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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR SUBLINEAR ELLIPTIC PROBLEMS

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Abstract. We study the following nonlinear Dirichlet boundary value problem:

$$-\Delta u = g(x, u), \quad u \in H_0^1(\Omega),$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with a smooth boundary $\partial\Omega$ and $g \in \mathbf{C}(\Omega \times \mathbb{R})$ is a function satisfying $\lim_{|t|\to 0} \frac{g(x,t)}{t} = \infty$ for all $x \in \Omega$. Under appropriate assumptions, we prove the existence of infinitely many solutions when $g(x, t)$ is not odd in t.

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1. Introduction. We study the following nonlinear Dirichlet boundary value problem:

$$-\Delta u = g(x, u), \quad u \in H_0^1(\Omega), \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with a smooth boundary $\partial\Omega$ and $g \in \mathbf{C}(\Omega \times \mathbb{R})$ is a function satisfying $\lim_{|t|\to 0} \frac{g(x,t)}{t} \to \infty$ for all $x \in \Omega$. This kind of problem arises in many physical and mechanical problems, and was investigated by several authors (see [**1**, **4**, **7**, **8**, **9**, **11**, **12**]). In [**1**], Brezis and Ambrosetti considered the problem

$$-\Delta u = \mu|u|^{q-1}u + \nu|u|^{p-1}u, \quad u \in H_0^1(\Omega), \tag{1.2}$$

where $0 < q < 1 < p < 2^* - 1$ but $0 < \mu \ll \nu = 1$. In [**4**], Bartsh and Willem established the existence of infinitely many solutions of Problem (1.2) for every $\mu > 0$ and $\nu \in \mathbb{R}$. In [**7**], the conditions of $\nu = 1, 0 < q < 1, p = 2^* - 1$ have been considered by Garcia and Peral. In these papers, oddness of $g(x, t)$ in t plays a crucial role to ensure the existence of infinitely many solutions and the global property of $g(x, t)$ was used in an essential way to derive multiplicity results of solutions with negative energy.

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In 2001, Wang [**11**] considered the problem

$$-\Delta u = \lambda |u|^{q-1}u + f(x,u), \quad u \in H_0^1(\Omega), \tag{1.3}$$

where $\lambda > 0$, $0 < q < 1$, $f(x,t) \in \mathbf{C}(\Omega \times \mathbb{R}, \mathbb{R})$ is odd in t for $|t|$ small and

$$f(x,t) = o(|t|^q), \quad \text{as} \quad |t| \to 0, \qquad \text{uniformly in } x \in \Omega.$$

The oddness of $f(x,t)$ in t is still indispensable despite there is no condition imposed on $f(x,t)$ for t large. In [**8**], Hirano made a breakthrough. He considered the nonlinear function $g(x,t)$ which is not necessarily odd in t. His conclusions based on the following condition:

(**A**) there exist positive numbers p, q and a such that $0 < q < 1 < p < 2^* - 1$, $N(1-q)/(1+q) < p-1$ and that

$$\limsup_{|t|\to 0} \left| \frac{g(x,t) - a|t|^{q-1}t}{|t|^p} \right| < \infty \quad \text{uniformly in } x \in \Omega.$$

Under this condition, he got infinitely many solutions. In his arguments, the strict inequality $N(1-q)/(1+q) < p-1$ is essential. At the end of [**8**], the author proposed an *open* problem: whether the condition $N(1-q)/(1+q) < p-1$ can be removed or not? In the present paper, we are going to give a partial answer to this question under suitable conditions. Precisely, we consider the case: $N(1-q)/(1+q) = p-1$.

To introduce our assumptions, we put, for simplicity, $H = H_0^1(\Omega)$, and denote by $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ the sequence of eigenvalues of the problem: $-\Delta v = \lambda v$, $v \in H$; $|\cdot|_q$ stands for the norm of $L^q(\Omega)$ for $q > 1$; $||\cdot||$ stands for the norm of H defined by $||z||^2 = |\nabla z|_2^2$ for $z \in H$. It is known (cf. [**6**]) that there exists $T > 0$ such that $\lim_{k\to\infty} \lambda_k / k^{2/N} = T$. We now impose the following conditions on $g \in \mathbf{C}(\Omega \times \mathbb{R})$:

