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SEMILOCAL E-CONVEXITY AND SEMILOCAL E-CONVEX PROGRAMMING

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In this paper, a new type of generalised convexity—semilocal E-convexity is introduced by combining the concepts of the semi-E-convexity in X.S. Chen [*J. Math. Anal. Appl.* 275(2002), 251-262] and semilocal convexity in G.M. Ewing [*SIAM. Rev.* 19(1977), 202-220], and some of its basic characters are discussed. By utilising the new concepts, we derive some optimality conditions and establish some duality results for the inequality constrained optimisation problem.

1. INTRODUCTION

Convexity and generalised convexity play a key role in many aspects of optimisation, such as optimality conditions, saddle-point theorems, duality theorems, theorems of alternatives, and convergence of optimisation algorithms, so the research on convexity and generalised convexity is one of the important aspects in mathematical programming. During the past several decades, various significant generalisations of convexity have been presented. Youness [**13**] brought forward the concepts of E-convex sets, E-convex functions and E-convex programming by using a point to point map E, discussed some of their basic properties, and established some optimality results on E-convex programming. Chen [**15**] introduced a class of semi-E-convex function and also discussed its basic properties. Jian [**16**] introduced (E,F)-convex sets, (E,F)-convex functions and (E,F)-convex programming by extending the definitions of E-convex sets, E-convex functions and E-convex programming, discussed some of their properties and also gave some examples to show that some results in [**13**] are incorrect also see [**14, 15**]. Recently, in [**18**], Jian, Hu, Zheng and Tang introduced a new class of non-convex functions, which are called semi-(E,F)-convex (quasi-semi-(E,F)-convex, pseudo-semi-(E,F)-convex) functions, and some of their basic characters were discussed, some sufficient conditions of optimality and duality theorems for the associated generalised convex programming were studied.

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Another generalisation of convexity, which was known as semilocal convexity, was introduced by Ewing [**1**], where the concept is applied to provide sufficient optimalty conditions in variational and control problems. Generalisations of semilocal convex functions and their properties have been investigated by Kaur [**6**] and Kaul, Kaur [**4, 5**]. In [**7**], a theorem of the alternatives is derived for semilocal convex functions defined on the local starshaped sets, and the result is applied to constrained minimisation problems to obtain optimality conditions and duality theorems. By using these concepts in Suneja and Gupta [**8**], some optimality conditions and duality results for a scalar valued nonlinear programming are obtained. These results are further extended in [**9**] for a multiple objective programming problems. In [**10**], optimality conditions and duality results were obtained for nonlinear programming involving semilocal preinvex and related functions. These results are extended in [**11**] for a multiple objective programming problems. In [**12, 17**], Lyall, Suneja, Aggarwal and Preda discussed the optimality conditions and duality results for fractional single (multiple) objective programming involving semilocal preinvex and related functions, respectively.

In this paper, based on the semi-E-convexity and semilocal convexity, we bring forward a new type of convexity – semilocal E-convexity, and discuss some of its basic properties, and establish the optimality conditions and duality results for the generalised convex programming.

The remainder of this paper is organised as follows. The preliminary results which will be used in the paper are stated in Section 2. In Section 3, we shall introduce the local starshaped E-convex sets and semilocal E-convex functions and discuss some of their properties. In Section 4, we study optimality conditions and duality results for semilocal E-convex programming.

2. Preliminaries

In this section, we review the related definitions and results which will be used in the paper.

Definition 2.1: ([**13**]) A set $M \subseteq R^n$ is said to be an E-convex set if there is a point to point map E: $M \to R^n$ such that $\lambda E(x) + (1-\lambda)E(y) \in M$, $\forall\, x, y \in M$, $\forall\, \lambda \in [0,1]$.

Definition 2.2: ([**13**]) A function f: $M \to R$ is said to be an E-convex function on a set $M \subseteq R^n$ if there is a point to point map E: $M \to R^n$ such that M is an E-convex set and

$$f\big(\lambda E(x) + (1-\lambda)E(y)\big) \leqslant \lambda f\big(E(x)\big) + \big(1-\lambda\big)f\big(E(y)\big),\quad \forall\, x, y \in M,\quad \forall\, \lambda \in [0,1].$$

Definition 2.3: ([**15**]) A function f: $M \to R$ is said to be a semi-E-convex function on a set M if there is a point to point map $E: M \to R^n$ such that M is an E-convex set and

$$f\big(\lambda E(x) + (1-\lambda)E(y)\big) \leqslant \lambda f(x) + (1-\lambda)f(y), \quad \forall\, x,y \in M, \quad \forall\, \lambda \in [0,1].$$

DEFINITION 2.4: ([**1**]) A set $M \subseteq R^n$ is said to be a local starshaped set if corresponding to each pair of points $x, y \in M$, there is a maximal positive number $a(x,y) \leqslant 1$ such that $\lambda x + (1-\lambda)y \in M, \ \forall \ \lambda \in \big(0, a(x,y)\big)$.

DEFINITION 2.5: ([**1**]) A function f: $R^n \to R$ is said to be a semilocal convex function on a local starshaped set $M \subseteq R^n$ if corresponding to each pair of points $x, y \in M$, there is a positive number $d(x,y) \leqslant a(x,y)$ such that $f\big(\lambda x + (1-\lambda)y\big) \leqslant \lambda f(x) + (1-\lambda)f(y), \ \forall \ \lambda \in \big(0, d(x,y)\big)$.

DEFINITION 2.6: ([**2**]) A vector function f: $X_0 \to R^k$ is said to be a convexlike function if for any $x, y \in X_0 \subseteq R^n$ and $0 \leqslant \lambda \leqslant 1$, there is $z \in X_0$ such that $f(z) \leqslant \lambda f(x) + (1-\lambda)f(y)$, where the inequalities are taken component-wise.

LEMMA 2.1. ([**3**]) *Let S be a nonempty set in R^n and $\psi : S \to R^k$ be a convexlike function. Then either $\psi(x) < 0$ has a solution $x \in S$ or $\lambda^T\psi(x) \geqslant 0$ for all $x \in S$ and some $\lambda \in R^k$, $\lambda \geqslant 0$ and $\lambda \neq 0$, but both alternatives are never true.*

3. LOCAL STARSHAPED E-CONVEX SETS AND SEMILOCAL E-CONVEX FUNCTIONS

In this section, we present the definitions of local starshaped E-convex sets and semilocal E-convex functions, respectively, and discuss their basic properties.

(1) LOCAL STARSHAPED E-CONVEX SETS.

