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Abstract

We show that the virtual cohomological dimension of a Coxeter group is essentially the regularity of the Stanley–Reisner ring of its nerve. Using this connection between geometric group theory and commutative algebra, as well as techniques from the theory of hyperbolic Coxeter groups, we study the behavior of the Castelnuovo–Mumford regularity of square-free quadratic monomial ideals. We construct examples of such ideals which exhibit arbitrarily high regularity after linear syzygies for arbitrarily many steps. We give a doubly logarithmic bound on the regularity as a function of the number of variables if these ideals are Cohen–Macaulay.

1. Introduction

Castelnuovo–Mumford regularity captures the complexity of finitely generated graded R-modules, where $R = \Bbbk[x_1, \dots, x_n]$ is a standard graded polynomial ring in n variables over a field $\Bbbk$. We focus on the case of modules of the kind R/I, where I is a homogeneous ideal of R. A fundamental question is how big the regularity of R/I can be, when $I \subseteq R$ is generated in fixed degree. The following are some important results in this area.

(i) For any $d \geqslant 2$, Mayr and Meyer [MM82] provided ideals $I \subseteq R$ generated in degrees less than or equal to d for which $\operatorname{reg} R/I$ is doubly exponential in the number of variables n, as explained by Bayer and Stillman in [BS88].

(ii) Caviglia and Sbarra showed in [CS05] that $\operatorname{reg} R/I \leqslant (2d)^{2^{n-2}}$ provided I is generated in degrees less than or equal to d.

(iii) Ananyan and Hochster [AH16] proved that, if I is generated by r forms of degrees less than or equal to d, then $\operatorname{projdim} R/I \leqslant \phi(r, d)$ provided the characteristic of $\Bbbk$ is zero or larger than d (here ϕ is a function not depending on the number of variables n). This solves Stillman's conjecture [PS09] in characteristic zero or greater than d. By a result of Caviglia (see, for example, [MS13, Theorem 2.4]), projective dimension can be equivalently replaced by regularity in the above statement.

(iv) McCullough and Peeva provided in [MP18] examples of homogeneous prime ideals $\mathfrak{p} \subseteq R$ such that $\operatorname{reg} R/\mathfrak{p}$ is not bounded by any polynomial function in the multiplicity. In particular, this shows that the Eisenbud–Goto conjecture [EG84] is false.

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(v) Caviglia, Chardin, McCullough, Peeva and the third named author noticed in [CCMPV19] that, if $\Bbbk$ is algebraically closed, there exists a function $\phi(e)$ bounding reg $R/\mathfrak{p}$ from above whenever $\mathfrak{p}$ is a homogeneous prime ideal of multiplicity e.

The Castelnuovo–Mumford regularity of R/I can be read off the graded Betti numbers β_{ij} of R/I as $\operatorname{reg} R/I = \max\{j-i : \beta_{ij} \neq 0\}$ (see § 2.4 for preliminaries on commutative algebra). The Mayr–Meyer ideals have the property that $\beta_{2j} \neq 0$ for a certain $j > d^{2^{n/10}}$. That is, their eventually high regularity is visible early in the resolution, indicating a possible connection between different homological degrees. Part of the purpose of the present paper is to investigate the possibilities for such connections. Specifically, we study the behavior of the regularity of free resolutions that stay linear until a certain homological degree. As an example of questions concerning the limit behavior of regularity consider the following open problem.

Question 1.1. Is there a family of quadratically generated ideals $\{I_n \subseteq R = \Bbbk[x_1, \dots, x_n]\}_{n\in\mathbb{N}}$ with linear syzygies such that

$$\lim_{n\to\infty} \frac{\operatorname{reg} R/I_n}{n} > 0?$$

Following Green and Lazarsfeld [GL86, § 3a], we say that, given an integer $p \geqslant 1$, R/I *satisfies property* N_p if $\beta_{ij} = 0$ for all $1 \leqslant i \leqslant p$ and $j \neq i+1$. So R/I satisfies property N_1 if and only if I is quadratically generated, it satisfies property N_2 if and only if I is quadratically generated and has linear first syzygies, and so on. The *Green–Lazarsfeld index* of R/I, denoted by $\operatorname{index} R/I$, is the largest p such that R/I satisfies N_p, where, by convention, $\operatorname{index} R/I = \infty$ if I has a 2-linear resolution, and $\operatorname{index} R/I = 0$ if I is not quadratically generated. If I is a Mayr–Meyer ideal, then its Green–Lazarsfeld index is at most 1.

As a consequence of Eisenbud and Schreyer's construction of pure modules, a syzygy degree that appears in a free resolution can be unrelated to all earlier parts of the resolution [ES09]. A construction due to Ullery shows that for any $p, k \in \mathbb{N}$, $k > p+1$, there is even a homogeneous ideal $I \subseteq R$ such that R/I satisfies N_p and $\beta_{p+1,k} \neq 0$; see [Ull14]. These constructions need a large number of variables in R, though, and are not efficient enough for Question 1.1. Due to the flexibility of resolutions of general ideals, it is interesting to look at more restricted classes. For example, Koszul algebras cannot exhibit extremal behavior as above. It is known, however, that for all $p \geqslant 2$ there exist families of homogeneous ideals $I_n \subseteq R$ such that R/I_n is a Koszul algebra satisfying N_p and $\lim_{n\to\infty}(\operatorname{reg} R/I_n)/\sqrt[p]{n} > 0$ [ACI13, Section 6].

In this paper we are interested in monomial ideals. Here the situation is even more rigid as the following result by Dao, Huneke and Schweig [DHS13] illustrates. If $I \subseteq R$ is a square-free monomial ideal such that R/I satisfies N_p for some $p \geqslant 2$, then

$$\operatorname{reg} R/I \leqslant \log_{(p+3)/2}\left(\frac{n-1}{p}\right) + 2.$$

In particular, a family which gives a positive answer to Question 1.1 cannot consist of monomial ideals. In § 4 we derive a new doubly logarithmic bound when R/I is Cohen–Macaulay.

The main motivation for the present paper is the following question.

Question 1.2. Fix an integer $p \geqslant 2$. Is there a bound $r(p)$ (independent of n) such that $\operatorname{reg} R/I \leqslant r(p)$ for all monomial ideals $I \subset R$ such that R/I satisfies N_p?

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For $p = 2$, a negative answer has been given by the authors in [CKV16]. If R/I is Gorenstein, the answer is positive by [CKV16, Theorem 4]. If R/I is Cohen–Macaulay, then the answer is unknown (Question 4.4). In this paper we give a negative answer for arbitrary p and begin the search for constructions that realize the negative answer with as few variables as possible.

These investigations lead us to the consideration of a connection between square-free monomial ideals and Coxeter groups. It starts from the observation that square-free monomial ideals with property N_2 correspond to right-angled hyperbolic Coxeter groups (see § 2.3.1). The study of the geometry and topology of such groups contains many ideas that we feel can be useful for commutative algebra. In § 5 we start to develop this connection, proving as a cornerstone the following identity of homological invariants (Theorem 5.2):

$$\operatorname{vcd} W = \max_{\operatorname{char} \Bbbk} \{\operatorname{reg} \Bbbk[\mathcal{N}]\}, \tag{1}$$

where W is a Coxeter group with nerve $\mathcal{N}(W)$, $\operatorname{vcd} W$ is the virtual cohomological dimension of W, and $\Bbbk[\mathcal{N}(W)]$ is the Stanley–Reisner ring of the simplicial complex $\mathcal{N}(W)$. As the regularity of a Stanley–Reisner ring depends only on the characteristic of the field, the maximum is taken over all possible characteristics, choosing one field for each.

We see (1) as a general tool to transfer results from Coxeter group theory to combinatorial commutative algebra and vice versa. For example, when $p = 2$, Question 1.2 is equivalent to the following question of Gromov,

Question 1.3. Is there a global bound on the virtual cohomological dimension of hyperbolic right-angled Coxeter groups?

In fact, a right-angled Coxeter group W is hyperbolic if and only if $\Bbbk[\mathcal{N}(W)]$ satisfies N_2. As an immediate consequence of (1) and the bound of [DHS13] we get the following result.

Corollary 1.4. *If W is a hyperbolic right-angled Coxeter group with n generators, then*

$$\operatorname{vcd} W \leqslant \log_{5/2}\left(\frac{n-1}{2}\right) + 2.$$

Gromov's question had already been answered negatively in [JŚ03]. Later on, new examples were constructed by Osajda in [Osa13b]. Let $2^{(\Delta)}$ denote the face complex of a simplicial complex Δ (Definition 2.8). Exploiting ideas from Osajda's construction and (1), we prove the following theorem.

Theorem 6.11. *Let $I = I_\Delta \subseteq R$ be a square-free quadratic monomial ideal. If $\operatorname{char}(\Bbbk) = 0$, then there are a positive integer N and a square-free monomial ideal $I' = I_{\Delta'} \subseteq R' = \Bbbk[y_1, \ldots, y_N]$ such that:*

(i) $\operatorname{reg} R'/I' = \operatorname{reg} R/I + 1$;

(ii) $\operatorname{index} R'/I' = \operatorname{index} R/I$;

(iii) *for each vertex v of Δ', $\operatorname{lk}_{\Delta'} v = 2^{(\Delta)}$.*

As a corollary, we get a negative answer to Question 1.2.

Corollary 6.12. *For any positive integers p and r, there exists a square-free monomial ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_{N(p,r)}]$, such that R/I satisfies N_p and $\operatorname{reg} R/I = r$.*

The proofs of these statements are contained in § 6. The crux of the corollary is that the number of indeterminates $N(p, r)$ depends on the desired r and p. In § 7 we give an explicit upper bound for the minimal number of variables in the corollary (Theorem 7.4).