(**B**) there exist positive numbers p, q, a and δ such that $0 < q < 1 < p < 2^* - 1$, $N(1-q)/(1+q) = p-1$, and

$$\limsup_{|t|\to 0} \left| \frac{g(x,t) - a|t|^{q-1}t}{|t|^p} \right| < \delta < a^{\frac{2-p}{2-q}} \min\{b_0, b_1\}$$

uniformly in $x \in \Omega$, where

$$b_0 := \frac{(1+p)(1+q)}{4N(2p+3q+7)} T^{\frac{N}{2}} M^{-(1+p)} |\Omega|^{-\frac{p-1}{2}},$$

$$b_1 := M^{-\frac{2(p-q)}{1-q}} |\Omega|^{-\frac{(p-1)(p-q)}{(p+1)(1-q)}} \left(\left(\frac{1-q}{p-1} \right)^{\frac{p-1}{p-q}} + \left(\frac{1-q}{p-1} \right)^{\frac{q-1}{p-q}} \right)^{-\frac{p-q}{1-q}},$$

and $M > 0$ is the best constant satisfying $|z|_{p+1} \le M||z||$ for all $z \in H$, and $|\Omega|$ denotes the Lebesgue measure of Ω.

The main result is the following:

THEOREM 1.1. *Suppose that* (**B**) *holds. Then, problem (1.1) possesses a sequence of weak solutions* $(u_n) \in H$ *such that* $||u_n||_{L^\infty(\Omega)} \to 0$ *as* $n \to \infty$. *Moreover,* $J(u_n) < 0$ *and*

$J(u_n) \to 0$ as $n \to \infty$, where,

$$J(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 dx - \int_\Omega G(x, v)dx, \quad v \in H,$$

with $G(x, t) = \int_0^t g(x, s)ds$.

REMARK 1.1. If (**A**) holds, we put $p_1 = N(1 - q)/(1 + q) + 1$, then $p_1 < p$. Hence, $\limsup_{|t|\to 0} \left|\frac{g(x,t)-a|t|^{q-1}t}{|t|^{p_1}}\right| = 0$, which implies (**B**). Then, the results of [**8**] are contained in our conclusions.

2. Proof of Theorem 1.1. Let $g(x, t) = a|t|^{q-1}t + f(x, t)$, $\lambda = a^{\frac{1}{2-q}}$, $u = \lambda v$, then the problem (1.1) is equivalent to $-\Delta v = |v|^{q-1}v + a^{-\frac{2}{2-q}} f(x, a^{\frac{1}{2-q}} v)$, $v \in H$. Hence, for simplicity we assume that $a = 1$, and (**B**) accordingly becomes (**C**):

(**C**) there exist positive numbers p, q and b such that $0 < q < 1 < p < 2^* - 1$, $N(1 - q)/(1 + q) = p - 1$, and

$$\limsup_{|t|\to 0} \left|\frac{g(x, t) - |t|^{q-1}t}{|t|^p}\right| < b < \min\{b_0, b_1\}$$

uniformly in $x \in \Omega$.

Take a cut off function $\varphi(t)$:

$$\begin{cases} \varphi(t) = 1, & \text{for all } |t| \le t_0, \\ 0 < \varphi(t) < 1, & t_0 < |t| < 2t_0, \\ \varphi(t) = 0, & \text{otherwise.} \end{cases}$$

Let $\tilde{g}(x, t) = \varphi(t)g(x, t) + (1 - \varphi(t))|t|^{q-1}t$ for $(x, t) \in \Omega \times \mathbb{R}$. Then, $\tilde{g}(x, t) \in \mathbf{C}(\Omega \times \mathbb{R})$ and satisfying the following condition (**D**):

(**D**) there exist $t_0 > 0$ small and $b > 0$ such that $b < \min\{b_0, b_1\}$,

$$|t|^{q-1}t - b|t|^p \le \tilde{g}(x, t) \le |t|^{q-1}t + b|t|^p \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

and that $\tilde{g}(x, t) = |t|^{q-1}t$ for all $x \in \Omega$ with $t \ge 2t_0$.

Throughout the rest paper, we consider the problem: $-\Delta u = \tilde{g}(x, u)$, $u \in H$ under (**D**). Equivalently, we first consider the problem (1.1) under the condition (**E**) :

(**E**) there exist $t_0 > 0$ small and $b > 0$ such that $b < \min\{b_0, b_1\}$,

$$|t|^{q-1}t - b|t|^p \le g(x, t) \le |t|^{q-1}t + b|t|^p \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

and that $g(x, t) = |t|^{q-1}t$ for all $x \in \Omega$ with $t \ge 2t_0$.