DEFINITION 3.1: A set $M \subseteq R^n$ is said to be local starshaped E-convex, if there is a map $E : M \to R^n$such that corresponding to each pair of points $x, y \in M$, there is a maximal positive number $a(x,y) \leqslant 1$ satisfies

$$\lambda E(x) + (1-\lambda)E(y) \in M, \quad \forall \ 0 < \lambda < a(x,y). \tag{3.1}$$

REMARK 3.1. From this definition, we know that there is a maximal positive number $a(x,y) \leqslant 1$ for each pair of points $x, y \in M$ satisfying (3.1) if and only if there is a positive number $a(x,y) \leqslant 1$ for each pair of points $x, y \in M$ satisfying (3.1).

REMARK 3.2. Every E-convex set is local starshaped E-convex, but the converse is not necessary true, which is shown in Example 3.1 as follows.

REMARK 3.3. Every local starshaped set is local starshaped E-convex, but the converse is not necessary true, as shown in Example 3.1.

We give an example of a local starshaped E-convex set, which is neither E-convex nor local starshaped.

EXAMPLE 3.1. Let $S_1 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(2,3) + \lambda_3(0,2)\}$,

$$S_2 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(-2,-3) + \lambda_3(0,-4)\}$$

and $M = S_1 \cup S_2$, where $\lambda_1, \lambda_2, \lambda_3 \geqslant 0$ and $\sum_{i=1}^{3} \lambda_i = 1$. Let $E : R^2 \to R^2$ be defined as

$$E(x,y) = \begin{cases} (0,3), & \text{if } (x,y) = (2,3); \\ (0,y-1), & \text{if } (x,y) \in S_1 \backslash \{(2,3),(0,0)\}; \\ (0,y), & \text{if } (x,y) \in S_2. \end{cases}$$

It is clear that M is a local starshaped E-convex set. However, by letting $x = (2,3)$, $y = (0,0)$, $\lambda = 3/4$, we have $\lambda E(x) + (1-\lambda)E(y) = (0, 9/4) \notin M$. That is, M is not a E-convex set. Similarly, we take $x = (2,3)$, $y = (0,-4)$, there is no a maximal positive number $a(x,y) \leqslant 1$ such that $\lambda x + (1-\lambda)y \in M$, $\forall\ \lambda \in (0, a(x,y))$. So M is not a local starshaped set.

PROPOSITION 3.1. *If M_1 and M_2 are two local starshaped E-convex sets, then $M_1 \cap M_2$ is also a local starshaped E-convex set. Furthermore, the intersection of finite local starshaped E-convex sets is also a local starshaped E-convex set.*

The proof is obvious and is omitted.

REMARK 3.4. Even if M_1 and M_2 are all local starshaped E-convex sets, $M_1 \cup M_2$ is not necessarily local starshaped E-convex. See the following example.

EXAMPLE 3.2. Let $E : R^2 \to R^2$ be defined as $E(x,y) = (2y/3 - x/3, y/3 + 4x/3)$. Consider the two sets $M_1 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(2,1) + \lambda_3(0,3)\}$,

$$M_2 = \{(x,y) \in R^2 : (x,y) = \lambda_1(0,0) + \lambda_2(0,-3) + \lambda_3(-2,-1)\}$$

where $\lambda_1, \lambda_2, \lambda_3 \geqslant 0$ and $\sum_{i=1}^{3} \lambda_i = 1$. The two sets M_1 and M_2 are both local starshaped E-convex. However, by letting $x = (2,1)$, $y = (0,-3)$, there is no a maximal positive number $a(x,y) \leqslant 1$ such that $\lambda E(x) + (1-\lambda)E(y) \in M_1 \cup M_2$, $\forall\ \lambda \in (0, a(x,y))$. So $M_1 \cup M_2$ is not a local starshaped E-convex set.

PROPOSITION 3.2. *Suppose that $E : R^n \to R^n$ be a linear map.*

(i) *If $M \subseteq R^n$ is a local starshaped E-convex set and $\alpha \in R$, then αM is a local starshaped E-convex set.*

(ii) *If $M_1, M_2 \subseteq R^n$ be local starshaped E-convex sets, then $M_1 + M_2$ is a local starshaped E-convex set.*

(iii) *If $M_1, M_2, \ldots, M_r$ are all local starshaped E-convex sets, then, for each $(\alpha_1, \alpha_2, \ldots, \alpha_r) \in R^r$, $\sum_{i=1}^{r} \alpha_i M_i$ is also a local starshaped E-convex set.*

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PROOF: (i) Let $x^1, x^2 \in \alpha M$, then $x^1 = \alpha y^1, x^2 = \alpha y^2$, where $y^1, y^2 \in M$. Noting that M is a local starshaped E-convex set, there is a maximal positive number $a(y^1, y^2) \leqslant 1$ such that $\lambda E(y^1) + (1-\lambda)E(y^2) \in M, \ \forall\ 0 < \lambda < a(y^1, y^2)$. Let $a(x^1, x^2) = a(y^1, y^2)$, then

$$\begin{aligned}\lambda E(x^1) + (1-\lambda)E(x^2) &= \lambda E(\alpha y^1) + (1-\lambda)E(\alpha y^2) \\ &= \alpha\big(\lambda E(y^1) + (1-\lambda)E(y^2)\big) \in \alpha M, \ \forall\ 0 < \lambda < a(x^1, x^2).\end{aligned}$$

Thus αM is a local starshaped E-convex set from Remark 3.1.

(ii) Let $p + q, x + y \in M_1 + M_2$, where $p, x \in M_1$ and $q, y \in M_2$. In view of M_1, M_2 being local starshaped E-convex sets, there are two maximal positive numbers $a(p, x) \leqslant 1$ and $a(q, y) \leqslant 1$ such that

$$\begin{aligned}&\lambda E(p) + (1-\lambda)E(x) \in M_1, \quad \forall\ 0 < \lambda < a(p, x), \\ &\lambda E(q) + (1-\lambda)E(y) \in M_2, \quad \forall\ 0 < \lambda < a(q, y).\end{aligned}$$

Let $a(p+q, x+y) = \min\{a(p, x), a(q, y)\} > 0$, then, for $\forall\ 0 < \lambda < a(p+q, x+y)$, we have $\lambda E(p+q) + (1-\lambda)E(x+y) = \big(\lambda E(p) + (1-\lambda)E(x)\big) + \big(\lambda E(q) + (1-\lambda)E(y)\big) \in M_1 + M_2$, since E is a linear map. Thus $M_1 + M_2$ is a local starshaped E-convex set.

