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Abstract

Let G be a split reductive group. We introduce the moduli problem of *bundle chains* parametrizing framed principal G-bundles on chains of lines. Any fan supported in a Weyl chamber determines a stability condition on bundle chains. Its moduli stack provides an equivariant toroidal compactification of G. All toric orbifolds may be thus obtained. Moreover, we get a canonical compactification of any semisimple G, which agrees with the wonderful compactification in the adjoint case, but which in other cases is an orbifold. Finally, we describe the connections with Losev–Manin's spaces of weighted pointed curves and with Kausz's compactification of GL_n.

Introduction

Let G be a split reductive group over a field k. Describing an equivariant compactification of G with smooth orbit closures is in some sense an old problem, with its roots in the construction of complete quadrics and complete collineations by Chasles, Schubert, Zeuthen, and others in the 19th century [Lak87]. Moduli problems compactifying the classical groups were introduced by Semple [Sem48, Sem51, Sem52], Kleiman and Thorup [KT88], and Kausz [Kau00, Kau05] in the 20th century. In this paper we embark on a new approach, for general G, using a moduli problem involving principal bundles on chains of lines. This naturally leads us to construct toroidal compactifications which, rather than being schemes, in general are orbifolds, that is, smooth tame stacks.

The best solutions in the category of schemes are in the two extreme cases where G is either abelian or has trivial center. The first is provided by the theory of toric varieties. The second appears in two influential papers of De Concini and Procesi [DCP82, DCP83], where a so-called *wonderful compactification* of a reductive group with trivial center is introduced and the toroidal compactifications lying over it are classified.

The wonderful compactification has many attractive properties. Most notably, it has finitely many $G \times G$ orbits, each of whose closures is smooth. However, the approach of De Concini and Procesi requires trivial center. It is not immediately clear even what the wonderful compactification of SL_n should be.

An intriguing hint was dropped by Tonny Springer in his 2006 ICM address [Spr06]. He pointed out that an arbitrary semisimple G may be compactified by taking the normalization in the function field of G of the wonderful compactification $\overline{G/Z_G}$. This is known in the theory

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of spherical varieties as the canonical embedding [Pez10]. Spherical varieties provide many compactifications of reductive groups, but no canonical smooth compactification of a semisimple group.

Indeed, the canonical embedding already fails to be smooth for $G = SL_n$. However, it has only finite quotient singularities and is therefore the variety underlying an orbifold or smooth tame stack. This suggests that it represents a moduli problem whose objects have finite automorphism groups.

A clue as to the nature of this moduli problem came from a 1999 paper of the second author [Tha99]. There it was shown that the wonderful compactification $\overline{PGL_n}$ may be realized as the locus of genus 0, degree n stable maps to $Gr(n, 2n + k)$ passing through two general points. Since this holds for any $k \geqslant 0$, we may fancifully view this as a locus in the (nonexistent) space of genus 0, degree n stable maps to BGL_n. Less fancifully, it still parametrizes a family of genus 0 curves equipped with a vector bundle framed at two points. In fact, all of the curves in question turn out to be *chains* of lines, so we might refer to the objects parametrized as *framed bundle chains*.

This picture extends to the setting of principal G-bundles for any split reductive G. In fact, the structure group of any such bundle on a chain reduces to a maximal torus T [MT12, Teo02], just as was shown by Grothendieck [Gro57] and Harder [Har68] for the projective line. The isomorphism classes therefore turn out to be parametrized by 1-parameter subgroups of T. But when framings at two marked points are introduced, continuous parameters appear.

A key insight was Charles Cadman's suggestion that everything be made equivariant for an action of $\mathbb{G}_m$, the multiplicative group of the field. In retrospect, this seems natural by analogy with the relative stable maps of Li [Li01], in which equivariant chains also loom large. Li was in turn inspired by the work of Gieseker [Gie84], generalized by Nagaraj and Seshadri [NS99] and Schmitt [Sch04], on vector bundles over semistable models of a nodal curve. Their work has some features in common with the present paper, but because of the new element of $\mathbb{G}_m$-equivariance, the relationship is not transparent.

The moduli problem

To be specific, we study the moduli problem of *framed bundle chains*, defined to consist of the following: (a) a chain of projective lines, of any length, with the multiplicative group $\mathbb{G}_m$ acting with weight 1 on every component; (b) a principal G-bundle on this chain, with a lifting of the $\mathbb{G}_m$-action to it; (c) $\mathbb{G}_m$-invariant trivializations of this bundle at the two smooth points $p_\pm$ fixed by $\mathbb{G}_m$. The stack of such objects is neither separated nor of finite type, but this may be cured by imposing something like a stability condition, as follows.

Thanks to the aforementioned reduction to T, those bundles on a chain with n nodes that admit a framing are classified (up to a Weyl group action) by a sequence of 1-parameter subgroups $\beta_1, \dots, \beta_n$ of T. At the ith node, β_i indicates the weight of the action on the fiber over the node. We call $\beta = (\beta_1, \dots, \beta_n)$ the *splitting type*.

The stability condition depends on a choice of a simplicial fan Σ supported on a Weyl chamber and a choice of lattice points $\beta_1, \dots, \beta_N$, one on each ray of the fan. A bundle is defined in (4.1) to be Σ-*stable* if the components of its splitting type span a cone in Σ and form a subsequence of $\beta_1, \dots, \beta_N$ in the correct order (see Figure 1). This bounds the length of the chain.

The substack representing Σ-stable bundle chains is then proved to be a smooth separated tame Artin stack of finite presentation, which is proper if Σ covers the entire Weyl chamber. When the ground field is $\mathbb{C}$, this means that it is a complex orbifold (smooth, Hausdorff, and second countable), which is compact if Σ covers the Weyl chamber.

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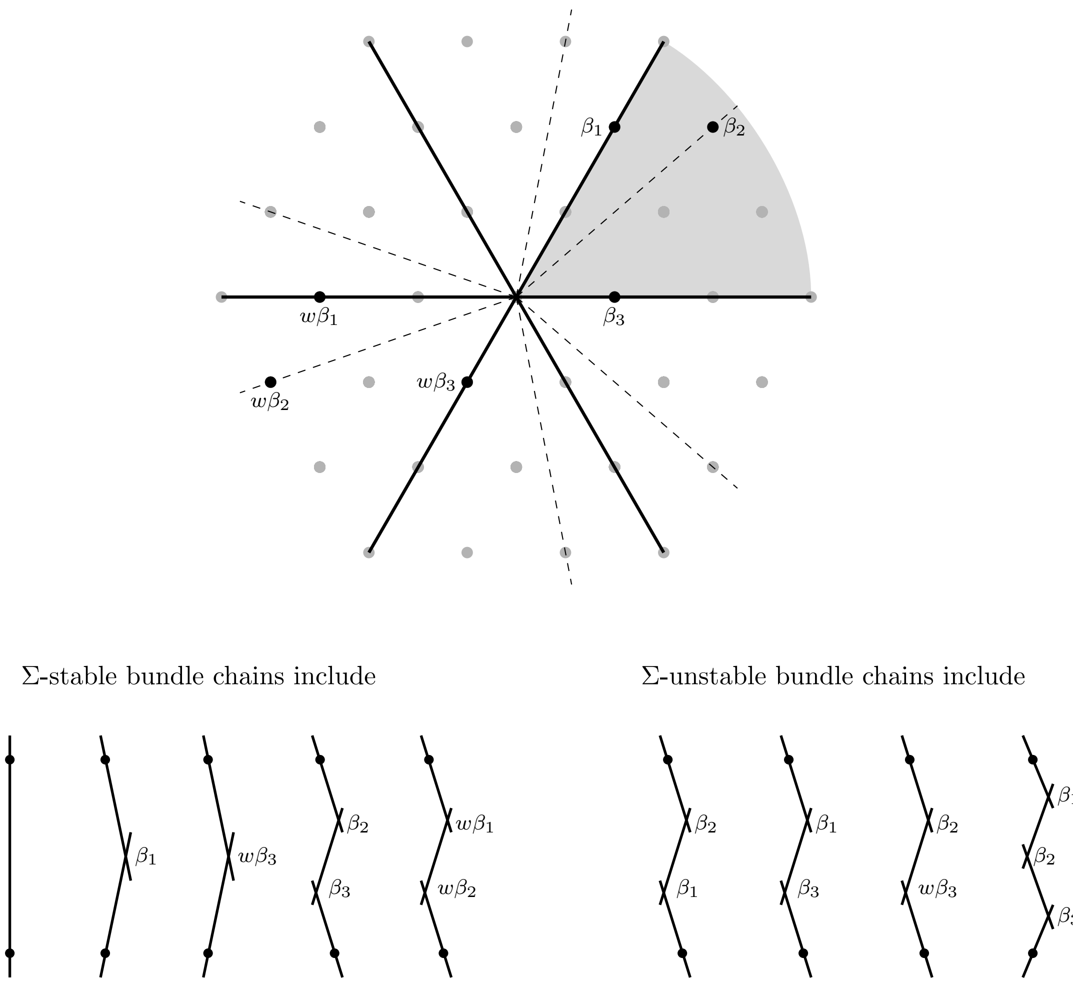

Figure 1. A stacky fan Σ supported in the positive Weyl chamber of $G = PGL_3$ (shaded in gray), as well as the complete fan $W\Sigma$. The action of a single element $w \in W$ is indicated. Ordering the rays of Σ determines a stability condition on bundle chains, as illustrated by the examples.

When G is a torus, this gives a modular interpretation to all toric orbifolds. On the other hand, when G is semisimple, then taking Σ to be the Weyl chamber itself leads to a canonical choice of orbifold compactification. If G moreover has trivial center, this is the familiar wonderful compactification, but if not, it has nontrivial orbifold structure except for $G = SL_2$. In general, it is a smooth stack whose coarse moduli space is an arbitrary toroidal compactification, with finite quotient singularities, of an arbitrary reductive group.

Summary of the paper

The paper is organized as follows. In § 1 we state the basic facts about chains of projective lines and line bundles over them. We give an explicit atlas for the smooth Artin stack $\mathfrak{C}$ parametrizing chains with two marked points, and we describe a canonical action of the multiplicative group on any family of such chains. In § 2 we introduce *bundle chains*, our main objects of study. These are

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principal G-bundles on a chain of projective lines. We classify them up to isomorphism and study their deformation theory. Then in § 3 we prove that bundle chains are parametrized by a smooth Artin stack $\mathfrak{B}$. To get a tame or Deligne–Mumford stack $\mathcal{M}$, we must make three adjustments: rigidify to get rid of an overall $\mathbb{G}_m$-action, add framings of our chains at two points, and impose a mild open condition. To get a separated stack $\mathcal{M}(\Sigma)$ of finite type (that is, an orbifold), one must go further and impose the stability condition described above, determined by a stacky fan Σ. This is carried out in § 4.

We then turn to an alternative construction of the moduli stack as a global quotient of an algebraic monoid by a torus action. This is based on Vinberg's construction [Vin95] of the wonderful compactification of an adjoint group in the same manner. In § 5 we review Vinberg's construction and show that it is a geometric invariant theory quotient. Then in § 6 we hybridize it with Cox's construction [Cox95] of toric varieties as global quotients of affine spaces. As a result, $\mathcal{M}(\Sigma)$ is expressed in § 7 as a geometric quotient $S_{G,\beta}/\!/\mathbb{G}_\beta$ for a certain algebraic monoid $S_{G,\beta}$ acted on by a torus $\mathbb{G}_\beta$. When it is projective, it is a geometric invariant theory quotient.

The case where Σ has a single cone covering the entire Weyl chamber gives a canonical orbifold compactification of a semisimple group. In § 8 we show that its coarse moduli space agrees with the space proposed by Springer.

The final sections describe a few connections between our construction and related ideas in the literature. In § 9 we explain where the coarse moduli spaces $M(\Sigma)$ of our stacks fall in the classification of spherical varieties. In § 10 we discuss the relationship with the moduli problems represented by toric varieties and studied by Losev and Manin [LM00]. Finally, in § 11, we show how the compactification of the general linear group given by Kausz [Kau00] fits into our picture.

Notation

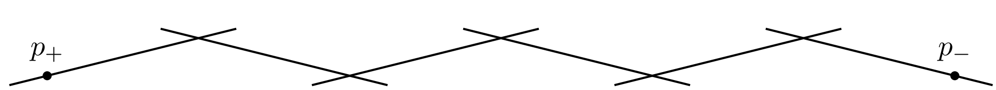
Angle brackets are used for the nonnegative span of a set. That is, the set of linear combinations of elements of S with nonnegative rational coefficients is denoted by $\langle S\rangle$. Also, the diagonal in $S \times S$ is denoted by Δ_S. Notation related to the reductive group G and its characters and cocharacters is summarized in (2.1). The center of G is denoted Z_G. For a cocharacter λ, we denote by t^λ the value of the corresponding 1-parameter subgroup on $t \in \mathbb{G}_m$. This has the virtue of yielding $t^{\lambda+\lambda'} = t^\lambda t^{\lambda'}$.

1. Chains and the stack of chains

Let k be a field. All schemes and stacks throughout will be over k.

The objects of our moduli problem will be principal bundles with reductive structure group over a chain of projective lines, equivariant for the multiplicative group action, and trivialized at the two endpoints. For brevity, we will refer to them as *framed bundle chains.*

Definition 1.1. The *standard chain with n nodes* over a field extension K/k is the nodal curve C_n each of whose $n+1$ irreducible components is isomorphic to the projective line $\mathbb{P}^1_K$, with nodes where $[0,1]$ in the $i-1$th line is glued to $[1,0]$ in the ith line. Its *endpoints* are $p_+ := [1,0]$ in the first line and $p_- := [0,1]$ in the last line. They are fixed points of the *standard action* $\mathbb{G}_m \circlearrowleft C_n$ given by $t \cdot [u,v] := [u,tv]$ on each component.



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Remark 1.2. We will be concerned with $\mathbb{G}_m$-equivariant principal bundles over C_n, or equivalently, principal bundles over the stack $[C_n/\mathbb{G}_m]$. Regarding line bundles, the following are easily verified.

(a) The map $\mathrm{Pic}[C_n/\mathbb{G}_m] \to \mathbb{Z}^{n+2}$ taking an equivariant line bundle to its weights on the fixed points is an isomorphism.

(b) Denote by $\mathcal{O}(b_0 \mid b_1 \mid \ldots \mid b_{n+1})$ the line bundle corresponding to $(b_0, \ldots, b_{n+1}) \in \mathbb{Z}^{n+2}$ under this isomorphism. Then the dualizing sheaf is $\omega_{C_n} \cong \mathcal{O}(-1 \mid 0 \mid \ldots \mid 0 \mid 1)$. That is, $\omega_{C_n} \cong \mathcal{O}(-p)$, where $p := p_+ + p_-$.

(c) For $1 \leqslant i \leqslant j \leqslant n$, there is a section of $\mathcal{O}(b_0 \mid \ldots \mid b_{n+1})$ over $[C_n/\mathbb{G}_m]$ whose support is the ith through jth components if and only if $b_{i-1} > 0 = b_i = \cdots = b_{j-1} > b_j$, except that $b_{i-1} = 0$ is allowed if $i = 1$ and $b_j = 0$ is allowed if $j = n+1$, in which cases, the section is nonvanishing at p_+ and p_-, respectively.

Definition 1.3. A *chain* over a scheme X is an algebraic space C, flat, proper, and finitely presented over X, equipped with two sections $p_\pm : X \to C$, such that every geometric fiber of $C \to X$ is isomorphic to a standard chain.

Since we choose no polarization on our chains, a flat family of chains need not be projective, or even a scheme, just an algebraic space.

Example 1.4. The *versal chain* $\mathbb{C}_n \to \mathbb{A}^n$ is defined recursively for any $n \geqslant 0$ by blowing up as follows. Let $\mathbb{C}_0 := \mathbb{P}^1$ with $p_+ := [1, 0]$ and $p_- := [0, 1]$. Then, given $\mathbb{C}_n$, let $\mathbb{C}_{n+1} := \mathrm{Bl}(\mathbb{C}_n \times \mathbb{A}^1, p_-(\mathbb{A}^n) \times 0)$ with $p_\pm$ defined as the proper transforms of their counterparts on $\mathbb{C}_n$. The following properties are now easily verified.

(a) The obvious action $\mathbb{G}_m^{n+1} \circlearrowright \mathbb{P}^1 \times \mathbb{A}^n$ induces an action $\mathbb{G}_m^{n+1} \circlearrowright \mathbb{C}_n$ in which the action of $\mathbb{G}_m \times 1^n$ is the standard action on each fiber, while the action of $1 \times \mathbb{G}_m^n$ lies over the obvious action on $\mathbb{A}^n$.

