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TOTALLY SYMMETRIC SURFACES OF CONSTANT MEAN CURVATURE IN HYPERBOLIC 3-SPACE

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Abstract

We detail a construction of totally symmetric surfaces of constant mean curvature $0 \le H < 1$ in hyperbolic 3-space of sectional curvature -1 via the generalized Weierstrass type representation.

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1. Introduction

It has been recognized that the generalized Weierstrass type representation for constant mean curvature (CMC for short) surfaces in $\boldsymbol{R}^3$ is a powerful tool for producing CMC surfaces with nontrivial topologies [1, 3, 5, 7, 11]. The heart of the construction is a second-order linear ordinary differential equation (ODE) of one complex variable, which is derived from a CMC surface, with an additional 'loop parameter'. More precisely, for the construction of CMC surfaces with nontrivial topologies the monodromy group of the complex linear ODE needs to be a unitary loop subgroup, which will be referred to as the 'unitarizability problem'. The unitarizability problem is an essential part of the construction of CMC surfaces via the generalized Weierstrass type representation.

The Gauss hypergeometric equation is a second-order linear ODE whose solutions have three regular singularities on the extended complex plane $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, and its monodromy group is completely understood. Thus it was natural to expect that one can construct corresponding CMC surfaces which are well defined on the thrice punctured extended complex plane $\hat{\boldsymbol{C}}$. It turns out that such CMC surfaces exist and are called CMC *trinoids* [5, 11]. The asymptotic behaviours of the CMC trinoid around the punctures, which are referred to as the *ends*, had been expected to be geometrically 'nice', since those ends correspond to regular singularities of the Gauss hypergeometric equation. In fact, it is shown that each end of the CMC trinoids is

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asymptotic to half of a CMC surface of revolution [5, 8]. Moreover, the generalized Weierstrass type representation can be generalized for CMC surfaces in S^3, and in $\boldsymbol{H}^3$ with mean curvature $H > 1$, [11]. Similar to the $\boldsymbol{R}^3$ case, there also exist CMC trinoids in S^3, and in $\boldsymbol{H}^3$ with mean curvature $H > 1$, [11].

Recently, in [2, 9], the generalized Weierstrass type representation for CMC surfaces in $\boldsymbol{H}^3$ with mean curvature $0 \leq H < 1$ has been established. It is shown that there also exist CMC surfaces using the Gauss hypergeometric equation [2]. The difference between CMC trinoids in $\boldsymbol{R}^3$ and CMC surfaces in $\boldsymbol{H}^3$ via the Gauss hypergeometric equation is the existence of singular curves. In fact, CMC trinoids in $\boldsymbol{R}^3$ do not have any singularity except the three regular singularities, which correspond to the ends of CMC trinoids, of the Gauss hypergeometric equation. In contrast, those CMC surfaces in $\boldsymbol{H}^3$ have singular curves, where the surfaces tends to infinity, in addition to three regular singularities. Moreover, until now, we have only been able to show the existence of the particular type of such surfaces such that each end is the same type. Therefore, it is natural to call such a surface the *totally symmetric surface*.

This paper gives a detailed computation for the construction of totally symmetric surfaces in $\boldsymbol{H}^3$ with mean curvature $0 \leq H < 1$, [2]. Similar to the $\boldsymbol{R}^3$ case, for the construction of CMC surfaces in $\boldsymbol{H}^3$ with mean curvature $0 \leq H < 1$, a monodromy group of the complex linear ODE needs to be a particular real loop subgroup, which is referred to as the 'realizability problem'. The method for the realizability problem is similar to the method for the unitarizability problem of CMC trinoids in $\boldsymbol{R}^3$; however, the resulting conditions are different [4].

For simplicity of presentation, we assume the following.

(1) The eigenvalues of the monodromy matrices are even in the loop parameter.
(2) The loop parameter is in S^1.

Assumptions (1) and (2) are closely related. If the eigenvalue is odd in the loop parameter, then the loop parameter needs to be chosen in S^r, the circle of radius $0 < r < 1$, to solve the realizability problem. However, for most cases, eigenvalues of monodromy matrices are even with respect to the loop parameter, so these are not severe restrictions.

2. Preliminaries

In this section we give basic notation and results.

2.1. Basic definitions. Let $G^{\boldsymbol{C}}$ be a semisimple complex Lie group. The twisted loop group is defined as

$$\Lambda G^{\boldsymbol{C}}_{\sigma} = \{g : S^1 \to G^{\boldsymbol{C}} \mid g \text{ is continuous and } g(-\lambda) = \sigma g(\lambda)\}, \tag{2.1}$$

where σ is an involution. More strictly, we assume that the coefficients of all $g \in \Lambda G^{\boldsymbol{C}}_{\sigma}$ are in the *Wiener algebra*

$$\mathcal{W} = \left\{ f(\lambda) = \sum_{n\in\boldsymbol{Z}} f_n \lambda^n : S^1 \to \boldsymbol{C};\ \sum_{n\in\boldsymbol{Z}} |f_n| < \infty \right\}.$$

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The Wiener algebra $\mathcal{W}$ is a Banach algebra relative to the norm $\|f\| = \sum |f_n|$, and $\mathcal{W}$ consists of continuous functions. Thus $\Lambda G_\sigma^{\boldsymbol{C}}$ is a Banach Lie group. We denote its Lie algebra by $\Lambda \mathfrak{g}_\sigma^{\boldsymbol{C}}$. Define $\mathcal{W}^+$ and $\mathcal{W}^-$ to be the subalgebras of $\mathcal{W}$ which consists of loops whose Fourier expansions have only nonnegative, respectively nonpositive, powers of λ.