(iii) This conclusion is a direct consequence of the conclusions (i),(ii). ∎

PROPOSITION 3.3. *If $E : R^n \to R^n$ is a linear map, $M \subseteq R^n$ is a local starshaped E-convex set, $d \in R^n$ and $E(d) = d$, then $M + d$ is also a local starshaped E-convex set.*

PROOF: Let $x^1, x^2 \in M + d$, then $x^1 = y^1 + d$ and $x^2 = y^2 + d$, where y^1, $y^2 \in M$. Noting that M is a local starshaped E-convex set, there exists a maximal positive number $a(y^1, y^2) \leqslant 1$ corresponding to y^1, y^2 such that $\lambda E(y^1) + (1-\lambda)E(y^2) \in M$ for $0 < \lambda < a(y^1, y^2)$. Let $a(x^1, x^2) = a(y^1, y^2)$, then

$$\begin{aligned}\lambda E(x^1) + (1-\lambda)E(x^2) &= \lambda E(y^1 + d) + (1-\lambda)E(y^2 + d) \\ &= \lambda E(y^1) + (1-\lambda)E(y^2) + d \in M + d, \quad \forall\ 0 < \lambda < a(y^1, y^2).\end{aligned}$$

Thus $M + d$ is a local starshaped E-convex set from Remark 3.1. ∎

(2) SEMILOCAL E-CONVEX FUNCTIONS.

DEFINITION 3.2: A function $f : R^n \to R$ is said to be semilocal E-convex on a local starshaped E-convex set $M \subseteq R^n$ if for each pair of points $x^1, x^2 \in M$ (with a maximal positive number $a(x^1, x^2) \leqslant 1$ satisfying (3.1)), there exists a positive number $d(x^1, x^2) \leqslant a(x^1, x^2)$ satisfying $f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) \leqslant \lambda f(x^1) + (1-\lambda)f(x^2)$, $\forall\ 0 < \lambda < d(x^1, x^2)$. If $f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) \geqslant \lambda f(x^1) + (1-\lambda)f(x^2)$, $\forall\ 0 < \lambda < d(x^1, x^2)$, then f is called as a semilocal E-concave function on M. If the inequalities above are strict for any $x_1, x_2 \in M$ and $x_1 \neq x_2$, then f is a strictly semilocal E-convex (E-concave) function.

A vector function $f : R^n \longrightarrow R^k$ is said to be semilocal E-convex on a local starshaped E-convex set $M \subseteq R^n$ if for each pair of points $x^1, x^2 \in M$ (with a maximal positive number $a(x^1,x^2) \leqslant 1$ satisfying (3.1)), there exists a positive number $d(x^1,x^2) \leqslant a(x^1,x^2)$ satisfying

$$f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) \leqslant \lambda f(x^1) + (1-\lambda) f(x^2), \quad \forall\, 0 < \lambda < d(x^1,x^2),$$

where the inequalities are taken component-wise. The definition of semilocal E-concave or strictly semilocal E-convex of a vector function $f : R^n \longrightarrow R^k$ is similar to the one for a vector semilocal E-convex function.

REMARK 3.5: Every semilocal convex function on a local starshaped set M is a semilocal E-convex function, where E is an identity map and $d(x^1,x^2) = a(x^1,x^2)$ for each pair of points $x^1, x^2 \in M$. Every semi-E-convex function on an E-convex set M is a semilocal E-convex function, where $d(x^1,x^2) = a(x^1,x^2) = 1$ for each pair of points $x^1, x^2 \in M$. But their converses is not necessary true.

We give below an example of semilocal E-convex function, which is neither a semi-E-convex function nor a semilocal convex function.

EXAMPLE 3.3. Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} -x+2, & \text{if } x < 0; \\ -x+1, & \text{if } 0 \leqslant x \leqslant 1; \\ 0, & \text{if } 1 < x \leqslant 2; \\ 1, & \text{if } x > 2, \end{cases}$$

and the map $E : R \to R$ be defined as

$$E(x) = \begin{cases} 0, & \text{if } x < 0; \\ x, & \text{if } 0 \leqslant x \leqslant 1 \text{ or } x > 2; \\ 1, & \text{if } 1 < x \leqslant 2. \end{cases}$$

It is not difficult to verify that f is a semilocal E-convex function on the local starshaped E-convex set R. However, by letting $x = 1.5$, $y = 8$, $\lambda = 0.8$, we have $f\big(\lambda E(x) + (1 - \lambda)E(y)\big) = f(2.4) = 1 > 0.2 = \lambda f(x) + (1-\lambda) f(y)$. That is, f is not a semi-E-convex function on the E-convex set R. Similarly, we take $x = 2$, $y > 2$, there is no $d(x,y) \in \big(0, a(x,y) = 1\big)$ such that $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$, $\forall\ \lambda \in \big(0, d(x,y)\big)$. So f is not a semilocal convex function on R.

The following two results can be proved easily.

PROPOSITION 3.4. *If f and g are both semilocal E-convex functions on the local starshaped E-convex set $M \subseteq R^n$, then $\alpha f + \beta g (\alpha \geqslant 0, \beta \geqslant 0)$ is also a semilocal E-convex function.*

PROPOSITION 3.5. *Let f be a semilocal E-convex function from the local starshaped E-convex set $M \subseteq R^n$ to R, and let φ be a non-decreasing and convex function from R to R. Then $h(x) = \varphi\big(f(x)\big)$ is a semilocal E-convex function on M.*

DEFINITION 3.3: The set $G = \{(x, \alpha) : x \in M \subseteq R^n,\ \alpha \in T \subseteq R\}$ is said to be a local starshaped E-convex set corresponding to R^n if there are a map $E : R^n \to R^n$ and a maximal positive number $a\big((x^1, \alpha_1), (x^2, \alpha_2)\big) \leqslant 1$ for each $(x^1, \alpha_1), (x^2, \alpha_2) \in G$ such that

$$(\lambda E(x^1) + (1 - \lambda)E(x^2), \lambda\alpha_1 + (1 - \lambda)\alpha_2) \in G, \quad \forall\, 0 < \lambda < a\big((x^1, \alpha_1), (x^2, \alpha_2)\big).$$

THEOREM 3.1. *Let $M \subseteq R^n$ be a local starshaped E-convex set. Then f is a semilocal E-convex function on M if and only if its epigraph*

$$G_f = \{(x, \alpha) : x \in M,\quad f(x) \leqslant \alpha,\quad \alpha \in R\}$$

is a local starshaped E-convex set corresponding to R^n.