The following § 2 contains some preliminaries that we hope will be useful to readers not already initiated into commutative algebra and geometric group theory. Section 3 gathers some new homological properties of Stanley–Reisner rings inspired by the developments in this paper, but potentially useful beyond. In § 4 we prove a new doubly logarithmic upper bound on the regularity of Stanley–Reisner rings of complexes with top homology and property N_p (Theorem 4.2), which yields the same bound for all Cohen–Macaulay Stanley–Reisner rings with property N_p (Corollary 4.3). Section 5 establishes the fundamental equality (1). Finally, §§ 6 and 7 give Theorem 6.11 and an upper bound on the number of variables necessary for arbitrary regularity with property N_p (Theorem 7.4).

2. Preliminaries

As this paper touches upon the somewhat separate topics of geometric group theory, commutative algebra, and combinatorics, we introduce some preliminaries first.

2.1 Cell complexes

A *poset* is a partially ordered set $(\mathcal{P}, \leqslant)$. For every element $p \in \mathcal{P}$ we define the subposets $\mathcal{P}_{\leqslant p} = \{q \in \mathcal{P} : q \leqslant p\}$ and $\mathcal{P}_{\geqslant p} = \{q \in \mathcal{P} : q \geqslant p\}$. We do not assume that $\mathcal{P}$ is finite.

Definition 2.1. An *(abstract) convex cell complex* is a poset $\mathcal{P}$ that satisfies the following two conditions.

(i) For each $p \in \mathcal{P}$, the subposet $\mathcal{P}_{\leqslant p}$ is isomorphic to the poset of faces of some finite convex polytope (including the empty face).

(ii) For any $p_1, p_2 \in \mathcal{P}$ the poset $\mathcal{P}_{\leqslant p_1} \cap \mathcal{P}_{\leqslant p_2}$ contains a greatest element.

The elements of $\mathcal{P}$ are called *faces*, and the maximal elements are called *facets*. If each of the convex polytopes in condition (i) are simplices (respectively, cubes), then $\mathcal{P}$ is an *abstract simplicial complex* (respectively, an *abstract cubical complex*).

Conditions (i) and (ii) imply that, if $\mathcal{P} \neq \emptyset$, it has a unique minimal element $\hat{0}$, and a well-defined rank function. The minimal element corresponds to the empty face, and the rank function defines the dimension of a face: $\dim(p) = \operatorname{rank}(p) - 1$. The zero-dimensional faces are called *vertices* and the one-dimensional faces are called *edges*. The 1-*skeleton* of a complex $\mathcal{P}$ is the subposet of elements of rank at most 2. We also interpret faces as finite sets of vertices: $F = \{\text{rank-1 elements of } \mathcal{P}_{\leqslant F}\}$. In this interpretation, the partial order is inclusion of sets. This way, a cell complex is a collection of finite subsets of a (possibly infinite) vertex set. A cell complex is thus a simplicial complex if the collection is closed under taking subsets. We can always speak of the cardinality of a face; however, the rank corresponds to the cardinality of faces only for simplicial complexes. *Non-faces* are collections of vertices which do not correspond to any face. These can also be ordered by inclusion, and minimal non-faces are well defined.

A convex cell complex is a *cell* if it has a unique maximal element F_0; the *boundary* of the cell is the poset $\mathcal{P} \setminus \{F_0\}$. The subcomplex of $\mathcal{P}$ *induced* by a non-empty subset V of its vertex set is $\mathcal{P}|_V = \bigcup_{p \in V} \mathcal{P}_{\geqslant p} \cup \{\hat{0}\}$. Some authors use the term *full subcomplex* for our induced subcomplexes. Not all subcomplexes are induced (e.g. the boundary of the triangle is not an induced subcomplex of the triangle, but all edges are induced subcomplexes). A cell complex is *locally finite* if $\mathcal{P}_{\geqslant p}$ is a finite poset for every $\hat{0} \neq p \in \mathcal{P}$.

Definition 2.2. Let $\mathcal{P}$ be an abstract convex cell complex and $F \in \mathcal{P}$ a face. The *link* $\operatorname{lk}_{\mathcal{P}} F$ *of* F *in* $\mathcal{P}$ is the abstract convex cell complex $\mathcal{P}_{\geqslant F}$.

Remark 2.3. If F is a vertex, Definition 2.2 yields what is commonly known as the spherical link at the vertex. Here we prefer a combinatorial definition as we do not think of our complexes as embedded in a metric space.

Remark 2.4. If $\mathcal{P}$ is a cubical or a simplicial complex, then every link is a simplicial complex. If $\mathcal{P}$ is locally finite, then the link is a finite complex.

Example 2.5. The link of each vertex in the three-dimensional cube is a triangle. The link of each vertex of the octahedron is a square.

2.1.1 *Simplicial complexes.* It is easy to check that, for simplicial complexes, all the 'usual' definitions agree with the ones given above. A simplicial complex Δ is *flag*, if all the minimal non-faces have cardinality 2. Equivalently, no induced subcomplex is the boundary of a simplex. For any integer $k \geqslant 3$, the *k-cycle* is the one-dimensional simplicial complex with vertex set $\{v_i\}_{i=0,\ldots,k-1}$ and edge set $\{\{v_i, v_{i+1(\mathrm{mod}\ k)}\}\}_{i=0,\ldots,k-1}$. The following property of simplicial complexes is essential to this paper, as it has interpretations in both commutative algebra and Coxeter group theory.

Definition 2.6. Let $k \geqslant 4$ be an integer. A simplicial complex is *k-large* if it is flag and does not have any induced j-cycles for $j < k$.

A cubical or simplicial complex is *locally k-large* if all its vertex links are k-large. In the literature, 5-largeness is sometimes referred to as *flag-no-square* or *Siebenmann's condition*. We stress here that all k-large complexes must be flag, and that an induced cycle contains no diagonals.

Example 2.7. Let Δ be the boundary of the octahedron, that is, Δ has vertex set $\{\pm v_i\}_{i=1,2,3}$ and 8 two-dimensional facets: $\{\pm v_1, \pm v_2, \pm v_3\}$. This complex is flag, because the minimal non-faces are $\{+v_i, -v_i\}$, but it is not 5-large, because the vertex subset $\{\pm v_1, \pm v_2\}$ induces a 4-cycle. Adding the edges $\{+v_i, -v_i\}$ for $i = 1, 2, 3$ to Δ, we obtain a simplicial complex without induced 4-cycles, but it is not flag.

Definition 2.8. Let Δ be a simplicial complex. The *face complex* $2^{(\Delta)}$ is the simplicial complex whose vertex set is the set of non-empty faces of Δ and where $F_1, \ldots, F_s \in \Delta$ form a face of $2^{(\Delta)}$ if and only they are all contained in a single face of Δ.

Example 2.9. The face complex of a d-simplex is the $(2^{d+1} - 2)$-simplex.

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2.2 Coxeter groups

We use the notation from Davis's book [Dav08]. A *Coxeter system* is a pair (W, S) consisting of a finitely generated group W and a finite set of distinct generators $S = \{s_1, \ldots, s_n\}$, all different from the identity, such that W is presented as

$$W = \langle s_1, \ldots, s_n : (s_i s_j)^{m_{ij}} = e \rangle$$

for $m_{ij} \in \mathbb{N} \cup \{\infty\}$ with $m_{ii} = 1$, and $m_{ij} \geqslant 2$ for $i \neq j$. The case $m_{ij} = \infty$ means no relation. If a group has a presentation as above, then it is a *Coxeter group*, and S is a set of *Coxeter generators*. The finite Coxeter groups have been classified by Coxeter [Cox35]. The matrix $M = (m_{ij})_{ij}$ is the *Coxeter matrix* of (W, S). If $m_{ij} \in \{1, 2, \infty\}$, then the Coxeter group (or Coxeter system) is *right-angled*. The elements of S are *letters*, and the elements of W are *words*.

A *special subgroup* of W is a subgroup W_T generated by a subset $T \subseteq S$ of the Coxeter generators. In particular, the trivial subgroup is special. By [Dav08, Theorem 4.1.6], (W_T, T) is a Coxeter system for all $T \subseteq S$. A subset $T \subseteq S$ is *spherical* if W_T is finite. In this case, W_T and the words in it are also called spherical. A *spherical coset* is a coset of a spherical subgroup. All spherical cosets are finite. Clearly, being spherical is closed under taking subsets.

DEFINITION 2.10. The *nerve* $\mathcal{N}(W, S)$ of a Coxeter system (W, S) is the simplicial complex consisting of the spherical sets ordered by inclusion.

The nerve of a Coxeter system is always a finite simplicial complex, with the Coxeter generators as vertices.

Remark 2.11. There is a one-to-one correspondence between right-angled Coxeter groups and flag simplicial complexes given as follows. Every flag simplicial complex Δ is the nerve of a right-angled Coxeter group $\mathcal{W}(\Delta)$: the off-diagonal entries of the Coxeter matrix of $\mathcal{W}(\Delta)$ are $m_{ij} = 2$ whenever $\{i, j\} \in \Delta$ and $m_{ij} = \infty$ otherwise. Conversely, if (W, S) is right-angled, and $T \subseteq S$, such that any two elements are connected by an edge in the nerve, then $W_T \cong (\mathbb{Z}/2\mathbb{Z})^{|T|}$.

Remark 2.12. In a right-angled Coxeter group a word is spherical if and only if it can be written with letters that commute pairwise. In particular, if the presentation is reduced (i.e. no subword is equal to the word), then each letter appears at most once.

Example 2.13. Not every simplicial complex is the nerve of a Coxeter system. The smallest counterexample occurs on five vertices and is given by the complex with facets $\{123, 145, 245, 345\}$. This can be confirmed using the classification of finite Coxeter groups.