Here, we use many marks as the paper [**8**]. For each $k \ge 1$, we denote by D^k and S^{k-1} the unit disk and the unit sphere of k dimensional Euclidian space, respectively. And we denote by $\langle \cdot, \cdot \rangle$ the inner product in H. For subsets A, B of H with $B \subset A$, we denote by $\pi_k(A, B)$ the k-relative homotopy group (cf. [**10**]). We denote by $B(r)$ the open ball of H centred at 0 with radius r. For each functional $F : H \to R$ and $a \in R$, F_a stands for the level set defined by $F_a = \{v \in H : F(v) \le a\}$. We defined a functional

$I : H \to R$ by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{q+1}\int_\Omega |u|^{q+1} dx, \quad \text{for all } u \in H.$$

By [**8**], we know there exists some $N_0(v) > 0$ satisfies

$$I\big(N_0(v)v\big) = \min\big\{I(tv) : t \geq 0\big\} < 0,$$

and one can see that for each $v \in H\backslash\{0\}$,

$$\big|\big|N_0(v)v\big|\big|^2 = \int_\Omega \big|N_0(v)v\big|^{q+1} dx \quad \text{and} \quad I\big(N_0(v)v\big) = \frac{q-1}{2(q+1)}\big|\big|N_0(v)v\big|\big|^2.$$

We define functionals $J : H \to R$ and $\widehat{J} : H \to R$ by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G\big(x, u(x)\big) dx \quad \text{for } u \in H$$

and

$$\widehat{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{1}{q+1}|u|^{q+1} + \frac{b}{p+1}|u|^{p+1} dx \quad \text{for } u \in H.$$

Lemma 2.1. *Under condition* (**E**)*, the functional* $J(u)$ *is coercive and the* (PS) *condition holds.*

Proof. It is easy to show that there exists $C > 0$ such that $G(x, v) \leq C + \frac{1}{q+1}|v|^{q+1}$, then $+\infty > J(u_n) \geq \frac{1}{2}||u_n||^2 - C|\Omega| - \frac{1}{q+1}|u_n|_{q+1}^{q+1}$. Therefore, $\{u_n\}$ is bounded in H. Next, we may assume $u_n \rightharpoonup u$ in H, $u_n \to u$ in $L^{p+1}(\Omega)$. By (**E**), we have $g(x, u_n) \to g(x, u)$ in $L^{\frac{p+1}{p}}$ (see theorem A.2 of [**12**]).
Since $\langle J'(u_n) - J'(u), u_n - u\rangle \to 0$ and by Hölder inequality, we have

$$\Big|\int_\Omega \big(g(x, u_n) - g(x, u)\big)(u_n - u)dx\Big| \leq \big|g(x, u_n) - g(x, u)\big|_{\frac{p+1}{p}}\big|u_n - u\big|_{p+1} \to 0.$$

Therefore,

$$||u_n - u||^2 = \langle J'(u_n) - J'(u), u_n - u\rangle + \int_\Omega \big(g(x, u_n) - g(x, u)\big)(u_n - u)dx = o(1).$$

The proof is completed. □

Lemma 2.2. *For each* $v \in H\backslash\{0\}$*, there exists a unique positive number* $N(v) < +\infty$ *such that* $\widehat{J}\big(N(v)v\big)$ *is a local minimum value of* $\{\widehat{J}(tv) : t \geq 0\}$ *with related to t and* $\widehat{J}(tv)$ *decreases on* $\big(0, N(v)\big]$.

Remark 2.1. Since here $\widehat{J}$ has different form from [**8**], $b < b_1$ is needed to make sure that $N(v) < +\infty$, otherwise, $N(v)$ may not be well defined. Moreover, here we will indicate that the similar Lemma still holds.