PROOF: Suppose that f is semilocal E-convex and $(x^1, \alpha_1),\ (x^2, \alpha_2) \in G_f$, then $x^1,\ x^2 \in M,\ \ f(x^1) \leqslant \alpha_1,\ \ f(x^2) \leqslant \alpha_2$. In view of M being a local starshaped E-convex set, there is a maximal positive number $a(x^1, x^2) \leqslant 1$ such that $\lambda E(x^1) + (1 - \lambda)E(x^2) \in M,\ \ \forall\, 0 < \lambda < a(x^1, x^2)$. In addition, taking into account f being a semilocal E-convex function, there is a positive number $d(x^1, x^2) \leqslant a(x^1, x^2)$ such that

$$\begin{aligned} f\big(\lambda E(x^1) + (1 - \lambda)E(x^2)\big) &\leqslant \lambda f(x^1) + (1 - \lambda)f(x^2) \\ &\leqslant \lambda\alpha_1 + (1 - \lambda)\alpha_2, \qquad \forall\, 0 < \lambda < d(x^1, x^2). \end{aligned}$$

That is $\big(\lambda E(x^1) + (1 - \lambda)E(x^2), \lambda\alpha_1 + (1 - \lambda)\alpha_2\big) \in G_f,\ \ \forall\, 0 < \lambda < d(x^1, x^2)$. Thus $G_f = \{(x, \alpha) : x \in M,\ \ f(x) \leqslant \alpha,\ \ \alpha \in R\}$ is a local starshaped E-convex set corresponding to R^n.

Conversely, if G_f is a local starshaped E-convex set corresponding to R^n, then for any points $\big(x^1, f(x^1)\big),\ \big(x^2, f(x^2)\big) \in G_f$, there is a maximal positive number

$$a\Big(\big(x^1, f(x^1)\big), \big(x^2, f(x^2)\big)\Big) \leqslant 1$$

such that

$$\big(\lambda E(x^1) + (1-\lambda)E(x^2), \lambda f(x^1) + (1-\lambda)f(x^2)\big) \in G_f,\ \ \forall\, 0 < \lambda < a\Big(\big(x^1, f(x^1)\big), \big(x^2, f(x^2)\big)\Big).$$

That is $\lambda E(x^1) + (1 - \lambda)E(x^2) \in M$,

$$f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) \leqslant \lambda f(x^1) + (1-\lambda)f(x^2), \quad \forall\lambda \in \Big(0, a\Big(\big(x^1, f(x^1)\big), \big(x^2, f(x^2)\big)\Big)\Big).$$

Thus M is local starshaped E-convex and f is a semilocal E-convex function on M. ∎

THEOREM 3.2. *If f is a semilocal E-convex function on a local starshaped E-convex set $M \subseteq R^n$, then the level set $S_\alpha = \{x \in M :\ f(x) \leqslant \alpha\}$ is a local starshaped E-convex set for each $\alpha \in R$.*

PROOF: Let $x^1,\ x^2 \in S_\alpha$. Then $x^1,\ x^2 \in M$ and $f(x^1) \leqslant \alpha,\ f(x^2) \leqslant \alpha$. Taking into account M being a local starshaped E-convex set, there is a maximal positive number $a(x^1,x^2) \leqslant 1$ such that $\lambda E(x^1) + (1-\lambda)E(x^2) \in M, \quad \forall\, 0 < \lambda < a(x^1,x^2)$. In addition, in view of the semilocal E-convexity of f, there is a positive number $d(x^1,x^2) \leqslant a(x^1,x^2)$ such that

$$\begin{aligned} f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) &\leqslant \lambda f(x^1) + (1-\lambda) f(x^2) \\ &\leqslant \lambda\alpha + (1-\lambda)\alpha = \alpha, \qquad \forall\, 0 < \lambda < d(x^1,x^2). \end{aligned}$$

This is $\lambda E(x^1) + (1-\lambda)E(x^2) \in S_\alpha, \quad \forall\, 0 < \lambda < d(x^1,x^2)$. Thus S_α is a local starshaped E-convex set for each $\alpha \in R$. ∎

REMARK 3.6. The converse of Theorem 3.2 is not necessary true, see the following example.

EXAMPLE 3.4. Let maps $E,\ f : R \to R$ be defined as

$$E(x) = \begin{cases} 1, & \text{if } 1 \leqslant x \leqslant 4; \\ 1 + (2/\pi)\arctan(1-x), & \text{if } x < 1; \\ (4/\pi)\arctan(x-4) - 1, & \text{if } x > 4. \end{cases}$$

$$f(x) = \begin{cases} 2, & \text{if } x < 1 \text{ or } x > 4; \\ x-3, & \text{if } 1 \leqslant x < 2; \\ 3-x, & \text{if } 2 \leqslant x \leqslant 3; \\ x-3, & \text{if } 3 < x \leqslant 4. \end{cases}$$

Obviously, the set R is a local starshaped E-convex set. The level sets can be given as

$$S_\alpha = \begin{cases} R, & \text{if } \alpha \geqslant 2; \\ [1,4], & \text{if } 1 < \alpha < 2; \\ [1,2) \cup [3-\alpha, 3+\alpha], & \text{if } 0 \leqslant \alpha \leqslant 1; \\ [1,2), & \text{if } -1 \leqslant \alpha < 0; \\ [1, 3+\alpha], & \text{if } -2 \leqslant \alpha < -1; \\ \phi(\textit{empty set}), & \text{if } \alpha < -2. \end{cases}$$

It is not difficult to verify that the set S_α is local starshaped E-convex for each $\alpha \in R$. But the function $f(x)$ is not a semilocal E-convex function on R since

$$f\big(\lambda E(2) + (1-\lambda)E(5)\big) = f(\lambda) = 2 > \lambda f(2) + (1-\lambda) f(5) = 2 - \lambda, \quad 0 < \lambda < 1.$$

THEOREM 3.3. *Let f be a real-valued function defined on a local starshaped E-convex set $M \subseteq R^n$. Then f is a semilocal E-convex function if and only if for each pair of points $x^1,\ x^2 \in M$ (with a maximal positive number $a(x^1,x^2) \leqslant 1$ satisfying (3.1)), there exists a positive number $d(x^1,x^2) \leqslant a(x^1,x^2)$ such that $f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) < \lambda\alpha + (1-\lambda)\beta,\ \forall\, 0 < \lambda < d(x^1,x^2)$. whenever $f(x^1) < \alpha$ and $f(x^2) < \beta$.*

PROOF: " $\Rightarrow$ " Let x^1, $x^2 \in M$ and α, $\beta \in R$ such that $f(x^1) < \alpha$ and $f(x^2) < \beta$. In view of the local starshaped E-convexity of M, there is a maximal positive number $a(x^1, x^2) \leqslant 1$ such that $\lambda E(x^1) + (1-\lambda)E(x^2) \in M$, $\forall\, 0 < \lambda < a(x^1, x^2)$. In addition, taking into account the semilocal E-convexity of f, there is a positive number $d(x^1, x^2) \leqslant a(x^1, x^2)$ such that

$$\begin{aligned} f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) &\leqslant \lambda f(x^1) + (1-\lambda) f(x^2) \\ &< \lambda\alpha + (1-\lambda)\beta, \qquad \forall\, 0 < \lambda < d(x^1, x^2). \end{aligned}$$