(b) Let E_i be the proper transform in $\mathbb{C}_n$ of the exceptional divisor of the ith blow-up (the one giving rise to $\mathbb{C}_i$), and let $\mathbb{L}_i := \mathcal{O}(-E_i)$ over $\mathbb{C}_n$. Also let $\mathbb{L}_0 := \mathcal{O}(p_+)$ and $\mathbb{L}_{n+1} := \mathcal{O}(-p_-)$. Then (i) $\mathbb{G}_m^{n+1}$ acts naturally on $\mathbb{L}_i$, that is, $\mathbb{L}_i \in \mathrm{Pic}[\mathbb{C}_n/\mathbb{G}_m^{n+1}]$; (ii) the restriction of $\mathbb{L}_i$ to the central fiber $[C_n/\mathbb{G}_m]$ is $\mathcal{O}(0 \mid \cdots \mid 0 \mid 1 \mid 0 \mid \cdots \mid 0)$, where 1 appears at the ith position; (iii) the jth factor of $\mathbb{G}_m^n$ acts on the fiber of $\mathbb{L}_i$ over $p_+(0)$ trivially and on that over $p_-(0)$ with weight $-\delta_{i,j}$; (iv) the $\mathbb{L}_i$ freely generate the kernel of the restriction $\mathrm{Pic}[\mathbb{C}_n/\mathbb{G}_m^{n+1}] \to \mathrm{Pic}[p_+(0)/\mathbb{G}_m^n]$.

(c) For any $S \subset \{1, \ldots, n\}$, let $U_S := \{x \in \mathbb{A}^n \mid x_i \neq 0 \text{ if } i \notin S\}$. Then the morphism $\pi_S : U_S \to \mathbb{A}^{|S|}$ given by projection on the coordinates in S is a geometric quotient by a subtorus of $\mathbb{G}_m^n$, inducing a $\mathbb{G}_m^n$-equivariant isomorphism of chains $\mathbb{C}_n|_{U_S} \cong \pi_S^* \mathbb{C}_{|S|}$ under which the line bundles $\mathbb{L}_i$ correspond.

The following statement and proof are well known (indeed, the stack described is an open substack of the stack $\mathfrak{M}_{0,2}$ of prestable curves with two smooth marked points), but we include them for completeness.

Theorem 1.5. *The category of chains (and isomorphisms thereof) is a smooth Artin stack* $\mathfrak{C}$ *with atlas* $\coprod_{n\geqslant 0} \mathbb{C}_n : \coprod_{n \geqslant 0} \mathbb{A}^n \to \mathfrak{C}$.

Proof. We must prove three things: (1) that $\mathfrak{C}$ is a stack, namely (a) a category fibered in groupoids, such that (b) étale descent data are effective and (c) automorphisms are a sheaf; (2) that the diagonal $\mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ is representable, separated, and finitely presented; and (3) that the atlas stated is surjective and smooth, so that the stack is algebraic. The smoothness of $\mathfrak{C}$

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then follows directly from that of $\mathbb{A}^n$ [LM00, 4.7.1], as local properties by definition hold for an algebraic stack if and only if they hold for an atlas.

Step 1. That $\mathfrak{C}$ is (a) a category fibered in groupoids is immediate from the existence of fibered products in the category of algebraic spaces [dJo, 02X2], [Knu71, II 1.5]. To prove (b), observe that since a chain is of finite type over its base, étale descent data are effective for chains [dJo, 04UD], and the property that the geometric fibers are standard chains is preserved by étale descent, since for algebraically closed K, étale $X' \to X$, and $\operatorname{Spec} K \to X$ a geometric point, $X' \times_X \operatorname{Spec} K$ is étale over $\operatorname{Spec} K$, hence a disjoint union of copies of $\operatorname{Spec} K$, so there is a section $\operatorname{Spec} K \to X' \times_X \operatorname{Spec} K$, identifying each geometric fiber of a descended chain $C \to X$ with some fiber of the original $C' \to X'$. That (c) isomorphisms between chains $C_1, C_2 \to X$ constitute a sheaf also follows from descent for algebraic spaces, since any automorphism can be identified with its graph, a closed algebraic subspace of $C_1 \times_X C_2$. Hence $\mathfrak{C}$ is a stack.

Step 2. The requisite properties of the diagonal are verified exactly as by Fulghesu [Ful05, 1.9] and Hall [Hal10, § 3].

Step 3. Finally, we must show that $\coprod_{n\geqslant 0} \mathbb{A}^n \to \mathfrak{C}$ is smooth and surjective on geometric points. The surjectivity is obvious, since the standard chain with n nodes appears over the origin of $\mathbb{A}^n$. The smoothness is by definition [LM00, 3.10.1] a matter of showing, for any chain $C \to X$ inducing $X \to \mathfrak{C}$, that $\mathbb{A}^n \times_{\mathfrak{C}} X \to X$ is smooth. By noetherian reduction we may assume X locally noetherian. By passing to an étale cover and using descent [Vis05, 1.15], [Knu71, II 3.2], we may also assume that C is a scheme. Then it suffices to show [EGA, IV 17.14.2] that it is locally of finite presentation and formally smooth in the sense that a lift always exists in the following diagram, where A is an Artinian K-algebra for some field extension K/k and $J \subset A$ is an ideal with $A/J \cong K$.

$$\begin{tikzcd} \operatorname{Spec} A/J \arrow[r] \arrow[d] & \mathbb{A}^n \times_{\mathfrak{C}} X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

It is locally of finite presentation by the corresponding property of the diagonal established above: since $\mathbb{A}^n \times_{\mathfrak{C}} X \cong (\mathbb{A}^n \times X) \times_{\mathfrak{C}\times\mathfrak{C}} \mathfrak{C}$, it is locally of finite presentation over $\mathbb{A}^n \times X$ and hence over X.

To show the formal smoothness, observe that a lift consists of a morphism $\operatorname{Spec} A \to \mathbb{A}^n$ extending the given $\operatorname{Spec} A/J \to \mathbb{A}^n$ and an isomorphism of the two chains on $\operatorname{Spec} A$ pulled back from $\mathbb{A}^n$ and from X, extending the given one on $\operatorname{Spec} A/J$. But we know the following.

(a) The set of isomorphism classes of chains over $\operatorname{Spec} A$ extending a given chain over $\operatorname{Spec} A/J$ is nonempty and is acted on transitively by $\operatorname{Ext}^1(\Omega, \mathcal{O}(-p))$, the space of first-order deformations of C_n with marked points $p = p_+ + p_-$. See Sernesi [Ser06, 2.3.4, 2.4.1, 2.4.8]. The addition of the two marked points does not affect the proofs.

(b) The set of morphisms $\operatorname{Spec} A \to \mathbb{A}^n$ extending the given $\operatorname{Spec} A/J \to \mathbb{A}^n$ is acted on transitively (and freely) by the tangent space to $\mathbb{A}^n$ at the image of $\operatorname{Spec} A/\mathfrak{m}$. This is elementary.

(c) The natural map from the latter to the former is given, in terms of these actions, by the derivative or Kodaira–Spencer map of $\mathbb{C}_n$. Indeed, by Schlessinger's condition H_2 (which Sernesi calls H_ε), chains over $\operatorname{Spec} A \otimes_K K[\varepsilon]/(\varepsilon^2)$ naturally correspond to pairs of chains over $\operatorname{Spec} A$ and $\operatorname{Spec} K[\varepsilon]/(\varepsilon^2)$ with an isomorphism of the central fiber. The actions of (a) and (b) are then given by base change by the morphism $\operatorname{Spec} A \to \operatorname{Spec} A \otimes_K K[\varepsilon]/(\varepsilon^2)$ taking ε to a generator

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of J. This clearly commutes with the map (b) $\to$ (a) induced by $\mathbb{C}_n$; but the map induced on $\operatorname{Spec} K[\varepsilon]/(\varepsilon^2)$ is exactly the Kodaira–Spencer map.

Finally, the family $\mathbb{C}_n \to \mathbb{A}^n$ has surjective Kodaira–Spencer map at every point. Indeed, the space of first-order deformations is $\operatorname{Ext}^1(\Omega, \mathcal{O}(-p)) = H^0(\mathcal{E}xt^1(\Omega, \mathcal{O}(-p)))$ as in Sernesi [Ser06, 1.1.11]. This has a basis consisting of elements smoothing a single node, and moving along the coordinate axes in $\mathbb{A}^n$ carries out these deformations. □

THEOREM 1.6. *Let $C \to X$ be a chain. There exists an unique $\mathbb{G}_m$-action on C lying over the trivial action on X and restricting to the standard action on each geometric fiber. Moreover, for any line bundle $L \to C$, there is an unique lifting of this action to L that acts trivially on $p_+^* L$.*

This canonical $\mathbb{G}_m$-action will be called the *uniform action*.

Proof. Let the *standard equivariant chain with n nodes* be the standard chain equipped with the standard $\mathbb{G}_m$-action, and let an *equivariant chain* be a chain equipped with a $\mathbb{G}_m$-action lying over the trivial action on the base and restricting to the standard action on each geometric fiber. Then the category of equivariant chains is again an Artin stack $\tilde{\mathfrak{C}}$. Indeed, most of the proof runs exactly parallel to that of Theorem 1.5. To complete Step 2, observe that an equivariant isomorphism of chains is just an ordinary isomorphism that satisfies a closed condition, namely that it intertwines the two $\mathbb{G}_m$-actions. Hence for any two equivariant chains $C_1, C_2 \to X$, the corresponding morphism $X \times_{\tilde{\mathfrak{C}} \times \tilde{\mathfrak{C}}} \tilde{\mathfrak{C}} \to X \times_{\mathfrak{C} \times \mathfrak{C}} \mathfrak{C}$ is a closed immersion. Hence it is separated [EGA, I 5.5.1] and of finite type [EGA, I 6.3.4]. By noetherian reduction we may assume X is locally noetherian. Since $\mathfrak{C} \to \mathfrak{C} \times \mathfrak{C}$ is of finite type by Step 2 of Theorem 1.5, $X \times_{\mathfrak{C} \times \mathfrak{C}} \mathfrak{C} \to X$ is of finite type [EGA, I 6.3.4]. Then $X \times_{\mathfrak{C} \times \mathfrak{C}} \mathfrak{C}$ is also locally noetherian [EGA, I 6.3.7], and hence our closed immersion is finitely presented [EGA, IV 1.6.1]. Therefore the composition $X \times_{\tilde{\mathfrak{C}} \times \tilde{\mathfrak{C}}} \tilde{\mathfrak{C}} \to X$ is separated and finitely presented, so the same is true of the diagonal $\tilde{\mathfrak{C}} \to \tilde{\mathfrak{C}} \times \tilde{\mathfrak{C}}$.

There is a forgetful morphism $\tilde{\mathfrak{C}} \to \mathfrak{C}$, and clearly it suffices to show it is an isomorphism. It is finitely presented, since the same is true of $\tilde{\mathfrak{C}} \to \operatorname{Spec} k$ and of the diagonal of $\mathfrak{C} \to \operatorname{Spec} k$. It is universally injective and surjective simply because every standard chain over every field admits an unique $\mathbb{G}_m$-action isomorphic to the standard action. Thanks to the fundamental property of étale morphisms [EGA, IV 17.9.1], it remains only to show that it is étale. Since it is finitely presented, it suffices to show that it is formally étale for morphisms from Artinian K-algebras. As in the proof of Theorem 1.5, it suffices to show that the derivative or Kodaira–Spencer map is an isomorphism. The first-order deformations are now $\operatorname{Ext}^1_{[C_n/\mathbb{G}_m]}(\Omega, \mathcal{O}(-p))$. This may be shown to equal $\operatorname{Ext}^1_{C_n}(\Omega, \mathcal{O}(-p))^{\mathbb{G}_m}$ via the spectral sequence of the hypercovering associated to the presentation $\mathbb{G}_m \times C_n \rightrightarrows C_n$. But $\operatorname{Ext}^1_{C_n}(\Omega, \mathcal{O}(-p)) = H^0(\mathcal{E}xt^1_{C_n}(\Omega, \mathcal{O}(-p)))$, and $\mathcal{E}xt^1_{C_n}(\Omega, \mathcal{O}(-p))$ is a sum of skyscraper sheaves supported at the nodes and acted on trivially by $\mathbb{G}_m$. Hence $\operatorname{Ext}^1_{C_n}(\Omega, \mathcal{O}(-p))^{\mathbb{G}_m} = \operatorname{Ext}^1_{C_n}(\Omega, \mathcal{O}(-p))$ as desired.

To prove the last statement, let $\phi : \mathbb{G}_m \times C \to \mathbb{G}_m \times C$ be given by $\phi(z, c) = (z, z \cdot c)$, acting on the second factor as above. To lift the action to L means to find an isomorphism $\phi : L \to L$ satisfying the obvious commutative diagram. From the classification of line bundles on standard chains (see Remark 1.2(a)), it is clear that there exists an action over each $\mathbb{G}_m \times \{s\}$. So $L^{-1} \otimes \phi^* L$ is trivial on the fibers of the projection $\mathrm{id} \times \pi : \mathbb{G}_m \times C \to \mathbb{G}_m \times X$, and hence the direct image $\pi_*(L^{-1} \otimes \phi^* L)$ is a line bundle on $\mathbb{G}_m \times X$. It is canonically trivialized on $1 \times X$, hence trivial. So there is an isomorphism $L \cong \phi^* L$ over $\mathbb{G}_m \times C$ restricting to the identity on $1 \times C$. In fact any such isomorphism satisfies the commutative diagram: after all, over each $\mathbb{G}_m \times \{s\}$, one isomorphism does, and any two such isomorphisms differ by multiplication by a morphism $\mathbb{G}_m \to \mathbb{G}_m$, which is of the form $z \mapsto z_0 z^n$ for some $z_0 \in \mathbb{G}_m$ and $n \in \mathbb{Z}$. Because

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the restriction to $1 \times C$ is required to be the identity, $z_0 = 1$, so all such isomorphisms differ by tensoring by a character of $\mathbb{G}_m$ and hence all satisfy the diagram. Tensoring by a further such character, one may arrange that the action on $p_+^* L$ is trivial. $\square$

2. Bundle chains

We now introduce our main objects of study.

Notation 2.1. Let G be a split reductive algebraic group over k, T a split maximal torus, $W := N(T)/T$ the Weyl group, $\mathrm{V} := \mathrm{Hom}(T, \mathbb{G}_m)$ the character lattice, $\Lambda := \mathrm{Hom}(\mathbb{G}_m, T)$ the cocharacter lattice, B the Borel subgroup corresponding to a choice of simple roots α_i, and $\Lambda_{\mathbb{Q}}^+ \subset \Lambda \otimes \mathbb{Q}$ the positive Weyl chamber.

Definition 2.2. A *principal G-bundle E* over a scheme (or stack) X is a G-torsor locally trivial in the étale topology. Its *adjoint bundle* $\mathrm{Ad}\, E$ is the associated bundle with fiber G, but with transition functions given by conjugation (instead of multiplication) by the transition functions of E. Likewise, $\mathrm{ad}\, E$ is the associated vector bundle with fiber $\mathfrak{g}$.

Definition 2.3. A *G-bundle chain* over a scheme (or stack) X is a chain $C \to X$ equipped with a principal G-bundle E over the stack $[C/\mathbb{G}_m]$, where the action of $\mathbb{G}_m$ on C is the uniform one of Theorem 1.6. (Equivalently, this is a $\mathbb{G}_m$-equivariant principal G-bundle over C.) Likewise, a *framed bundle chain* is a bundle chain E equipped with ($\mathbb{G}_m$-invariant) trivializations of $p_\pm^* E$.

Remark 2.4. We shall be interested only in bundle chains where the restriction of the bundle to each geometric fiber is *rationally trivial*, that is, trivial on the generic point of each component. This notion will play a minor role, as it is automatic if k is algebraically closed of characteristic zero [Ste65, 1.9], if G is a torus [Mil80, III 4.9], or if the bundle chain is framed [BN09, 1.1]. It is also clearly an open condition, as the deformation space $H^1(\mathrm{Spec}\, K(t), \mathrm{ad}\, E)$ of a trivial bundle E over a generic point $\mathrm{Spec}\, K(t)$ of a chain certainly vanishes. So it will not affect our deformation theory arguments.

Example 2.5. Suppose the chain is simply $C = \mathbb{P}^1 \times X \to X$. Then any framed bundle chain over C must be trivial as a bundle, as we will see in Corollary 2.11 below. The framing at p_+ fixes a global trivialization, from which the framing at p_- differs by a morphism $X \to G$. Hence the moduli space of such framed bundle chains is nothing but G. The moduli stacks we eventually study will compactify this locus.

Example 2.6. Suppose the chain is a standard chain $C_n \to \mathrm{Spec}\, K$. A T-bundle over C_n is essentially an r-tuple of line bundles and as such, according to Remark 1.2(a), is determined up to isomorphism by the weights of the $\mathbb{G}_m$-action at the fixed points of C_n. Let $\beta = (\beta_1, \ldots, \beta_n) \in \Lambda^n$, let $F(\beta)$ be the T-bundle whose weights at $p_\pm$ are 0 and whose weights at the nodes are $\beta_1, \ldots, \beta_n$, and let $E(\beta)$ be the associated G-bundle. Then $E(\beta)$ defines a bundle chain over $\mathrm{Spec}\, K$.