We will need to consider two subgroups of $\Lambda G_\sigma^{\boldsymbol{C}}$, the *twisted plus loop group* and the *minus loop group*. Let $D = \{\lambda \in \boldsymbol{C} \mid |\lambda| < 1\}$ be a radius-one disk, and let $E = \{\lambda \in \boldsymbol{C} \mid 1 < |\lambda|\} \cup \{\infty\}$ the exterior of the radius-one disk. We denote the closure of D and E by $\overline{D}^{\mathrm{cl}}$ and $\overline{E}^{\mathrm{cl}}$, respectively. Moreover, let B be a subgroup of $G^{\boldsymbol{C}}$. Then

$$\begin{aligned} \Lambda_B^+ G_\sigma^{\boldsymbol{C}} &= \{g \in \Lambda G_\sigma^{\boldsymbol{C}} \mid g \text{ can be holomorphically extended to } D \\ &\qquad \text{and takes values in } B \text{ at } \lambda = 0\}, \\ \Lambda_B^- G_\sigma^{\boldsymbol{C}} &= \{g \in \Lambda G_\sigma^{\boldsymbol{C}} \mid g \text{ can be holomorphically extended to } E \\ &\qquad \text{and takes values in } B \text{ at } \lambda = \infty\}. \end{aligned}$$

If $B = \{\mathrm{id}\}$, we write the subscript $*$ instead of B, and if $B = G^{\boldsymbol{C}}$ we abbreviate $\Lambda_B^+ G_\sigma^{\boldsymbol{C}}$ and $\Lambda_B^- G_\sigma^{\boldsymbol{C}}$ by $\Lambda^+ G_\sigma^{\boldsymbol{C}}$ and $\Lambda^- G_\sigma^{\boldsymbol{C}}$, respectively.

2.2. Loop groups factorizations. From now on, we consider only $\Lambda \mathrm{SL}_2 \boldsymbol{C}_\sigma$ as $\Lambda G_\sigma^{\boldsymbol{C}}$, where the involution σ is given by $\mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We introduce some useful notation. For $f \in \mathcal{W}$, the $\sharp$ and $\flat$ operations are defined as follows:

$$f^\sharp = \overline{f(i/\bar{\lambda})}^{-1} \quad \text{and} \quad f^\flat = \overline{f(i/\bar{\lambda})}.$$

For $g \in \Lambda G_\sigma^{\boldsymbol{C}}$, the $\sharp$ operation is defined as follows:

$$g^\sharp = \mathrm{Ad}(\mathcal{R}) \overline{g(i/\bar{\lambda})}^{t-1} \quad \text{with } \mathcal{R} = \begin{pmatrix} 1/\sqrt{i} & 0 \\ 0 & \sqrt{i} \end{pmatrix}.$$

The subgroup ΛG_σ of $\Lambda G_\sigma^{\boldsymbol{C}}$ is defined as

$$\Lambda G_\sigma = \{g \in \Lambda G_\sigma^{\boldsymbol{C}} \mid g^\sharp = g\},$$

which is a real form of $\Lambda G_\sigma^{\boldsymbol{C}}$. It is easy to check that, for $g = (g_{ij}) \in \Lambda G_\sigma^{\boldsymbol{C}}$, the condition $g^\sharp = g$ can be computed via the entries of g as $g_{11} = g_{22}^\flat$ and $g_{21} = -i g_{12}^\flat$.

REMARK 2.1. The loop subgroup ΛG_σ is not defined by any finite-dimensional real form G of $G^{\boldsymbol{C}}$, that is,

$$\Lambda G_\sigma \neq \{g : S^1 \to G \mid g(-\lambda) = \sigma g(\lambda)\}.$$

The real form naturally induces an Iwasawa decomposition of $\Lambda G_\sigma^{\boldsymbol{C}}$.

THEOREM 2.2 [2]. *$\Lambda G_\sigma^{\boldsymbol{C}}$ can be factorized as*

$$\Lambda G_\sigma^{\boldsymbol{C}} = \bigcup_{n=0}^{\infty} \Lambda G_\sigma \omega_n \Lambda^+ G_\sigma^{\boldsymbol{C}},$$

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where ω_n is the middle term of the form $\omega_n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}$ for $n = 2k$ and $\omega_n = \begin{pmatrix} 0 & \lambda^{-n} \\ -\lambda^n & 0 \end{pmatrix}$ for $n = 2k+1$. Moreover, $\Lambda G_\sigma \Lambda^+ G_\sigma^{\boldsymbol{C}} \cup \Lambda G_\sigma \omega_1 \Lambda^+ G_\sigma^{\boldsymbol{C}}$ is the open dense subset of $\Lambda G_\sigma^{\boldsymbol{C}}$, which is called the big cell.

We quote the Birkhoff decomposition for functions, which will be used later.

LEMMA 2.3 [6]. *$h \in \mathcal{W}$ can be factorized as*

$$h(\lambda) = c\lambda^m h_+(\lambda) h_-(\lambda),$$

where $h_\pm \in \mathcal{W}^\pm$ are nonvanishing holomorphic functions on D and E, respectively, with $h_+(0) = h_-(\infty) = 1$, c is a nonzero constant and $m \in \boldsymbol{Z}$.

2.3. Generalized Weierstrass type representation. In this subsection, we briefly explain the generalized Weierstrass type representation for CMC surfaces in $\boldsymbol{H}^3$ with mean curvature $0 \le H < 1$. We refer the reader to [2] for more details.

Step 1. Let η be a holomorphic potential of the form

$$\eta = \sum_{j=-1}^{\infty} \eta_j \lambda^j,$$

where η_{odd} (respectively η_{even}) is an off-diagonal (respectively diagonal) $\mathfrak{sl}_2\boldsymbol{C}$-valued holomorphic 1-form defined on $\Sigma \subset \boldsymbol{C}$, and the upper right entry of η_{-1} does not have a zero on Σ.