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Proof. Let $v \in H\backslash\{0\}$ such that $||v|| = 1$. We put $c = \int_\Omega |v|^{q+1}dx$. Then,

$$\begin{aligned} f(t) &:= \frac{d}{dt}\widehat{J}(tv) \\ &= t||v||^2 - t^q\int_\Omega |v|^{q+1}dx - bt^p\int_\Omega |v|^{p+1}dx \\ &= t||v||^2 - ct^q - bt^p|v|_{p+1}^{p+1} \\ &= t\Big[\big(||v||^2 - bt^{p-1}|v|_{p+1}^{p+1}\big) - \frac{c}{t^{1-q}}\Big]. \end{aligned}$$

For any $c_1, c_2 > 0$, let $h(t) := c_1t^{p-1} + c_2t^{q-1}$, then $h'(t) = (p-1)c_1t^{p-2} - (1-q)c_2t^{q-2}$, so there exists unique $t_v = \left(\frac{c_2(1-q)}{c_1(p-1)}\right)^{\frac{1}{p-q}} > 0$ such that $h'(t_v) = 0$, and one can see that $h''(t_v) > 0$. This implies that

$$\min\{h(t) : t > 0\} = h(t_v) = \left(\left(\frac{1-q}{p-1}\right)^{\frac{p-1}{p-q}} + \left(\frac{1-q}{p-1}\right)^{\frac{q-1}{p-q}}\right) c_1^{\frac{1-q}{p-q}} c_2^{\frac{p-1}{p-q}}.$$

Since Ω is bounded in $\mathbb{R}^N$ and $q+1,\ p+1 < 2^*,\ ||v|| = 1$, then $|v|_{p+1}^{p+1} < M^{p+1}$ and $|v|_{q+1}^{q+1} < M^{q+1}|\Omega|^{\frac{p-q}{p+1}}$. We let $c_1 = b|v|_{p+1}^{p+1}$ and $c_2 = |v|_{q+1}^{q+1}$. Recalling that $b < b_1$, we have

$$\left(\left(\frac{1-q}{p-1}\right)^{\frac{p-1}{p-q}} + \left(\frac{1-q}{p-1}\right)^{\frac{q-1}{p-q}}\right) c_1^{\frac{1-q}{p-q}} c_2^{\frac{p-1}{p-q}} < 1.$$

On the other hand, $h(t) \to +\infty$ as $t \to 0$ or $t \to +\infty$, so, there exist two numbers $0 < t_0 < t_v < t_1$ such that $h(t_0) = h(t_1) = 1$ and $h'(t_0) < 0,\ h'(t_1) > 0$. Therefore, $f'(t_0) = \frac{d^2}{dt^2}\widehat{J}(tv)\big|_{t=t_0} > 0$ and $f'(t_1) = \frac{d^2}{dt^2}\widehat{J}(tv)\big|_{t=t_1} < 0$. This implies the uniqueness of $N(v) = t_0$ and the proof is completed. □

REMARK 2.2. (improvement of Remark 1 in **[8]**) One can see that N_0 and N are continuous. Since functions I and $\widehat{J}$ are even, we have that N_0 and N are even functions. Moreover, by the definition of $N_0(v)$ and $N(v)$, $N(tv) = \frac{1}{t}N(v)$ and $N_0(tv) = \frac{1}{t}N_0(v)$ for all $t > 0$.

Next, as in **[8]**, we put $\beta_k = \min_{h\in\Gamma_k}\max_{x\in S^{k-1}} I(h(x))$, where $\Gamma_k = \{h \in C(S^{k-1}, H) : h(x) = -h(-x) \quad \text{for } x \in \mathrm{S}^{k-1}\}$. Then, we have the improvement of Lemma 2.2 in **[8]** as the following:

LEMMA 2.3. *(cf. [8]) Each β_k is negative and there exist $k_0 \geq 1$ such that*

$$\beta_k \geq -\Big(\frac{1-q}{2(1+q)}T^{-\frac{1+q}{1-q}}|\Omega|\Big)k^{\frac{2(q+1)}{N(q-1)}} \quad \textit{for } k \geq k_0.$$

Especially, for any $k \geq 1$ we have $\beta_{k+1} \geq \beta_k$.

Proof. The conclusion that each β_k is negative and there exist some $C_0 > 0, k_0 \geq 1$ such that $\beta_k \geq -C_0k^{\frac{2(q+1)}{N(q-1)}}$ *for* $k \geq k_0$ has been proved by Lemma 2.2 of **[8]**. Calculate carefully step by step, we can find an appropriate value that $C_0 = \frac{1-q}{2(1+q)}T^{-\frac{1+q}{1-q}}|\Omega|$. It is trivial that $\beta_k \leq \beta_{k+1}$. □

We put $\alpha = \frac{p-1}{2}$ and $\gamma = \frac{2(q+1)}{N(1-q)}$, then $\gamma\alpha = 1$. By Lemma 2.3 we can write $\beta_k = -m_k k^{-\gamma}$. For k large enough, $0 < m_k \le C_0$, $m_{k+1} \le m_k(\frac{k+1}{k})^{\gamma}$. We also put $c := \big\|N_0(v)v\big\|^2 = \big|N_0(v)v\big|_{q+1}^{q+1} = \frac{2(q+1)}{q-1} I\big(N_0(v)v\big)$.