DEFINITION 2.14. The *Davis complex* of a Coxeter system (W, S) is the cell complex $\Sigma(W, S)$ given by the poset of spherical cosets.

Remark 2.15. The link of any vertex w of $\Sigma(W, S)$ is the poset of spherical cosets wW_T for all spherical subsets T. It is thus isomorphic to the nerve $\mathcal{N}(W, S)$.

Remark 2.16. Davis and Januszkiewicz have discovered a link between Stanley–Reisner theory and Coxeter groups that is different from the developments in our paper. The cohomology ring $H^*(W, \mathbb{F}_2)$ is isomorphic to the Stanley–Reisner ring $\mathbb{F}_2[\mathbb{N}(W, S)]$. However, this connection is a characteristic-2 phenomenon, as otherwise the product in the cohomology ring need not be commutative. See [DJ91, Theorem 4.11].

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2.3 Geometric group theory

Let Γ be a simple graph on a (possibly infinite) vertex set V. Given two vertices $v, w \in V$, a *path* e from v to w is a subset $\{v = v_0, v_1, v_2, \dots, v_k = w\} \subseteq V$, such that $\{v_i, v_{i+1}\}$ is an edge for all $i = 0, \dots, k-1$. The *length* of a path is $\ell(e) = k$. The *distance* between v and w is

$$d(v, w) := \min\{\ell(e) : e \text{ is a path from } v \text{ to } w\}.$$

If $W \subseteq V$ is a set of vertices, then $d(v, W) := \min\{d(v, w), w \in W\}$. A path e from v to w is a *geodesic path* if $\ell(e) = d(v, w)$. A *geodesic triangle* of vertices v_1, v_2, v_3 consists of three geodesic paths e_i from v_i to $v_{i+1 (\text{mod } 3)}$ for $i = 1, 2, 3$. For a real number $\delta \geqslant 0$, a geodesic triangle e_1, e_2, e_3 is δ-*slim* if $d(v, e_i \cup e_j) \leqslant \delta$ for all $v \in e_k$ and $\{i, j, k\} = \{1, 2, 3\}$. The graph Γ is δ-*hyperbolic* if each geodesic triangle of Γ is δ-slim, and *hyperbolic* if it is δ-hyperbolic for some $\delta \geqslant 0$.

2.3.1 *Hyperbolic groups.* Let G be a group and S a set of distinct generators of G, not containing the identity. The *Cayley graph* $\mathrm{Cay}(G, S)$ is the simple graph with vertex set G and edges $\{g, gs\}$ for all $g \in G$ and $s \in S$. For example, the vertices of the Davis complex $\Sigma(W, S)$ of a Coxeter system (W, S) are the elements of W and the edges are the cosets of the spherical subgroups W_{s_i}. Therefore the 1-skeleton of $\Sigma(W, S)$ is the Cayley graph of (W, S). Gromov proved that if $\mathrm{Cay}(G, S)$ is hyperbolic for some finite set of generators S then it is hyperbolic for any finite set of generators S [Dav08, Theorem 12.3.5].

This justifies the definition of hyperbolic groups in the following way.

Definition 2.17. A group G is *hyperbolic* if $\mathrm{Cay}(G, S)$ is a hyperbolic graph for some (equivalently for any) finite set of generators S.

It is easy to check that $\mathbb{Z}^2$ is not hyperbolic. Therefore, if G contains $\mathbb{Z}^2$ as a subgroup, then G cannot be hyperbolic. By work of Moussong, for a Coxeter group (W, S) this can be reversed:

$$W \text{ is hyperbolic} \iff \mathbb{Z}^2 \not\subseteq W.$$

Combining results of Siebenmann [Dav08, Lemma I.6.5] and Moussong [Dav08, Lemma 12.6.2], if (W, S) is right-angled then

$$W \text{ is hyperbolic} \iff \mathcal{N}(W, S) \text{ has no induced 4-cycles.} \tag{2}$$

2.3.2 *Cohomological dimension.* The *cohomological dimension* of a group G is

$$\mathrm{cd}\, G = \sup\{n : H^n(G; M) \neq 0 \text{ for some } \mathbb{Z}G\text{-module } M\},$$

where $H^n(G; M)$ is the nth group cohomology of G with values in M (see [Dav08, Appendix F] for equivalent definitions and some properties). If G has non-trivial torsion, then $\mathrm{cd}\, G = +\infty$ (see [Dav08, Lemma F.3.1]). Therefore the notion is of no interest for groups with torsion, but this can be rectified. A group G is *virtually torsion-free* if it has a finite-index subgroup which is torsion-free. It follows from a result of Serre [Dav08, Theorem F.3.4] that, if Γ and Γ' are two finite-index torsion-free subgroups of G, then $\mathrm{cd}\, \Gamma = \mathrm{cd}\, \Gamma'$. Thus the following notion is well defined.

Definition 2.18. Let G be a virtually torsion-free group, and Γ some (equivalently any) finite-index torsion-free subgroup of G. The *virtual cohomological dimension* of G is

$$\mathrm{vcd}\, G = \mathrm{cd}\, \Gamma.$$

Each non-trivial Coxeter group has torsion but, admitting a faithful linear representation (see [Dav08, Corollary D.1.2]), it is virtually torsion-free. Thus the virtual cohomological dimension is always well-defined for a Coxeter group. By [Dav08, Corollary 8.5.5], and using [Mun84, Lemma 70.1] to avoid geometric realizations, the vcd of a Coxeter group (W,S) can be read off the nerve $\mathcal{N}(W,S)$, namely,

$$\operatorname{vcd} W = \max\{i : \widetilde{H}^{i-1}(\mathcal{N}(W,S)\setminus\sigma;\mathbb{Z}) \neq 0 \text{ for some } \sigma \in \mathcal{N}(W,S)\}, \tag{3}$$

where $\mathcal{N}(W,S)\setminus\sigma$ is the restriction of $\mathcal{N}(W,S)$ to $S\setminus\sigma$, and $\widetilde{H}^i$ denotes the reduced simplicial cohomology modules.

2.4 Commutative algebra

Let n be a positive integer, $R=\Bbbk[x_1,\dots,x_n]$ the polynomial ring in n variables over a field $\Bbbk$, and $\mathfrak{m}=(x_1,\dots,x_n)$ its irrelevant ideal. Any quotient R/I by some homogeneous ideal $I\subseteq R$ has a minimal graded free resolution,

$$0 \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{kj}} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{2j}} \to \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1j}} \to R \to R/I \to 0.$$

The *Betti number* β_{ij} is the number of minimal generators of degree j of the free module in homological degree i in the resolution. It is independent of the particular minimal resolution and can be computed as $\beta_{i,j}(R/I)=\dim_{\Bbbk}\operatorname{Tor}_i(R/I,\Bbbk)_j$.

Definition 2.19. The *Castelnuovo–Mumford regularity* of R/I is

$$\operatorname{reg}(R/I)=\max\{j-i : \beta_{i,j}(R/I)\neq 0\}.$$

If $H^i_{\mathfrak{m}}$ denotes local cohomology with support in $\mathfrak{m}$, [Eis95, Proposition 20.16] and Grothendieck duality imply $\operatorname{reg}(R/I)=\max\{j+i : H^i_{\mathfrak{m}}(R/I)_j\neq 0\}$.

Definition 2.20. For any positive integer p, the $\Bbbk$-algebra R/I *satisfies property* N_p if

$$\beta_{i,j}(R/I)=0 \quad \forall i=1,\dots,p \text{ and } j\neq i+1.$$

Definition 2.21. Let Δ be a finite simplicial complex with vertex set $[n]=\{1,\dots,n\}$. The *Stanley–Reisner ring* of Δ, denoted by $\Bbbk[\Delta]$, is the quotient of R by the square-free monomial ideal

$$I_\Delta=\left(\prod_{i\in A} x_i : A\subseteq[n] \text{ and } A\notin\Delta\right).$$

The ideal I_Δ is the *Stanley–Reisner ideal* of Δ.

There is a one-to-one correspondence between simplicial complexes and ideals generated by square-free monomials. From the definition it follows that a simplicial complex is flag if and only if its Stanley–Reisner ideal is quadratic. The N_p property for Stanley–Reisner rings was characterized combinatorially in [EGHP05, Theorem 2.1].