Step 2. Solve the ODE

$$dC = C\eta,$$

for some initial condition $C(z_*)$.

Step 3. Perform the Iwasawa decomposition in Theorem 2.2 on the big cell:

$$C = FV_+, \quad F \in \Lambda G_\sigma \quad \text{and} \quad V_+ \in \Lambda^+ G_\sigma^{\boldsymbol{C}}$$

or

$$C = F\omega_1 V_+, \quad F \in \Lambda G_\sigma \quad \text{and} \quad V_+ \in \Lambda^+ G_\sigma^{\boldsymbol{C}},$$

where $\omega_1 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}$.

Step 4. Plug F or $F\omega_1$ into the Sym formula

$$\psi = F \begin{pmatrix} e^{-q/2} & 0 \\ 0 & e^{q/2} \end{pmatrix} F^* \Big|_{\lambda = e^{-q/2}} \tag{2.2}$$

or

$$\psi = -F\omega_1 \begin{pmatrix} e^{-q/2} & 0 \\ 0 & e^{q/2} \end{pmatrix} \omega_1^* F^* \Big|_{\lambda = e^{-q/2}}, \tag{2.3}$$

respectively, where X^* denotes $\bar{X}^t$. Then ψ is a constant mean curvature immersion in $\boldsymbol{H}^3$ with mean curvature $H = \tanh q$.

REMARK 2.4. As explained in [2, Section 9.4], the resulting immersion ψ by the Sym formula (2.2) for F can be extended meromorphically to $\boldsymbol{C}^2$. Then restricting ψ again

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to the other component of the big cell, the Sym formula ψ with $F\omega_1$ is obtained as in (2.3), and the resulting immersion can be shown in $\boldsymbol{H}^3$ of the lower half.

2.4. Monodromy problem. It is known that, for a CMC surface defined on a noncompact Riemann surface Σ, there exists a holomorphic potential η which is well defined on Σ, [2, Section 8.2]. Thus constructing a noncompact CMC surface, one can chose a holomorphic potential η which is well defined on a noncompact Riemann surface Σ. It is well known that solutions of the ODE $dC = C\eta$ are not well defined on Σ, that is, solutions have monodromy matrices. Then the following lemma holds.

LEMMA 2.5. *Let γ be an element of $\pi_1(\Sigma)$, the fundamental group of Σ, and $M = (\gamma^* C)C^{-1}$ the monodromy matrix. Assume that:*

(1) $M \in \Lambda G_\sigma$;
(2) $M|_{\lambda = e^{-q/2}} = \mathrm{id}$ *or* $-\mathrm{id}$.

Then the resulting immersion ψ obtained by the generalized Weierstrass type representation has γ as an element of the fundamental group of ψ.

PROOF. From the first condition, one can easily see that the Iwasawa decomposition for $\gamma^* C$ can be computed as

$$\gamma^* C = (MF)\omega_n V_+, \quad MF \in \Lambda G_\sigma \quad \text{and} \quad V_+ \in \Lambda^+ G_\sigma^{\boldsymbol{C}},$$

where $C = F\omega_n V_+$ is the Iwasawa decomposition of C. Thus $\gamma^* F = MF$ and $\gamma^*\psi = M\psi M^*|_{\lambda = e^{-q/2}}$. Therefore, the second condition implies that $\gamma^*\psi = \psi$. □

2.5. Eigenvalues of surfaces of revolution. Set

$$\eta = \frac{1}{z} A\, dz \quad \text{with } A = \begin{pmatrix} c & a\lambda^{-1} + b\lambda \\ -a\lambda + b\lambda^{-1} & -c \end{pmatrix},$$

where $a, b, c \in \boldsymbol{R}$ and $c^2 - a^2 + b^2 + ab(e^q - e^{-q}) = 1/4$. Then the ODE $dC = C\eta$ with the initial condition $C(z_* = 1) = \mathrm{id}$ can be solved explicitly as $C = \exp(\log zA)$. The monodromy matrix around $z = 0$ is $M = \exp(2\pi i A) \in \Lambda G_\sigma$. Then a straightforward computation shows that the eigenvalues of M are

$$\mu_\pm = \exp(\pm 2\pi i \sqrt{-\det A}) = \exp(\pm 2\pi i \sqrt{c^2 - a^2 + b^2 + ab(\lambda^{-2} - \lambda^2)}). \quad (2.4)$$

Thus $\mu_\pm|_{\lambda = e^{-q/2}} = -1$ and $M|_{\lambda = e^{-q/2}} = -\mathrm{id}$. From Lemma 2.5, the resulting surface ψ closes up.

Then the following lemma holds.

LEMMA 2.6. *Let $\mu_\pm$ be eigenvalues defined in (2.4). Assume that $c^2 - a^2 + b^2 \neq 0$. Then $\mu_\pm$ satisfy*

$$\mu_\pm^\sharp = \mu_\pm \quad \text{on } S^1.$$

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PROOF. Since the $c^2 - a^2 + b^2 < 0$ case is similar, we consider only the $c^2 - a^2 + b^2 > 0$ case. A direct computation shows that

$$\begin{aligned} -\det A &= c^2 - a^2 + b^2 + ab(\lambda^{-2} - \lambda^2) \\ &= p + 2iab \sin\theta, \end{aligned}$$

where $p = c^2 - a^2 + b^2 > 0$ and $\lambda^{-2} = e^{i\theta} \in S^1$. Since $-\det A$ is never zero by the assumption, $\sqrt{-\det A}$ is well defined on S^1. Set $X = \sqrt{-\det A}$. The positivity of p implies that X^2 takes values in the right half space $\check{\boldsymbol{H}}$ of $\boldsymbol{C}$, and X also takes values in $\check{\boldsymbol{H}}$. It is easy to check that $X^{2\flat} = X^2$ on S^1, thus $X = X^\flat$ or $X = -X^\flat$. Since $X^\flat$ takes values in $\check{\boldsymbol{H}}$, however, we conclude that $X^\flat = X$. Therefore,

$$\mu_\pm^\sharp = \exp(\pm 2\pi i X^\flat) = \exp(\pm 2\pi i X) = \mu_\pm.$$

This completes the proof. □

REMARK 2.7. Lemma 2.6 does not hold for the $c^2 - a^2 + b^2 = 0$ case, since $\det A$ has a zero on S^1. A similar case is the round cylinder, as a CMC surface in $\boldsymbol{R}^3$, [4].

