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INDUCING CHARACTERS AND NILPOTENT INJECTORS

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Let G be a finite group and let N be a normal subgroup of G. If G/N is solvable and H/N is a nilpotent injector of G/N, then there exists a canonical basis of the complex space of the class functions of G which vanish off the G-conjugates of H.

1. INTRODUCTION

Let G be a finite group and let $\mathrm{cf}(G)$ be the space of complex class functions of G. Let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G. A subgroup H of G is *good* if there exists a basis $P(G \mid H)$ of the subspace $\mathrm{cf}(H)^G$ of the class functions of G induced from the class functions of H satisfying:

(I) for each $\eta \in P(G \mid H)$, there is $\gamma \in \mathrm{Irr}(H)$ such that $\eta = \gamma^G$; and

(D) for every $\alpha \in \mathrm{Irr}(H)$, we have α^G is a nonnegative linear combination of $P(G \mid H)$.

This basis (if it exists) is unique and does not depend on H but on its G-conjugacy class.

There are examples of good subgroups. For instance, if $H \triangleleft G$, then it easy to show that H is good (although this is already false if $H \triangleleft\triangleleft G$.) If G is p-solvable and H is a p-complement of G, then $P(G \mid H)$ exists and it is the set of projective indecomposable characters by Fong theory. If G is π-separable and H is a Hall π-subgroup of G, then $P(G \mid H)$ also exists by Isaacs π-theory. (It is interesting to notice that, in general, a Sylow p-subgroup of a finite group G need not to be good as shown by A_9 and $p = 3$.) If G/N is π-separable and H/N is a Hall π-subgroup of G/N, we have recently proved that $P(G \mid H)$ exists (see [**6**, Theorem A]). If H is a nilpotent injector of the solvable group G, then $P(G \mid H)$ exists (see [**3**, Theorem (3.1)]).

In this note we find a new class of good subgroups.

THEOREM A. *Let $N \triangleleft G$ and suppose that G/N is a solvable group. If H/N is a nilpotent injector of G/N, then H is good.*

As pointed out in [**5**], good subgroups provide a canonical partition of $\mathrm{Irr}(G)$ which behaves like the Brauer p-blocks of G.

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2. Preliminaries: Good bases

In this section we are going to review the notation and terminology of good bases. We refer the reader to [**4**, Section 2] for more details.

Let G be a finite group and let $\mathrm{cf}(G)$ be the space of complex class functions defined on G. If $H \subseteq G$, let $\mathrm{cf}(H)^G = \{\gamma^G \mid \gamma \in \mathrm{cf}(H)\}$.

Definition 2.1: Let H be a subgroup of G. We say that H is *good* if there exists a basis $P(G \mid H)$ of $\mathrm{cf}(H)^G$ satisfying:

(I) for each $\eta \in P(G \mid H)$, there is $\gamma \in \mathrm{Irr}(H)$ such that $\eta = \gamma^G$; and

(D) for every $\alpha \in \mathrm{Irr}(H)$, we have

$$\alpha^G = \sum_{\eta \in P(G|H)} a_\eta \eta$$

for nonnegative integers a_η.

In this case, we say that $P(G \mid H)$ is the *good basis* of the space $\mathrm{cf}(H)^G$. (Good bases are unique by [**3**, Theorem (2.2)].)

Now, let N be a normal subgroup of G contained in H. If $\theta \in \mathrm{Irr}(N)$, then we write $\mathrm{Irr}(G \mid \theta)$ for the set of irreducible constituents of θ^G. Also, $\mathrm{cf}(G \mid \theta)$ is the $\mathbb{C}$-span of the set $\mathrm{Irr}(G \mid \theta)$. We denote by

$$\mathrm{vcf}(G \mid H, \theta) = \mathrm{cf}(H)^G \cap \mathrm{cf}(G \mid \theta).$$

Let Θ be a complete set of representatives of the orbits of the action of G on $\mathrm{Irr}(N)$. We know that

$$\mathrm{cf}(H)^G = \bigoplus_{\theta \in \Theta} \mathrm{vcf}(G \mid H, \theta)$$

by [**4**, Lemma (2.2)].

We define good bases "over" an irreducible character of a certain normal subgroup.

Definition 2.2: Let $N \triangleleft G$, let $\theta \in \mathrm{Irr}(N)$ and let $N \subseteq H \subseteq G$. A basis $\mathcal{B}$ of $\mathrm{vcf}(G \mid H, \theta)$ is *good* if satisfies the following conditions:

(I) If $\eta \in \mathcal{B}$, then there exists $\alpha \in \mathrm{Irr}(H \mid \theta)$ such that $\alpha^G = \eta$.

(D) If $\gamma \in \mathrm{Irr}(H \mid \theta)$, then $\gamma^G = \sum_{\eta \in \mathcal{B}} a_\eta \eta$ for uniquely determined integers a_η.

Good bases "over" irreducible characters are necessarily unique (by the same argument as in [**3**, Theorem (2.2)]). We shall denote by $P(G \mid H, \theta)$ the unique good basis (if it exists) of $\mathrm{vcf}(G \mid H, \theta)$.