Theorem 2.22. *The Stanley–Reisner ring* $\Bbbk[\Delta]$ *satisfies* N_p *if and only if* Δ *is* $(p+3)$*-large.*

The Castelnuovo–Mumford regularity of $\Bbbk[\Delta]$ can be computed from the reduced singular cohomology of either induced subcomplexes or links of Δ. More precisely, Hochster's formula for graded Betti numbers [MS05, Corollary 5.12] gives

$$\operatorname{reg}\Bbbk[\Delta]=\max\{i : \widetilde{H}^{i-1}(\Delta|_A;\Bbbk)\neq 0 \text{ for some } A\subseteq[n]\}. \tag{4}$$

On the other hand, by Hochster's formula for local cohomology [MS05, Theorem 13.13],

$$\operatorname{reg}\Bbbk[\Delta]=\max\{i : \widetilde{H}^{i-1}(\operatorname{lk}_\Delta\sigma;\Bbbk)\neq 0 \text{ for some } \sigma\in\Delta\}. \tag{5}$$

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3. Homological remarks on Stanley–Reisner rings

In this section Δ is a d-dimensional simplicial complex on n vertices, and a face of Δ is identified with its set of vertices. We use some standard algebraic topology (see, for example, [Mun84, § 5]). For $r \leqslant d$ let $C_r(\Delta;\Bbbk)$ be the $\Bbbk$-vector space spanned by the r-dimensional faces of Δ. Let $\partial_r : C_r(\Delta;\Bbbk) \to C_{r-1}(\Delta;\Bbbk)$ be the boundary operator and $Z_r(\Delta;\Bbbk) = \operatorname{Ker}\partial_r$ the subspace spanned by the cycles in $C_r(\Delta;\Bbbk)$. Also write $B_r(\Delta;\Bbbk) = \operatorname{im}\partial_{r+1}$ for the subspace spanned by the boundaries in $C_r(\Delta;\Bbbk)$. An r-cycle C is *non-trivial* if $C \notin B_r(\Delta;\Bbbk)$. A non-trivial r-cycle C is *vertex-minimal* if there is no non-trivial r-cycle C' with $V(C') \subsetneq V(C)$, where for $C = \sum_{i=1}^{l} c_i F_i \in C_r(\Delta;\Bbbk)$ we set $V(C) = \bigcup_{i=1}^{l} F_i$. For every $v \in V(C)$ define

$$C_v = \sum_{F_i \ni v} c_i (F_i \setminus v) \in C_{r-1}(\operatorname{lk}_\Delta v;\Bbbk).$$

If $C \in Z_r(\Delta;\Bbbk)$, then all codimension-1 faces containing v sum to zero when applying ∂, so $\partial(\sum_{F_i\ni v} c_iF_i) = C_v$; therefore $\partial(C_v) = \partial^2(\sum_{F_i\ni v} c_iF_i) = 0$ in $C_{r-2}(\Delta;\Bbbk)$ and thus, since the differentials of $(C_i(\operatorname{lk}_\Delta v;\Bbbk))_i$ are just the restrictions of the differentials of $(C_i(\Delta;\Bbbk))_i$, we get $C_v \in Z_{r-1}(\operatorname{lk}_\Delta v;\Bbbk)$.

Lemma 3.1. *Let $C = \sum_{i=1}^{l} c_iF_i$ be a non-trivial r-cycle in Δ and $v \in V(C)$ a vertex.*

(i) *If $r = d$, then C_v is a non-trivial $(d-1)$-cycle in $\operatorname{lk}_\Delta v$.*

(ii) *If C is vertex-minimal, then C_v is a non-trivial $(r-1)$-cycle in $(\operatorname{lk}_\Delta v)|_{V(C)}$.*

Proof. (i) is clear. For (ii), there is no harm in assuming that, in the linear order given to the vertices of Δ, v comes first in $V(C)$. Assume there exists $B_v = b_1G_1 + \cdots + b_sG_s$ with $b_i \in \Bbbk$, such that the G_i are r-dimensional faces in $(\operatorname{lk}_\Delta v)|_{V(C)}$ and $\partial(B_v) = C_v$. Consider the $\Bbbk$-linear combination of $(r+1)$-faces in Δ defined as $B = b_1(G_1 \cup v) + \cdots + b_s(G_s \cup v)$. Then

$$\partial(B) = -\sum_{F_i\ni v} c_iF_i + B_v.$$

So $A = C + \partial(B)$ is a non-trivial r-cycle of Δ (otherwise C would be trivial). However, $v \notin V(A) \subsetneq V(C)$, a contradiction.

Proposition 3.2. *There exists a vertex $v \in \Delta$ such that*

$$\operatorname{reg}\Bbbk[\operatorname{lk}_\Delta v] \geqslant \operatorname{reg}\Bbbk[\Delta] - 1.$$

Proof. Let V' be a subset of the vertex set of Δ such that $\Gamma = \Delta|_{V'}$ has non-trivial rth homology with coefficients in $\Bbbk$, where $\operatorname{reg}\Bbbk[\Delta] = r + 1$. Let C be a vertex-minimal non-trivial r-cycle of Γ. By Lemma 3.1(ii), C_v is a non-trivial $(r-1)$-cycle in $(\operatorname{lk}_\Gamma v)|_{V(C)}$, for all $v \in V(C)$. Since $(\operatorname{lk}_\Gamma v)|_{V(C)} = (\operatorname{lk}_\Delta v)|_{V(C)}$, the proposition follows. □

Proposition 3.3. *If $I \subseteq R$ is a homogeneous (not necessarily monomial) ideal such that $\sqrt{I}$ is a square-free monomial ideal, then for any $i \in \mathbb{N}, j \in \mathbb{Z}$ the map of $\Bbbk$-vector spaces*

$$H^i_{\mathfrak{m}}(R/I)_j \to H^i_{\mathfrak{m}}(R/\sqrt{I})_j$$

is surjective. In particular, $\operatorname{reg} R/\sqrt{I} \leqslant \operatorname{reg} R/I$ *and* $\operatorname{projdim} R/\sqrt{I} \leqslant \operatorname{projdim} R/I$.

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Proof. Let A denote R/I localized at $\mathfrak{m}$. If $\operatorname{char} \Bbbk > 0$, then the quotient by the nilradical $A_{\mathrm{red}} = A/\sqrt{(0)}$ is F-pure. By [Sch09, Theorem 6.1], in characteristic zero, A_{red} is DuBois.

So in each case, by [MSS17, Lemma 3.3, Remark 3.4], the map

$$H^i_{\mathfrak{m}}(R/I) = H^i_{\mathfrak{m}A}(A) \to H^i_{\mathfrak{m}A}(A_{\mathrm{red}}) = H^i_{\mathfrak{m}}(R/\sqrt{I})$$

is surjective for any $i \in \mathbb{N}$. Since the above map is homogeneous, we conclude. □

Given two polynomial rings $R = \Bbbk[x_1, \dots, x_n]$ and $R' = \Bbbk[y_1, \dots, y_m]$, a map of $\Bbbk$-algebras $f : R \to R'$ is a *monomial map* if $f(x_i)$ is a monomial in $\{y_1, \dots, y_m\}$ for all $i = 1, \dots, n$.

LEMMA 3.4. *If $f : R \to R'$ is a monomial map and $I \subseteq R$ is a monomial ideal, then*

$$\operatorname{projdim} R'/\sqrt{f(I)R'} \leqslant \operatorname{projdim} R/I.$$

Proof. Since $\sqrt{I}$ is a square-free monomial ideal, $\operatorname{projdim} R/I \geqslant \operatorname{projdim} R/\sqrt{I}$ by Proposition 3.3. By a classical result of Lyubeznik (see the main theorem of [Lyu84]) the projective dimension of $R/\sqrt{I}$ equals $\operatorname{cd}(R, I)$, the cohomological dimension of I. Since the computation of local cohomology is independent of the base ring [BS13, Theorem 4.2.1], $\operatorname{cd}(R', f(I)R') \leqslant \operatorname{cd}(R, I)$. Again using [Lyu84], $\operatorname{cd}(R', f(I)R') = \operatorname{projdim} R'/\sqrt{f(I)R'}$. □

PROPOSITION 3.5. *If $2^{(\Delta)}$ is the face complex of Δ, then* $\operatorname{reg} \Bbbk[\Delta] = \operatorname{reg} \Bbbk[2^{(\Delta)}]$.

Proof. Clearly $\operatorname{reg} \Bbbk[\Delta] \leqslant \operatorname{reg} \Bbbk[2^{(\Delta)}]$ by Definition 2.8 and (4). Let Γ and Γ' be the Alexander duals of Δ and $2^{(\Delta)}$, respectively. Then, by the Eagon–Reiner theorem [MS05, Theorem 5.63], $\operatorname{projdim} \Bbbk[\Gamma] - 1 = \operatorname{reg} \Bbbk[\Delta]$ and $\operatorname{projdim} \Bbbk[\Gamma'] - 1 = \operatorname{reg} \Bbbk[2^{(\Delta)}]$. It can be checked that I_Γ is the following ideal of $R = \Bbbk[x_1, \dots, x_n]$:

$$I_\Gamma = \left(\prod_{i \in [n] \setminus \tau} x_i : \tau \text{ is a facet of } \Delta \right).$$

For $I_{\Gamma'} \subseteq R' = \Bbbk[y_\sigma : \sigma \in \Delta]$, we have that

$$I_{\Gamma'} = \left(\prod_{\substack{\sigma \in \Delta \\ \sigma \not\subseteq \tau}} y_\sigma : \tau \text{ is a facet of } \Delta \right).$$

The map $R \xrightarrow{f} R'$ defined by

$$x_i \mapsto \prod_{\sigma \in \Delta, i \in \sigma} y_\sigma$$

gives $I_{\Gamma'} = \sqrt{f(I_\Gamma)R'}$, so that the result follows from Lemma 3.4. □

4. Regularity from top homology

The main result of this section is an improvement of the [DHS13] bound in the case that Δ is a Cohen–Macaulay complex. In this case, a doubly logarithmic bound for the regularity as a function of the number of vertices is possible (Corollary 4.3). The underlying Theorem 4.2 uses similar techniques to the proof of [CKV16, Theorem 7]. We use the following technical lemma, the proof of which is a routine computation using the inequality $(i-1)(i+1) < i^2$ several times.

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Lemma 4.1. *For any integer $k \geqslant 3$ we have*

$$\prod_{i=0}^{k-3}(k-i)^{2^i} < 12^{2^{k-3}}.$$

Theorem 4.2. *Let Δ be a simplicial complex of dimension d on n vertices that is $(p+3)$-large for some $p \geqslant 2$, and has non-trivial top homology. If $f_i(\Delta)$ is the number of i-dimensional faces of Δ, then*

$$f_d(\Delta) > \left(\frac{p^2+6p+9}{12}\right)^{2^{d-2}} \quad \text{and} \quad f_0(\Delta) > \left(\frac{p^2+6p+9}{12}\right)^{2^{d-3}}.$$

Proof. For every d-dimensional simplicial complex Δ with non-trivial top homology we define

$$v_d(\Delta) = \min\{\text{number of vertices in a top-dimensional cycle in } \Delta\},$$
$$s_d(\Delta) = \min\{\text{number of facets in a top-dimensional cycle in } \Delta\}.$$

Minimizing over all d-dimensional $(p+3)$-large complexes with non-trivial top homology, let

$$v_d = \min\{v_d(\Delta) : \Delta\ (p+3)\text{-large, with non-trivial top homology}\},$$
$$s_d = \min\{s_d(\Delta) : \Delta\ (p+3)\text{-large, with non-trivial top homology}\}.$$

This implies, in particular, that $v_1 = s_1 = p+3$.