3. Realizability of loops

3.1. Realizability of one loop.

DEFINITION 3.1. Let L be an element in $\Lambda G_\sigma^{\boldsymbol{C}}$. Then L is realizable by S or dressing, if $SLS^{-1} \in \Lambda G_\sigma$ for some $S \in \Lambda G_\sigma^{\boldsymbol{C}}$, that is, $(SLS^{-1})^\sharp = SLS^{-1}$.

THEOREM 3.2. *Let $L \in \Lambda G_\sigma^{\boldsymbol{C}} \setminus \{\pm \mathrm{id}\}$ be a holomorphic loop around S^1 such that $TLT^{-1} = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \in \Lambda G_\sigma$ for some $T \in \Lambda G_\sigma^{\boldsymbol{C}}$. Moreover, assume that L is realizable by $S \in \Lambda G_\sigma^{\boldsymbol{C}}$. Then S has the form $S = U\omega_n G_+ T$, where $U \in \Lambda G_\sigma$, $\omega_n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}$ for $n = 2k$ or $\omega_n = \begin{pmatrix} 0 & \lambda^{-n} \\ -\lambda^n & 0 \end{pmatrix}$ for $n = 2k+1$ and a diagonal $G_+ \in \Lambda^+ G_\sigma^{\boldsymbol{C}}$.*

PROOF. First, we note that the assumptions imply that $v - v^{-1}$ equals zero at finitely many points at most and $v^\flat = v^{-1}$. Let $ST^{-1} = U\omega_n G_+$ be the Iwasawa decomposition of ST^{-1}, that is, $U \in \Lambda G_\sigma$, $\omega_n = \begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{pmatrix}$ for $n = 2k$ or $\omega_n = \begin{pmatrix} 0 & \lambda^{-n} \\ -\lambda^n & 0 \end{pmatrix}$ for $n = 2k+1$ and $G_+ \in \Lambda^+ G_\sigma^{\boldsymbol{C}}$. Set $G_+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We show that $b = c = 0$. A straightforward computation shows that

$$\begin{aligned} U^{-1}SLS^{-1}U &= \omega_n G_+ \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} G_+^{-1} \omega_n^{-1}, \\ &= \omega_n \begin{pmatrix} v + bc(v - v^{-1}) & ab(v^{-1} - v) \\ cd(v - v^{-1}) & v^{-1} + bc(v^{-1} - v) \end{pmatrix} \omega_n^{-1}. \end{aligned}$$

We recall that for a loop $X = (x_{ij}) \in \Lambda G_\sigma^{\boldsymbol{C}}$, $X \in \Lambda G_\sigma \Leftrightarrow x_{22} = x_{11}^\flat$ and $x_{21} = -ix_{12}^\flat$, and $\omega_n^\sharp = \pm\omega_n$. Thus $U^{-1}SLS^{-1}U \in \Lambda G_\sigma$ is equivalent to

$$bc = (bc)^\flat, \quad cd = -i(ab)^\flat \quad \text{on } S^1. \tag{3.1}$$

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Since $x^\flat = \overline{x(i/\bar{\lambda})}$ and b, c are holomorphic on D, bc has a holomorphic extension to E. Therefore bc is holomorphic on $\hat{\boldsymbol{C}}$ except S^1. However, bc is continuous on S^1, thus bc can be extended holomorphically to $\hat{\boldsymbol{C}}$ [10, p. 289]. Therefore bc is constant. $bc|_{\lambda=0} = 0$ implies that $b = 0$ or $c = 0$. Also, $ad - bc = 1$ implies that $ad = 1$. Therefore the second equation in (3.1) implies that $b = c = 0$. This completes the proof. □

Moreover, we have the following lemma.

LEMMA 3.3. *Let $L = (l_{ij}) \in \Lambda G_\sigma^{\boldsymbol{C}}$ be holomorphic around S^1. Assume that $l_{12} = 0$ and $l_{22} = l_{11}^\flat$. Then the following statements hold.*

(1) *If $l_{11} \equiv l_{22}$, then L is realizable via dressing if and only if $L = \pm\mathrm{id}$.*
(2) *If $l_{11} \not\equiv l_{22}$, then L is realizable via dressing if $l_{21}/(l_{22} - l_{11})$ does not have any pole on S^1.*

PROOF. (1) From $l_{11} \equiv l_{22}$ and $l_{22} = l_{11}^{-1}$, we have $l_{11} = l_{22} = \pm 1$. Thus the claim follows.

(2) We set

$$T = \begin{pmatrix} 1 & 0 \\ l_{21}/(l_{22} - l_{11}) & 1 \end{pmatrix}.$$

Since $l_{21}/(l_{22} - l_{11})$ does not have any pole on S^1, then $T \in \Lambda G_\sigma^{\boldsymbol{C}}$. It is easy to see that $TLT^{-1} = \left(\begin{smallmatrix} l_{11} & 0 \\ 0 & l_{22} \end{smallmatrix}\right) \in \Lambda G_\sigma$, since l_{11} and l_{22} are even in λ and $l_{22} = l_{11}^\flat$. □

REMARK 3.4.