" $\Leftarrow$ " We shall apply Theorem 3.1 to prove $f(x)$ is a semilocal E-convex function on M. Let $(x^1, \alpha) \in G_f$ and $(x^2, \beta) \in G_f$. Then x^1, $x^2 \in M$ and $f(x^1) \leqslant \alpha$, $f(x^2) \leqslant \beta$, where $\alpha, \beta \in R$. So $f(x^1) < \alpha + \varepsilon$ and $f(x^2) < \beta + \varepsilon$ hold for any $\varepsilon > 0$. According to the hypothesis, we know that, for x^1, $x^2 \in M$(with a maximal positive number $a(x^1, x^2) \leqslant 1$ satisfying (3.1)), there exists a positive number $d(x^1, x^2) \leqslant a(x^1, x^2)$ such that

$$f\big(\lambda E(x^1) + (1-\lambda)(x^2)\big) < \lambda\alpha + (1-\lambda)\beta + \varepsilon, \quad \forall\, 0 < \lambda < d(x^1, x^2).$$

Let $\varepsilon \to 0^+$, then

$$f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) \leqslant \lambda\alpha + (1-\lambda)\beta, \quad \forall\, 0 < \lambda < d(x^1, x^2).$$

That is

$$(\lambda E(x^1) + (1-\lambda)E(x^2), \lambda\alpha + (1-\lambda)\beta) \in G_f, \quad \forall\, 0 < \lambda < d(x^1, x^2).$$

So G_f is a local starshaped E-convex set, and it shows that f is a semilocal E-convex function on M from Theorem 3.1. ∎

4. Semilocal E-convex Programming

In this section, we shall study the necessary and sufficient optimality conditions and dualities for semilocal E-convex programming.

(1) SEMILOCAL E-CONVEX PROGRAMMING. First, a new definition is given below.

DEFINITION 4.1: The programming (P) min $\{f(x) : x \in M\}$ is said to be a semilocal E-convex programming if there is a map $E : M \to R^n$ such that the feasible set M is a local starshaped E-convex set and the objective function f is a semilocal E-convex function on M.

THEOREM 4.1. *For a semilocal E-convex programming (P), the following statements are true.*

(i) *The optimal solution set Ω for (P) is a local starshaped E-convex set.*
(ii) *If x^0 is a local minimum for (P) and $E(x^0) = x^0$, then x^0 is a global minimum for (P).*

(iii) *If the real-valued function f is a strictly semilocal E-convex function on M, then the global optimal solution for (P) is unique.*

PROOF: (i) Suppose that $x^1,\ x^2 \in \Omega$. Then $x^1, x^2 \in M$ and $f(x^1) = f(x^2)$. Taking into account M being a local starshaped E-convex set, there is a maximal positive number $a(x^1,x^2) \leqslant 1$ such that $\lambda E(x^1) + (1-\lambda)E(x^2) \in M,\quad \forall\, 0 < \lambda < a(x^1,x^2)$. In addition, in view of f being a semilocal E-convex function, there is a positive number $d(x^1,x^2) \leqslant a(x^1,x^2)$ such that $f\big(\lambda E(x^1)+(1-\lambda)E(x^2)\big) \leqslant \lambda f(x^1)+(1-\lambda)f(x^2) = f(x^1),\quad \forall\, 0 < \lambda < d(x^1,x^2)$. From the optimality of x^1, we have $f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) \geqslant f(x^1)$. Thus $f\big(\lambda E(x^1) + (1-\lambda)E(x^2)\big) = f(x^1)$, that is, $\lambda E(x^1) + (1-\lambda)E(x^2) \in \Omega$, for each $\lambda \in \big(0, a(x^1,x^2)\big)$. This shows Ω is a local starshaped E-convex set.

(ii) Suppose that $N_\varepsilon(x^0)$ is a neighbourhood of x^0 with radius $\varepsilon > 0$, and f is minimised at x^0 in $N_\varepsilon(x^0) \cap M$. For $x, x^0 \in M$, there is a maximal positive number $a(x,x^0) \leqslant 1$ such that $\lambda E(x) + (1-\lambda)E(x^0) \in M,\quad \forall\, 0 < \lambda < a(x,x^0)$. Furthermore, there is a positive number $d(x,x^0) \leqslant a(x,x^0)$ such that $f\big(\lambda E(x) + (1-\lambda)E(x^0)\big) \leqslant \lambda f(x) + (1-\lambda) f(x^0),\quad \forall\, 0 < \lambda < d(x,x^0)$. In view of $E(x^0) = x^0$, so for $\lambda > 0$ and sufficient small, $\lambda E(x) + (1-\lambda)E(x^0) \in N_\varepsilon(x^0) \cap M$. So $f(x^0) \leqslant f\big((1-\lambda)E(x^0) + \lambda E(x)\big) \leqslant (1-\lambda) f(x^0) + \lambda f(x)$, that is, $f(x^0) \leqslant f(x)$, this shows that x^0 is a global minimum for (P).

(iii) By contradiction, suppose that $x^1, x^2 \in M$ are two global optimal solutions for (P) and $x^1 \neq x^2$. For $x^1, x^2 \in M$, there is a maximal positive number $a(x^2,x^1) \leqslant 1$ such that $\lambda E(x^2) + (1-\lambda)E(x^1) \in M,\quad \forall\, 0 < \lambda < a(x^2,x^1)$. Furthermore, there is a positive number $d(x^2,x^1) \leqslant a(x^2,x^1)$ such that $f\big(\lambda E(x^2)+(1-\lambda)E(x^1)\big) < \lambda f(x^2)+(1-\lambda)f(x^1) = f(x^1),\quad \forall\ 0 < \lambda < d(x^2,x^1)$. This contradicts x^1 is a global optimal solution for (P). Hence the global optimal solution for (P) is unique. ∎

THEOREM 4.2. *Let $u \in M$ and $E(u) = u$. If f is differentiable on M, then u is a minimum for the semilocal E-convex programming (P) if and only if u satisfies the inequality $\nabla f(u)^T(E(v) - u) \geqslant 0$ for all $v \in M$.*

PROOF: Suppose that u is a minimum for (P). Since M is a local starshaped E-convex set, for each $v \in M$, there is a maximal positive number $a(v,u) \leqslant 1$ such that $\lambda E(v) + (1-\lambda)E(u) \in M,\quad \forall\, 0 < \lambda < a(v,u)$. From the differentiability of f and $E(u) = u$, we have

$$
\begin{aligned}
f(u) \leqslant f\big(\lambda E(v) + (1-\lambda)E(u)\big) &= f\Big(E(u) + \lambda\big(E(v) - E(u)\big)\Big)\\
&= f(u) + \lambda \nabla f(u)^T(E(v) - u) + o(\lambda).
\end{aligned}
$$

Dividing the inequality above by λ and passing to the limit $\lambda \to 0^+$, we have $\nabla f(u)^T(E(v) - u) \geqslant 0$.