LEMMA 2.4. *(cf. [8]) For each $v \in H\backslash\{0\}$ with $\big|I(N_0(v)v)\big|$ sufficiently small,*

$$J\big(N_0(v)v\big) \le \left(1 - \left(\left(\frac{2(q+1)}{1-q}\right)^{\frac{p+1}{2}} \frac{1}{p+1} bM^{p+1}\right)\big|I\big(N_0(v)v\big)\big|^{\frac{p-1}{2}}\right) I\big(N_0(v)v\big). \quad (2.1)$$

Proof. Lemma 2.3 of **[8]** has showed that

$$J\big(N_0(v)v\big) \le \left(1 - C_1\big|I\big(N_0(v)v\big)\big|^{\frac{p-1}{2}}\right) I\big(N_0(v)v\big).$$

Calculate carefully, we can find an appropriate value

$$C_1 = \left(\frac{2(q+1)}{1-q}\right)^{\frac{p+1}{2}} \frac{1}{p+1} bM^{p+1}.$$

□

REMARK 2.3. *Lemma 2.4 is the inequality of (2.8) in [**8**], here we find a suitable value of C_1. Since there exist some mistakes in the proof of (2.9) in [**8**], next, we will give another way to prove more than (2.9) of [**8**] by the lemma 2.5.*

LEMMA 2.5. *For each $v \in H\backslash\{0\}$ with $\big|I(N_0(v)v)\big|$ sufficiently small,*

$$\left(1 + C_2\big|I\big(N_0(v)v\big)\big|^{\frac{p-1}{2}}\right) I\big(N_0(v)v\big) \le \widehat{J}\big(N(v)v\big) \le I\big(N_0(v)v\big)$$
$$\le \left(1 - C_2\big|\widehat{J}\big(N(v)v\big)\big|^{\frac{p-1}{2}}\right)\widehat{J}\big(N(v)v\big). \quad (2.2)$$

where

$$C_2 := \left(\frac{2(q+1)}{1-q}\right)^{\frac{p-1}{2}} \frac{4b(p+q+3)}{(1-q)(1+p)} M^{p+1}$$

REMARK 2.4. (2.8),(2.9) of **[8]** is the source of our inspiration. However, essentially, (2.2) is different from (2.8), (2.9) of **[8]**. We obtain that the control of $I(N_0(v)v)$ and $\widehat{J}(N(v)v)$ is mutual. Note (2.11) of **[8]**, we could delete (2.11) and take $\lambda^2\big(\frac{q-1}{2(q+1)}||v||^2 - \frac{2b}{p+1}M^{p+1}||v||^{p+1}\big)$ in place of (2.12) in **[8]**. Furthermore, our goal is to point out that C_2 in (2.2) can be taken some sensible values under (**E**).

Proof. Let $v \in H\backslash\{0\}$ such that $N(v) = 1$ and suppose that $|I(N_0(v)v)|$ is sufficiently small, since $I(N_0(v)v) = \frac{q-1}{2(q+1)}||N_0(v)v||^2$, then, c is sufficiently small.
Let

$$f(t) := \frac{d}{dt}\widehat{J}(tv) = t||v||^2 - t^q\int_\Omega |v|^{q+1}\,dx - bt^p\int_\Omega |v|^{p+1}dx.$$

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Since $N(v) = 1$, we have $f(1) = 0$, that is, $||v||^2 - |v|_{q+1}^{q+1} - b|v|_{p+1}^{p+1} = 0$. Then,

$$\begin{aligned} N_0(v)^2||v||^2 &= N_0(v)^{q+1}|v|_{q+1}^{q+1} \\ &= N_0(v)^{q+1}\big(||v||^2 - b|v|_{p+1}^{p+1}\big) \\ &< N_0(v)^{q+1}||v||^2, \end{aligned}$$

which implies

$$N_0(v) < 1. \tag{2.3}$$

On the other hand, since $||N_0(v)v||^2 = |N_0(v)v|_{q+1}^{q+1}$, we have

$$\begin{aligned} ||v||^2 - N_0(v)^{1-q}||v||^2 &= ||v||^2 - |v|_{q+1}^{q+1} \\ &= b|v|_{p+1}^{p+1} \\ &\leq bM^{p+1}||v||^{p+1}, \end{aligned}$$

which implies

$$N_0(v)^{p-1} - N_0(v)^{p-q} \leq bM^{p+1}c^{\frac{p-1}{2}}. \tag{2.4}$$

Then, we have one of the following holds :

(1) $0 < N_0(v)$ is sufficiently small;

(2) $0 < 1 - N_0(v)$ is sufficiently small.