Example 2.7. Again let $\beta = (\beta_1, \ldots, \beta_n) \in \Lambda^n$. Then *versal bundle chains* $\mathbb{E}(\beta)$ and $\mathbb{F}(\beta)$ over $\mathbb{A}^n$, with structure groups G and T respectively, are defined as follows. Let $\mathbb{C}_n \to \mathbb{A}^n$ be the versal chain of Example 1.4, $\mathbb{L}_1, \ldots, \mathbb{L}_n \in \mathrm{Pic}[\mathbb{C}_n/\mathbb{G}_m^{n+1}]$ as described there. Then $\mathbb{L}_1^\times \times \cdots \times \mathbb{L}_n^\times$ is a principal $\mathbb{G}_m^n$-bundle. Regarding $\beta \in \Lambda^n$ as an element of $\mathrm{Hom}(\mathbb{G}_m^n, T)$, let $\mathbb{F}(\beta)$ be the associated T-bundle and $\mathbb{E}(\beta)$ the associated G-bundle. Then $\mathbb{F}(\beta)$ and $\mathbb{E}(\beta)$ are bundle chains over $[\mathbb{A}^n/\mathbb{G}_m^n]$.

Regarded as bundle chains over $\mathbb{A}^n$, both $\mathbb{F}(\beta)$ and $\mathbb{E}(\beta)$ may of course be framed at $p_\pm$. It is convenient to single out a choice of *conventional framings* $\Psi_\pm : \mathbb{A}^n \times G \to p_\pm^* \mathbb{E}(\beta)$ as follows.

Since $\mathbb{L}_i = \mathcal{O}(-E_i)$, the functions 1 and x_i constitute nowhere vanishing sections of $p_+^*\mathbb{L}_i$ and $p_-^*\mathbb{L}_i$ respectively. They induce framings of $\mathbb{L}_1^\times \times \cdots \times \mathbb{L}_n^\times$ and hence of $\mathbb{F}(\beta)$ and $\mathbb{E}(\beta)$.

Over the open set $x_1 \cdots x_n \neq 0$, all of the above bundles are canonically trivial. Relative to this trivialization, the conventional framings may be expressed as 1 and $x^\beta := \prod x_i^{\beta_i}$ respectively. Therefore, the conventional framings are *not* both $\mathbb{G}_m^n$-equivariant: Ψ_+ is, but Ψ_- is not.

Definition 2.8. The *automorphism group* of E is $\Gamma(\operatorname{Ad} E)$, cf. [Bri11]. However, the notation $\operatorname{Aut} E$ is reserved for the *framed automorphism group*, that is, the subgroup of $\Gamma(\operatorname{Ad} E)$ preserving the framings at $p_\pm$. Likewise, $\operatorname{Aut} C$ denotes the framed automorphisms of the chain C, namely those fixing p_+ and p_-, and $\operatorname{Aut}(C, E)$ denotes the framed automorphisms of the bundle chain, so that

$$1 \longrightarrow \operatorname{Aut} E \longrightarrow \operatorname{Aut}(C, E) \longrightarrow \operatorname{Aut} C.$$

Definition 2.9. For a subgroup $\iota : H \to G$, a *reduction of structure group* of a principal G-bundle $E \to X$ is a principal H-bundle $F \to X$ and an isomorphism $F_\iota \cong E$, where F_ι is the associated G-bundle $(F \times G)/H$. Two reductions are *isomorphic* if there is an isomorphism of the two H-bundles making the obvious triangle commute. Isomorphism classes of reductions then correspond to sections of the associated G/H-bundle $E/H \to X$. For example, when $G = GL_r$ and the maximal torus T consists of diagonal matrices, a reduction to T is equivalent to a splitting as a sum of line bundles.

Theorem 2.10. *Every rationally trivial G-bundle chain over* $\operatorname{Spec} K$ *admits a reduction of structure group to T. Its isomorphism class is unique modulo the actions of $\Gamma(\operatorname{Ad} E)$ and the Weyl group W.*

Proof. See another paper by the authors [MT12, 6.4]. □

Corollary 2.11. *Let $\bar{H}^1(X, G)$ denote the set of isomorphism classes of rationally trivial principal G-bundles on X. Then there is a natural bijection $\bar{H}^1([C_n/\mathbb{G}_m], G) = \Lambda^{n+2}/W$. The G-bundles admitting a framing correspond to $n+2$-tuples whose first and last coordinates vanish, hence are in bijection with Λ^n/W. That is, they are the bundles $E(\beta)$ defined in Example 2.6.*

Proof. By Remark 1.2(a), $\operatorname{Pic}[C_n/\mathbb{G}_m] \cong \mathbb{Z}^{n+2}$, and by Theorem 90 [Mil80, III 4.9] all line bundles are rationally trivial. Hence $\bar{H}^1([C_n/\mathbb{G}_m]; T) \cong \Lambda^{n+2}$. The first statement then follows from the theorem. A bundle corresponding to $(\beta_0, \ldots, \beta_{n+1}) \in \Lambda^{n+2}$ admits a framing if and only if the action of $\mathbb{G}_m$ at the endpoints is trivial, that is, $\beta_0 = \beta_{n+1} = 0$. □

Let Ω be the Kähler differentials on C_n, and let $\mathcal{S}$ denote the sheaf on C_n of G-invariant Kähler differentials on the total space of a framed bundle chain E. There is then a short exact sequence of sheaves on $[C_n/\mathbb{G}_m]$

$$0 \longrightarrow \Omega \longrightarrow \mathcal{S} \longrightarrow \operatorname{ad} E \longrightarrow 0$$

in which the three nonzero terms control the deformation theory of the chain C_n, of the bundle chain (C_n, E), and of the bundle E respectively. More precisely, let $T^0_{C_n}$, $T^1_{C_n}$, $T^2_{C_n}$ denote first-order endomorphisms, deformations, and obstructions, respectively, of the chain $[C_n/\mathbb{G}_m]$ together with its two marked points. (For simplicity, this notation suppresses any mention of the marked points or of the $\mathbb{G}_m$-equivariance.) Likewise, let $T^i_{C_n,E}$ denote the corresponding spaces for the framed bundle chain E over $[C_n/\mathbb{G}_m]$, and let T^i_E denote the corresponding spaces for the framed bundle E over the fixed base $[C_n/\mathbb{G}_m]$.

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Proposition 2.12. *For $i = 0, 1, 2$, there are natural isomorphisms*

$$T^i_{C_n} = \mathrm{Ext}^i_{C_n}(\Omega, \mathcal{O}(-p))^{\mathbb{G}_m}, \quad T^i_{C_n,E} = \mathrm{Ext}^i_{C_n}(\mathcal{S}, \mathcal{O}(-p))^{\mathbb{G}_m}, \quad T^i_E = \mathrm{Ext}^i_{C_n}(\mathrm{ad}\, E, \mathcal{O}(-p))^{\mathbb{G}_m}.$$

In particular all obstructions vanish.

Proof. The proofs that $T^i_{C_n} = \mathrm{Ext}^i_{[C_n/\mathbb{G}_m]}(\Omega, \mathcal{O}(-p))$, and similarly for the other two cases, are straightforward following Sernesi [Ser06, 2.4.1, 3.3.11]. But the latter agrees with the $\mathbb{G}_m$-invariant part of $\mathrm{Ext}^i_{C_n}$. This can be shown by computing $\mathrm{Ext}^i_{[C_n/\mathbb{G}_m]}$ via the spectral sequence associated to the hypercovering of the presentation $\mathbb{G}_m \times C_n \rightrightarrows C_n$. □

Any bundle chain possesses a trivial 1-parameter group of automorphisms, namely the $\mathbb{G}_m$-action itself. Hence $\dim T^0_{C_n,E} \geqslant 1$. The following result explains when equality holds, that is, when there are no other infinitesimal automorphisms.

Theorem 2.13. *Let $\beta = (\beta_1, \dots, \beta_n)$ and $E = E(\beta)$. Then:*

(a) $\dim T^0_E = 0$ *if and only if $\beta_1, \dots, \beta_n$ lie in a common Weyl chamber;*
(b) $\dim T^0_{C_n,E} = 1$ *if and only if $\beta_1, \dots, \beta_n$ lie in a common Weyl chamber and are linearly independent in $\Lambda \otimes k$;*
(c) $\dim \mathrm{Aut}(C_n, E) = 1$ *if and only if $\beta_1, \dots, \beta_n$ lie in a common Weyl chamber and are linearly independent in Λ.*

Conditions (b) and (c) are, of course, equivalent in characteristic zero. In positive characteristic the β_i may be dependent in $\Lambda \otimes k$ but not in Λ. In this case $\mathrm{Aut}(C_n, E)$ will not be smooth.

Proof. (a) We have $T^0_E = \mathrm{Ext}^0(\mathrm{ad}\, E, \mathcal{O}(-p)) = H^0(\mathrm{ad}\, E(-p))$. Since E reduces to T, $\mathrm{ad}\, E$ splits as a sum of line bundles

$$\mathrm{ad}\, E \cong \mathcal{O}^r \oplus \bigoplus_{\alpha \in \Phi} L_\alpha, \tag{2.14}$$

where r is the rank of G and Φ is the set of roots. Of course $H^0(\mathcal{O}^r(-p)) = 0$. It is easily seen that $L_\alpha \cong \mathcal{O}(0 \mid \alpha \cdot \beta_1 \mid \cdots \mid \alpha \cdot \beta_n \mid 0)$ and hence $L_\alpha(-p) \cong \mathcal{O}(-1 \mid \alpha \cdot \beta_1 \mid \cdots \mid \alpha \cdot \beta_n \mid 1)$. Because of the weights ± 1 at $p_\pm$, any section of $L_\alpha(-p)$ vanishes on the first and last components. By Remark 1.2(c), $L_\alpha(-p)$ has a nonzero section if and only if $\alpha \cdot \beta_i > 0 > \alpha \cdot \beta_j$ for some $i < j$. On the other hand, if α is a root, then so is $-\alpha$. So $H^0(\bigoplus L_\alpha(-p)) = 0$ if and only if, for all α and all i, j, $\alpha \cdot \beta_i$ and $\alpha \cdot \beta_j$ are not of opposite sign. This is equivalent to all β_i lying in a common Weyl chamber.

(b) Consider the long exact sequence

$$0 \longrightarrow T^0_E \longrightarrow T^0_{C_n,E} \longrightarrow T^0_{C_n} \longrightarrow T^1_E \longrightarrow T^1_{C_n,E} \longrightarrow T^1_{C_n} \longrightarrow 0.$$

The 1-dimensional subspace $S \subset T^0_{C_n,E}$ arising from the $\mathbb{G}_m$-action injects into $T^0_{C_n}$. Hence $\dim T^0_{C_n,E} = 1$ if and only if $T^0_E = 0$ and $T^0_{C_n}/S \to T^1_E$ is injective. The former is covered by (a), so it remains to consider the latter.

The connecting homomorphism in the sequence above may be described as follows. Let $D = \mathrm{Spec}\, k[\epsilon]/(\epsilon^2)$. For $v \in T^0_{C_n}$, an automorphism of $C_n \times_k D$ is given by $\mathrm{id} + \epsilon v$. The pullback of $E \times_k D$ by this automorphism is a bundle chain over D whose isomorphism class determines an element of T^1_E.

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It is tempting to infer that the connecting homomorphism is zero. After all, a first-order deformation cannot change the weights of the $\mathbb{G}_m$-action on a line bundle, so line bundles and hence T-bundles on $[C_n/\mathbb{G}_m]$ are rigid, but any G-bundle E reduces to a T-bundle, so the connecting homomorphism for G factors through the trivial one for T. However, this ignores the framing. The deformation class of the unframed bundle is indeed zero in $H^1(\operatorname{ad} E)$ by the above argument, but the framing at p may deform nontrivially. From the long exact sequence of

$$0 \longrightarrow \operatorname{ad} E(-p) \longrightarrow \operatorname{ad} E \longrightarrow \operatorname{ad} E \otimes \mathcal{O}_p \longrightarrow 0,$$

namely

$$0 \longrightarrow T^0_E \longrightarrow H^0(\operatorname{ad} E) \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \longrightarrow T^1_E \longrightarrow H^1(\operatorname{ad} E) \longrightarrow 0, \tag{2.15}$$

it follows that the connecting homomorphism lifts to $\mathfrak{g} \oplus \mathfrak{g}$. It will be injective if and only if the image of its lift intersects that of $H^0(\operatorname{ad} E)$ trivially.

A basis for $T^0_{C_n}$ consists of the infinitesimal generators e_i for the $\mathbb{G}_m$-actions on each component of C_n. The action on the ith component lifts to act on the line bundle $\mathcal{O}(0 \mid b_1 \mid \cdots \mid b_n \mid 0)$ with weights b_{i-1} and b_i on the two adjacent nodes. Since the action on all other components is trivial, this same lifting has weights b_{i-1} and b_i on p_+ and p_-, respectively.

A T-bundle is essentially an r-tuple of line bundles, so the same reasoning applies to the T-bundle to which E reduces. This has weights $0, \beta_1, \ldots, \beta_n, 0 \in \Lambda$. Hence the image of e_i in $\mathfrak{g} \oplus \mathfrak{g}$ is $(\beta_{i-1}, \beta_i) \in \mathfrak{t} \oplus \mathfrak{t} \subset \mathfrak{g} \oplus \mathfrak{g}$.

As for the image of $H^0(\operatorname{ad} E)$, it is now clear that only the part meeting $\mathfrak{t} \oplus \mathfrak{t}$ is relevant. Since this comes from $\mathcal{O}^r$ in the splitting (2.14), it is simply the diagonal $\Delta_{\mathfrak{t}} \subset \mathfrak{t} \oplus \mathfrak{t}$. The composite map $T^0_{C_n} \to (\mathfrak{t} \oplus \mathfrak{t})/\Delta_{\mathfrak{t}} \cong \mathfrak{t}$ is then given by $e_i \mapsto \beta_i - \beta_{i-1}$.

Since S is spanned by $e_1 + \cdots + e_n$, this defines an injection $T^0_{C_n}/S \to (\mathfrak{t} \oplus \mathfrak{t})/\Delta_{\mathfrak{t}}$ if and only if $\beta_1, \ldots, \beta_n$ are linearly independent in $\mathfrak{t} = \Lambda \otimes k$.

(c) There is a short exact sequence

$$1 \longrightarrow \operatorname{Aut} E \longrightarrow \operatorname{Aut}(C_n, E) \longrightarrow \operatorname{Aut} C_n,$$

where as usual $\operatorname{Aut} E$ denotes the framed, $\mathbb{G}_m$-equivariant automorphisms of E lying over the identity on C_n.

The standard action of $\mathbb{G}_m$ on C_n lifts to E, so there is always a subgroup $\mathbb{G}_m \subset \operatorname{Aut}(C_n, E)$ lying over the diagonal in $\operatorname{Aut} C_n \cong \mathbb{G}_m^{n+1}$. On the other hand, the Lie algebra of $\operatorname{Aut} E$ is T^0_E, and $\operatorname{Aut} E$ is connected [MT12, 6.7]. Hence $\operatorname{Aut} E$ is trivial if and only if $T^0_E = 0$. Using (a), if the β_i do not lie in a common Weyl chamber, then certainly $\dim \operatorname{Aut}(C_n, E) > 1$, while if they do, then $\operatorname{Aut}(C_n, E)$ is a subgroup of $\operatorname{Aut} C_n \cong \mathbb{G}_m^{n+1}$. It suffices to show that, in the latter case, the β_i are dependent in Λ if and only if $\operatorname{Aut}(C_n, E)/\mathbb{G}_m$ contains a torus. For any positive-dimensional subgroup of a torus contains a torus [DG70, IV 1.1.7].

Indeed, we will show that the 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_m^{n+1}$ given by $t^\lambda := (t^{a_1}, \ldots, t^{a_n})$ for $a_i \in \mathbb{Z}$ lifts to $\operatorname{Aut}(C_n, E)$ if and only if $\sum_{i=0}^n a_i(\beta_{i+1} - \beta_i) = 0$. So there is always the diagonal $(1, \ldots, 1)$, but there will be further subgroups if and only if the β_i are dependent.

Let $S \subset \operatorname{Aut} E$ be the 2-dimensional torus generated by λ and the diagonal. A lifting of λ to E is the same as a G-bundle over $[C_n/S]$ whose pullback to $[C_n/\mathbb{G}_m]$ is E. The structure group of such a bundle reduces to T [MT12, 6.4]. It is elementary that a line bundle $\mathcal{O}(b_0 \mid \cdots \mid b_{n+1})$ admits a λ-action preserving the framing at $p_\pm$ if and only if $\sum_{i=0}^n a_i(b_{i+1} - b_i) = 0$. Consequently, the T-bundle $\mathbb{F}(\beta)$ admits a λ-action preserving the framing if and only if $\sum_{i=0}^n a_i(\beta_{i+1} - \beta_i) = 0$, as desired. $\square$

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Corollary 2.16. *Condition* (a) *is equivalent to* $H^1(\operatorname{ad} E) = 0$ *and to* $\operatorname{Aut} E = 1$.