Conversely, if $\nabla f(u)^T(E(v) - u) \geqslant 0$, since f is a semilocal E-convex function on M, there is a positive number $d(v,u) \leqslant a(v,u)$ such that $f\big(\lambda E(v) + (1-\lambda)E(u)\big)$

$\leqslant \lambda f(v) + (1-\lambda) f(u), \quad \forall\, 0 < \lambda < d(v,u)$, which together with $E(u) = u$ implies $f(v) - f(u) \geqslant \big(f\big(u + \lambda(E(v) - u)\big) - f(u)\big)/\lambda$. Passing to the limit $\lambda \to 0^+$ in the inequality above, one has $f(v) - f(u) \geqslant \nabla f(u)^T (E(v) - u) \geqslant 0$, this shows that u is a minimum for (P). ∎

In the sequel discussion, we consider the following inequality constrained optimisation problem:

$$(\mathrm{P_g}) \qquad \begin{array}{ll} \min & f(x) \\ \text{subject to} & g_i(x) \leqslant 0, \quad i \in I \\ & x \in M. \end{array}$$

Denote the feasible set of $(\mathrm{P_g})$ by $M_g = \{x \in M \mid g_i(x) \leqslant 0,\ i \in I\}$, where $I = \{1, \dots, m\}$, and $M \subseteq R^n$ is a local starshaped E-convex and open set.

If the constraint functions $g_i (i \in I)$ are all semilocal E-convex on M, then, from Theorem 3.2 and Proposition 3.1, we can conclude that the feasible set M_g is a local starshaped E-convex set. Furthermore, from Theorem 4.1, we can obtain the following theorem easily.

THEOREM 4.3. *Suppose that f, g_i $(i \in I)$ are all semilocal E-convex functions on M. Then*

(i) *M_g is a local starshaped E-convex set;*
(ii) *The optimal solution set Ω_g for (P_g) is a local starshaped E-convex set;*
(iii) *If x^0 is a local minimum for (P_g) and $E(x^0) = x^0$, then x^0 is a global minimum for (P_g);*
(iv) *If the real-valued function f is a strictly semilocal E-convex function on M, then the global optimal solution for (P_g) is unique.*

(2) NECESSARY OPTIMALITY CONDITIONS. For convenience of discussion, for a subset $J \subseteq I$, the following notation is used:

$$g(x) = (g_i(x),\ i \in I), \quad g_J(x) = (g_j(x),\ j \in J)$$

For $x^* \in M_g$, denote $I(x^*) = \big\{i \in I : g_i(x^*) = 0\big\}, \quad \overline{I}(x^*) = I \setminus I(x^*)$.

To discuss the necessary optimality conditions for the corresponding programming, we give a lemma as follows.

LEMMA 4.1. *Let x^* be a local optimal solution for (P_g). Assume that g_j is continuous at x^* for any $j \in \overline{I}(x^*)$ and $f, g_{I(x^*)}$ possess the directional derivatives at x^* along the direction $(E(x) - x^*)$ for each $x \in M$. Then the system*

$$\begin{cases} f'(x^*; E(x) - x^*) & < 0 \\ g'_{I(x^*)}(x^*; E(x) - x^*) & < 0 \end{cases}$$

has no solution in M, where $f'(x^*; d)$ denotes the directional derivative of f at x^* along the direction d and $g'_{I(x^*)}(x^*; d) = \big(g'_i(x^*; d), i \in I(x^*)\big)$.

The proof of this lemma is similar to the one of [**17**, Lemma 13].

THEOREM 4.4. *Let x^* be a local optimal solution for (P_g). Suppose that functions g_j are continuous at x^* for any $j \in \overline{I}(x^*)$, and f, g possess the directional derivatives with respect to $(E(x)-x^*)$ at x^* for each $x \in M$. If $(f'(x^*;E(x)-x^*), g'_{I(x^*)}(x^*;E(x)-x^*))$ is a convexlike function and $E(x^*) = x^*$, then there are $\lambda_0^* \in R$, $\overline{\lambda}^* \in R^m$ such that*

$$\lambda_0^* f'(x^*;E(x)-x^*) + \overline{\lambda}^{*T} g'(x^*;E(x)-x^*) \geqslant 0, \quad \forall \ x \in M, \tag{4.1}$$

$$\overline{\lambda}^{*T} g(x^*) = 0, \quad g(x^*) \leqslant 0, \quad 0 \neq (\lambda_0^*, \overline{\lambda}^*) \geqslant 0. \tag{4.2}$$

Additionally, if g is a semilocal E-convex function and there is $\widehat{x} \in M$ such that $g(\widehat{x}) < 0$, then there exists $\lambda^* \in R^m$ such that

$$f'(x^*;E(x)-x^*) + \lambda^{*T} g'(x^*;E(x)-x^*) \geqslant 0, \quad \forall \ x \in M, \tag{4.3}$$

$$\lambda^{*T} g(x^*) = 0, \quad g(x^*) \leqslant 0, \quad \lambda^* \geqslant 0. \tag{4.4}$$

PROOF: Define vector function $\psi(x) = (f'(x^*;E(x)-x^*), g'_{I(x^*)}(x^*;E(x)-x^*))$. Then $\psi(x)$ is a convexlike function. By Lemma 4.1, the system $\psi(x) < 0$ has no solution in M. So, from Lemma 2.1, there is $\lambda_0^* \in R$, $\overline{\lambda}^*_{I(x^*)} \in R^{|I(x^*)|}$ such that

$$\lambda_0^* f'(x^*;E(x)-x^*) + \overline{\lambda}^{*\,T}_{I(x^*)} g'_{I(x^*)}(x^*;E(x)-x^*) \geqslant 0, \quad \forall \ x \in M, \quad 0 \neq (\lambda_0^*, \ \overline{\lambda}^*_{I(x^*)}) \geqslant 0.$$

Thus, by letting $\overline{\lambda}^* = (\overline{\lambda}^*_{I(x^*)}, 0_{\overline{I}(x^*)})$, one further has

$$\lambda_0^* f'(x^*;E(x)-x^*) + \overline{\lambda}^{*T} g'(x^*;E(x)-x^*) \geqslant 0, \quad \forall \ x \in M,$$
$$\overline{\lambda}^{*T} g(x^*) = 0, \quad g(x^*) \leqslant 0, \quad 0 \neq (\lambda_0^*, \ \overline{\lambda}^*) \geqslant 0.$$