(1) In [4, Proposition 2.32], it is shown that the condition (2) in Lemma 3.3 also gives the necessary condition of the unitarizability of the lower triangular loop. To prove this, the reality of bc of G_+ as in Theorem 3.2 is important [4, proof of Theorem 2.16(iii)]. Since bc of G_+ in Theorem 3.2 is not real for our case, it is not clear that it also gives the necessary condition.
(2) If L is the upper triangular loop, that is, $l_{21} = 0$ in Lemma 3.3, then the statements also hold under the replacement of $l_{21}/(l_{22} - l_{11})$ by $-l_{12}/(l_{22} - l_{11})$.

3.2. Simultaneous realization.

LEMMA 3.5. *Let h be a finite, nonvanishing function on S^1. Moreover, assume that $h = h^\flat$ on S^1. Then $h = \pm k_+^n (k_+^n)^\flat$ for some $k_+ \in \mathcal{W}^+$ which is finite, non-vanishing on S^1 and nonvanishing holomorphic on D. Moreover, if h is even, then k_+ is even.*

PROOF. Let $h = c\lambda^m h_+ h_-$ be the Birkhoff decomposition on S^1 in Lemma 2.3, where $c \in \boldsymbol{C} \setminus \{0\}$, $m \in \boldsymbol{Z}$, and $h_\pm \in \mathcal{W}^\pm$ are nonvanishing holomorphic on D and E, respectively. Since $h = h^\flat$, we have $m = 0$, $c \in \boldsymbol{R} \setminus \{0\}$ and $h_- = h_+^\flat$. Since h_+ never vanishes on $\overline{D}^{\mathrm{cl}}$, we can take $k_+ = (c^{1/2} h_+)^{1/n}$ if $c > 0$ and $k_+ = ((-c)^{1/2} h_+)^{1/n}$ if $c < 0$. Thus, $h = k_+^n (k_+^n)^\flat$ if $c > 0$ and $h = -k_+^n (k_+^n)^\flat$ if $c < 0$. Uniqueness of the Birkhoff decomposition implies that if h is even, then k_+ is also even. This completes the proof. □

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THEOREM 3.6. *Let $L_j \in \Lambda G_\sigma^{\mathbf{C}} \setminus \{\pm \mathrm{id}\}$, $(j = 1, 2)$, be holomorphic loops around S^1 such that*

$$T_j L_j T_j^{-1} = \begin{pmatrix} v_j & 0 \\ 0 & v_j^{-1} \end{pmatrix} \in \Lambda G_\sigma$$

for some $T_j \in \Lambda G_\sigma^{\mathbf{C}}$, $(j = 1, 2)$. Set

$$T_1 T_2^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Assume that none of a, b, c, d is identically zero on S^1. Then L_1 and L_2 are simultaneously realizable via dressing if and only if

$$\frac{d}{a^\flat} = \pm f f^\flat, \quad i\frac{c}{b^\flat} = \pm g g^\flat,$$

for some $f, g \in \mathcal{W}^+$ such that f, g are finite, nonvanishing on S^1 and nonvanishing holomorphic on D, the signs in the left- and right-hand side of the above equation must be in the same order.

PROOF. ($\Rightarrow$) Let S be a loop such that $S L_j S^{-1} \in \Lambda G_\sigma (j = 1, 2)$. Then from Theorem 3.2, S can be decomposed as

$$S = U_1 \omega_{1,n} G_1 T_1 = U_2 \omega_{2,n} G_2 T_2,$$

where

$$U_j \in \Lambda G_\sigma, \quad G_j = \begin{pmatrix} k_j & 0 \\ 0 & k_j^{-1} \end{pmatrix} \in \Lambda^+ G_\sigma^{\mathbf{C}} \tag{3.2}$$

and $\omega_{j,n} = \left(\begin{smallmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{smallmatrix}\right)$ for $n = 2k$ or $\omega_{j,n} = \left(\begin{smallmatrix} 0 & \lambda^{-n} \\ -\lambda^n & 0 \end{smallmatrix}\right)$ for $n = 2k + 1$ $(j = 1, 2)$. Since

$$U_1^{-1} U_2 = \omega_{1,n} G_1 T_1 T_2^{-1} G_2^{-1} \omega_{2,n}^{-1} \in \Lambda G_\sigma$$

and $\omega_{j,n}^\sharp = \pm \omega_{j,n}$,

$$d k_1^{-1} k_2 = \pm (a k_1 k_2^{-1})^\flat, \quad c k_1^{-1} k_2^{-1} = \mp i (b k_1 k_2)^\flat,$$

where the signs in the left- and right-hand side of the above equation must be in the same order. This is equivalent to

$$\frac{d}{a^\flat} = \pm \frac{k_1 k_1^\flat}{k_2 k_2^\flat}, \quad i\frac{c}{b^\flat} = \pm k_1 k_1^\flat k_2 k_2^\flat.$$

Therefore $d/a^\flat$ and $ic/b^\flat$ have the form $d/a^\flat = \pm f f^\flat$ and $ic/b^\flat = \pm g g^\flat$, where $f = k_1/k_2$, $g = k_1 k_2 \in \mathcal{W}^+$. From (3.2), f and g are finite, nonvanishing on S^1 and nonvanishing holomorphic on D.