Proof. Since the dualizing sheaf is $\omega \cong \mathcal{O}(-p)$, by Serre duality $H^1(\operatorname{ad} E)^* \cong T^0_E = 0$. This is the Lie algebra of $\operatorname{Aut} E$. On the other hand, $\operatorname{Aut} E$ is connected [MT12, 6.7], and a connected group scheme over a field is trivial if and only if its Lie algebra is trivial. □

Corollary 2.17. *If condition* (c) *holds, then* $\operatorname{Aut}(C_n, E)/\mathbb{G}_m$ *is a subgroup scheme of* $\mathbb{G}_m^n$*, finite over* k.

Hence in characteristic zero it is simply a finite group.

Proof. It is 0-dimensional and lies in the exact sequence

$$1 \longrightarrow \operatorname{Aut} E \longrightarrow \operatorname{Aut}(C_n, E)/\mathbb{G}_m \longrightarrow \operatorname{Aut} C_n/\mathbb{G}_m.$$

But $\operatorname{Aut} E = 1$ by Corollary 2.16, so the last arrow has trivial scheme-theoretic kernel, hence is a closed immersion [DG70, II 5.5.1b], and $\operatorname{Aut} C_n/\mathbb{G}_m \cong \mathbb{G}_m^n$. □

Corollary 2.18. *For* (b) *to hold is an open condition in families, and likewise for* (c).

Proof. By noetherian reduction, every bundle chain is obtained by base change from one with a locally noetherian base. The statement for (b) then follows immediately from the usual semicontinuity [EGA, III 7.7.5]. As for (c), the group scheme $\operatorname{Aut}(C, E)$ is of finite type over its base, say by Theorem 3.1 below. Hence the dimension at the identity section of $\operatorname{Aut}(C, E)$ is semicontinuous [EGA, IV 13.1.3]. □

Proposition 2.19. *For* $\beta = (\beta_1, \dots, \beta_n) \in \Lambda^n$*, let* $L \subset G$ *be the Levi subgroup centralizing every* β_i*, and let* $U_+, U_- \subset G$ *be the unipotent subgroups consisting of root spaces with some* $\alpha \cdot \beta_i < 0$ *(respectively* > 0*). Then evaluation at* $p_\pm$ *embeds the automorphism group* $\Gamma(\operatorname{Ad} E(\beta))$ *in* $G \times G$ *with image* $\Delta_L \ltimes (U_+ \times U_-)$.

Proof. By definition, the kernel of the evaluation map is $\operatorname{Aut} E(\beta)$, which is trivial by Corollary 2.16. Hence the evaluation is a closed immersion [DG70, II 5.5.1b]. It suffices, then, to show that the image of the derivative is the Lie algebra of the subgroup stated. This follows from the splitting (2.14) together with the observation that if α is a root of the Levi factor then the line bundle L_α is trivial, whereas, thanks to Remark 1.2(c), if α is a root of $U_\pm$, then L_α has a section nonvanishing at $p_\pm$ but vanishing at $p_\mp$. □

3. The stack of bundle chains

Let $\mathfrak{B}$ denote the category of G-bundle chains such that $\mathbb{G}_m$ acts trivially over p and $H^1(\operatorname{ad} E) = 0$ on each geometric fiber. As we have seen, for a chain over $\operatorname{Spec} K$, these conditions are equivalent to having the isomorphism class of $E(\beta)$, where $\beta = (\beta_1, \dots, \beta_n) \in \Lambda^n$ and β_i lie in a common Weyl chamber. For each such sequence β of weights, the versal family $\mathbb{E}(\beta) \to \mathbb{C}_n$ of Example 2.7 is an object of $\mathfrak{B}$ over $\mathbb{A}^n$.

Theorem 3.1. *The category* $\mathfrak{B}$ *is a smooth Artin stack containing* $BG \times B\mathbb{G}_m$ *as the dense open locus where the length of the chain is* $n = 1$*. An atlas is given by* $\bigsqcup_\beta \mathbb{E}(\beta) : \bigsqcup_\beta \mathbb{A}^{n(\beta)} \to \mathfrak{B}$*, where* β *runs over all finite sequences in* Λ *lying in a common Weyl chamber.*

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Proof. We must prove three things: (1) that $\mathfrak{B}$ is a stack, namely (a) a category fibered in groupoids, such that (b) descent data are effective and (c) automorphisms are a sheaf; (2) that the diagonal $\mathfrak{B} \to \mathfrak{B} \times \mathfrak{B}$ is (a) representable, (b) separated, and (c) finitely presented; and (3) that the atlas stated is surjective and smooth, so that the stack is algebraic.

Step 1. That all morphisms over the identity are isomorphisms is implicit in the definition of the category. Also, pullbacks exist and are unique up to isomorphism. Hence $\mathfrak{B}$ is a category fibered in groupoids, so (a) is true.

Next, suppose an étale morphism $\tilde{X} \to X$ and descent data for a bundle chain $\tilde{E} \to \tilde{C} \to \tilde{X}$ are given. Since a bundle chain is equipped with actions of $\mathbb{G}_m$ on $\tilde{E}$ and $\tilde{C}$, it is understood that the descent data preserve these actions. Since algebraic spaces of finite type are a stack [dJo, 04UD], $\tilde{E} \to \tilde{C}$ descend to algebraic spaces $E \to C \to X$, and the $\mathbb{G}_m$-actions descend as well. By Theorem 1.5 C is a chain over X. A principal G-bundle such as $\tilde{E}$ is by definition locally trivial in the étale topology, so the same is automatically true of $E \to C$. Hence (b) is true.

To see that automorphisms are a sheaf, observe that an automorphism of a bundle chain $E \to C \to X$ is an automorphism of the total space E subject to the closed condition that it preserve the projection to X and commute with the action of $\mathbb{G}_m \times G$. Again since algebraic spaces of finite type are a stack, automorphisms of such algebraic spaces are a sheaf, and hence the same is true of automorphisms of a bundle chain. This proves (c).

Step 2. Let E_1, E_2 be bundle chains over Y, corresponding to a morphism $Y \to \mathfrak{B} \times \mathfrak{B}$. It suffices to show that $\mathrm{Hom}(E_1, E_2)$ is represented by a scheme X, separated and finitely presented over Y. The families C_1, C_2 of chains underlying E_1, E_2 are induced by two morphisms $Y \to \mathfrak{C}$. Hence by Theorem 1.5 $\mathrm{Hom}(C_1, C_2)$ is represented by an algebraic space Z, separated and finitely presented over Y, with a tautological morphism $g : C_1 \times_Y Z \to C_2$. The G-bundle (or bitorsor) $\mathrm{Hom}(E_1, g^* E_2)$ over $C_1 \times_Y Z$ is locally trivial in the étale topology, hence separated and finitely presented over $C_1 \times_Y Z$. Isomorphisms $E_1 \to E_2$ are naturally equivalent to sections of this bundle, and as such are represented by an algebraic space, separated and locally of finite presentation over $C_1 \times_Y Z$ [Art69, 6.2], [Art74, Appendix].

All that remains for Step 2 is to show that the diagonal is quasi-compact. This is more easily accomplished after Step 3.

Step 3. Surjectivity follows from Corollary 2.16, so it remains only to prove smoothness.

Let E be a family of bundle chains over Y, corresponding to a morphism $Y \to \mathfrak{B}$. It suffices to show that $Y \times_{\mathfrak{B}} \mathbb{A}^n \to Y$ is smooth. By noetherian reduction we may assume that Y is locally noetherian. Since smoothness descends under étale covers [Vis05, 1.15] we may also assume that Y is a scheme. By Step 2 $Y \times_{\mathfrak{B}} \mathbb{A}^n$ is locally of finite presentation over $Y \times \mathbb{A}^n$ and hence over Y. So it suffices to show [EGA, IV 17.5.4] that, for a local Artinian k-algebra A, any ideal $J \subset A$, and any square

$$\begin{tikzcd} \operatorname{Spec} A/J \arrow[r] \arrow[d] & Y \times_{\mathfrak{B}} \mathbb{A}^n \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a diagonal arrow making the diagram commute. For such an A, we know $\mathfrak{m}^n = 0$ for some $n > 0$, and $J/\mathfrak{m}J, \mathfrak{m}J/\mathfrak{m}^2 J, \ldots, \mathfrak{m}^{n-1}J$ are finite-dimensional vector spaces over $A/\mathfrak{m}$. It therefore suffices to assume $\mathfrak{m}J = 0$. The map $\operatorname{Spec} A \to Y$ gives a family of bundle chains over A.

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Another such family may be obtained as follows. Since $\mathbb{A}^n$ is smooth and hence formally smooth over k, the morphism $\operatorname{Spec} A/J \to \mathbb{A}^n$ extends to $\operatorname{Spec} A \to \mathbb{A}^n$ by definition [EGA, IV 17.1.1, 17.3.1]. Pull back to $\operatorname{Spec} A$ the versal family $\mathbb{E}(\beta)$ of bundle chains over $\mathbb{A}^n$ from Example 2.7.

We now have two families of bundle chains over A with an isomorphism of their restrictions to A/J. It suffices to show that they are isomorphic over A. However, exactly as in Hartshorne [Har10, 6.2a] or Sernesi [Ser06, 1.2.12], the set of isomorphism classes of liftings of the bundle chain from A/J to A is a torsor for $H^1(\operatorname{ad} E) \otimes_{A/\mathfrak{m}} J = 0$. Hence the two liftings are isomorphic. This completes Step 3.

To finish Step 2, since $\bigsqcup_n \mathbb{A}^n \to \mathfrak{B}$ is surjective and smooth, base changing by $\bigsqcup_m \mathbb{A}^m \times \bigsqcup_n \mathbb{A}^n \to \mathfrak{B} \times \mathfrak{B}$, it suffices to show that $\mathbb{A}^m \times_{\mathfrak{B}} \mathbb{A}^n \to \mathbb{A}^m \times \mathbb{A}^n$ is quasi-compact [Vis05, 1.15]. Since $\mathbb{A}^m \times \mathbb{A}^n$ is affine, it suffices to show that $\mathbb{A}^m \times_{\mathfrak{B}} \mathbb{A}^n$ is quasi-compact [EGA, I 6.6.1]. Express $\mathbb{A}^m$ as a disjoint union over all $S \subset \{1, \dots, m\}$ of $\mathbb{A}^m_S$, the locus where $x_i = 0$ if and only if $i \in S$. Then $\mathbb{E}(\beta)|_{\mathbb{A}^m_S}$ is a constant family, so $\mathbb{A}^m_S \times_{\mathfrak{B}} \mathbb{A}^n_T \cong \mathbb{A}^m_S \times \mathbb{A}^n_T \times H$ where H (if nonempty) is the automorphism group of a bundle chain over $\operatorname{Spec} k$. The latter is an affine variety over k [MT12, 6.7]. Hence $\mathbb{A}^m_S \times_{\mathfrak{B}} \mathbb{A}^n_T$ is quasi-compact. But $\mathbb{A}^m \times_{\mathfrak{B}} \mathbb{A}^n$ is the image of a surjective continuous morphism from $\bigsqcup_{S,T} \mathbb{A}^m_S \times_{\mathfrak{B}} \mathbb{A}^n_T$, so it is quasi-compact. □

With the stacks $\mathfrak{C}$ and $\mathfrak{B}$ in hand, it is now easy to prove the following proposition.

Proposition 3.2. *Let $\mathbb{E} := \mathbb{E}(\beta) \to \mathbb{C}_n$ as in Example 2.7. Then the stack H_β of automorphisms of $\mathbb{E}$ lying over the identity on $\mathbb{C}_n$ is represented over $\mathbb{A}^n$ by a smooth closed subgroup scheme of $\mathbb{A}^n \times G \times G$.*

Proof. To establish that H_β is represented by a smooth scheme, it suffices to show that the natural morphism $\mathbb{A}^n \times_{\mathfrak{B}} \mathbb{A}^n \to \mathbb{A}^n \times_{\mathfrak{C}} \mathbb{A}^n$ is smooth in a neighborhood of the diagonal $\mathbb{A}^n \to \mathbb{A}^n \times_{\mathfrak{C}} \mathbb{A}^n$; for $H_\beta \to \mathbb{A}^n$ is nothing but the base change by this diagonal.

Since $\mathbb{A}^n \to \mathfrak{B}$ and $\mathbb{A}^n \to \mathfrak{C}$ are components of the atlases for $\mathfrak{B}$ and $\mathfrak{C}$, they are smooth; hence $\mathbb{A}^n \times_{\mathfrak{B}} \mathbb{A}^n$ and $\mathbb{A}^n \times_{\mathfrak{C}} \mathbb{A}^n$ are smooth over $\mathbb{A}^n$, and hence over k. Also, since the diagonals of $\mathfrak{B}$ and $\mathfrak{C}$ are separated and finitely presented, $\mathbb{A}^n \times_{\mathfrak{B}} \mathbb{A}^n$ and $\mathbb{A}^n \times_{\mathfrak{C}} \mathbb{A}^n$ are separated and finitely presented over $\mathbb{A}^n \times \mathbb{A}^n$ and hence over k. In other words, they are smooth varieties.

The dimension of $\mathfrak{C}$ is -1, so the obvious morphism $\mathbb{A}^n \times \mathbb{G}_m^{n+1} \to \mathbb{A}^n \times_{\mathfrak{C}} \mathbb{A}^n$ coming from the $\mathbb{G}_m^{n+1}$-action on $\mathbb{C}_n$ is an open immersion. The image of the diagonal $\mathbb{A}^n \to \mathbb{A}^n \times_{\mathfrak{C}} \mathbb{A}^n$ lies in this open set.

Since the action of $\mathbb{G}_m^{n+1}$ on $\mathbb{C}_n$ lifts to $\mathbb{E}$, there are compatible actions of $\mathbb{G}_m^{n+1}$ on $\mathbb{A}^n \times_{\mathfrak{B}} \mathbb{A}^n$ and $\mathbb{A}^n \times_{\mathfrak{C}} \mathbb{A}^n$. On the open set $\mathbb{A}^n \times \mathbb{G}_m^{n+1}$ mentioned above, the action is by multiplication on the second factor.

Since we have a morphism of smooth varieties, to show it is smooth over $\mathbb{A}^n \times \mathbb{G}_m^{n+1}$, it suffices to show that its derivative is everywhere surjective. The compatible group actions show that the image of the derivative contains the tangent space to $\mathbb{G}_m^{n+1}$. But the smoothness of $\mathbb{A}^n \times_{\mathfrak{B}} \mathbb{A}^n$ over $\mathbb{A}^n$ shows that the image of the derivative also contains a complement to this tangent space.

Consequently, H_β is represented by a smooth variety, indeed a group scheme smooth over $\mathbb{A}^n$. Now let us show that this variety admits a closed immersion (of group schemes) into $\mathbb{A}^n \times G \times G$. The immersion is given by restricting automorphisms of $\mathbb{E}$ to the endpoints $p_\pm(\mathbb{A}^n)$, at which $\mathbb{E}$ is given the conventional framings of Example 2.7. This defines the morphism $H_\beta \to \mathbb{A}^n \times G \times G$. To show it is a closed immersion, it suffices [DG70, II 5.5.1b] to prove that it has trivial scheme-theoretic kernel, hence [DG70, II 5.1.4] that the kernel is connected with trivial Lie algebra. That the Lie algebra is trivial follows from (2.15), while connectedness is proved in another paper of the authors [MT12, 6.7]. □

Remark 3.3. Since H_β is a smooth family of groups over $\mathbb{A}^n$, with connected fibers [MT12], it is connected. Over the locus $U \subset \mathbb{A}^n$ where $x_1 \cdots x_n \neq 0$, it is easy to identify. For $\mathbb{E}$ is trivial there, as explained in Example 2.7. Hence $H_\beta|_U \cong U \times G$. Its immersion into $\mathbb{A}^n \times G \times G$, determined by the conventional framings of Example 2.7, is $(x, g) \mapsto (x, g, x^{-\beta} g\, x^\beta)$, where $x^\beta = \prod x_i^{\beta_i}$ as in Example 2.7.

To get a Deligne–Mumford or tame Artin stack, we need to adjust $\mathfrak{B}$ in three ways: (1) rigidify to get rid of the uniform $\mathbb{G}_m$-action; (2) pass to the $G \times G$-bundle parametrizing bundle chains equipped with framings at $p_\pm$; and (3) restrict to the open substack where Theorem 2.13(c) holds. Of these, (3) is self-explanatory. Also, (2) is straightforward since $\mathfrak{B}$ clearly admits a universal family $\pi : C \to \mathfrak{B}$ with sections $p_\pm : \mathfrak{B} \to C$ and a universal G-bundle $E \to C$; the stack we want is nothing but $p_+^* E \times_{\mathfrak{B}} p_-^* E$. It remains to explain (1), which runs along the lines laid down by Abramovich *et al.* [ACV03].

Since the standard $\mathbb{G}_m$-action on chains acts on all of our G-bundles by hypothesis, we may rigidify our moduli problem to remove the common $\mathbb{G}_m$-symmetry. This may be accomplished, as in the theory of stable vector bundles, by introducing an equivalence on families slightly broader than isomorphism. Strictly speaking we need to take the stack associated to this prestack [ACV03, § 5.1].

