} A \equiv (0,0,0,0) \pmod d \quad \text{and} \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = d.$$

Given such an $\boldsymbol{\alpha}_{ss} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, we define

$$e_A = \frac{d}{\gcd(\alpha_0, \alpha_1, \alpha_2, \alpha_3, d)}. \tag{2.1}$$

Then many properties of $\tilde{X}_A$ can be determined by arithmetic conditions modulo e_A.

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Proposition 2.2 *Let X_A be a weighted Delsarte K3 surface with matrix A. Then the minimal resolution $\tilde{X}_A$ of X_A is a supersingular K3 surface (i.e. $\rho(\tilde{X}_A) = 22$) if and only if $p^\mu \equiv -1 \pmod{e_A}$ for some integer $\mu \geq 1$.*

Proof See [6], Lemma 2.2. ■

Remark 2.1 For Fermat surfaces and their finite quotients, the slopes of their Newton polygons can be calculated using quantities related with $\mathfrak{A}(F_d)$ or $\mathfrak{A}(X_A)$ (*cf.* [10], [11]). These slopes determine the height of the formal Brauer groups of $K3$ surfaces (*cf.* [2], [11]).

On the other hand, as we see in Proposition 2.2, the set $\mathfrak{A}(X_A)$ also contains the information about $\tilde{X}_A$ being supersingular (*i.e.* $\rho = 22$) or not; this is based on the characterization of supersingular Fermat surfaces given in [9]. Combining these two data, one sees that a $K3$ surface $\tilde{X}_A$ is supersingular if and only if the height of its formal Brauer group is infinite.

3 Height of the Formal Brauer Groups

We apply the results of Yui [11] to compute the height of the formal Brauer groups of weighted Delsarte $K3$ surfaces. Our algorithm of computing the height is essentially the same as in [11]; thus, we explain only the additional content relevant to our surfaces. For details about the algorithm, the reader is referred to [11].

Recall that p is the characteristic of the ground field k, $\boldsymbol{\alpha}_{ss}$ is the element in $\mathfrak{A}(X_A)$ with $\|\boldsymbol{\alpha}_{ss}\| = 0$ and that e_A is the integer defined in (2.1). Write $d = |\det A|$. Let f_d be the order of p modulo d. Put

$$H = \{p^i \pmod{d} \mid 0 \leq i < f_d\}.$$

For $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathfrak{A}(X_A)$, define

$$A_H(\boldsymbol{\alpha}) = \sum_{t \in H} \|t\boldsymbol{\alpha}\|$$

Lemma 3.1 *Let X_A be a weighted Delsarte K3 surface with matrix A. Let f_d and f_e be the orders of p modulo d and e_A, respectively. Then*

$$A_H(\boldsymbol{\alpha}_{ss}) = \begin{cases} f_d & \text{if } p^\mu \equiv -1 \pmod{e_A} \text{ for some } \mu \\ \frac{f_d}{f_e}(f_e - 1) & \text{otherwise.} \end{cases}$$

Proof For $(t, d) = 1$, we have

$$\|t\boldsymbol{\alpha}_{ss}\| = \begin{cases} 0 & \text{if } t \equiv 1 \pmod{d} \\ 1 & \text{if } t \not\equiv \pm 1 \pmod{d} \\ 2 & \text{if } t \equiv -1 \pmod{d} \end{cases}$$

From the definition of e_A, $p^i\boldsymbol{\alpha}_{ss} = t\boldsymbol{\alpha}_{ss}$ if and only if $p^i \equiv t \pmod{e_A}$. Hence

$$\sum_{i=0}^{f_e-1} \|p^i\boldsymbol{\alpha}_{ss}\| = \begin{cases} f_e & \text{if } p^\mu \equiv -1 \pmod{e_A} \text{ for some } \mu \\ f_e - 1 & \text{otherwise.} \end{cases}$$

Therefore the asserted formula follows from the equality:

$$A_H(\boldsymbol{\alpha}_{ss}) = \frac{f_d}{f_e} \sum_{i=0}^{f_e-1} \|p^i\boldsymbol{\alpha}_{ss}\|. \qquad \blacksquare$$

Theorem 3.2 *Let X_A be a weighted Delsarte K3 surface with matrix A. Write $\tilde{X}_A$ for the minimal resolution of X_A. Assume that there is no integer $\mu \geq 1$ such that $p^\mu \equiv -1 \pmod{e_A}$. Then the height of the formal Brauer group of $\tilde{X}_A$ is equal to the order of p modulo e_A.*

Proof Write h for the height of the formal Brauer group of $\tilde{X}_A$. By Proposition 2.2, the non-existence of $\mu \geq 1$ with $p^\mu \equiv -1 \pmod{e_A}$ implies that $\tilde{X}_A$ is not supersingular. Hence h is finite (*cf.* Remark 2.1) and h can be calculated in the same way as [11] (see Proposition 4.6.1) by the formula

$$h = f_d/(f_d - A_H(\boldsymbol{\alpha}_{ss}))$$

where f_d is the order of p modulo $d = |\det A|$. If f_e denotes the order of p modulo e_A, then Lemma 3.1 gives $A_H(\boldsymbol{\alpha}_{ss}) = f_d(f_e - 1)/f_e$. Hence we obtain $h = f_e$. $\blacksquare$

4 Examples

We give examples of $K3$ surfaces with $h = 5, 8$ and 9. The height 7 can not be realized by this method; see Remark 4.1 for its arithmetic explanations.