($\Leftarrow$) Conversely, assume that $d/a^\flat = \pm f f^\flat$ and $ic/b^\flat = \pm g g^\flat$, where $f, g \in \mathcal{W}^+$ are finite, nonvanishing on S^1 and nonvanishing holomorphic on D. First, we can rephrase $f f^\flat$ and $g g^\flat$ as

$$f f^\flat = \pm \frac{1}{m} \hat{f} \hat{f}^\flat, \quad g g^\flat = \pm n \hat{g} \hat{g}^\flat$$

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such that

$$\hat{f}|_{\lambda=0} = \hat{f}^\flat|_{\lambda=\infty} = \hat{g}|_{\lambda=0} = \hat{g}^\flat|_{\lambda=\infty} = 1,$$

where n and m are positive constants. Since $\hat{f}$, $\hat{g}$ are nonvanishing functions on $\overline{D}^{\mathrm{cl}}$, we have $\hat{f} = \exp h_+$, $\hat{g} = \exp l_+$ for some $h_+, l_+ \in \mathcal{W}^+$. Thus the square roots of $\hat{f}$ and $\hat{g}$ are well defined. Set

$$k_1 = \left(\frac{n}{m}\right)^{1/4} (\hat{f}\hat{g})^{1/2}, \quad k_2 = (nm)^{1/4}\left(\frac{\hat{g}}{\hat{f}}\right)^{1/2}.$$

It is easy to see that k_1, k_2 are finite, nonvanishing on S^1 and nonvanishing holomorphic on D. Moreover, $k_1|_{\lambda=0} > 0$, $k_2|_{\lambda=0} > 0$. Set

$$K_j = \begin{pmatrix} k_j & 0 \\ 0 & k_j^{-1} \end{pmatrix} \quad (j = 1, 2)$$

and $\omega = \mathrm{id}$ for $d/a^\flat = ff^\flat$ and $ic/b^\flat = gg^\flat$, or $\omega = \left(\begin{smallmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{smallmatrix}\right)$ for $d/a^\flat = -ff^\flat$ and $ic/b^\flat = -gg^\flat$. Then a straightforward computation shows that

$$\omega K_1 T_1 T_2^{-1} K_2^{-1} = \omega \begin{pmatrix} ak_1k_2^{-1} & bk_1k_2 \\ ck_1^{-1}k_2^{-1} & dk_1^{-1}k_2 \end{pmatrix}.$$

It is easy to check that $dk_1^{-1}k_2 = \pm(ak_1k_2^{-1})^\flat$ and $ck_1^{-1}k_2^{-1} = \mp i(bk_1k_2)^\flat$ hold. Since $(k_1k_1^\flat)^2 = icd/(ab)^\flat$ and $(k_2k_2^\flat)^2 = i(ac)^\flat/(bd)$ on S^1, k_1 and k_2 are even in λ (see Lemma 3.5). Thus $G_j = K_j \in \Lambda^+ G_\sigma^{\mathbf{C}}$. By using $\omega^\sharp = \pm\omega$, we obtain $U = \omega G_1 T_1 T_2^{-1} G_2^{-1} \in \Lambda G_\sigma$. Therefore, L_1 and L_2 are simultaneously realizable by the dressing $S = \omega G_1 T_1 = U G_2 T_2$. □

COROLLARY 3.7. *Let*

$$L_1 = \begin{pmatrix} v_1 & 0 \\ p & v_1^{-1} \end{pmatrix}, \; L_2 = \begin{pmatrix} v_2 & q \\ 0 & v_2^{-1} \end{pmatrix} \in \Lambda G_\sigma^{\mathbf{C}}.$$

Assume that $v_j^\flat = v_j^{-1}$, v_j, p and q are holomorphic around S^1 and none of $v_j - v_j^{-1}$, p and q is identically zero on S^1. Then L_1 and L_2 are simultaneously realizable if the following three conditions hold:

(1) $\frac{p}{v_1^{-1}-v_1}$, $\frac{-q}{v_2^{-1}-v_2}$ *are finite on S^1;*

(2) $\frac{pq}{(v_1^{-1}-v_1)(v_2^{-1}-v_2)} + 1 = \pm ff^\flat$ *for some $f \in \mathcal{W}^+$ which is finite, nonvanishing on S^1 and nonvanishing holomorphic on D;*

(3) $-i\frac{p}{q^\flat}\left(\frac{v_2^{-1}-v_2}{v_1^{-1}-v_1}\right) = \pm gg^\flat$ *for some $g \in \mathcal{W}^+$ which is finite, nonvanishing on S^1 and nonvanishing holomorphic on D. Here the sign is chosen in same order as in (2).*

PROOF. From Lemma 3.3 and Remark 3.4, the lower and upper triangular loops are realizable if $p/(v_1^{-1} - v_1)$ and $-q/(v_2^{-1} - v_2)$ are finite on S^1, which is satisfied by

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the first condition. Set

$$T_1 = \begin{pmatrix} 1 & 0 \\ \dfrac{p}{v_1^{-1} - v_1} & 1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & \dfrac{-q}{v_2^{-1} - v_2} \\ 0 & 1 \end{pmatrix}.$$

It is clear that $T_1, T_2 \in \Lambda G_\sigma^{\boldsymbol{C}}$ and

$$T_j L_j T_j^{-1} = \begin{pmatrix} v_j & 0 \\ 0 & v_j^{-1} \end{pmatrix}.$$

A straightforward computation shows that

$$T_1 T_2^{-1} = \begin{pmatrix} 1 & \dfrac{q}{v_2^{-1} - v_2} \\ \dfrac{p}{v_1^{-1} - v_1} & \dfrac{pq}{(v_1^{-1} - v_1)(v_2^{-1} - v_2)} + 1 \end{pmatrix}.$$

By Theorem 3.6, L_1 and L_2 are simultaneously realizable if and only if the second and third conditions in the corollary are satisfied. □

REMARK 3.8. Condition (3) in Corollary 3.7 can be rephrased as

$$-i\,\frac{p}{q^\flat}\left(\frac{v_2^{-1} - v_2}{v_1^{-1} - v_1}\right) = \frac{-pq}{(v_1^{-1} - v_1)(v_2^{-1} - v_2)} \cdot \frac{(v_2^{-1} - v_2)(v_2^{-1} - v_2)^\flat}{iqq^\flat}.$$