Fix a complex Δ satisfying the hypotheses of the theorem. Let $C \in \Delta$ be a top-dimensional cycle with $s_d(\Delta)$ facets. For every vertex $v \in C$, the link $\operatorname{lk}_\Delta v$ is a $(d-1)$-dimensional simplicial complex with non-trivial top homology by item (i) in Lemma 3.1. Furthermore, $\operatorname{lk}_\Delta v$ is $(p+3)$-large. Counting codimension-1 faces in $\Delta|_{V(C)}$ with multiplicity, we get

$$s_d(\Delta) \geqslant \frac{1}{d+1}\sum_{v\in V(C)} s_{d-1}(\operatorname{lk}_{\Delta|_{V(C)}} v).$$

Fix $v \in V(C)$. Every facet F of the link of v in $\Delta|_{V(C)}$ is contained in at least two facets of C only one of which can contain v. Thus, a map associating to $F \in \operatorname{lk}_{\Delta|_{V(C)}} v$ a vertex $w \neq v$, with $F \cup \{w\} \in \Delta$, is well defined:

$$\Phi_v : \mathcal{F}(\operatorname{lk}_{\Delta|_{V(C)}} v) \longrightarrow V(\Delta) \setminus V(\operatorname{star}_{\Delta|_{V(C)}} v).$$

We claim that Φ_v is injective. To see this, let $F_1, F_2 \in \mathcal{F}(\operatorname{lk}_{\Delta|_{V(C)}} v)$ be distinct faces such that $F_1 \cup \{w\}$ and $F_2 \cup \{w\}$ are faces of Δ. Since Δ is flag, there exist $v_1 \in F_1$ and $v_2 \in F_2$ such that v, v_1, w, v_2, v is a 4-cycle and $\{v_1, v_2\} \notin \Delta$. Since $\operatorname{lk}_{\Delta|_{V(C)}} v$ is flag, also $\{v, w\} \notin \Delta$. Because of this contradiction, Φ_v is injective. The injectivity yields $v_d(\Delta) \geqslant s_{d-1} + v_{d-1} + 1$ and then, putting together the above inequalities,

$$s_d > \frac{s_{d-1}^2}{d+1}, \quad v_d > s_{d-1}.$$

Now, since $s_1 = p+3$,

$$s_d > \frac{(p+3)^{2^{d-1}}}{\prod_{i=0}^{d-2}(d+1-i)^{2^i}}.$$

Finally, by Lemma 4.1,

$$f_d(\Delta) \geqslant s_d > \frac{(p+3)^{2^{d-1}}}{12^{2^{d-2}}} = \left(\frac{p^2+6p+9}{12}\right)^{2^{d-2}}. \qquad \square$$

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Corollary 4.3. *Let $I \subseteq R$ be a square-free monomial ideal such that R/I is a Cohen–Macaulay ring satisfying property N_p, for $p \geqslant 2$. Then*

$$\operatorname{reg} R/I \leqslant \log_2 \log_{(p^2+6p+9)/12} n + 3.$$

Proof. Let Δ be a simplicial complex on n vertices such that $I = I_\Delta$. By Hochster's formula for local cohomology [MS05, Theorem 13.13],

$$\operatorname{reg} \Bbbk[\Delta] = \max\{i : \widetilde{H}^{i-1}(\operatorname{lk}_\Delta \sigma; \Bbbk) \neq 0 : \sigma \in \Delta\}.$$

Let $\sigma \in \Delta$ attain the maximum. Because Δ is Cohen–Macaulay, $\operatorname{lk}_\Delta \sigma$ has non-trivial top homology. Therefore $\operatorname{reg} \Bbbk[\Delta] - 1 = \dim \operatorname{lk}_\Delta \sigma =: d$. Since Δ is $(p+3)$-large, so is $\operatorname{lk}_\Delta \sigma$. Hence, by Theorem 4.2,

$$n \geqslant f_0(\operatorname{lk}_\Delta \sigma) > \left(\frac{p^2+6p+9}{12}\right)^{2^{d-3}}$$

and the conclusion follows. □

Corollary 4.3 motivates us to ask Question 1.2 again with a Cohen–Macaulay restriction. In this case the answer is not known.

Question 4.4. Fix an integer $p \geqslant 2$. Is there a global bound $r(p)$ (independent of n) such that $\operatorname{reg} R/I \leqslant r(p)$ for all monomial ideals $I \subseteq R$ for which R/I satisfies N_p and is Cohen–Macaulay?

5. Virtual cohomological dimension meets regularity

The main theorem of this section establishes a new connection between Coxeter groups and commutative algebra. Its proof is by a cohomology computation using two spectral sequences associated to a double complex. A reference and our source of notation is [GM03, ch. III].

Fix a ring A. For any finite double complex $L = (L^{p,q})_{(p,q)\in\mathbb{N}^2}$ of A-modules, there are two spectral sequences both converging to the cohomology of the diagonal complex SL of L, whose entries are $SL^n = \bigoplus_{p+q=n} L^{p,q}$. We denote these spectral sequences by $({}^{I}E_r^{p,q})$ and $({}^{II}E_r^{p,q})$. Both converge to ${}^{I}E^k = {}^{II}E^k = H^k(SL)$. By [GM03, III.7, Proposition 10], ${}^{I}E_2^{p,q}$ is isomorphic to $H_I^p(H_{II}^{\bullet,q}(L^{\bullet,\bullet}))$ (vertical cohomology of horizontal cohomology), while ${}^{II}E_2^{p,q}$ is isomorphic to $H_{II}^p(H_I^{q,\bullet}(L^{\bullet,\bullet}))$ (horizontal cohomology of vertical cohomology).

For alignment with existing notation it is convenient to let Δ be a simplicial complex with $n+1$ vertices $V = \{0, \ldots, n\}$. For any $s < n$ and any $i \in \{0, \ldots, s\}$, denote $\Delta^i = \Delta|_{V\setminus\{i\}}$. For any sequence of integers $0 \leqslant a_0 < \cdots < a_p \leqslant s$, let

$$\Delta^{a_0,\ldots,a_p} = \bigcap_{k=0}^{p} \Delta^{a_k}.$$

Then $\Delta^{a_0,\ldots,a_p}$ equals the induced subcomplex $\Delta|_{V\setminus\{a_0,\ldots,a_p\}}$. The first notation, however, is more natural for our purposes. For example, if $\{0, \ldots, s\}$ is not a face of Δ, then $\{\Delta^i\}_{i=0,\ldots,s}$ forms a closed cover of Δ, that is, $\bigcup_{i=0}^{s} \Delta^i = \Delta$. Denote by $\mathbf{C}^\bullet(\Delta, A)$ the cochain complex of a simplicial complex Δ with coefficients in the ring A. Consider the double complex of A-modules $\mathcal{C}(A) = (\mathcal{C}^{p,q}(A))_{(p,q)\in\mathbb{N}^2}$ with

$$\mathcal{C}^{p,q}(A) = \bigoplus_{a_0<\cdots<a_p} \mathbf{C}^q(\Delta^{a_0,\ldots,a_p}; A), \quad 0 \leqslant p \leqslant s, 0 \leqslant q \leqslant \dim \Delta, \tag{6}$$

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where the direct sum runs over all sequences of $p+1$ integers $0 \leqslant a_0 < \cdots < a_p \leqslant s$. Throughout we use the standard convention that all modules with indices outside of defined bounds are zero. The vertical maps $\mathcal{C}^{p,q}(A) \to \mathcal{C}^{p,q+1}(A)$ are just the maps defined for each direct summand in the cochain complex $\mathbf{C}(\Delta^{a_0,\ldots,a_p}, A)$. The rows

$$0 \to \mathcal{C}^{0,\bullet}(A) \xrightarrow{d^1} \mathcal{C}^{1,\bullet}(A) \xrightarrow{d^2} \cdots \xrightarrow{d^{s-1}} \mathcal{C}^{s-1,\bullet}(A) \xrightarrow{d^s} \mathcal{C}^{s,\bullet}(A) \to 0 \tag{7}$$

are defined by mapping an element $\alpha = (\alpha_{a_0,\ldots,a_p})_{a_0<\cdots<a_p} \in \mathcal{C}^{p,q}(A)$ to $d^{p+1}(\alpha) \in \mathcal{C}^{p+1,q}(A)$, whose $(b_0, \ldots, b_{p+1})$th component is

$$\sum_{k=0}^{p+1} (-1)^k (\alpha_{b_0,\ldots,\widehat{b_k},\ldots,b_{p+1}})|_{\mathbf{C}^q(\Delta^{b_0,\ldots,b_{p+1}})}.$$

A routine computation confirms that this defines a double complex. The vertical cohomology is by definition the direct sum of the cohomologies of the corresponding $\Delta^{a_0,\ldots,a_p}$. The horizontal cohomology is non-trivial only in cohomological degree zero according to the following lemma, whose proof is standard.