But by Lemma 2.2 we have $N(v) = t_0 < t_v$, that is,

$$1 = N(v) < \left(\frac{(1-q)|v|_{q+1}^{q+1}}{(p-1)b|v|_{p+1}^{p+1}}\right)^{\frac{1}{p-q}}.$$

Then,

$$\begin{aligned} (1-q)|v|_{q+1}^{q+1} &> (p-1)b|v|_{p+1}^{p+1} \\ &= (p-1)(||v||^2 - |v|_{q+1}^{q+1}), \end{aligned}$$

which implies $N_0(v) > \left(\frac{p-1}{p-q}\right)^{\frac{1}{1-q}}$. Therefore, case (1) is impossible, then $1 - N_0(v)$ is sufficiently small, and we have

$$\begin{aligned} \frac{1}{2}(1 - N_0(v)^{1-q}) &< (1 - N_0(v)^{1-q})N_0(v)^{p-1} \\ &= N_0(v)^{p-1} - N_0(v)^{p-q} \\ &\leq bM^{p+1}c^{\frac{p-1}{2}}. \end{aligned}$$

Thus, we get

$$\big(1 - 2bM^{p+1}c^{\frac{p-1}{2}}\big)^{\frac{1}{1-q}} < N_0(v) < 1. \tag{2.5}$$

Note that $\widehat{J}(tv) \le I(tv)$ always holds. Then, by (2.5) and lemma 2.2,

$$\widehat{J}(v) \le \widehat{J}(N_0(v)v) \le I(N_0(v)v) < 0.$$

That is,

$$\widehat{J}(N(v)v) \le \widehat{J}(N_0(v)v) \le I(N_0(v)v) < 0.$$

Next, we take $v \in H\backslash\{0\}$ such that $N_0(v) = 1$, and put $\lambda = N(v)$. By Remark 2.2, we have that $N_0(v)v = N_0(\lambda v)\lambda v$ and $N(\lambda v) = 1$. Thus $c = ||N_0(v)v||^2 = ||N_0(\lambda v)\lambda v||^2$, then by (2.5),

$$\left(1 - 2bM^{p+1}c^{\frac{p-1}{2}}\right)^{\frac{1}{1-q}} < N_0(\lambda v) = \frac{N_0(v)}{\lambda} = \frac{1}{\lambda} < 1.$$

When c is sufficiently small, we have that

$$1 < \lambda < \left(1 + b' M^{p+1}c^{\frac{p-1}{2}}\right)^{\frac{1}{1-q}}, \tag{2.6}$$

where $b' := \frac{3}{2}b > b$. Then, we have $1 < \lambda^{p-1} < 2$ and by the Taylor formula that

$$\lambda^2 \le 1 + \frac{4b}{1-q}M^{p+1}c^{\frac{p-1}{2}}. \tag{2.7}$$

Hence,

$$\begin{aligned}\widehat{J}(\lambda v) &= \frac{1}{2}||\lambda v||^2 - \frac{\lambda^{q+1}}{q+1}\int_\Omega |v|^{q+1} - \frac{\lambda^{p+1}b}{p+1}\int_\Omega |v|^{p+1}\\ &= \lambda^2\left(\frac{1}{2}c - \frac{\lambda^{q-1}}{q+1}c - \frac{\lambda^{p-1}b}{p+1}\int_\Omega |v|^{p+1}\right)\\ &\ge \lambda^2\left(\frac{1}{2}c - \frac{1}{q+1}c - \frac{2b}{p+1}\int_\Omega |v|^{p+1}\right)\\ &= \lambda^2\left(\frac{q-1}{2(q+1)}||v||^2 - \frac{2b}{p+1}M^{p+1}||v||^{p+1}\right).\end{aligned}$$