Since q is an odd function, $(iqq^\flat)^\flat = iqq^\flat$ holds. Then using Lemma 3.5,

$$\frac{(v_2^{-1} - v_2)(v_2^{-1} - v_2)^\flat}{iqq^\flat} = \pm ll^\flat, \tag{3.3}$$

for some $l \in \mathcal{W}^+$ which is finite, nonvanishing on S^1 and nonvanishing holomorphic on D. Therefore, the third condition in Corollary 3.7 can be replaced by:

(3)′ $\frac{\pm pq}{(v_1^{-1} - v_1)(v_2^{-1} - v_2)} = \pm gg^\flat$ for some $g \in \mathcal{W}^+$ which is finite, non-vanishing on S^1 and non-vanishing holomorphic on D.

Here, the sign is determined from (3.3).

4. Totally symmetric surfaces

In this section, we show the existence of the totally symmetric CMC surfaces in $\boldsymbol{H}^3$ with mean curvature $0 \leq H < 1$.

THEOREM 4.1. *Let η be the following holomorphic potential:*

$$\eta = -\begin{pmatrix} 0 & \lambda^{-1} \\ \lambda X^2 & 0 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda X^2 & 0 \end{pmatrix} \frac{dz}{z-1} + \begin{pmatrix} 0 & 0 \\ \lambda(X^2 - 1/4) & 0 \end{pmatrix} dz, \tag{4.1}$$

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where $a, b \in \boldsymbol{R}$, $X = \sqrt{b^2 - a^2 + ab(\lambda^2 - \lambda^{-2})}$ *with* $b^2 - a^2 + ab(e^q - e^{-q}) = 1/4$, $q \in \boldsymbol{R}_{\geqq 0}$. *Assume that*

$$b^2 \not\equiv a^2 \bmod \boldsymbol{S} \quad \textit{and} \quad -1 + 4\sin^2(\pi X) \neq 0 \quad \textit{on } \lambda \in S^1, \tag{4.2}$$

where $\boldsymbol{S} = \{y \in \boldsymbol{N} \mid x^2 = y \text{ for some } x \in \boldsymbol{Z}\}$. *Then there exists an immersion, possibly singular, from* $\hat{\boldsymbol{C}} \setminus \{0, 1, \infty\}$ *into* $\boldsymbol{H}^3$ *with constant mean curvature* $0 \leq H = \tanh q < 1$ *via the generalized Weierstrass type representation.*

PROOF. Set

$$\eta = \begin{pmatrix} 0 & \eta_{12} \\ \eta_{21} & 0 \end{pmatrix} dz \quad \text{and} \quad C = \begin{pmatrix} y_1'/\eta_{12} & y_1 \\ y_2'/\eta_{12} & y_2 \end{pmatrix},$$

where y_1 and y_2 are the fundamental solutions of

$$y'' - \frac{\eta_{12}'}{\eta_{12}} y' - \eta_{12}\eta_{21} y = 0. \tag{4.3}$$

Then it is easy to see that C satisfies $dC = C\eta$. Since

$$\eta_{12} = -\frac{\lambda^{-1}}{z} + \frac{\lambda^{-1}}{z-1} \quad \text{and} \quad \eta_{21} = \frac{-\lambda X^2}{z} + \frac{\lambda X^2}{z-1} + \lambda(X^2 - 1/4),$$

we compute

$$-\frac{\eta_{12}'}{\eta_{12}} = \frac{1}{z} + \frac{1}{z-1}$$

and

$$-\eta_{12}\eta_{21} = -\frac{X^2}{z^2} - \frac{X^2}{(z-1)^2} + \left(\frac{1}{4} + X^2\right)\left(-\frac{1}{z} + \frac{1}{z-1}\right).$$

Setting $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$, $a_1 = a_2 = 1$, $a_3 = 0$, $b_1 = b_2 = -X^2$, $b_3 = 1/4 - X^2$ and $c_2 = -c_1 = 1/4 + X^2$,

$$-\frac{\eta_{12}'}{\eta_{12}} = \sum_{j=1}^{2} \frac{a_j}{z - z_j}, \quad -\eta_{12}\eta_{21} = \sum_{j=1}^{2} \left\{ \frac{b_j}{(z - z_j)^2} + \frac{c_j}{z - z_j} \right\}$$

and

$$\sum_{j=1}^{2} c_j = 0, \quad \sum_{j=1}^{3} a_j = 2, \quad b_3 = \sum_{j=1}^{2} (b_j + c_j z_j).$$

These are equations given in [5, Section 3.3]. It is easy to see that (4.3) is equivalent to the Gauss hypergeometric equation. Let γ_1, γ_2 and γ_3 be the simple closed loops around $z = z_1$, z_2 and z_3 from some base point z_*, respectively. Moreover, let M_1, M_2 and M_3 be monodromy matrices corresponding to γ_1, γ_2 and γ_3, respectively, that is, $\gamma_j^* C = M_j C$ ($j = 1, 2, 3$). It is well known that

$$M_1 M_2 M_3 = \text{id},$$

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since $\gamma_1\gamma_2\gamma_3 = 1$. Following the discussion in [5, Section 3.7], the monodromy matrices for a suitable fundamental system of (4.3) can be computed explicitly as

$$M_1 = \begin{pmatrix} e^{2\pi i r_{1,+}} & 0 \\ (e^{2\pi i r_{1,+}} - e^{2\pi i r_{1,-}})d_1\lambda^{-1} & e^{2\pi i r_{1,-}} \end{pmatrix},$$
$$M_2 = \begin{pmatrix} e^{2\pi i r_{2,-}} & (e^{2\pi i r_{2,+}} - e^{2\pi i r_{2,-}})d_2\lambda \\ 0 & e^{2\pi i r_{2,+}} \end{pmatrix},$$