The next objective is to show that $\lambda_0^* \neq 0$, if this was not true, we have from above $\overline{\lambda}^{*T} g'(x^*;E(x)-x^*) \geqslant 0, \ \ \forall \ x \in M, \ \ \overline{\lambda}^{*T} g(x^*) = 0, \ \ g(x^*) \leqslant 0, \ \ 0 \neq \overline{\lambda}^* \geqslant 0$. Taking into account g being semilocal E-convex and $E(x^*) = x^*$, we have from Proposition 4.1(i)(which will be obtained in the next section), $g(\widehat{x}) - g(x^*) \geqslant g'(x^*;E(\widehat{x})-x^*)$, so $\overline{\lambda}^{*T} g(\widehat{x}) \geqslant 0$. But this contradicts the fact that $g(\widehat{x}) < 0$ and $\overline{\lambda}^* > 0$. Thus $\lambda_0^* > 0$. Dividing (4.1) and the first equality of (4.2) by λ_0^* and let $\lambda^* = \overline{\lambda}^*/\lambda_0^*$. We know that (4.3) and (4.4) hold. The whole proof is completed. ∎

(3) SUFFICIENT OPTIMALITY CONDITIONS. To discuss the sufficient optimality conditions for the problem (P_g), we further extend the concept of semilocal E-convex function as follows.

DEFINITION 4.2: A real-valued function f defined on a local starshaped E-convex set $M \subseteq R^n$ is said to be quasi-semilocal E-convex if for all $x^1, x^2 \in M$ (with a maximal positive number $a(x^1,x^2) \leqslant 1$ satisfying (3.1)) satisfying $f(x^1) \leqslant f(x^2)$, there is a positive number $d(x^1,x^2) \leqslant a(x^1,x^2)$ such that $f(\lambda E(x^1) + (1-\lambda)E(x^2)) \leqslant f(x^2), \quad \forall\ 0 < \lambda < d(x^1,x^2)$.

The definition of quasi-semilocal E-convex of a vector function $f : R^n \longrightarrow R^k$ is similar to the one for a vector semilocal E-convex function.

DEFINITION 4.3: A real-valued function f defined on a local starshaped E-convex set $M \subseteq R^n$ is said to be pseudo-semilocal E-convex if for all $x^1, x^2 \in M$ (with a maximal positive number $a(x^1, x^2) \leqslant 1$ satisfying (3.1)) satisfying $f(x^1) < f(x^2)$, there are a positive number $d(x^1, x^2) \leqslant a(x^1, x^2)$ and a positive number $b(x^1, x^2)$ such that $f(\lambda E(x^1) + (1-\lambda)E(x^2)) \leqslant f(x^2) - \lambda b(x^1, x^2), \quad \forall\, 0 < \lambda < d(x^1, x^2)$.

The definition of pseudo-semilocal E-convex of a vector function $f : R^n \longrightarrow R^k$ is similar to the one for a vector semilocal E-convex function.

REMARK 4.1. Every semilocal E-convex function on a local starshaped E-convex set M is both a quasi-semilocal E-convex function and a pseudo-semilocal E-convex function.

For convenience of discussion, we give the following proposition.

PROPOSITION 4.1. *Let f be a real-valued function on a local starshaped E-convex set $M \subseteq R^n$, and f possesses directional derivative with respect to the direction $(E(x^1) - x^2)$ at x^2 for $x^1, x^2 \in M$. If $E(x^2) = x^2$, then the following statements hold true.*

(i) *If f is semilocal E-convex, then $f'(x^2; E(x^1) - x^2) \leqslant f(x^1) - f(x^2)$.*

(ii) *If f is quasi-semilocal E-convex, then $f(x^1) \leqslant f(x^2)$ implies that $f'(x^2; E(x^1) - x^2) \leqslant 0$.*

(iii) *If f is pseudo-semilocal E-convex, then $f(x^1) < f(x^2)$ implies that $f'(x^2; E(x^1) - x^2) < 0$.*

The proof is elementary by using the associated definitions and is omitted.

THEOREM 4.5. *Let $x^* \in M_g$ and $E(x^*) = x^*$. Assume that f, g possess directional derivatives with respect to the direction $(E(x) - x^*)$ at $x^*$for each $x \in M$, and suppose that there is $\lambda^* \in R^m$ such that (4.3) and (4.4) hold. If f is pseudo-semilocal E-convex on M and $g_{I(x^*)}$ is quasi-semilocal E-convex on M, then x^* is an optimal solution for (P_g).*

PROOF: In view of $g_{I(x^*)}(x) \leqslant g_{I(x^*)}(x^*) = 0, \quad \forall\ x \in M_g$, it follows from Proposition 4.1(ii) that $g'_{I(x^*)}(x^*; E(x) - x^*) \leqslant 0, \quad \forall\ x \in M_g$, which together with $\lambda^*_{I(x^*)} \geqslant 0$ implies ${\lambda^*_{I(x^*)}}^T g'_{I(x^*)}(x^*; E(x) - x^*) \leqslant 0, \quad \forall\ x \in M_g$. Furthermore, using $\lambda^{*T} g(x^*) = 0, \ g(x^*) \leqslant 0$ and $\lambda^* \geqslant 0$, we get $\lambda^{*T} g'(x^*; E(x) - x^*) \leqslant 0, \quad \forall\ x \in M_g$. So we have from the given conditions $f'(x^*; E(x) - x^*) \geqslant 0, \quad \forall\ x \in M_g$. Therefore, from Proposition 4.1(iii), this implies $f(x) \geqslant f(x^*), \ \forall x \in M_g$. Hence x^* is an optimal solution for (P_g). ∎

THEOREM 4.6. *Let $x^* \in M_g$ and $E(x^*) = x^*$. Assume that f, g possesses directional derivative with respect to the direction $(E(x) - x^*)$ at x^* for each $x \in M$, and there is $\lambda^* \in R^m$ such that (4.3) and (4.4) hold. If $(f + {\lambda^*_{I(x^*)}}^T g_{I(x^*)})$ is pseudo-semilocal E-convex on M, then x^* is an optimal solution for (P_g).*

PROOF: Taking into account $\lambda^{*T}g(x^*) = 0$, $g(x^*) \leqslant 0$, $\lambda^* \geqslant 0$, we have from the given conditions $(f + \lambda^{*}_{I(x^*)}{}^T g_{I(x^*)})'(x^*; E(x) - x^*) = f'(x^*; E(x) - x^*) + \lambda^{*T}g'(x^*; E(x) - x^*) \geqslant 0$, $\forall\ x \in M_g$. So we obtain from Proposition 4.1(iii),

$$\left(f + \lambda^{*}_{I(x^*)}{}^T g_{I(x^*)}\right)(x) \geqslant \left(f + \lambda^{*}_{I(x^*)}{}^T g_{I(x^*)}\right)(x^*),\ \forall\ x \in M_g.$$

The inequality above together with $\lambda^{*}_{I(x^*)}{}^T g_{I(x^*)}(x^*) = 0$, yields $f(x) + \lambda^{*}_{I(x^*)}{}^T g_{I(x^*)}(x) \geqslant f(x^*)$, $\forall\ x \in M_g$. In view of $\lambda^{*}_{I(x^*)} \geqslant 0$ and $g_{I(x^*)}(x) \leqslant 0$, we get $f(x) \geqslant f(x^*)$, $\forall x \in M_g$. Hence x^* is an optimal solution for (P_g). ∎

The next conclusion is a direct corollary of Theorem 4.5 or Theorem 4.6.