Then,

$$\widehat{J}(\lambda v) \ge \lambda^2\left(1 + c_1||v||^{p-1}\right)I(v), \tag{2.8}$$

where $c_1 = \frac{4b(q+1)}{(1-q)(1+p)}M^{p+1}$. By (2.7), (2.8) and Taylor formula, we have $(1 + c_2|I(v)|^{\frac{p-1}{2}})I(v) \le \widehat{J}(\lambda v) < 0$, here we let

$$c_2 := \left(\frac{2(q+1)}{1-q}\right)^{\frac{p-1}{2}}\frac{4b(p+q+3)}{(1-q)(1+p)}M^{p+1}.$$

We put $C_2 = c_2$, then,

$$(1 + C_2|I(N_0(v)v)|^{\frac{p-1}{2}})I(N_0(v)v) \le \widehat{J}(N(v)v) \le I(N_0(v)v).$$

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Let $x = |I(N_0(v)v)|,\ y = |\widehat{J}(N(v)v)|$, we have $0 < x \leq y \leq (1 + C_2 x^{\alpha})x$. Then, $\left|\frac{y-x}{y^{\alpha+1}}\right| \leq C_2$. Hence, $\widehat{J}(N(v)v) \leq I(N_0(v)v) \leq (1 - C_2|\widehat{J}(N(v)v)|^{\frac{p-1}{2}})\widehat{J}(N(v)v)$. □

REMARK 2.5. By (2.3), we know that if $N(v) = 1$, then $N_0(v) < 1$. By (2.6), if $N_0(v) = 1$, then $N(v) > 1$.

Let

$$C_3 := 2(C_1 + C_2) = \left(\frac{2(q+1)}{1-q}\right)^{\frac{p-1}{2}} \frac{8p + 12q + 28}{(1-q)(1+p)} bM^{p+1} > 0,$$

then similarly to **[8]**, we can choose $m > 0$, such that for each $t \in (-m, 0)$,

$$\left(1 - C_2\left[(1 - C_1|t|^{\alpha})|t|\right]^{\alpha}\right)(1 - C_1|t|^{\alpha})t < (1 - C_3|t|^{\alpha})t.$$

Indeed, if we let

$$h(t) := \frac{1 - \left(1 - C_2\left[(1 - C_1|t|^{\alpha})|t|\right]^{\alpha}\right)(1 - C_1|t|^{\alpha})}{|t|^{\alpha}},$$

then we have $\lim_{t\to 0} h(t) = C_1 + C_2 > 0$. Hence, we can find $m > 0$ such that for each $t \in (-m, 0)$, $0 < h(t) < C_3$, then we have

$$\left(1 - C_2\left[(1 - C_1|t|^{\alpha})|t|\right]^{\alpha}\right)(1 - C_1|t|^{\alpha})t < (1 - C_3|t|^{\alpha})t. \tag{2.9}$$

REMARK 2.6. Assume that **(E)** holds, we have that $C_3 C_0^{\alpha} < \frac{\gamma}{2}$ and $C_3 C_0^{\alpha}\alpha < \frac{1}{2}$.

LEMMA 2.6. *$\forall\, 0 < \theta$ there exists a sequence $\{k_i\} \subset N$ such that $k_i \to \infty$ as $i \to \infty$, and $m_{k_i+1} < m_{k_i}\left(\frac{k_i+1}{k_i}\right)^{\theta}$*

Proof. Suppose that, if there exists some $k_0 \geq 1$ such that, $m_{k+1} \geq m_k\left(\frac{k+1}{k}\right)^{\theta}$ for all $k \geq k_0$. Then, $\frac{m_{nk_0}}{m_{k_0}} \geq n^{\theta}$, which implies when n is large enough $m_{nk_0} > m_{k_0}n^{\theta} \to +\infty$. This is a contradiction to $m_k \leq C_0$. □

LEMMA 2.7. *There exists a sequence $\{k_i\} \subset N$ such that $k_i \to \infty$ as $i \to \infty$, and*

$$\beta_{k_i+1} > \left(1 - C_3|\beta_{k_i}|^{\alpha}\right)\beta_{k_i} \quad \textit{for all } i \geq 1$$

REMARK 2.7. When $\frac{1}{\alpha} < \gamma$, the same conclusion has been showed in **[8]** by an useful inequality of **[3]** which depends on the strict inequality $\frac{1}{\alpha} < \gamma$ holds. In **[8]**, the author used the truncation approach. Thanks to the strict inequality $\frac{1}{\alpha} < \gamma$, it play an crucial role in the process of dealing with inequalities that the exponent can be share a very little with the cut off level t_0 small. At the present paper, we still use the truncation approach, and consider that $g(x, t) = |t|^{q-1}t$ for all $x \in \Omega$ with $t \geq 2t_0$. But the real essential condition for problem (1.1) is that b is small enough since we cannot share any part from the exponent. Hence there are many inequalities seem like those in **[8]**, but there are a lot of differences in the process of dealing with these inequalities.