where

$$r_{j,\pm} = \tfrac{1}{2}\{1 - a_j \pm \sqrt{(1-a_j)^2 - 4b_j}\}$$

and

$$d_1 = \frac{\Gamma(1-\delta_1)\Gamma(\delta_2)}{\Gamma(\delta_2-\alpha)\Gamma(\delta_2-\beta)}, \quad d_2 = \frac{\Gamma(1-\delta_2)\Gamma(\delta_1)}{\Gamma(\delta_1-\alpha)\Gamma(\delta_1-\beta)},$$

with

$$\begin{aligned} \delta_1 &= 1 + \sqrt{(1-a_1)^2 - 4b_1} = 1 + 2X, \quad \delta_2 = 1 + \sqrt{(1-a_2)^2 - 4b_2} = 1 + 2X, \\ \alpha &= \tfrac{1}{2}\{\delta_1 + \sqrt{(1-a_2)^2 - 4b_2} + \sqrt{(1-a_3)^2 - 4b_3}\} = \tfrac{1}{2}(1+6X), \\ \beta &= \tfrac{1}{2}\{\delta_1 + \sqrt{(1-a_2)^2 - 4b_2} - \sqrt{(1-a_3)^2 - 4b_3}\} = \tfrac{1}{2}(1+2X). \end{aligned}$$

Since $\delta_1 = \delta_2$ and $r_{1,\pm} = r_{2,\pm}$,

$$d = d_1 = d_2 = \frac{-1}{2\sin \pi X}, \quad e^{2\pi i r_\pm} = e^{2\pi i r_{1,\pm}} = e^{2\pi i r_{2,\pm}}.$$

The first condition in Corollary 3.7 can be computed as

$$\frac{p}{v_1^{-1} - v_1} = -d\lambda^{-1}, \quad \frac{-q}{v_2^{-1} - v_2} = -d\lambda.$$

Since d is finite on S^1 by the assumption, the first condition is satisfied.

The third condition in Corollary 3.7 can be computed as

$$-i\frac{p}{q^\flat}\left(\frac{v_2^{-1} - v_2}{v_1^{-1} - v_1}\right) = 1.$$

Thus the third condition is clearly satisfied.

Since $X^\flat = X$ (see the proof of Lemma 2.6), we see that $d^\flat = d$. A direct computation shows that the second condition can be rephrased as

$$\frac{pq}{(v_1^{-1} - v_1)(v_2^{-1} - v_2)} + 1 = -d^2 + 1 = \frac{-1 + 4\sin^2 \pi X}{4\sin^2 \pi X}. \tag{4.4}$$

From the assumption, the denominator and numerator of (4.4) never vanish on S^1. Therefore, one can rephrase (4.4) as

$$-d^2 + 1 = h^2 \quad \text{with } h = \frac{\sqrt{-1 + 4\sin^2 \pi X}}{2\sin \pi X}.$$

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Since h is a finite, nonvanishing function on S^1, using the Birkhoff decomposition in Lemma 3.5 for h, we have $h = \epsilon k_+ k_+^\flat$, where $\epsilon \in \{\pm 1\}$ and k_+ is an element in $\mathcal{W}^+$ which is finite and nonvanishing on S^1 and nonvanishing holomorphic on D. Thus

$$-d^2 + 1 = h^2 = k_+^2 k_+^{\flat 2}.$$

Finally, setting $f = k_+^2$, we have $-d^2 + 1 = f f^\flat$. Thus the second condition is satisfied, and from Corollary 3.7, M_1 and M_2 can be simultaneously realizable via some dressing $S \in \Lambda G_\sigma^{\boldsymbol{C}}$.

Let $S \in \Lambda G_\sigma^{\boldsymbol{C}}$ be the dressing which realizes M_1 and M_2 simultaneously. It is easy to see that M_3 is also realizable by S, since $M_1 M_2 M_3 = \mathrm{id}$. Then from Theorem 3.2, $S M_j S^{-1} \in \Lambda G_\sigma$ $(j = 1, 2)$ can be computed as

$$S M_j S^{-1} = U_j G_{j+} \begin{pmatrix} e^{2\pi i r_+} & 0 \\ 0 & e^{2\pi i r_-} \end{pmatrix} G_{j+}^{-1} U_j^{-1},$$

where $U_j \in \Lambda G_\sigma$ and $G_{j+} \in \Lambda^+ G_\sigma^{\boldsymbol{C}}$. From the assumption, $S M_j S^{-1}|_{\lambda = e^{-q/2}} = -\mathrm{id}$, $(j = 1, 2)$ and $S M_3 S^{-1}|_{\lambda = e^{-q/2}} = \mathrm{id}$. Therefore, from Lemma 2.5, γ_1, γ_2 and γ_3 are elements of the fundamental group of the resulting CMC immersion ψ. This completes the proof. □

REMARK 4.2.

(1) It is known that CMC immersions from $\hat{\boldsymbol{C}} \setminus \{0, 1, \infty\}$ into $\boldsymbol{R}^3$ can be constructed from potentials η of the form (4.1). However, in this case a and b need to be chosen differently; see [5, 11].

(2) It could be possible to construct nontotally symmetric trinoids from potentials η of the form (4.1) by choosing the different X in the coefficient matrix of dz/z, $dz/(1 - z)$ and dz, respectively.

(3) Since the Iwasawa decomposition Theorem 2.2 is not global in general, the CMC surfaces constructed by this method have singularities, where the surfaces tend to infinity. These singularities are different from the three regular singularities of the Gauss hypergeometric equation.

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