Lemma 5.1. *For each $q \in \{0, \ldots, \dim \Delta\}$ we have*

$$H^p(\mathcal{C}^{\bullet,q}(A)) = \begin{cases} \mathbf{C}^q\left(\bigcup_i \Delta^i; A\right) & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

Theorem 5.2. *Let (W, S) be a Coxeter group and $\mathcal{N}$ its nerve. Then*

$$\operatorname{vcd} W = \max_{\operatorname{char} \Bbbk} \{\operatorname{reg} \Bbbk[\mathcal{N}]\}.$$

Proof. In § 2, equations (3) and (4) present interpretations of both invariants in terms of the reduced simplicial cohomology of $\mathcal{N}$, namely,

$$\operatorname{vcd} W = \max\{i : \widetilde{H}^{i-1}(\mathcal{N}|_{S\setminus\sigma}; \mathbb{Z}) \neq 0 \text{ for some } \sigma \in \mathcal{N}\},$$
$$\operatorname{reg} \Bbbk[\mathcal{N}] = \max\{i : \widetilde{H}^{i-1}(\mathcal{N}|_U; \Bbbk) \neq 0 \text{ for some } U \subseteq S\}.$$

Therefore the result is a consequence of the following claim.

Claim. Let Δ be a simplicial complex on $V = \{0, \ldots, n\}$ and A be a ring. Then

$$\begin{aligned} &\max\{i : H^i(\Delta|_{V\setminus\sigma}; A) \neq 0 \text{ for some } \sigma \in \Delta\} \\ &\quad = \max\{i : H^i(\Delta|_{V'}; A) \neq 0 \text{ for some } V' \subseteq V\}. \end{aligned} \tag{8}$$

Clearly the left-hand side is less than or equal to the the right-hand side. To see that equality holds, let r be the maximum on the right and choose $V' \subseteq V$ such that $H^r(\Delta|_{V'}; A) \neq 0$. If $V \setminus V' \in \Delta$ we have nothing to prove, so assume that $V \setminus V' \notin \Delta$ (in particular, $|V \setminus V'| \geqslant 2$). We can (and will) also assume that $H^i(\Delta|_U; A) = 0$ for all $i \geqslant r$ and $V' \subsetneq U \subseteq V$.

After a potential renumbering we can assume that $V \setminus V' = \{0, \ldots, s\}$. For any $i \in \{0, \ldots, s\}$, let $\Delta^i = \Delta|_{V\setminus\{i\}}$ and consider the double complex defined in (6). By Lemma 5.1, $({}^I E_r^{p,q})$ stabilizes at the second page and

$$H^p\left(\bigcup_{i=0}^{s} \Delta^i; A\right) = {}^I E_2^{p,0} = {}^I E_\infty^{p,0} = {}^I E^p.$$

Since $\{0,\dots,s\}$ is not a face of Δ, we have $\Delta = \bigcup_{i=0}^{s}\Delta^i$. Now consider the spectral sequence $({}^{II}E_r^{p,q})$. From the maximality assumption on V' it follows that

$$H^r(\Delta|_{V'};A) = H^r(\Delta^{0,\dots,s};A) = {}^{II}E_2^{r,s}.$$

In particular, if $r' > r$ or $s' > s$, then ${}^{II}E_2^{r',s'} = 0$, since it is a subquotient of

$$\bigoplus_{0\leqslant a_0<\cdots<a_{s'}\leqslant s} H^{r'}(\Delta^{a_0,\dots,a_{s'}};A) = 0.$$

We have ${}^{II}E_2^{r,s} = {}^{II}E_\infty^{r,s} = {}^{II}E^{r+s}$, from which we conclude that

$$H^{r+s}(\Delta;A) = {}^{I}E^{r+s} = {}^{II}E^{r+s} = H^r(\Delta|_{V'};A) \neq 0.$$

Since $s > 0$ (because $|V \setminus V'| \geqslant 2$), we obtain a contradiction to the maximality of r and V'. □

6. Arbitrary large regularity with property N_p

We now prove Theorem 6.11. To this end, for each k-large simplicial complex Δ we construct a k-large simplicial complex $\mathrm{S}(\Delta,k)$ such that, in characteristic zero, $\operatorname{reg}\Bbbk[\mathrm{S}(\Delta,k)] = \operatorname{reg}\Bbbk[\Delta]+1$ (Lemma 6.6). This uses a construction based on a detour through geometric group theory and is inspired by the work of Osajda [Osa13b, § 4].

We need to make a few definitions. The first turns a cell complex into a simplicial complex.

DEFINITION 6.1. The *thickening* of a convex cell complex $\mathcal{P}$ is the simplicial complex $\mathrm{Th}(\mathcal{P})$, with the same vertex set as $\mathcal{P}$, obtained by turning all cells into simplices. In particular, $\{v_1,\dots,v_s\}$ is a face of $\mathrm{Th}(\mathcal{P})$ if there is a face of $\mathcal{P}$ that contains $\{v_1,\dots,v_s\}$.

Example 6.2. The thickening of the d-dimensional cube is the (2^d-1)-simplex.

The thickening induces a distance between the vertices of a convex cell complex that counts the minimal number of maximal cells one needs to pass to get from one vertex to another. Namely, for two vertices $v, w \in \mathcal{P}$, the distance $d(v,w)$ is the length of a shortest path connecting v and w in the 1-skeleton of the thickening $\mathrm{Th}(\mathcal{P})$.

A step in our construction is taking a finite quotient of an infinite cubical complex. We clarify here how this is intended. Let G be a group acting on the vertex set $V(\mathcal{P})$ of a convex cell complex $\mathcal{P}$ such that for every face $F=\{v_1,\dots,v_k\}\in\mathcal{P}$ and every $g\in G$ we have

$$g\cdot F = \{g\cdot v_1,\dots,g\cdot v_k\}\in\mathcal{P}.$$

This induces an action of G on $\mathcal{P}$. The *displacement* of the action of G on $\mathcal{P}$ is the minimum distance between the elements in the orbit of a vertex. We can take the quotient $\mathcal{P}/G$, which is in general only a set.

Remark 6.3. If the displacement of the action is at least 2, then $\mathcal{P}/G$ is a poset with the inclusion given by $\widehat{F'} \subseteq \widehat{F}$ if there exists $g \in G$ such that $g\cdot F' \subseteq F$. If the displacement of the action is at least 3, then $\mathcal{P}/G$ is a convex cell complex.

An example of such a group action is that of the subgroup of some Coxeter group on the vertices of the Davis complex. In this case the displacement of the action coincides with the displacement of the subgroup as defined below.

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Definition 6.4. Let W be a Coxeter group. The *displacement* of an element $w \in W$ is the distance $d(e, w)$ of w to the identity in the (1-skeleton of the) thickening $\mathrm{Th}(\Sigma)$. The *displacement of a subgroup* $H \subseteq W$ is the minimal displacement among its non-trivial elements.

Let Δ be a k-large simplicial complex for an integer $k \geqslant 4$. We introduce an iterative construction which produces a new k-large simplicial complex $\mathrm{S}(\Delta, k)$. It works as follows.

(i) Let W be the right-angled Coxeter group with nerve Δ.
(ii) Let Σ be the Davis complex of W.
(iii) Let $Y = \mathrm{Th}(\Sigma)$ be the thickening of Σ.
(iv) Pick a torsion-free finite-index subgroup $H \subseteq W$ with displacement at least k.
(v) Let $\mathrm{S}(\Delta, k)$ be the quotient Y/H.

Since Δ is flag, there is a right-angled Coxeter group $\mathcal{W}(\Delta)$ as described in Remark 2.11. The group H in (iv) exists because W is virtually torsion-free [Dav08, Corollary D.1.4] and residually finite [Dav08, § 14.1]. In § 7, we take a constructive approach and find a concrete H using representations of W in $\mathrm{GL}_n(\mathbb{Z})$. The resulting complex $\mathrm{S}(\Delta, k)$ evidently depends on the choice of H in step (iv). However, the desired properties of $\mathrm{S}(\Delta, k)$, such as Lemma 6.6, do not depend on this choice.

Lemma 6.5. *In the above situation,* $2^{(\Delta)} = \mathrm{lk}_{\mathrm{S}(\Delta,k)}\, v$ *for any vertex* v.

Proof. After unraveling definitions, it is evident that if Σ is a cubical complex and $v \in \Sigma$ is a vertex, then $2^{(\mathrm{lk}_\Sigma v)} = \mathrm{lk}_{\mathrm{Th}(\Sigma)}\, v$. If Σ is the Davis complex of a right-angled Coxeter group with nerve Δ, then by Remark 2.15, $\mathrm{lk}_\Sigma\, v = \Delta$ for any vertex $v \in \Sigma$. □

Lemma 6.6. *If* $\Bbbk$ *is a field of characteristic zero and* $k \geqslant 4$*, then* $\operatorname{reg} \Bbbk[\mathrm{S}(\Delta, k)] = \operatorname{reg} \Bbbk[\Delta] + 1$.

Proof. By the previous lemma $2^{(\Delta)} = \mathrm{lk}_{\mathrm{S}(\Delta,k)}\, v$ for any vertex v, and by Proposition 3.2 there exists a vertex v such that $\operatorname{reg} \Bbbk[\mathrm{lk}_{\mathrm{S}(\Delta,k)}\, v] \geqslant \operatorname{reg} \Bbbk[\mathrm{S}(\Delta, k)] - 1$. Since $\operatorname{reg} \Bbbk[2^{(\Delta)}] = \operatorname{reg} \Bbbk[\Delta]$ by Proposition 3.5, it follows that $\operatorname{reg} \Bbbk[\mathrm{S}(\Delta, k)] \leqslant \operatorname{reg} \Bbbk[\Delta] + 1$.