Next is a key definition in order to find good bases for the subspace $\mathrm{cf}(H)^G$.

Definition 2.3: Suppose that $N \triangleleft G$ is contained in $H \subseteq G$. Let $\theta \in \mathrm{Irr}(N)$ and and write $T = I_G(\theta)$ for the stabiliser of θ in G. We say that θ is H-*good* (with respect to G), if for every $g \in G$, we have that $H^g \cap T$ is contained in some T-conjugate of $H \cap T$.

3. Proof of Theorem A

Let G be a solvable group. We recall that a nilpotent injector is a maximal nilpotent subgroup of G containing $\mathbf{F}(G)$. Any two of them are G-conjugate [**2**].

We need the following new property of the nilpotent injectors.

Theorem 3.1. *Let G be a solvable group and let T be a subgroup of G. Then there exists a nilpotent injector H of G such that for every $g \in G$*

$$H^g \cap T \subseteq (H \cap T)^t,$$

for some $t \in T$.

Proof: We argue by induction on $|G|$. Write $F = \mathbf{F}(G)$. We claim that we may assume that $FT = G$. Otherwise, we have that there exists a nilpotent injector J of FT such that for every $x \in FT$

$$J^x \cap T \subseteq (J \cap T)^t,$$

for some $t \in T$. Now, by [**2**, Theorem 2(b)], we know that there exists a nilpotent injector H of G such that

$$H \cap FT = J.$$

Let $g \in G$. We have that $H^g \cap T = H^g \cap FT \cap T$. Now, by [**2**, Theorem 2(c)], there exists $y \in FT$ such that $H^g \cap FT \subseteq J^y$. Then, we have that there exists an element $t \in T$ such that

$$H^g \cap T = H^g \cap FT \cap T \subseteq J^y \cap T \subseteq (J \cap T)^t.$$

Now, since $H \cap FT = J$, it follows that

$$H^g \cap T \subseteq (J \cap T)^t = (H \cap FT \cap T)^t = (H \cap T)^t,$$

as claimed.

Therefore, we assume that $G = FT$. Let H be a nilpotent injector of G. For every $g \in G$, we write $g = ft$ with $f \in F$ and $t \in T$. We have that

$$H^g \cap T = H^t \cap T = (H \cap T)^t$$

and the proof of the theorem is complete. $\square$

Corollary 3.2. *Let N be a normal subgroup of a solvable group G. Suppose that $\theta \in \mathrm{Irr}(N)$ and let $T = I_G(\theta)$ be the stabiliser of θ in G. Then there exists a nilpotent injector H/N of G/N such that θ is H-good.*

Proof: The proof easily follows from Theorem (3.1). $\square$

Now, let $N \triangleleft G$ and let $\theta \in \mathrm{Irr}(G)$ be invariant in G. Under these hypotheses we say that (G, N, θ) is a *character triple*. For the definition and main properties of isomorphisms of character triples we refer the reader to [**1**, Chapter 11].

We are now ready to prove Theorem A.

Theorem 3.3. *Let G/N be a solvable group. If H/N is a nilpotent injector of G/N, then H is good.*

Proof Given $\theta \in \mathrm{Irr}(N)$, by Corollary (3.2) we know that there exists $x \in G$ such that θ is H^x-good. It follows that $\theta^{x^{-1}}$ is H-good. Hence, we may find a complete set Θ of representatives of the orbits of the action of G on $\mathrm{Irr}(N)$ such that each $\theta \in \Theta$ is H-good. Now, we are going to prove that there exists a good basis of $\mathrm{vcf}(G \mid H, \theta)$ for every $\theta \in \Theta$.

We fix $\theta \in \Theta$. Since there is a "Clifford correspondence" for good bases over $\theta \in \Theta$ [**4**, Lemma (2.10)], we may assume that θ is G-invariant. Hence (G, N, θ) is a character triple. By [**1**, Theorem (11.28)], there exists an isomorphic character triple (G^*, N^*, θ^*) such that N^* is a central subgroup of G^*. Since $N^* \subseteq \mathbf{Z}(G^*)$, we have that $\mathbf{F}(G^*/N^*) = \mathbf{F}(G^*)/N^*$. Now, since H^*/N^* is nilpotent if and only if H^* is nilpotent, it easily follows that H^*/N^* is a nilpotent injector of G^*/N^* if and only if H^* is a nilpotent injector of G^*. We know that $P(G^* \mid H^*, \theta^*)$, the good basis of $\mathrm{vcf}(G^* \mid H^*, \theta^*)$ exists by [**3**, Theorem (3.1)] and [**6**, Theorem (2.4)]. Now, by [**6**, Lemma (3.4)], it follows that $P(G \mid H, \theta)$ is the good basis of $\mathrm{vcf}(G \mid H, \theta)$. We conclude that $\bigcup_{\theta \in \Theta} P(G \mid H, \theta)$ is $P(G \mid H)$ by [**4**, Lemma (2.9)]. □

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