Proof. Let $\theta = \frac{\gamma}{2} > 0$, by lemma 2.6,

$$\begin{aligned}\frac{-m_{k_i+1} + m_{k_i}\left(\frac{k_i+1}{k_i}\right)^{\gamma}}{m_{k_i}^{\alpha+1}k_i^{-\gamma\alpha}}\frac{k_i^{\gamma}}{(k_i+1)^{\gamma}} &= \frac{-\frac{m_{k_i+1}}{m_{k_i}} + \left(\frac{k_i+1}{k_i}\right)^{\gamma}}{m_{k_i}^{\alpha}k_i^{-\gamma\alpha}}\frac{k_i^{\gamma}}{(k_i+1)^{\gamma}} \\ &> \frac{\left(\frac{k_i+1}{k_i}\right)^{\gamma} - \left(\frac{k_i+1}{k_i}\right)^{\frac{\gamma}{2}}}{k_i^{-\gamma\alpha}}\frac{1}{m_{k_i}^{\alpha}}\left(\frac{k_i}{k_i+1}\right)^{\gamma} \\ &> \frac{\gamma}{2C_0^{\alpha}} \text{ when } k_i \text{ is large enough.}\end{aligned}$$

By Remark 2.6, $\frac{\gamma}{2C_0^{\alpha}} > C_3$, which implies that when k_i is large enough,

$$\beta_{k_i+1} > (1 - C_3|\beta_{k_i}|^{\alpha})\beta_{k_i}.$$

□

LEMMA 2.8. *(cf. [8]) For each $i \geq 1$, there exist $\varepsilon_i > 0$ and $c < 0$ such that $\beta_{k_i} + \varepsilon_i < c$ and $\pi_{k_i}(J_c, J_{\beta_{k_i}+\varepsilon}) \neq \{0\}$ for $0 < \varepsilon < \varepsilon_i$.*

Proof. The details are similar to that of Lemma 2.6 in [**8**]. □

REMARK 2.8. Since $N(v)v$ is a local minimizer of $\widehat{J}(tv)$ in this paper but it is a global minimizer in [**8**], $\widehat{J}(tv)$ decreases on $(0, N(v)]$ and Remark 2.5 are necessary. If not, the similar inequality of (2.15) in [**8**] may not hold. In fact replaces $\widehat{J}(N(v)v)$ by $J(N(v)v)$, one can show that (2.2) still holds, but we can not ensure that $J(tv)$ decreases on $(0, N(v)]$. In another word , we can not ensure $J(N(f(z))f(z)) \leq d_k$. Therefore, we have to introduce the functional $\widehat{J}$.

PROOF OF THEOREM 1.1. Next, we can get sequences $\{k_i\}, \{c_i\}, \{e_i\}$ and $\{\varepsilon_i\}$ such that $\beta_{k_i} + \varepsilon_i < e_i < 0$ are regular values of J, where $c_i = \min_{h\in[\sigma]}\sup_{x\in S_+^{k_i}} J(h(x))$ with $[\sigma] \in \pi_{k_i}(J_{e_i}, J_{\beta_{k_i}+\varepsilon_i})$ nontrivial, $c_i > \beta_{k_i} + \varepsilon_i$ and J satisfies the $(PS)_{c_i}$ condition. Therefore, we by Theorem 1.4 of [**5**] , there exists a sequence of critical points $\{u_i\} \subset H$ of J with critical value $\{c_i\}$ and $c_i \in (\beta_{k_i} + \varepsilon_i, 0)$. Then, we have $c_i \to 0$. Hence, under condition (**E**), the conclusions of theorem 1.1 hold. Equivalently, we have discussed the problem: $-\Delta u = \widetilde{g}(x, u)$ under (**D**). The remaining work we need to do is the bootstrap argument. At last, we can find that $\lim_{i\to\infty}|u_i|_{\infty} = 0$. Recalling that $\widetilde{g}(x, t) = g(x, t)$ for all $x \in \Omega$ with $|t| \leq t_0$, we obtain that u_i is a solution of Problem (1.1) for i is large enough (the details see [**8**]). Hence, theorem 1.1 holds. □

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