Example 4.1 ($h = 5$). Assume $p \neq 2, 3, 5$. Let $Q = (1, 1, 1, 3)$, $m = 6$ and

$$A = \begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Then X_A is a weighted Delsarte surface in $\mathbb{P}^3(1, 1, 1, 3)$ defined by the equation

$$x_0^5x_1 + x_1^5x_2 + x_2^3x_3 + x_3^2 = 0$$

of degree 6. We see that X_A is quasi-smooth and in general position relative to $\mathbb{P}^3(Q)_{\text{sing}}$. As $\mathbb{P}^3(Q)_{\text{sing}} \cap X_A = \varnothing$, X_A is already smooth. The equality $m = 6 =$

$q_0 + q_1 + q_2 + q_3$ then implies that $\tilde{X}_A = X_A$ is a $K3$ surface. (This surface is also considered in [6], Example 3.2.) As in [6], we find $e_A = 5^2$. Hence

$$\rho(\tilde{X}_A) = \begin{cases} 2 & \text{if } p \equiv 1, 6, 11, 16, 21 \pmod{25} \\ 22 & \text{otherwise.} \end{cases}$$

When $\rho(\tilde{X}_A) = 2$, the formal Brauer group of $\tilde{X}_A$ has a finite height. Using Theorem 3.2, we obtain

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{25} \\ 5 & \text{if } p \equiv 6, 11, 16, 21 \pmod{25} \end{cases}$$

Example 4.2 ($h = 8$) Assume $p \neq 2, 3$. Let $Q = (1, 1, 3, 4)$, $m = 9$ and

$$A = \begin{bmatrix} 9 & 1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Then X_A is a weighted Delsarte surface in $\mathbb{P}^3(Q)$ defined by the equation

$$x_0^8 x_1 + x_1^6 x_2 + x_3^3 + x_3^2 x_0 = 0$$

of degree 9. It can be seen that X_A is quasi-smooth and in general position relative to $\mathbb{P}^3(Q)_{\text{sing}}$. Since $m = 9 = q_0 + q_1 + q_2 + q_3$, the minimal resolution $\tilde{X}_A$ of X_A is a $K3$ surface. We find $e_A = 2^5$. Hence

$$\rho(\tilde{X}_A) = \begin{cases} 6 & \text{if } p \not\equiv -1 \pmod{32} \\ 22 & \text{if } p \equiv -1 \pmod{32}. \end{cases}$$

When $\rho(\tilde{X}_A) = 6$, the formal Brauer group of $\tilde{X}_A$ has a finite height. Using Theorem 3.2, we obtain

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{32} \\ 2 & \text{if } p \equiv \pm 15 \pmod{32} \\ 4 & \text{if } p \equiv \pm 7, \pm 9 \pmod{32} \\ 8 & \text{if } p \equiv \pm 3, \pm 5, \pm 11, \pm 13 \pmod{32}. \end{cases}$$

Example 4.3 ($h = 9$) Assume $p \neq 2, 3$. Let $Q = (1, 1, 1, 1)$, $m = 4$ and

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

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Then X_A is a (weighted) Delsarte surface in the usual projective space $\mathbb{P}^3$ defined by the equation:

$$x_0^4 + x_0x_1^3 + x_1x_2^3 + x_2x_3^3 = 0.$$

We see that X_A is smooth in $\mathbb{P}^3$ and $X_A = \tilde{X}_A$ is a $K3$ surface. (This surface is also considered in [8], Example 6.) We find $e_A = 3^3$. Hence

$$\rho(X_A) = \begin{cases} 4 & \text{if } p \equiv 1, 4, 7, 10, 13, 16, 19, 22, 25 \pmod{27} \\ 22 & \text{otherwise.} \end{cases}$$

When $\rho(X_A) = 4$, the formal Brauer group of X_A has a finite height. Using Theorem 3.2, we obtain

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{27} \\ 3 & \text{if } p \equiv 10, 19 \pmod{27} \\ 9 & \text{if } p \equiv 4, 7, 13, 16, 22, 25 \pmod{27}. \end{cases}$$

We give another $K3$ surface with $h = 9$, for which the condition on the modulus of p is different from the case above.

Example 4.4 ($h = 9$) Assume $p \neq 2, 3, 5, 19$. Let $Q = (1, 1, 1, 2)$, $m = 5$ and

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Then X_A is a weighted Delsarte surface in $\mathbb{P}^3(Q)$ defined by the equation:

$$x_0^4x_1 + x_1^4x_2 + x_2^3x_3 + x_3^2x_0 = 0.$$

We can check that X_A is a weighted Delsarte $K3$ surface and find $e_A = 19$. Hence

$$\rho(\tilde{X}_A) = \begin{cases} 4 & \text{if } p \equiv 1, 4, 5, 6, 7, 9, 11, 16, 17 \pmod{19} \\ 22 & \text{otherwise.} \end{cases}$$

When $\rho(\tilde{X}_A) = 4$, the formal Brauer group of $\tilde{X}_A$ has a finite height. Using Theorem 3.2, we obtain

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{19} \\ 3 & \text{if } p \equiv 7, 11 \pmod{19} \\ 9 & \text{if } p \equiv 4, 5, 6, 9, 16, 17 \pmod{19}. \end{cases}$$

Remark 4.1 The case $h = 7$ may not be realized by this method since there is no integer d satisfying the following two conditions: (i) $\phi(d) \leq 20$, where ϕ is Euler's ϕ-function, (ii) there is some prime p having the order 7 in the multiplicative group $(\mathbb{Z}/d\mathbb{Z})^*$ (*i.e.* $\phi(d)$ is divisible by 7).

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