Given a chain $C \to X$ and a line bundle $L \to X$, define a new chain, denoted $C \otimes L$, to be the geometric quotient $(C \times_B L^\times)/\mathbb{G}_m$. That this satisfies the requirements of being a chain over X is easy to check: indeed, over each local trivialization of L, it is isomorphic to C. Thanks to the $\mathbb{G}_m$-equivariance, there is a canonical equivalence between bundle chains over C and those over $C \otimes L$. The desired prestack is the one parametrizing bundle chains up to this equivalence.

Let $\mathcal{M}$ be the stack obtained from $\mathfrak{B}$ by performing operations (1–3) above. It is the stack associated to the prestack parametrizing bundle chains, framed at $p_\pm$, satisfying Theorem 2.13(c), modulo the equivalence described above. It is easily verified that the $\mathbb{G}_m$-action on $\mathcal{M}$ induced by the standard $\mathbb{G}_m$-action is trivial. The same is true, by the way, of the stack $\mathcal{N}$ of rigidified chains obtained from $\mathfrak{B}$ by performing operation (1), and there is a natural morphism $\mathcal{M} \to \mathcal{N}$.

By definition the *inertia stack* is $I\mathcal{M} := \mathcal{M} \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M}$. It is naturally isomorphic to the stack whose objects are pairs consisting of an object of $\mathcal{M}$ and an automorphism of that object, and whose morphisms are the obvious ones. The two projections induce naturally 2-isomorphic maps $I\mathcal{M} \to \mathcal{M}$, which we regard as the same.

Theorem 3.4. *The stack $\mathcal{M}$ has finite inertia: that is, $I\mathcal{M} \to \mathcal{M}$ is finite.*

Proof. The fiber of $I\mathcal{M} \to \mathcal{M}$ over a bundle chain $E \to C \to X$ is represented by the scheme $\mathrm{Aut}(C, E)$. We must show it is finite over X. That it is quasi-finite follows immediately from Theorem 2.13(c). It therefore suffices to show that $I\mathcal{M} \to \mathcal{M}$ is proper. This we accomplish with the valuative criterion.

Since $\mathcal{M}$ has a locally noetherian atlas (whose connected components are $\mathbb{A}^n \times G \times G$) it is locally noetherian by definition [LM00, 4.7.1]. Since it is algebraic, the diagonal $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable and finitely presented, hence so is its base change $I\mathcal{M} \to \mathcal{M}$. According to Laumon and Moret-Bailly [LM00, 7.8, 7.12], it suffices to show that for any complete discrete valuation ring R over k with fraction field F, any commutative diagram in the form of the right-hand square of

$$\begin{tikzcd}
\operatorname{Spec}\tilde{F} \arrow[r] \arrow[d] & \operatorname{Spec}F \arrow[r] \arrow[d] & I\mathcal{M} \arrow[d] \\
\operatorname{Spec}\tilde{R} \arrow[r] \arrow[urr, dashed] & \operatorname{Spec}R \arrow[r] & \mathcal{M}
\end{tikzcd}$$

may be extended by adding a left-hand square (where $\tilde{R}$ is a discrete valuation ring containing R whose residue field $\tilde{F}$ is an extension of F) and a morphism as indicated by the dashed arrow shown, unique up to 2-isomorphism. By the Cohen structure theorem [Har77, I 5.5] $R \cong K[[t]]$ and $F \cong K((t))$ for some extension K/k. Since $K[[t]] = \lim_{n\to\infty} K[t]/(t^n)$, the morphism $\operatorname{Spec} R \to \mathcal{M}$ may be lifted to $f : \operatorname{Spec} R \to \mathbb{A}^n \times G \times G$, a connected component of the smooth atlas for $\mathcal{M}$. Let $\mathbb{E}(\beta) \to \mathbb{C}_n \to \mathbb{A}^n$ be the corresponding bundle chain. Then $E := f^*\mathbb{E}(\beta)$ is a framed bundle chain over $K[[t]]$; we must show that any given automorphism of $E_\eta := E|_{K((t))}$ extends uniquely to all of E.

Let $i_1, \dots, i_m$ denote those coordinates of $f_1 : \operatorname{Spec} R \to \mathbb{A}^n$ which vanish identically. There may be no such coordinates, but if there are, then the chain $C := f^*\mathbb{C}_n$ will be reducible. In any case, its generic fiber C_η consists of $m+1$ lines, and C itself consists of $C = C^{(1)} \cup \cdots \cup C^{(m+1)}$ glued end to end.

The given automorphism of E_η is a $K((t))$-valued point of $\operatorname{Aut}(C_n, E)/\mathbb{G}_m$, which by Corollary 2.17 is diagonalizable and 0-dimensional, so it is of finite order, say n, prime to the characteristic. Consequently, E_η together with its given automorphism may be regarded as a bundle chain over $[C_{m+1}/(\mathbb{G}_m \times \mu_n)]$.

Over some extension $\tilde{K}$ of K, the structure group of such an object reduces to T. The proof of this is the same as [MT12, 4.3, 6.4], with two exceptions. Part B of 4.3 is replaced by an easy argument: having a $\mathbb{G}_m$-action trivializes any G-bundle over the dense $\mathbb{G}_m$-orbit in $[\mathbb{P}^1/\mathbb{G}_m]$, and the μ_n-action there, being essentially a homomorphism $\mu_n \to G$, may then be conjugated into T [MT12, 7.1], so the associated G/B-bundle has a section over this dense orbit, which extends to a regular section over $[\mathbb{P}^1/\mathbb{G}_m]$ by the valuative criterion. And in 6.4, the centralizer $Z(\chi(\mathbb{G}_m \times \mu_n))$ need not be connected, but as the centralizer of a torus in a reductive group, its identity component is reductive [CGP10, A.8.12], so, over some extension $\tilde{K}$, it has a Bruhat decomposition of the form BWB just like a (connected) reductive group.

Consequently, once E_η is reduced to T as a bundle chain over $[C_{m+1}/\mathbb{G}_m]$, the given automorphism is uniquely determined by its triviality over $p_\pm$ together with the automorphism of $C_\eta \cong C_{m+1}$ underlying it.

Suppose the latter is given by the $m+1$-tuple $(z_0, \dots, z_m)$ for $z_i \in K((t))^\times$. Each z_i extends to a uniform action on $C^{(i)}$ in the sense of Theorem 1.6. Indeed, each $C^{(i)}$ is the base change of a versal chain by $\pi f_1 : \operatorname{Spec} R \to \mathbb{A}^{k_j}$, where $k_j = i_{j+1} - i_j - 1$ and $\pi : \mathbb{A}^n \to \mathbb{A}^{k_j}$ is projection on the coordinates between i_j and i_{j+1}. Let $\mathbb{F}^{(i)}$ be the T-bundle over C_{k_j} associated to $\mathbb{L}_0^\times \times \cdots \times \mathbb{L}_{k_j+1}^\times$ via the homomorphism $(\beta_{i_j}, \dots, \beta_{i_{j+1}}) : \mathbb{G}_m^{k_j+2} \to T$, and let $\mathbb{E}^{(i)}$ be the associated G-bundle. Then $(\pi f_1)^*\mathbb{E}^{(i)} \cong E|_{C^{(i)}}$. Since the uniform action of $\mathbb{G}_m$ on $\mathbb{A}^{k_j}$ lifts to $\mathbb{F}^{(i)}$ and thence to $\mathbb{E}^{(i)}$, the uniform action of z_i on $C^{(i)}$ lifts to $E|_{C^{(i)}}$.

These actions have compatible weights at the endpoints of $C^{(i)}$. So they glue to give an automorphism of E, which reduces to T and hence extends the one given over E_η. Two such automorphisms agree on E_η and hence everywhere, giving the required uniqueness. □

An Artin stack with finite inertia is defined by Abramovich *et al.* [AOV08] to be *tame* if the push-forward of quasi-coherent sheaves to the coarse moduli space is an exact functor. They prove [AOV08, 3.2] that an Artin stack with finite inertia is tame if and only if every geometric point has linearly reductive stabilizer. In particular, an Artin stack over a field of characteristic zero is tame if and only if it is Deligne–Mumford.

Theorem 3.5. *The stack $\mathcal{M}$ is tame, and the open substack where condition Theorem 2.13(b) holds is Deligne–Mumford.*

Proof. If $E \to C$ is a bundle chain over a field, then there is a short exact sequence of group schemes

$$1 \longrightarrow \operatorname{Aut} E \longrightarrow \operatorname{Aut}(C, E) \longrightarrow \operatorname{Aut} C.$$

But $\operatorname{Aut} E$ is trivial by Corollary 2.16, so $\operatorname{Aut}(C, E) \to \operatorname{Aut} C$ is a closed immersion [DG70, II 5.5.1b]. Hence $\operatorname{Aut}(C, E)$ is a subgroup scheme of a torus, so it is diagonalizable and hence linearly reductive.

For a stack with finite inertia to be Deligne–Mumford, it suffices [AOV08] that every geometric point have reduced stabilizer. But that is exactly what is guaranteed by Theorem 2.13(b). $\square$

Proposition 3.6. *Neither $\mathfrak{B}$ nor $\mathcal{M}$ is separated unless $G = 1$.*

Proof. First let $G = \mathbb{G}_m$, so that $\Lambda = \mathbb{Z}$. Let x be the coordinate on $\mathbb{A}^1$, and consider the chain $\mathbb{C}_1 = \mathrm{Bl}(\mathbb{A}^1 \times \mathbb{P}^1, 0 \times [0, 1])$ over $\mathbb{A}^1$. For any $i \in \mathbb{Z}$, the functions 1 and x^i provide framings, at p_+ and p_- respectively, for the $\mathbb{G}_m$-bundle $\mathbb{L}_1^i = \mathcal{O}(-iE)$, where E is the exceptional divisor of the blow-up. Let $\mathbb{L}$ be the bundle chain $\mathbb{L}_1^2$ with this framing, but let $\mathbb{L}'$ be the base change of $\mathbb{L}_1$ by $x \mapsto x^2$ with the induced framing. Then the nonzero fibers of $\mathbb{L}$ and $\mathbb{L}'$ are isomorphic as framed bundle chains away from 0, but the fibers at 0 are not: rather, they are $\mathcal{O}(0 \mid 2 \mid 0)$ and $\mathcal{O}(0 \mid 1 \mid 0)$ respectively.

Consequently, $(\mathbb{L}, \mathbb{L}')$ defines a morphism $\mathbb{A}^1 \to \mathcal{M} \times \mathcal{M}$ such that $\mathbb{A}^1 \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M} = \mathbb{A}^1 \backslash 0$, which is not proper over $\mathbb{A}^1$. For $G = \mathbb{G}_m$, consequently, $\mathcal{M}$ is not proper over $\mathcal{M} \times \mathcal{M}$, and hence $\mathcal{M}$ is not separated. Exactly the same families define morphisms to $\mathfrak{B}$ and show that it is not separated either.

For general reductive $G \neq 1$, let $\beta : \mathbb{G}_m \to G$ be a nontrivial cocharacter. Extension of structure group by β converts the $\mathbb{G}_m$-example above into a G-example showing that $\mathfrak{B}$ and $\mathcal{M}$ are not separated in this case either. $\square$

Changing the framings at $p_\pm$ defines an action of $G \times G$ on $\mathcal{M}$. Each orbit is the locus where the isomorphism class of the unframed bundle chain is fixed. For example, the orbit of $E = \mathcal{O}_{\mathbb{P}^1}$ has isotropy group $\{(g, g^{-1}) \mid g \in G\}$ and may therefore be identified with G. Clearly this action lifts to the obvious action on $\mathbb{A}^{n(\beta)} \times G \times G$, the component of the atlas for $\mathcal{M}$ obtained from the versal chain $\mathbb{E}(\beta)$.

Proposition 3.7. *Every $G \times G$-orbit closure in $\mathcal{M}$ is smooth, and every intersection of two orbit closures is in a normal crossing. In particular, the complement $\mathcal{M}\backslash G$ of the dense orbit G is a divisor with normal crossings.*

Proof. The same is true of the preimages of the orbits in each component $\mathbb{A}^{n(\beta)} \times G \times G$ of the atlas for $\mathcal{M}$, since these preimages are nothing but the orbits of the usual action of $\mathbb{G}_m^{n(\beta)}$ on $\mathbb{A}^{n(\beta)}$. $\square$

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4. The stability condition

We now cut down our stack to a separated stack of finite presentation by imposing some sort of stability condition. This requires that we make some non-canonical choices.

First, as in toric geometry, choose a fan Σ, supported within $\Lambda_{\mathbb{Q}}^{+}$, consisting of rational simplicial cones. Also choose a nonzero lattice element $\beta_i \in \Lambda$ on each ray of Σ. The choice of a distinguished lattice point on each ray is precisely the additional structure making a fan into a *torsion-free stacky fan* as defined by Borisov *et al.* [BCS05], who established a correspondence between such objects and toric orbifolds. (The slightly more general notion introduced by Borisov–Chen–Smith of a *stacky fan*, possibly with torsion, will not be needed here.)

Second, choose an ordering $\beta_1, \dots, \beta_N \in \Lambda$ of these elements, and hence of all the rays of Σ. This induces an ordering of the rays of any $\sigma \in \Sigma$. Thus to any n-dimensional $\sigma \in \Sigma$ is associated an increasing function, by abuse of notation also denoted $\sigma : \{1, \dots, n\} \to \{1, \dots, N\}$, such that $\sigma = \langle \beta_{\sigma(1)}, \dots, \beta_{\sigma(n)} \rangle$.

Definition 4.1. A bundle chain over a field is said to be Σ-*stable* if it is isomorphic to the bundle chain $E(\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)})$ for some n-dimensional cone $\sigma \in \Sigma$. An arbitrary bundle chain is Σ-*stable* if its geometric fibers are Σ-stable.

Strictly speaking, this depends not only on the stacky fan Σ, but also on the ordering of its rays. Notice that a Σ-stable bundle chain automatically satisfies Theorem 2.13(c), hence belongs to $\mathcal{M}$.

Theorem 4.2. *The locus of Σ-stable bundle chains is an open $G\times G$-invariant substack $\mathcal{M}_G(\Sigma) \subset \mathcal{M}$, of finite presentation over k, which contains G as a dense open substack. It is always separated and is proper if and only if the support of Σ equals $\Lambda_{\mathbb{Q}}^{+}$.*

Proof. For any Σ-stable $E(\beta) \to C_n$ over k, the versal chain $\mathbb{E}(\beta)$ over $\mathbb{A}^n \times G \times G$ is also Σ-stable. The Σ-stable locus is therefore open.

Finitely many such families now constitute an atlas, one for each cone in Σ. The stack is therefore of finite presentation over k.

It is clear from this atlas that the smaller locus where the chain has no nodes is a dense open substack. There the bundle must be trivial, as it is acted on with weight 0 over $p_{\pm}$. A global trivialization is determined by the framing at p_+, relative to which the framing at p_- induces a natural isomorphism of this substack with G.

It remains to establish separation and properness. This will be accomplished by reducing to a split maximal torus $T \subset G$ and invoking the corresponding facts about toric stacks.

To prove separation, it suffices [LM00, 7.8] to show, for any complete discrete valuation ring R over k with fraction field F and any two morphisms $f_1, f_2 : \operatorname{Spec} R \to \mathcal{M}_G(\Sigma)$, that any 2-isomorphism $f_1|_{\operatorname{Spec} F} \to f_2|_{\operatorname{Spec} F}$ extends to a 2-isomorphism $f_1 \to f_2$. That is, if E_1 and E_2 are the framed bundle chains over $\operatorname{Spec} R$ induced by f_1 and f_2, then any isomorphism $E_1 \cong E_2$ over $\operatorname{Spec} F$ extends over $\operatorname{Spec} R$.

As in the proof of Theorem 3.4, $R \cong K[[t]]$ and $F \cong K((t))$ for some extension field K/k. Moreover, following Deligne–Mumford [DM69, 4.19], [EGA, II 7.2.3], one may assume that the image $f_1(\eta) \cong f_2(\eta)$ of $\eta = \operatorname{Spec} K((t))$ lies in the dense open substack $G \subset \mathcal{M}_G(\Sigma)$.

For any $f : \operatorname{Spec} K((t)) \to G$, there exists an unique 1-parameter subgroup $\lambda : \operatorname{Spec} K((t)) \to T$ in the positive Weyl chamber, and morphisms $\gamma, \delta : \operatorname{Spec} K[[t]] \to G$, such that $f = \gamma\lambda\delta^{-1}$. This was originally proved by Iwahori and Matsumoto [IM65]. It is best regarded as being essentially the Bruhat decomposition for the formal loop group $G((t))$ whose points are morphisms

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$\operatorname{Spec} K((t)) \to G$. To see this, let $\mathcal{G} = G((t)) \rtimes \mathbb{G}_m$, let $\mathcal{B}$ be the inverse image of the Borel B under the evaluation map $G[[t]] \rtimes \mathbb{G}_m \to G$, let $\mathcal{N} = \Lambda \rtimes (N(T) \times \mathbb{G}_m)$, and let $\mathcal{S}$ be the usual generators for the affine Weyl group $\tilde{W} = \Lambda \rtimes W$. Then $\mathcal{G}$, $\mathcal{B}$, $\mathcal{N}$, $\mathcal{S}$ satisfy [Kum02, 6.2.8] the requirements to be a Tits system [Kum02, 5.1.1], and, since $\Lambda^+ = W\backslash\tilde{W}/W$, we obtain [Kum02, 5.1.3c] the Bruhat decomposition

$$G((t)) = \bigsqcup_{\lambda \in \Lambda^+} G[[t]]\ \lambda\ G[[t]].$$

Taking $f = f_1$ in the above, and then replacing f_1 and f_2 by $\gamma^{-1} f_1 \delta$ and $\gamma^{-1} f_2 \delta$, we may assume that $f_i|_{\operatorname{Spec} K((t))}$ is a 1-parameter subgroup λ of T, indeed in the positive Weyl chamber.