To show $\operatorname{reg} \Bbbk[\mathrm{S}(\Delta, k)] \geqslant \operatorname{reg} \Bbbk[\Delta] + 1$, let $X = \Sigma/H$. Then $\mathrm{S}(\Delta, k)$ is the thickening of X. By Hochster's formula for graded Betti numbers and (8), we have that

$$\operatorname{reg} \Bbbk[\Delta] = \max\{i : \widetilde{H}^{i-1}(\Delta \setminus \sigma; \Bbbk) \neq 0 \text{ for some } \sigma \in \Delta\}.$$

Let $r = \operatorname{reg} \Bbbk[\Delta]$, and fix $\sigma \in \Delta$ for which $\widetilde{H}^{r-1}(\Delta \setminus \sigma; \Bbbk) \neq 0$. From now on the argument goes along the same lines of the proof leading to [Osa13a, Lemma 4.5]. With the same notation as used there, $\Delta \setminus \sigma$ deformation retracts onto $K^{S \setminus \sigma}$, where K is the subcomplex of Σ induced by the spherical words (including the identity) and, for any subset of generators $T \subseteq S$, K^T is the subcomplex induced by the spherical words containing some element of T. So we have

$$\widetilde{H}^r(K, K^{S \setminus \sigma}; \Bbbk) \neq 0.$$

Osajda produces a map of $\Bbbk$-vector spaces from the cocycles $Z^r(K, K^S; \Bbbk)$ to the cocycles $Z^r(X; \Bbbk)$. This uses the assumption $\mathrm{char}(\Bbbk) = 0$. One can check that the same rule defines a map of $\Bbbk$-vector spaces $Z^r(K, K^{S \setminus \sigma}; \Bbbk) \to Z^r(X \setminus A; \Bbbk)$, where $A = \{\widehat{w\sigma} : w \in W\}$ and $\widehat{w\sigma}$ is the class in X of $w\sigma \in \Sigma$. By the same argument used in [Osa13a, Lemma 4.5], the above map induces an injection

$$\widetilde{H}^r(K, K^{S \setminus \sigma}; \Bbbk) \hookrightarrow \widetilde{H}^r(X \setminus A; \Bbbk);$$

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in particular, $\widetilde{H}^r(X \setminus A; \Bbbk)$ is not zero. By [Mun84, Lemma 70.1], $\widetilde{H}^k(X \setminus A; \Bbbk) \cong \widetilde{H}^r(X_B; \Bbbk)$, where B are the vertices of X which are not in $\widehat{w\sigma}$ for any $w \in W$. Finally, the thickening of X_B is exactly $\mathrm{S}(\Delta, k)_B$, so

$$\widetilde{H}^r(\mathrm{S}(\Delta, k)_B; \Bbbk) \neq 0.$$

By Hochster's formula for graded Betti numbers $\operatorname{reg} \Bbbk[\mathrm{S}(\Delta, k)] \geqslant r + 1$. □

Remark 6.7. In the definition of cohomological dimension, $\mathbb{Z}$ could be replaced by a field $\Bbbk$ of characteristic zero. The resulting notion of *virtual rational cohomological dimension* $\operatorname{vcd}_{\mathbb{Q}} W$ of a virtually torsion-free group W does not depend on the choice of the field. This notion, however, differs from virtual cohomological dimension. Lemma 6.6, together with Hochster's formula for graded Betti numbers and (8), implies that

$$\operatorname{vcd}_{\mathbb{Q}} \mathcal{W}(\mathrm{S}(\Delta, k)) = \operatorname{vcd}_{\mathbb{Q}} \mathcal{W}(\Delta) + 1.$$

This conclusion for vcd does not follow from Lemma 6.6 because of the assumptions on $\Bbbk$.

Lemma 6.8. *If a cubical complex is locally k-large, then its thickening is locally k-large.*

Proof. Let Σ be a locally k-large cubical complex. As in the proof of Lemma 6.5, each vertex link $\operatorname{lk}_{\mathrm{Th}(\Sigma)} v$ is equal to $2^{(\operatorname{lk}_\Sigma v)}$. By a result of Haglund, a simplicial complex is k-large if and only if its face complex is k-large [JŚ10, Proposition B.1]. □

A proof of Lemma 6.8 also appears in [Osa13a, Lemma 6.7].

Lemma 6.9. *Let Σ be the Davis complex of $\mathcal{W}(\Delta)$, where Δ is k-large for $k \geqslant 4$. Then* $\mathrm{Th}(\Sigma)$ *is k-large.*

Proof. The Davis complex Σ is a deformation retract of its thickening $\mathrm{Th}(\Sigma)$ and, in particular, has the same homotopy type. Therefore $\mathrm{Th}(\Sigma)$ is simply connected. By Lemma 6.8, $\mathrm{Th}(\Sigma)$ is locally k-large. According to [JŚ06, Corollary 1.5], a simplicial complex is k-large if and only if all links are k-large and the systole (the length of the shortest non-contractible loop in the complex) is at least k. Since there are no non-contractible loops, the proof is complete. □

When forming the quotient of the thickening of the Davis complex modulo the finite-index torsion-free subgroup $H \subseteq W$ in step (v) of the construction, cycles are created. The quotient by a group of displacement k creates cycles of length k. By Remark 6.3, $k \geqslant 4$ implies the quotient is simplicial complex.

Lemma 6.10. *Let Σ be the Davis complex of $\mathcal{W}(\Delta)$, where Δ is k-large for $k \geqslant 4$. If $H \subseteq \mathcal{W}(\Delta)$ is a torsion-free subgroup of displacement at least k, then* $\mathrm{Th}(\Sigma)/H$ *is k-large.*

Proof. If $C \in \mathrm{Th}(\Sigma)/H$ is a cycle of length $l < k$, then it consists of disjoint orbits and thus there is a cycle of length l in $\mathrm{Th}(\Sigma)$. This is impossible since, by Lemma 6.9, $\mathrm{Th}(\Sigma)$ is k-large. □

We are now ready to prove the two main results of this section.

Theorem 6.11. *Let $I = I_\Delta \subseteq R = \Bbbk[x_1, \ldots, x_n]$ be a square-free quadratic monomial ideal. If the characteristic of $\Bbbk$ is zero, then there exist a positive integer N and a square-free monomial ideal $I' = I_{\Delta'} \subseteq R' = \Bbbk[y_1, \ldots, y_N]$ such that:*

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(i) $\operatorname{reg} R'/I' = \operatorname{reg} R/I + 1$;
(ii) $\operatorname{index} R'/I' = \operatorname{index} R/I$;
(iii) *for each vertex* v *of* Δ', $\operatorname{lk}_{\Delta'} v = 2^{(\Delta)}$.

Proof. Let $p = \operatorname{index} R/I$. Since $I = I_\Delta$ is quadratic, $p \geqslant 1$. Let $\Delta' = \mathrm{S}(\Delta, p+3)$. The first item is Lemma 6.6. The third item is Lemma 6.5. For the second item, Lemma 6.9 implies that $\operatorname{index} R'/I' \geqslant p$. If $\operatorname{index} R'/I' > p$, then $\operatorname{index} \Bbbk[2^{(\Delta)}] > p$ since if there are no induced $(p+3)$-cycles, then no link in Δ has an induced $(p+3)$-cycle. Furthermore, by [JŚ10, Proposition B.1], $\operatorname{index} \Bbbk[2^{(\Delta)}] = \operatorname{index} \Bbbk[\Delta] = \operatorname{index} R/I$. ☐

Corollary 6.12. *For any positive integers* p *and* r*, there exists a square-free monomial ideal* $I \subseteq R = \Bbbk[x_1, \ldots, x_{N(p,r)}]$*, such that* R/I *satisfies* N_p *and* $\operatorname{reg} R/I = r$.

Proof. Let Δ_2 be the $(p+3)$-cycle, and inductively $\Delta_r = \mathrm{S}(\Delta_{r-1}, p+3)$. Then Δ_r satisfies the conditions of the corollary if $\operatorname{char} \Bbbk = 0$. To see that the construction is independent of the field, assume that for some $\Bbbk$, $\operatorname{reg} \Bbbk[\Delta_r] > \operatorname{reg} \mathbb{Q}[\Delta_r]$. By Lemma 6.5 and Propositions 3.2 and 3.5, $\operatorname{reg} \Bbbk[\Delta_{r-1}] > \operatorname{reg} \mathbb{Q}[\Delta_{r-1}]$ and inductively $\operatorname{reg} \Bbbk[\Delta_2] > \operatorname{reg} \mathbb{Q}[\Delta_2]$, which is not the case. ☐

Remark 6.13. In [JŚ06, Corollary 19.2], Januszkiewicz and Świątkowski proved, for any $k \geqslant 6$ and $d \in \mathbb{N}$, the existence of a k-large orientable d-dimensional pseudomanifold. Together with [JŚ03, Theorem 1], this could be used to give a shorter proof of Corollary 6.12. We feel that such a proof would have been less insightful for commutative algebra.

Remark 6.14. The results in this section can also be used to strengthen a result of Nevo and Peeva who studied a question of Francisco, Hà and Van Tuyl. The latter noticed (unpublished) that if $I \subseteq R$ is a quadratic square-free monomial ideal such that I^s has a linear resolution for all $s \geqslant 2$, then R/I satisfies N_2, and wondered if the converse was true. In [NP13, Counterexample 1.10] Nevo and Peeva gave a square-free monomial ideal $I \subseteq R$ such that R/I has property N_2 but I^2 does not have a linear resolution. Using our results, this can be extended to N_p and any power as follows.

Corollary 6.15. *For any integers* $p, t \geqslant 2$ *there exists a square-free monomial ideal* $I \subseteq R$ *such that* R/I *has property* N_p *and* I^s *does not have a linear resolution for all* $1 \leqslant s \leqslant t$.