COROLLARY 4.1. *Let $x^* \in M_g$ and $E(x^*) = x^*$. Assume that f, g possess directional derivatives with respect to the direction $(E(x) - x^*)$ at x^* for each $x \in M$, and suppose that there is $\lambda^* \in R^m$ such that (4.3) and (4.4) hold. If f and $g_{I(x^*)}$ are semilocal E-convex functions on M, then x^* is an optimal solution for (P_g).*

(4) MOND–WEIR-LIKE TYPE DUALITY. Similar to Mond–Weir dual, we discuss a dual problem of P_g as follows:

$$(D_g)\qquad \begin{array}{ll} \max & f(u) \\ \text{subject to} & f'(u; E(x) - u) + \lambda^T g'(u; E(x) - u) \geqslant 0,\quad \forall\ x \in M. \\ & \lambda^T g(u) \geqslant 0, \\ & \lambda \in R^m,\quad \lambda \geqslant 0, \\ & u \in M. \end{array}$$

THEOREM 4.7. (Weak Duality) *Let x and (u, λ) be arbitrary feasible solutions of (P_g) and (D_g), respectively. If f and g are all semilocal E-convex functions, and they possess directional derivatives with respect to the direction $\big(E(x) - u\big)$ at u, where $E(u) = u$, $x \in M$, then $f(x) \geqslant f(u)$.*

PROOF: Taking into account f and g being a semilocal E-convex function and $E(u) = u$, we have from Proposition 4.1(i)

$$f(x) \geqslant f(u) + f'\big(u; E(x) - u\big),\quad g(x) \geqslant g(u) + g'\big(u; E(x) - u\big).$$

Combining the first constraint of D_g and the relationships above, one gets

$$f(x) \geqslant f(u) - \lambda^T g'\big(u; E(x) - u\big) \geqslant f(u) + \lambda^T\big(g(u) - g(x)\big).$$

So, in view of

$$\lambda \geqslant 0,\quad g(x) \leqslant 0,\quad \lambda^T g(u) \geqslant 0,$$

we have $f(x) \geqslant f(u)$. ∎

THEOREM 4.8. (Strong duality) *Suppose that x^* be an optimal solution for (P_g), $E(x^*) = x^*$, and $E(u) = u$ for any feasible point (u, λ) of (D_g). Assume that f and g are*

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semilocal E-convex on M and g_j is continuous at x^* for any $j \in \overline{I}(x^*)$, and they possess directional derivatives with respect to the direction $(E(x) - x^*)$ at x^* and the direction $(x^* - u)$ at u, respectively, where $x \in M$. Further assume that there is $\widehat{x} \in M$ such that $g(\widehat{x}) < 0$. If $\left(f'(x^*; E(x) - x^*), g'_{I(x^*)}(x^*; E(x) - x^*)\right)$ is a convexlike function, then there is $\lambda^* \in R^m$ such that (x^*, λ^*) is an optimal solution for (D_g).

PROOF: From the assumptions and Theorem 4.4, we can conclude that there is $\lambda^* \geqslant 0$ such that (x^*, λ^*) is a feasible point for $(\mathrm{D_g})$.

Suppose that (u, λ) is a feasible solution of $\mathrm{D_g}$. In view of f and g being a semilocal E-convex function on M and $E(u) = u$, we have from Proposition 4.1(i) $f(x^*) - f(u) \geqslant f'(u; x^* - u)$, $g(x^*) - g(u) \geqslant g'(u; x^* - u)$. Taking into account the first constraint condition of $(\mathrm{D_g})$ and the two inequalities above, one gets $f(x^*) - f(u) \geqslant -\lambda^T g'(u; x^* - u) \geqslant \lambda^T\left(g(u) - g(x^*)\right)$. Noting that $\lambda \geqslant 0$, $g(x^*) \leqslant 0$, and $\lambda^T g(u) \geqslant 0$, one knows $f(x^*) \geqslant f(u)$. Hence (x^*, λ^*) is an optimal solution for $(\mathrm{D_g})$. ∎

THEOREM 4.9. (Converse duality) *Assume that $x^* \in M_g$ and $(\overline{u}, \overline{\lambda})$ is a feasible point for (D_g). Further assume that f and g are semilocal E-convex functions on M, and f, g possess directional derivatives with respect to the direction $(E(x) - \overline{u})$ at $\overline{u}$ for each $x \in M$. If $f(x^*) = f(\overline{u})$ and $E(\overline{u}) = \overline{u}$, then x^* is an optimal solution for (P_g).*

PROOF: Taking into account f and g are semilocal E-convex and $E(\overline{u}) = \overline{u}$, one gets from Proposition 4.1(i)

$$f(x) - f(\overline{u}) \geqslant f'\left(\overline{u}; E(x) - \overline{u}\right), \quad g(x) - g(\overline{u}) \geqslant g'\left(\overline{u}; E(x) - \overline{u}\right), \quad \forall \ x \in M_g.$$

In view of $(\overline{u}, \overline{\lambda})$ being a feasible point for $(\mathrm{D_g})$, we have from the first constraint condition of $(\mathrm{D_g})$ and the two inequalities above

$$f(x) - f(\overline{u}) \geqslant -\overline{\lambda}^T g'\left(\overline{u}; E(x) - \overline{u}\right) \geqslant \overline{\lambda}^T\left(g(\overline{u}) - g(x)\right).$$

This together with $\overline{\lambda} \geqslant 0$, $g(x) \leqslant 0$, $f(x^*) = f(\overline{u})$ and $\overline{\lambda}^T g(\overline{u}) \geqslant 0$ shows that $f(x) \geqslant f(x^*)$. Hence x^* is an optimal solution for $(\mathrm{P_g})$. ∎

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