We claim that in fact f_1 and f_2 factor through $\mathcal{M}_T(\Sigma)$, the moduli stack with structure group T. To establish this, we will first lift them to the atlas for $\mathcal{M}_G(\Sigma)$, then adjust the lifts to go into the atlas for $\mathcal{M}_T(\Sigma)$.

The connected components of the atlas for $\mathcal{M}_G(\Sigma)$ are of the form $\mathbb{A}^n \times G \times G$ for $n = n(\beta)$. Choose a point $(x(0), g(0), h(0))$ in one of them lying over $f_1(0) \in \mathcal{M}_G(\Sigma)$. By a suitable choice of component we may assume that $x(0) = 0$.

Since $K[[t]]$ is a limit of Artinian K-algebras, we may lift f_1 to $\tilde{f}_1 = (x, g, h) : \operatorname{Spec} K[[t]] \to \mathbb{A}^n \times G \times G$. Then $\tilde{f}_1(\eta) \subset U \times G \times G$, where $U \subset \mathbb{A}^n$ is the locus where $x_1 \cdots x_n \neq 0$, for this is the inverse image of $G \subset \mathcal{M}_G(\Sigma)$. Indeed, using the conventional framings of Example 2.7 we see that the morphism $U \times G \times G \to G$ is $(x, g, h) \mapsto g^{-1} x^\beta h$.

As pointed out in Example 2.7, the action of $\mathbb{G}_m^n$ on $\mathbb{E}(\beta) \to \mathbb{C}_n \to \mathbb{A}^n$ fixes the conventional framing at p_+ but changes it at p_-. Hence there is an action of $\mathbb{G}_m^n$ on $\mathbb{A}^n \times G \times G$ that projects to the obvious action on the first factor, acts trivially on the second factor, acts nontrivially on the third factor, and lifts to the framed bundle chain $\mathbb{E}(\beta)$. It therefore preserves the morphism to $\mathcal{M}_G(\Sigma)$.

Hence we may adjust the lift $\tilde{f}_1$ by acting by any morphism $z : \operatorname{Spec} K[[t]] \to \mathbb{G}_m^n$. That is to say, $z\tilde{f}_1$ is another lift of f_1. We may thus assume that the first factor of $\tilde{f}_1$, namely $x : \operatorname{Spec} K((t)) \to U$, is a 1-parameter subgroup $t \mapsto (t^{i_1}, \ldots, t^{i_n})$, with each $i_j > 0$, when U is identified with $\mathbb{G}_m^n$.

We then have $\lambda = g^{-1}\, x^\beta\, h$ where both λ and x^β are 1-parameter subgroups in the positive Weyl chamber Λ^+. By the uniqueness in the Iwahori decomposition, it follows that $x^\beta = \lambda$. Hence $h = x^\beta\, g\, x^{-\beta}$.

Comparison with Remark 3.3 now reveals that (g, h) is a regular section over $\operatorname{Spec} K[[t]]$ of $\tilde{f}_1^* H_\beta \subset \operatorname{Spec} K[[t]] \times G \times G$. Acting by the inverse of this automorphism takes the framing to a trivial framing. Hence the pullbacks of $\mathbb{E}(\beta)$ by the morphisms $\tilde{f}_1 = (x, g, h)$ and $(x, 1, 1)$ are isomorphic as framed bundle chains over $\operatorname{Spec} K[[t]]$. The latter is therefore also a lift of f_1.

In other words, we may assume that $\tilde{f}_1 = (x, 1, 1)$ for some 1-parameter subgroup $x : \operatorname{Spec} K[[t]] \to \mathbb{G}_m^n$. Likewise, we may assume a similar lift for f_2. Thus the structure group of E_1 and E_2, even with their framings, is reduced to T. This establishes our claim that f_1 and f_2 factor through $\mathcal{M}_T(\Sigma)$.

But the latter is easily seen to be the toric stack $\mathcal{X}(\Sigma)$ associated to the stacky fan Σ by Borisov *et al.* [BCS05]. Indeed, the atlas consists of components $\mathbb{A}^n \times T \times T$, but by Remark 3.3, the automorphism group H_β reduces, when $G = T$, to the constant family consisting of the diagonal $\Delta_T \subset T \times T$. Since this acts by framed bundle automorphisms, we may descend to the quotient $(\mathbb{A}^n \times T \times T)/\Delta_T \cong \mathbb{A}^n \times T$. This is essentially the toric atlas of Borisov *et al.* [BCS05, 4.4]: indeed, there is a further automorphism group $\mathbb{G}_m^n$, and $[(\mathbb{A}^n \times T)/\mathbb{G}_m^n] = [\mathbb{A}^n/K]$, where

$K = \ker\beta : \mathbb{G}_m^n \to T$ is a torus times the finite group called $N(\sigma)$ by Borisov–Chen–Smith. (In positive characteristic this is a zero-dimensional subgroup scheme of $\mathbb{G}_m^n$.)

Since $\mathcal{X}(\Sigma) = \mathcal{M}_T(\Sigma)$ is tame, the natural morphism to $X(\Sigma)$, its coarse moduli space [BCS05, 3.7], is proper [Con05, 1.1]. Since $X(\Sigma)$, being a toric variety, is certainly separated [KKMS73, I § 2], it finally follows that $f_1 \cong f_2$ in $\mathcal{M}_T(\Sigma)$ and hence in $\mathcal{M}_G(\Sigma)$. Hence the latter is separated.

To prove properness, proceed as in the proof of separation and observe that the limits of all 1-parameter subgroups in the positive Weyl chamber Λ^+ exist in $X(\Sigma)$, and hence in $\mathcal{X}(\Sigma)$ which is proper over it, if and only if the support of Σ is $\Lambda^+_{\mathbb{Q}}$. □

A connected atlas for $\mathcal{M}_G(\Sigma)$ may be exhibited as follows. Let $\beta := (\beta_1, \dots, \beta_N)$, and let $\mathbb{E}(\beta)$ be the corresponding versal bundle chain. Also let $\mathbb{A}_\beta := \mathbb{A}^N$, and let $\mathbb{A}_\beta \times G \times G$ parametrize all framings at $p_\pm$ of bundle chains in $\mathbb{E}(\beta)$ by mapping $(x, g_+, g_-) \mapsto \Psi_\pm \circ g_\pm$, where $\Psi_\pm : \mathbb{A}_\beta \times G \to p_\pm^*\mathbb{E}(\beta)$ are the conventional framings of Example 2.7. The pullback $\tilde{\mathbb{E}}(\beta)$ of $\mathbb{E}(\beta)$ to $\mathbb{A}_\beta \times G \times G$ is then tautologically framed.

While $\tilde{\mathbb{E}}(\beta)$ is not Σ-stable everywhere, it is Σ-stable over an open subset. As in Example 1.4(c), for any $S \subset \{1, \dots, N\}$, let $U_S = \{x \in \mathbb{A}^N \mid x_i \neq 0 \text{ if } i \notin S\}$. Let $\mathbb{A}^0_\Sigma$ be the union of all U_S such that $\langle \beta_i \mid i \in S\rangle$ is a cone in Σ. This precisely covers the permitted splitting types, so the restriction of $\tilde{\mathbb{E}}(\beta)$ to $\mathbb{A}^0_\Sigma \times G \times G$ is Σ-stable.

Proposition 4.3. *The morphism $\mathbb{A}^0_\Sigma \times G \times G \to \mathcal{M}_G(\Sigma)$ defined by the framed bundle chain $\tilde{\mathbb{E}}(\beta)$ is smooth and surjective, providing a connected atlas for $\mathcal{M}_G(\Sigma)$.*

Proof. Surjectivity is clear, since the bundle chains that appear are precisely those permitted by Σ-stability. As for smoothness, observe that by Example 1.4(c), the projection $\pi_S : U_S \to \mathbb{A}^{|S|}$ induces an isomorphism $\mathbb{C}_N|_{U_S} \cong \pi_S^*\mathbb{C}_{|S|}$; since the line bundles $\mathbb{L}_i$ correspond under this isomorphism, it follows that $\mathbb{E}(\beta)|_{U_S} \cong \pi_S^*\mathbb{E}(\beta_S)$, where β_S is the subsequence of β indexed by S. The restriction of $\mathbb{A}^0_\Sigma \times G \times G \to \mathcal{M}_G(\Sigma)$ to $U_S \times G \times G$ therefore factors as a composition of π_S with the morphism $\mathbb{A}^{|S|} \times G \times G \to \mathcal{M}_G(\Sigma)$ provided by the obvious atlas for $\mathcal{M}_G(\Sigma)$, both of which are smooth. □

If the structure group G is a torus, then $\mathcal{M}_G(\Sigma)$ is a toric stack. Indeed, we have seen in the proof of Theorem 4.2 that it is the toric stack $\mathcal{X}(\Sigma)$ associated to the stacky fan Σ by Borisov *et al.* [BCS05], generalizing Cox [Cox95]. Moreover, the connected atlas of Proposition (4.3) coincides in that case with their global quotient construction.

Theorem 4.4. *The closure of T in $\mathcal{M}_G(\Sigma)$ is $\overline{T} = \mathcal{X}(W\Sigma)$, the toric stack associated to the Weyl-invariant stacky fan $W\Sigma$.*

Proof. First, $\mathcal{M}_T(W\Sigma) = \mathcal{X}(W\Sigma)$ by the remark above.

Second, note that since $E(w\beta) \cong E(\beta)$ as G-bundles, the framed T-bundle chains parametrized by $\mathcal{M}_T(W\Sigma)$ are all Σ-stable G-bundle chains, determining a representable morphism $\mathcal{M}_T(W\Sigma) \to \mathcal{M}_G(\Sigma)$.

Third, since a proper, unramified, and universally injective morphism is a closed embedding [dJo, 02K5], it suffices to show that this representable morphism enjoys these three properties.

Properness is immediate when the support of Σ is all of $\Lambda^+_{\mathbb{Q}}$, for then both $\mathcal{M}_T(W\Sigma)$ and $\mathcal{M}_G(\Sigma)$ are proper over k by Theorem 4.2. For general Σ, extend Σ to a stacky simplicial fan $\overline{\Sigma}$ whose support is $\Lambda^+_{\mathbb{Q}}$ and observe that by Corollary 2.11, the inverse image in $\mathcal{M}_T(W\overline{\Sigma})$ of $\mathcal{M}_G(\Sigma) \subset \mathfrak{M}_G(\overline{\Sigma})$ is $\mathcal{M}_T(W\Sigma)$.

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Since $\mathcal{M}_T(W\Sigma)$ and $\mathcal{M}_G(\Sigma)$ are separated and finitely presented over k, our morphism is finitely presented [EGA, IV 1.6.2], hence is unramified if and only if it is formally unramified. It therefore suffices, as in the proof of Theorem 1.5, to show that the derivative is everywhere injective. For this we recall the description of the deformation space in terms of the sheaf $\mathcal{S}$ in Proposition 2.12. Let $F \to C$ be a framed T-bundle chain over a geometric point of $\mathcal{M}_T(W\Sigma)$, and let $E \to C$ be the associated G-bundle chain in $\mathcal{M}_G(\Sigma)$. Let $\mathcal{S}_G$ and $\mathcal{S}_T$ denote the sheaves on C of G-invariant (respectively T-invariant) Kähler differentials on E (respectively F). Then there is a short exact sequence

$$0 \longrightarrow \bigoplus_{\alpha\in\Phi} L_\alpha \longrightarrow \mathcal{S}_G \longrightarrow \mathcal{S}_T \longrightarrow 0,$$

where Φ denotes the roots of G and L_α is the line bundle defined in the proof of Proposition 2.12. By that result, the relevant derivative is the induced map $\operatorname{Ext}^1_C(\mathcal{S}_T, \mathcal{O}(-p)) \to \operatorname{Ext}^1_C(\mathcal{S}_G, \mathcal{O}(-p))$. Since $H^0(L_\alpha(-p)) = 0$ as shown in the proof of Proposition 2.12, it follows that the derivative is injective.

Finally, to see that the morphism is universally injective, let $F, F' \to C_n$ be two $W\Sigma$-stable framed T-bundles over a field, and suppose their extensions E, E' to G are isomorphic as framed G-bundles. Let $g : E \to E'$ be the isomorphism taking framings to framings. By Corollary 2.11, the splitting types of E and E' agree modulo the Weyl group. That is, there is an isomorphism f of the associated $N(T)$-bundles, though not necessarily compatible with the framings. The automorphism $f^{-1} \circ g$ of E then preserves, at the points $p_\pm$, the $N(T)$-structure induced by F. Hence it must lie in $\Delta_{L\cap N(T)} \subset \Delta_L$ by Proposition 2.19, where L is the Levi subgroup described there. Since L centralizes each β_i, it is simply the automorphism globally induced by a fixed element of $L \cap N(T)$. The structure group of $f^{-1} \circ g$, and hence of g, thus reduces to $N(T)$ everywhere on the chain C_n, meaning that g, viewed as an isomorphism from the total space of E to that of E', takes the total space of the $N(T)$-bundle containing F to the one containing F'. The total spaces of these $N(T)$-bundles contain F and F' as connected components, and we know that $g(F) = F'$ since the framings lie there and are preserved by g. Hence the structure group of g reduces to T, that is, $F \cong F'$ as framed bundles. □

5. The Vinberg monoid

We now turn to a different description of our moduli stacks: as global quotients, by a torus action, of an open subset in an algebraic monoid. As an application, we deduce in § 7 necessary and sufficient conditions for their coarse moduli spaces to be projective.

Our description generalizes Vinberg's description [Vin95] of the wonderful compactification of an adjoint group as a quotient, by a torus action, of an open subset in a certain algebraic monoid. Vinberg calls it the *enveloping semigroup*, and we call it the *Vinberg monoid*. The construction of this monoid has been extended to reductive groups by Alexeev and Brion [AB04]. At least some parts have been extended to positive characteristic by Rittatore [Rit98, Rit01], using the theory of spherical varieties.

At this point we must assume that our ground field k is algebraically closed. All of the above references assume this, and the most comprehensive account, namely Vinberg's, also assumes that the characteristic is zero. We think it plausible that Vinberg's results listed below could be established for a split reductive group over an arbitrary field, using the work of Huruguen [Hur11b] on spherical varieties over arbitrary fields. If so, our subsequent results would go through.

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To state Vinberg's results, we first define some polyhedral cones. Let V be the character lattice of $T \subset G$. Let the Weyl group W act on $\mathrm{V}_{\mathbb{Q}} \oplus \mathrm{V}_{\mathbb{Q}}$ by $w(\mu, \nu) := (\mu, w\nu)$. Let the simple roots α_i and fundamental weights ϖ_j be indexed by $i, j \in \Omega$, the Dynkin diagram. Define K to be the polyhedral cone in $\mathrm{V}_{\mathbb{Q}} \oplus \mathrm{V}_{\mathbb{Q}}$ spanned by the simple roots in $\mathrm{V}_{\mathbb{Q}} \oplus 0$ and the diagonal, that is, $K = \langle (\alpha_i, 0) \mid i \in \Omega \rangle + \Delta_{\mathrm{V}_{\mathbb{Q}}}$. Its intersection $K^+ := K \cap (\mathrm{V}_{\mathbb{Q}} \oplus \mathrm{V}_{\mathbb{Q}}^+)$ with the positive Weyl chamber is the product of a simplicial cone, spanned by $(0, -\alpha_i)$ and (ϖ_j, ϖ_j), with a linear subspace. The union of the images of K^+ under the action of W is again a cone: indeed $WK^+ = \bigcap_{w \in W} wK$ [Ren05, 5.11].

Vinberg relates the faces of WK^+ to those of K^+ as follows. For $I, J \subset \Omega$, define cones

$$\begin{aligned} D_I = \langle \alpha_i \mid i \in I \rangle, \quad C_J = \langle \varpi_j \mid j \in J \rangle, \\ F_{I,J} = \{(\mu, \nu) \in \mathrm{V}_{\mathbb{Q}} \oplus \mathrm{V}_{\mathbb{Q}} \mid \mu - \nu \in D_I,\, \nu \in C_J + \Delta^W_{\mathrm{V}_{\mathbb{Q}}}\}. \end{aligned} \tag{5.1}$$

The $F_{I,J}$ are the faces of K^+. A face $F_{I,J}$ is said to be *essential* if no connected component of $\Omega \backslash J$ is contained in I when these are regarded as subsets of the Dynkin diagram. Vinberg proves that every face F of WK^+ satisfies $WF = WF_{I,J}$ for exactly one essential face $F_{I,J}$ of K^+.