Proof. Set $r = 2t$ and choose $I \subseteq R$ as in Corollary 6.12. Then $\operatorname{reg}(R/I) = 2t$, and $\operatorname{reg}(R/I^s) \geqslant 2t$ for all $s \geqslant 1$ by Proposition 3.3. ☐

Question 1.11 in [NP13] asks whether I_Δ^s has a linear resolution for $s \gg 0$ whenever $\Bbbk[\Delta]$ satisfies N_2. It remains open and the construction yielding Theorem 6.11 provides examples worth testing. For an experimental investigation with computer algebra, the number of variables involved would need a vast improvement, though.

7. Counting the number of vertices of $\mathrm{S}(\Delta, k)$

For complexity theory in commutative algebra a bound on the number of variables $N(r, p)$ in Corollary 6.12 is necessary. We now derive such a bound by controlling the choice of the torsion-free subgroup H in step (iv) of the construction of $\mathrm{S}(\Delta, k)$.

Each Coxeter group W can be embedded in $\mathrm{GL}_n(\mathbb{R})$ by means of its canonical representation $\rho : W \to \mathrm{GL}_n(\mathbb{R})$ [Dav08, Corollary 6.12.4]. This representation starts from the cosine matrix

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$C = (c_{ij})_{ij}$ of a Coxeter system whose entries are $c_{ij} = -\cos(\pi/m_{ij})$. A generator s_i is represented by the linear map $\rho(s_i) : x \mapsto x - 2\sum_j c_{ij}x_je_i$. As the order of every product of generators is 2 or ∞, right-angled Coxeter groups embed also in $\mathrm{GL}_n(\mathbb{Z})$. More specifically, since the cosine matrix has entries only $-1, 0, 1$, the canonical representation matrices use only $0, \pm 1, 2$. An easy computation using the definition of the linear map for one generator and $\cos(\pi/2) = 0$ shows that whenever $w = s_{i_1} \cdots s_{i_l}$ is a spherical word, it is represented by the linear map

$$\rho(w) : x \mapsto x - 2\sum_j c_{i_1j}x_je_{i_1} - \cdots - 2\sum_j c_{i_lj}x_je_{i_l}. \tag{9}$$

We have thus shown a simple fact about the entries of $\rho(w)$.

LEMMA 7.1. *Let $\mathcal{W}(\Delta)$ be a right-angled Coxeter group with nerve Δ and $d = \dim\Delta$. For each spherical word $w \in \mathcal{W}(\Delta)$ of length l, the matrix $\rho(w)$ uses only $0, \pm 1, 2$ for its entries and each of its columns has at most l entries equal to 2.*

We employ the projection $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$ to find finite-index torsion-free subgroups H as in step (iv) of the construction in § 6, so that the size of $\mathrm{S}(\Delta, k)$ can be controlled. To preserve k-largeness, we need to choose m so that no words of displacement less than k reduce to the identity modulo m. This requires information about the orders of elements of $\mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$.

Fix $k > 4$ and a k-large simplicial complex Δ of dimension d with n vertices. For any $m \geqslant 2$ consider the canonical homomorphism

$$\pi_m : \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z}).$$

Denote $\Gamma_m = \mathrm{Ker}(\pi_m)$ and let $\Xi_m = \Gamma_m \cap \rho(\mathcal{W}(\Delta)) \subseteq \rho(\mathcal{W}(\Delta))$ be the subgroup of $\rho(\mathcal{W}(\Delta))$ that lies in the kernel of π_m.

LEMMA 7.2. *Ξ_m is torsion-free if $m > 2$.*

Proof. It is well known that any torsion element in a right-angled Coxeter group has order 2 and is in fact conjugate to a spherical word. Let $w \in \Xi_m$ be an involution and write $w = g^{-1}sg$ with some spherical word s and $g \in \mathcal{W}(\Delta)$. Then $1 = \pi_m(w) = \pi_m(g)^{-1}\pi_m(s)\pi_m(g)$ implies $\pi_m(s) = 1$ which for $m > 2$ implies $s = 1$ (by Lemma 7.1) and finally $w = 1$. □

The subgroup to be used in step (iv) is $H(m) = \rho^{-1}(\Xi_m)$. Let $w \in \mathcal{W}(\Delta)$. As a function of the displacement and the dimension d of Δ, we determine an upper bound on $a(w) = \max\{|\rho(w)_{i,j}| : 1 \leqslant i, j \leqslant n\}$, the maximum absolute value of the entries of the corresponding matrix $\rho(w) \in \mathrm{GL}_n(\mathbb{Z})$.

LEMMA 7.3. *Let w be a word of displacement less than k. Then $a(w) < (2d + 3)^{k-1}$.*

Proof. A word of displacement less than k is a product of at most $k - 1$ spherical words. When w is a spherical word, it has length at most $d + 1$, and thus each column of $\rho(w)$ has at most $d + 1$ entries 2 and one entry 1 by Lemma 7.1. This yields the recursion $a(ws) \leqslant (2d + 3)a(w)$. Since $a(s) = 2$ for any spherical word, the bound follows. □

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Our aim is to pick an integer m so that any word in $H(m)$ has displacement at least $p+3$. Lemma 7.3 shows that $m=(2d+3)^{p+2}$ is sufficient. Given m, the number of vertices of $\mathrm{S}(\Delta,p)$ is bounded by the size of $\mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z})$, which is of the order m^{n^2}. Iterating the construction of $\mathrm{S}(\Delta,p)$, we achieve the desired bound for the number of variables needed in Corollary 6.12. To write it, we use Knuth's up arrow notation [Knu76] which is convenient for iterative constructions. Fortunately, we can limit ourselves to two up arrows which represent power towers. Specifically, $a\uparrow\uparrow b$ means $a^{a^{a^{\cdots}}}$ exactly b times.

Theorem 7.4. *For all p, there exists a family of ideals indexed by r realizing Corollary 6.12 with*

$$N(p,r+1) < (2(2\uparrow\uparrow(r-1))+1)^{(p+2)N(p,r)^2}.$$

Furthermore, if c_p is the smallest integer such that $2\uparrow\uparrow c_p > p+2$, then

$$N(p,r+1) < 2\uparrow\uparrow(r(r+c_p)).$$

Proof. Let Δ_2 be the $(p+3)$-cycle which implies $N(p,2)=p+3$. Let $\Delta_{r+1}=\mathrm{S}(\Delta_r,p)$, where the subgroup in step (iv) is chosen as $H(m_{r+1})$ with $m_{r+1}=(2d_r+1)^{p+2}$. Here $d_r=\dim\Delta_r+1$ and thus $d_2=2$. We have the recursion $d_{r+1}=2^{d_r}$, which yields $d_r=2\uparrow\uparrow(r-1)$. The number of vertices of Δ_{r+1} is bounded by the order of $\mathrm{GL}_n(\mathbb{Z}/m_{r+1}\mathbb{Z})$. Estimating this order as $m_{r+1}^{N(r,p)^2}$, we obtain the recursive bound.

For the second part we use the fact that removing parentheses from a power tower does not make the expression smaller by generalizations of $(2^2)^{(2^2)} < 2^{2^{2^2}}$. We thus get

$$\begin{aligned} N(p,r+1) &< (2\uparrow\uparrow r)^{(p+2)N(p,r)^2} \\ &< (2\uparrow\uparrow r)^{(2\uparrow\uparrow c_p)N(p,r)^2} \\ &< (2\uparrow\uparrow(r+c_p))^{N(p,r)^2}. \end{aligned}$$

Now by a simple induction, the structure of the expression on the right is continued exponentiation of 2 for at most $r(r+c_p)$ times, but with certain parentheses inside the tower. Removing the parentheses, we conclude. □

We hope that the bound in Theorem 7.4 can be improved significantly. To justify this hope we illustrate vast improvements in a simple example. Let Δ be the 5-cycle. The right-angled Coxeter group with nerve Δ has the following Coxeter and cosine matrices:

$$\begin{pmatrix} 1 & 2 & \infty & \infty & 2 \\ 2 & 1 & 2 & \infty & \infty \\ \infty & 2 & 1 & 2 & \infty \\ \infty & \infty & 2 & 1 & 2 \\ 2 & \infty & \infty & 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{pmatrix}.$$

The generators of the standard representation of this Coxeter group are

$$s_1 \mapsto \begin{pmatrix} -1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \ldots \quad s_5 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & -1 \end{pmatrix}.$$

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Table 1. Entry sizes in words.

Word length	1	2	3	4	5	6	7	8	9	10
Entry size	2	4	8	18	39	84	180	388	836	1801

Table 1 gives the maximum absolute value of entries of words of length l in $\mathcal{W}(\Delta)$. It shows that the prime number $p = 1811$ would certainly suffice to guarantee that no word of displacement less than or equal to 5 (which all have length less than or equal to 10) is in the kernel of the reduction modulo p. However, it can be checked algorithmically (we used the Coxeter group functionality in Sage [Sage13]) that no word of length at most 10 is in the kernel of the reduction modulo 7. In the reduction modulo 5, however, $(s_1s_3)^5$ maps to the identity. We also checked words of length 12 for the Coxeter group corresponding to the heptagon. There 7 is not large enough, as for example $(s_1s_3s_1s_5)^3$ goes to the identity.

In the example of the 5-cycle, the bound derived in Theorem 7.4 yields $N(2,3) < 5^{100}$, while using $m = 7$ yields $N(2,3) < 7^{25}$. In contrast, one can exhibit a 5-large triangulation of a 2-sphere with 12 vertices. Nevertheless, a good understanding of representations of Coxeter groups in finite characteristic should yield better estimates than Theorem 7.4.

The integer m_r used in the recursive construction of Δ_r in Theorem 7.4 currently depends on the dimension, which grows very quickly. It is conceivable that for each p there is a uniform bound, independent of r.

Question 7.5. Is there a bound for the integer m_r that depends only on p and not on r?

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