Definition 5.2. Let Z be a torus with a given isomorphism to the maximal torus T of G. The *Vinberg monoid* is an affine algebraic monoid S_G over k with group of units $(Z \times G)/Z_G$, where the center Z_G of G is antidiagonally included in $Z \times G$. It is defined to be the spectrum of the ring of matrix entries of representations of $(Z \times G)/Z_G$ whose highest weights lie in WK^+.

The Vinberg monoid is equipped with monoid homomorphisms $\pi : S_G \to \mathbb{A}$ and $\psi : \mathbb{A} \to S_G$ satisfying $\pi \circ \psi = \mathrm{id}_{\mathbb{A}}$. Here $\mathbb{A} := \operatorname{Spec} k[\alpha_i]$ is an affine space having Z/Z_G as its group of units. Vinberg [Vin95] proves the following.

(V.1) The restriction of π to the group of units is the projection to the torus Z/Z_G, and the restriction of ψ to Z/Z_G is given by $\psi(z) = (z, z^{-1})$.

(V.2) The closure in S_G of $(Z \times T)/Z_G$ is the affine toric variety $X((WK^+)^\vee)$.

(V.3) Each orbit of $Z \times G \times G$ in S_G (acting by left and right multiplication) contains exactly one orbit of $Z \times N(T)$ in $X((WK^+)^\vee)$. The orbits $O_{I,J} \subset S_G$ are therefore indexed by the essential faces.

(V.4) The morphism $U_- \times Z \times \mathbb{A} \times U_+ \to S_G$ given by $(g_-, z, x, g_+) \mapsto g_- \, z \, \psi(x) g_+$, where $U_\pm$ are the usual maximal unipotents in G, is an open immersion with image $S_G^0 := \bigcup_I O_{I,\Omega}$.

(V.5) This open subset has as geometric quotient $S_G^0/Z \cong G_{\mathrm{ad}}$, the wonderful compactification of the adjoint group $G_{\mathrm{ad}} := G/Z_G$.

In fact, this geometric quotient is actually a geometric invariant theory quotient, as we shall now prove. Since S_G is affine, we may choose as our ample line bundle the trivial bundle $\mathcal{O}$, but with the usual action twisted by a character $\rho \in \mathrm{V}$ of Z. The geometric invariant theory quotient is then $\operatorname{Proj} \bigoplus_{i \geqslant 0} k[S_G]^{i\rho}$, where $k[S_G]^{i\rho}$ is the space of regular functions with weight $i\rho$ for the usual action. In this setting, the Hilbert–Mumford numerical criterion may be stated as follows.

Lemma 5.3. *If a torus acts effectively on an affine variety, linearized on a trivial bundle by a character ρ, then x is semistable (respectively stable) if and only if for every nontrivial cocharacter λ such that $\lim_{t \to 0} t^\lambda \, x$ exists, $\rho \cdot \lambda \geqslant 0$ (respectively > 0).*

A proof of this version of the criterion is given by King [Kin94] and by Gulbrandsen *et al.* [GHH14].

Theorem 5.4. *When linearized by a character ρ in the interior of the positive Weyl chamber V^+, the Z-action on S_G has semistable (and stable) set S_G^0, and the geometric invariant theory quotient $S_G /\!/_\rho Z$ is the wonderful compactification $\overline{G}_{\mathrm{ad}}$.*

Proof. Since the stable and semistable sets are unions of orbits of $Z \times G \times G$, by (V.2) and (V.3) it suffices to consider points in the affine toric variety $X((WK^+)^\vee)$, the closure of the maximal torus $(Z \times T)/Z_G$. In particular, to every essential face $F_{I,J}$ of K^+ corresponds a Weyl orbit of faces of WK^+.

We will test for stability for the Z-action using the numerical criterion Lemma 5.3. If x lies in the torus orbit corresponding to a face F of WK^+, it is straightforward to check that $\lim_{t\to 0} t^\lambda x$ exists if and only if $F \cdot \lambda \geqslant 0$, meaning $\mu \cdot \lambda \geqslant 0$ for all $\mu \in F$. Hence the torus orbit is stable (respectively semistable) if and only if for all nontrivial $\lambda \in \Lambda$ with $F \cdot \lambda \geqslant 0$, one also has $\rho \cdot \lambda > 0$ (respectively $\geqslant 0$).

Now Z, its subgroups λ, and its character ρ are acted on trivially by W, whereas $WF = WF_{I,J}$ for every face F of WK^+. So the stability condition above holds for F if and only if it holds for $F_{I,J}$. Hence $O_{I,J}$ is semistable (respectively stable) if and only if for all nontrivial $\lambda \in \Lambda$ with $F_{I,J} \cdot \lambda \geqslant 0$, one also has $\rho \cdot \lambda \geqslant 0$ (respectively > 0).

Observe from (5.1) that $\mu \in D_I + C_J$ if and only if $(\mu, \nu) \in F_{I,J}$ for some ν. For any 1-parameter subgroup λ of the first factor Z of $Z \times T$, then, $F_{I,J} \cdot \lambda = (D_I + C_J) \cdot \lambda$.

If $J = \Omega$, it follows that $F_{I,J} \cdot \lambda \geqslant 0$ implies $\rho \cdot \lambda > 0$, for ρ is in the interior of the positive Weyl chamber V^+. Consequently, each $O_{I,\Omega}$, and hence all of S_G^0, is stable.

If $J \neq \Omega$, however, then from the definition of essential pair there is necessarily some $i \in \Omega$ contained in neither I nor J. In this case $-\alpha_i^\vee$ provides a destabilizing 1-parameter subgroup, since $\varpi_i \cdot \alpha_i^\vee = 1$ whereas for $j \neq i$, $\alpha_j \cdot \alpha_i^\vee \leqslant 0$ and $\varpi_j \cdot \alpha_i^\vee = 0$. Hence the complement of S_G^0 is unstable.

The geometric invariant theory quotient $S_G /\!/_\rho Z$ is therefore a geometric quotient S_G^0/Z, and as such agrees with $\overline{G}_{\mathrm{ad}}$ by (V.5).

As an alternative to the last step, we may assume that ρ is a regular dominant weight of G_{ad}, and consider the representation $\overline{R}^\rho : S_G \to \operatorname{End} V_\rho$ also defined by Vinberg. His work implies [Vin95, (57),(59)] that $\overline{R}^\rho(O_{I,J}) = 0$ if and only if $J \neq \Omega$. It immediately follows that $S_G^0 = (\overline{R}^\rho)^{-1}(\operatorname{End} V_\rho \backslash \{0\})$. We therefore get an embedding of $G/Z_G = G_{\mathrm{ad}}$ into $\mathbb{P}(\operatorname{End} V_\rho)$, which may be regarded as the geometric invariant theory quotient $\operatorname{End} V_\rho /\!/_\rho Z$. Hence S_G^0/Z will contain the closure of G_{ad} in $\mathbb{P}(\operatorname{End} V_\rho)$, which is well known to be the wonderful compactification of G_{ad} [EJ08, 2.1]. Since both have exactly 2^r orbits of $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$, where r is the rank of G_{ad}, they are isomorphic. □

6. The Cox–Vinberg hybrid

We aim to hybridize the Vinberg quotient, which realizes $\overline{G}_{\mathrm{ad}}$ as a torus quotient of the monoid S_G, with the Cox construction, which realizes toric orbifolds as torus quotients of affine spaces $\mathbb{A}_\beta$ [BCS05, FMN10]. The hybrid will realize $\mathcal{M}_G(\Sigma)$ as a torus quotient of a monoid $S_{G,\beta}$ obtained as the base change of S_G by a linear map of affine spaces $\mathbb{A}_\beta \to \mathbb{A}$. This was inspired by Brion's observation [Bri07] that spectra of Cox rings of spherical varieties, such as the wonderful compactification, are often base changes of the Vinberg monoid.

We briefly recall the Cox construction. Let Z be a torus and Λ its cocharacter lattice. In $\Lambda_{\mathbb{Q}}$, let Σ be a torsion-free stacky simplicial fan, that is, a simplicial fan equipped with a set of nonzero lattice elements $\beta_1, \ldots, \beta_N$ so that each ray of the fan contains exactly one β_i. Let

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$\mathbb{A}_\beta := \mathbb{A}^N$ and $\mathbb{G}_\beta := \mathbb{G}_m^N$. For any $\sigma \subset \{1,\dots,N\}$, let $U_\sigma := \{x \in \mathbb{A}_\beta \mid x_i \neq 0 \text{ if } i \notin \sigma\}$. Then let $\mathbb{A}^0_\beta$ be the union of all U_σ such that $\langle \beta_i \mid i \in \sigma\rangle$ is a cone in Σ. Though it is not explicitly indicated in the notation, this open subset $\mathbb{A}^0_\beta \subset \mathbb{A}_\beta$ depends on Σ as well as β.

Let $\phi_\beta : \mathbb{G}_\beta \to Z$ be given by $\phi_\beta(z_1,\dots,z_N) := \prod z_i^{\beta_i}$ (or more concisely, $\phi_\beta(z) = z^\beta$) and let $K_\beta := \ker \phi_\beta$. If the β_i span $\Lambda_\mathbb{Q}$, then

$$1 \to K_\beta \to \mathbb{G}_\beta \xrightarrow{\phi_\beta} Z \to 1. \tag{6.1}$$

In this case Borisov *et al.* [BCS05] show that $\mathcal{X}(\Sigma) := [\mathbb{A}^0_\beta/K_\beta]$ is a separated tame stack whose coarse moduli space is the toric variety $X(\Sigma)$. (They consider only characteristic zero, but this construction is general.)

Remark 6.2. Obviously $[\mathbb{A}^0_\beta/K_\beta] \cong [(\mathbb{A}^0_\beta \times Z)/\mathbb{G}_\beta]$ if the β_i span $\Lambda_\mathbb{Q}$. But even if they do not, so that (6.1) is not exact on the right, $[(\mathbb{A}^0_\beta \times Z)/\mathbb{G}_\beta]$ is still a toric stack $\mathcal{X}(\Sigma)$ agreeing with the stack of that name constructed by Fantechi *et al.* [FMN10, 7.12].

Now let Z be isomorphic to the maximal torus T of G, and suppose the support of Σ lies in the positive Weyl chamber $\Lambda^+_\mathbb{Q}$. Let $\pi : Z \to Z/Z_G$ be the projection, and let $\overline{\phi}_\beta := \pi \circ \phi_\beta : \mathbb{G}_\beta \to Z/Z_G$. Since $\beta_i \in \Lambda^+$, the group homomorphisms $\overline{\phi}_{\beta_i} : \mathbb{G}_m \to Z/Z_G$ extend to monoid homomorphisms $\mathbb{A}^1 \to \mathbb{A}$. Hence $\overline{\phi}_\beta$ extends to a monoid homomorphism $\mathbb{A}_\beta \to \mathbb{A}$.

Definition 6.3. Let the *Cox–Vinberg monoid* be the fibered product $S_{G,\beta} := \mathbb{A}_\beta \times_\mathbb{A} S_G$. Likewise, let $S^0_{G,\beta} := \mathbb{A}^0_\beta \times_\mathbb{A} S^0_G$.

The Cox–Vinberg monoid is a reductive monoid, flat over $\mathbb{A}_\beta$, with group of units $\mathbb{G}_\beta \times G$. Like the Vinberg monoid, it has a projection $\pi_\beta : S_{G,\beta} \to \mathbb{A}_\beta$ whose restriction to the group of units is $(z,g) \mapsto z$ and a section $\psi_\beta : \mathbb{A}_\beta \to S_{G,\beta}$ whose restriction to $\mathbb{G}_\beta \subset \mathbb{A}_\beta$ is $z \mapsto (z, z^{-\beta})$. By the way, choosing $\mathbb{A}_\beta \cong \mathbb{A}$ is generally not allowed: for then β_i must be (some permutation of) the fundamental coweights, and these are not in Λ unless the semisimple part of G has trivial center.

Theorem 6.4. *The stack $[S^0_{G,\beta}/\mathbb{G}_\beta]$ is canonically isomorphic to $\mathcal{M}_G(\Sigma)$.*

Proof. Identify $\mathbb{A}_\beta \times G \times G$ with the space parametrizing all framings at $p_\pm$ of bundle chains in $\mathbb{E}(\beta)$ via $(x, g_+, g_-) \mapsto \Psi_\pm \circ g_\pm$, where $\Psi_\pm : \mathbb{A}_\beta \times G \to p_\pm^*\mathbb{E}(\beta)$ are the conventional framings of Example 2.7. Then, over the base $\mathbb{A}_\beta$, the natural action of the group scheme H_β on $\mathbb{A}_\beta \times G \times G$ is

$$h(x, g_+, g_-) := (x, e_+(h)g_+, e_-(h)g_-),$$

where $e_\pm : H_\beta \to G$ are the evaluations at $p_\pm$ relative to $\Psi_\pm$. The pullback $\tilde{\mathbb{E}}(\beta)$ of $\mathbb{E}(\beta)$ to $\mathbb{A}_\beta \times G \times G$ is then tautologically framed, and the H_β-action lifts to the framed bundle chain $\tilde{\mathbb{E}}(\beta)$.

As seen in Proposition 3.2, $\pi \times e_+ \times e_- : H_\beta \to \mathbb{A}_\beta \times G \times G$ is a closed immersion of smooth group schemes over $\mathbb{A}_\beta$. Hence the quotient $J_\beta := (\mathbb{A}_\beta \times G \times G)/H_\beta$ is a smooth family of homogeneous spaces over $\mathbb{A}_\beta$, and in particular a smooth scheme. Since the H_β-action lifts to $\tilde{\mathbb{E}}(\beta)$, this framed bundle chain descends to the quotient. This determines a morphism $J_\beta \to \mathfrak{B}$.

Moreover, let $J^0_\beta := (\mathbb{A}^0_\beta \times G \times G)/H_\beta$ be the open subset of J_β lying over the aforementioned $\mathbb{A}^0_\beta \subset \mathbb{A}_\beta$. There, the framed bundle chain is Σ-stable and hence defines a morphism $J^0_\beta \to \mathcal{M}_G(\Sigma)$. This is surjective since the atlas $\mathbb{A}^0_\beta \times G \times G \to \mathcal{M}_G(\Sigma)$ of Proposition 4.3 factors through it. This also implies that it is smooth. For a morphism g is smooth provided that g is finitely presented, f is smooth and surjective, and $g \circ f$ is smooth [dJo, 02K5]. Apply this to $\mathbb{A}^0_\beta \times G \times G \to J^0_\beta \to \mathcal{M}_G(\Sigma)$, observing that since $J^0_\beta \to \operatorname{Spec} k$ and the diagonal $\mathcal{M}_G(\Sigma) \to \mathcal{M}_G(\Sigma) \times \mathcal{M}_G(\Sigma)$ are finitely presented, so is $J^0_\beta \to \mathcal{M}_G(\Sigma)$. It follows that $J^0_\beta \to \mathcal{M}_G(\Sigma)$ is an atlas.

The torus $\mathbb{G}_\beta$ acts on every ingredient in this recipe, hence on J^0_β and the framed bundle chain over it. The morphism $J^0_\beta \to \mathcal{M}_G(\Sigma)$ therefore descends to $[J^0_\beta/\mathbb{G}_\beta] \to \mathcal{M}_G(\Sigma)$. We claim this is an isomorphism. It suffices [LM00, 3.8] to show that the action morphism $J^0_\beta \times \mathbb{G}_\beta \to J^0_\beta \times_{\mathcal{M}_G(\Sigma)} J^0_\beta$ is an isomorphism. For this it suffices to show [EGA, IV 17.9.1] that it is étale and bijective on points over any field. That it is bijective is straightforward, since bundle chains of distinct splitting types lie over the distinct $\mathbb{G}_\beta$-orbits in $\mathbb{A}_\beta$, and the isomorphism classes of possible framings are bijectively parametrized by the points of J^0_β.

To show that it is étale, since source and target are smooth and finitely presented, it suffices to show that its derivative is an isomorphism on Zariski tangent spaces. Let $E \to C$ be a bundle chain in J^0_β. From the long exact sequence of

$$0 \longrightarrow \operatorname{ad} E(-p) \longrightarrow \operatorname{ad} E \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \longrightarrow 0$$

and Corollary 2.16, the tangent space to the fiber of $J^0_\beta \to \mathbb{A}_\beta$ is $H^1(\operatorname{ad} E(-p))$. So there are the following short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & H^1(\operatorname{ad} E(-p)) \arrow[r] \arrow[d] & T_{C,E}\, J^0_\beta \oplus T_z\mathbb{G}_\beta \arrow[r] \arrow[d] & T_C\mathbb{A}_\beta \oplus T_z\mathbb{G}_\beta \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Delta_{H^1(\operatorname{ad} E(-p))} \arrow[r] & T_{C,E}(J^0_\beta \times_{\mathcal{M}_G(\Sigma)} J^0_\beta) \arrow[r] & T_{C\times z\cdot C}(\mathbb{A}_\beta \times_{\mathcal{N}} \mathbb{A}_\beta) \arrow[r] & 0
\end{tikzcd}$$

The right-hand column is the derivative of the action $\mathbb{A}_\beta \times \mathbb{G}_\beta \to \mathbb{A}_\beta \times_{\mathcal{N}} \mathbb{A}_\beta$, where $\mathcal{N}$ is the stack of rigidified chains as in § 3. It is easy to check, using $T^1_C = \operatorname{Ext}^1(\Omega, \mathcal{O}(-p))$ as in Theorem 1.6 and Proposition 2.12, that this derivative is an isomorphism. Hence the middle column is an isomorphism as well. This completes the proof that $[J^0_\beta/\mathbb{G}_\beta] \to \mathcal{M}_G(\Sigma)$ is an isomorphism.

It suffices, finally, to exhibit a $\mathbb{G}_\beta$-equivariant isomorphism $J^0_\beta \to S^0_{G,\beta}$. Let $\mathbb{G}_\beta$ act on $\mathbb{A}^0_\beta \times G \times G$ by

$$z \cdot (x, g_+, g_-) = (zx,\, g_+,\, g_- z^{-\beta}).$$

Then the morphism $\mathbb{A}^0_\beta \times G \times G \to S^0_{G,\beta}$ is given by $(x, g_+, g_-) \mapsto g_+\psi_\beta(x)g_-^{-1}$ is $\mathbb{G}_\beta$-equivariant. Indeed, as a closed condition the equivariance may be verified on the group of units where $\psi_\beta(x) = (x, x^{-\beta})$. This morphism is smooth with image $S^0_{G,\beta}$ by (V.4). The inverse image of $\psi_\beta(\mathbb{A}^0_\beta)$ is therefore a smooth family of subgroups of $G \times G$. By (V.1), over a point $z \in \mathbb{G}_\beta \subset \mathbb{A}^0_\beta$, this group is exactly $(1 \times z^\beta)\, \Delta_G\, (1 \times z^{-\beta})$. By the last remark in Example 2.7, it therefore coincides with H_β.

Consequently, this morphism descends to a $\mathbb{G}_\beta$-equivariant, surjective morphism $J^0_\beta \to S^0_{G,\beta}$ of schemes over $\mathbb{A}^0_\beta$. It is universally injective, since on each fiber of J^0_β over $\mathbb{A}^0_\beta$ we have exactly divided by the stabilizer group of the transitive $G \times G$-action. And it is étale, since source and target are smooth of the same dimension, and the original $\mathbb{A}^0_\beta \times G \times G \to S^0_{G,\beta}$ is smooth, so the derivative is everywhere surjective. Again by the fundamental property of étale morphisms [EGA, IV 17.9.1], $J^0_\beta \to S^0_{G,\beta}$ is an isomorphism. □

Corollary 6.5. *If G has trivial center, Σ comprises the single cone $\Lambda^+_{\mathbb{Q}}$, and $\beta_i = \varpi_i^\vee$, the fundamental coweights, then the coarse moduli space of $\mathcal{M}_G(\Lambda^+_{\mathbb{Q}})$ is $M(\Lambda^+_{\mathbb{Q}}) \cong \overline{G}_{\mathrm{ad}}$, the wonderful compactification of $\overline{G}_{\mathrm{ad}} := G/Z_G$.*

Proof. In this case ϕ_β is an isomorphism and so $S_{G,\beta} \cong S_G$ as varieties with the action of a torus $\mathbb{G}_\beta \cong Z$. By (V.5), $S^0_G/Z \cong \overline{G}_{\mathrm{ad}}$ as a geometric quotient. □

Remark 6.6. Since $\mathbb{A}^0_\beta$ and hence $S^0_{G,\beta}$ do not depend on the choice of ordering of $\beta_1, \ldots, \beta_N$, by Theorem 6.4 the moduli stack $\mathcal{M}_G(\Sigma)$ does not depend on it either. In other words, for two different orderings, there is a functor taking Σ-stable bundle chains for one ordering to Σ-stable bundle chains for the other. We do not know an explicit description of this functor.

7. Functoriality

We now turn to the question of functoriality: given a homomorphism $f : G \to G'$ of split reductive groups, for which fans does it extend to a morphism $\mathcal{M}_G(\Sigma) \to \mathcal{M}_{G'}(\Sigma')$ of the compactifications? The answer turns out to be essentially the same as for toric stacks.

Let Σ, Σ' be torsion-free stacky fans supported on the positive Weyl chambers, with distinguished elements β_i and β'_j, respectively. By composing with inner automorphisms, we may suppose without loss of generality that $f(T) \subset T'$ and that $\langle f^*\alpha'_j \rangle \subset \langle \alpha_i \rangle$. Thus f determines a linear $f_* : \mathfrak{t} \to \mathfrak{t}'$ (well-defined modulo a subgroup of $W \times W'$). Say that f_* defines a *morphism of stacky fans* $W\Sigma \to W'\Sigma'$ if for every cone $\sigma \in W\Sigma$ there exists $\sigma' \in W'\Sigma'$ such that $f_*(\sigma) \subset \sigma'$ and if for every distinguished $\beta_i \in \sigma$, $f_*(\beta_i)$ is an integer combination of those $\beta'_j \in \sigma'$. (Because the fans are simplicial, these integers are unique and nonnegative.) Borisov *et al.* show [BCS05, 4.5] that a homomorphism $T \to T'$ of tori extends to a morphism $\mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma')$ of toric orbifolds if and only if its derivative $\mathfrak{t} \to \mathfrak{t}'$ defines a morphism of stacky fans $\Sigma \to \Sigma'$.

Theorem 7.1. *A homomorphism $f : G \to G'$ extends to a morphism of stacks $\mathcal{M}_G(\Sigma) \to \mathcal{M}_{G'}(\Sigma')$ if and only if $f_* : \mathfrak{t} \to \mathfrak{t}'$ defines a morphism of stacky fans $W\Sigma \to W'\Sigma'$.*

Proof. By Theorem 4.4, the closures of T and T' in $\mathcal{M}_G(\Sigma)$ and $\mathcal{M}_{G'}(\Sigma')$ are $\mathcal{X}(W\Sigma)$ and $\mathcal{X}(W'\Sigma')$ respectively, so the necessity is an immediate consequence of Borisov *et al.*'s criterion [BCS05, 4.5].

Conversely, suppose that f_* defines a morphism of stacky fans. Then each $f_*\beta_i = \sum_j a_{ij}\beta'_j$, where the integers a_{ij} satisfy $a_{ij} = 0$ unless $\beta'_j \in \sigma' \supset f_*\sigma \ni \beta_i$. Consider the monoid homomorphism $\rho : \mathbb{A}_\beta \to \mathbb{A}_{\beta'}$ given by $\rho(x_i) = \left(\prod_i x_i^{a_{ij}}\right)$. Thanks to the vanishing condition just mentioned, if $x_i \neq 0$ for $i \notin \sigma$, then $\prod_i x_i^{a_{ij}} \neq 0$ for $j \notin \sigma'$. That is, $\rho(U_\sigma) \subset U_{\sigma'}$, where U_σ and $U_{\sigma'}$ are the open sets in the Cox construction. Consequently $\rho(\mathbb{A}^0_\beta) \subset \mathbb{A}^0_{\beta'}$, where as before $\mathbb{A}^0_\beta = \bigcup_{\sigma\in\Sigma} U_\sigma$.

Furthermore, since $\langle f^*\alpha'_j \rangle \subset \langle \alpha_i \rangle$, the homomorphism $\bar{f} : Z/Z_G \to Z'/Z_{G'}$ induced by f extends to a monoid homomorphism $\bar{f} : \mathbb{A} \to \mathbb{A}'$. The square

$$\begin{tikzcd} \mathbb{A}_\beta \arrow[r, "\rho"] \arrow[d, "\overline{\phi}_\beta"'] & \mathbb{A}_{\beta'} \arrow[d, "\overline{\phi}_{\beta'}"] \\ \mathbb{A} \arrow[r, "\bar{f}"] & \mathbb{A}' \end{tikzcd}$$

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commutes, since

$$\begin{aligned}\overline{\phi}_{\beta'}(\rho(x_i)) &= \overline{\phi}_{\beta'}\left(\prod_i x_i^{a_{ij}}\right)\\ &= \prod_{i,j} x_i^{a_{ij}\beta'_j}\\ &= \prod_i x_i^{\sum_j a_{ij}\beta'_j}\\ &= \prod_i x_i^{f_*\beta_i}\\ &= f\left(\prod_i x_i^{\beta_i}\right)\\ &= \bar{f}(\overline{\phi}_\beta(x_i)).\end{aligned}$$

On the Cox side, therefore, we have a morphism $\mathbb{A}^0_\beta \to \mathbb{A}^0_{\beta'}$, equivariant for the actions of the groups of units $\mathbb{G}_\beta$ and $\mathbb{G}_{\beta'}$, and lying over the monoid homomorphism $\bar{f}$.

On the Vinberg side, it follows directly from the definition of the Vinberg monoid Definition 5.2 that $f : G \to G'$ extends to a homomorphism $S_f : S_G \to S_{G'}$. By (V.1) S_f takes $\psi(\mathbb{A})$ to $\psi'(\mathbb{A}')$ and hence $S^0_G = G\psi(\mathbb{A})G$ to $S^0_{G'} = G'\psi'(\mathbb{A}')G'$.

There is hence a morphism of stacks $\rho \times S_f : [(\mathbb{A}^0_\beta \times_{\mathbb{A}} S^0_G)/\mathbb{G}_\beta] \to [(\mathbb{A}^0_{\beta'} \times'_{\mathbb{A}} S^0_{G'})/\mathbb{G}_{\beta'}]$, which by Theorem 6.4 is exactly $\mathcal{M}_G(\Sigma) \to \mathcal{M}_{G'}(\Sigma')$. Restricting to the open subset $G \cong \mathbb{G}_\beta \times_{\mathbb{A}} (G \times Z)/Z_G \subset \mathbb{A}^0_\beta \times_{\mathbb{A}} S^0_G$, we recover the original homomorphism $f : G \to G'$. The homomorphism therefore extends as desired. □

Since the stacks in question represent moduli problems, there must be a functor associated to the morphism of stacky fans above, taking stable G-bundle chains to stable G'-bundle chains. Alas, we do not know an explicit description of this functor.

8. $M_G(\Sigma)$ as a geometric invariant theory quotient

In this section we show that if the coarse moduli space of the toric orbifold $\mathcal{M}_T(W\Sigma)$ is projective, then the same is true of $\mathcal{M}_G(\Sigma)$, and the quotient construction of Theorem 6.4 is a geometric invariant theory quotient. In fact we show a little more.

Slightly adapting the terminology of Cox *et al.* [CLS11] and Hausel and Sturmfels [HS02], we say a variety is *semiprojective* if it is projective over an affine variety. In particular, a proper semiprojective variety is projective. Additionally, a closed subvariety of a semiprojective variety is semiprojective. A variety is semiprojective if and only if it is Proj of an integral algebra of finite type over k, so any geometric invariant theory quotient of a semiprojective variety is semiprojective.

We say a rational polyhedral fan is a *normal fan* if it comprises the normal cones of some polyhedron, or, equivalently [CLS11, 7.2.4, 7.2.9], if its support is convex and admits a strictly convex piecewise linear function to $\mathbb{Q}$, linear on each cone of the fan. Then a fan Σ is a normal fan if and only if the toric variety $X(\Sigma)$ is semiprojective.

Theorem 8.1. *If Σ is a normal fan and $W\Sigma$ has convex support, then the coarse moduli space $M_G(\Sigma)$ of $\mathcal{M}_G(\Sigma)$ is semiprojective and indeed is a geometric invariant theory quotient $S_{G,\beta}/\!/\mathbb{G}_\beta$ with semistable (and stable) set $S^0_{G,\beta}$.*

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Corollary 8.2. *If Σ is a normal fan and has support equal to $\Lambda^+_{\mathbb{Q}}$, then the coarse moduli space $M_G(\Sigma)$ of $\mathcal{M}_G(\Sigma)$ is projective.*

Proof. It is proper by Theorem 4.2 and semiprojective by Theorem 8.1. □

Before proving the theorem, we pause to observe a purely polyhedral consequence.

Corollary 8.3. *If Σ is a normal fan and its support is the entire Weyl chamber $\Lambda^+_{\mathbb{Q}}$, then $W\Sigma$ is a normal fan.*

Proof. The support of $W\Sigma$ is then the entire $\Lambda_{\mathbb{Q}}$, hence convex, so $M_G(\Sigma)$ and hence $M_T(W\Sigma) \subset M_G(\Sigma)$ are semiprojective varieties. □

Lemma 8.4. *If $W\Sigma$ has convex support, then the projection of any element in $|\Sigma|$ onto any face of the positive Weyl chamber is also contained in $|\Sigma|$.*

Proof. The projection of any $x \in |\Sigma|$ onto any face of the Weyl chamber is contained in the convex hull of the Weyl orbit Wx, hence belongs to the convex W-invariant set $|W\Sigma|$. □

Proof of Theorem 8.1. Since $\mathcal{M}_G(\Sigma) \cong [S^0_{G,\beta}/G_\beta]$ as stacks, it suffices to find a linearization of the $\mathbb{G}_\beta$-action on $S_{G,\beta}$ for which the semistable (and stable) set is $S^0_{G,\beta}$. Since $S_{G,\beta}$ is affine, we may choose the trivial bundle as our ample bundle. However, $\mathbb{G}_\beta$ should act by a nontrivial character.

We will combine two characters adapted to the two factors of $S_{G,\beta} = \mathbb{A}_\beta \times_{\mathbb{A}} S_G$. First, let $\xi : \mathbb{G}_\beta \to \mathbb{G}_m$ be any character so that $(\mathbb{A}_\beta \times Z)/\!\!/_\xi\, \mathbb{G}_\beta = X(\Sigma)$ with semistable (and stable) set $\mathbb{A}^0_\beta \times Z$. Note that this is equivalent to $\mathbb{A}^0_\beta$ being the stable set for the K_β-action. Such a character ξ always exists when Σ is a normal fan [Dol03, Remark 12.1, p. 202]. Second, let $\rho : \mathbb{G}_\beta \to \mathbb{G}_m$ be the composition of $\phi_\beta : \mathbb{G}_\beta \to Z$ with a character $Z \to \mathbb{G}_m$ in the interior of the positive Weyl chamber. Then linearize the $\mathbb{G}_m$-action on $S_{G,\beta}$ 'asymptotically' by $\xi + m\rho$ for $0 \ll m \in \mathbb{Z}$. (Intuitively, the motivation is that $m\rho$ translates the moment polyhedron for the K_β-action so that all of its vertices lie in the positive Weyl chamber.)

By the numerical criterion Lemma 5.3, (x, y) is semistable (respectively stable) for the action of $\mathbb{G}_\beta$ if and only if for every nontrivial cocharacter λ such that $\lim_{t\to 0} t^\lambda (x, y)$ exists, $(\xi + m\rho) \cdot \lambda \geqslant 0$ (respectively > 0).

But suppose first that $(x, y) \in S^0_{G,\beta} = \mathbb{A}^0_\beta \times_{\mathbb{A}} S^0_G$, which means that $(x, 1) \in \mathbb{A}_\beta \times Z$ and $y \in S_G$ are stable points for the $\mathbb{G}_\beta$ actions linearized by ξ and ρ respectively. And suppose that $\lim_{t\to 0} t^\lambda (x, y)$ exists. Then certainly $\lim_{t\to 0} t^\lambda y$ exists, so $\rho \cdot \lambda > 0$. However, $\lim_{t\to 0} t^\lambda (x, 1)$ exists if and only if $\phi_\beta(t^\lambda) = 1$. In this case $\rho \cdot \lambda = 0$ and λ acts trivially on S_G, but since $(x, 1)$ is stable, $\xi \cdot \lambda > 0$. Hence $(\xi + m\rho) \cdot \lambda > 0$, as required. Otherwise, if $\phi_\beta(t^\lambda) \neq 1$, then since $1 \ll m$ and $\rho \cdot \lambda > 0$, again $(\xi + m\rho) \cdot \lambda > 0$. Hence (x, y) is a stable point for the Z-action on $S_{G,\beta}$ linearized by $\xi + m\rho$.

Now, conversely, suppose that $(x, y) \in S_{G,\beta} = \mathbb{A}_\beta \times_{\mathbb{A}} S_G$ is stable for the linearization $\xi + m\rho$, and let λ be a nontrivial cocharacter of $\mathbb{G}_\beta$. If $\phi_\beta(t^\lambda) = 1$, so that the 1-parameter subgroup corresponding to λ is contained in K_β, then $\rho \cdot \lambda = 0$ and λ acts trivially on S_G, so the linearization reduces to ξ on $\mathbb{A}_\beta$, showing that $x \in \mathbb{A}^0_\beta$, the stable set for the K_β-action.

What remains is to show that in this case y is also stable. In fact we will show, equivalently, that if y is unstable then (x, y) is unstable. The unstable sets in S_G and $S_{G,\beta}$ are unions of orbits of $Z \times G \times G$ and $\mathbb{G}_\beta \times G \times G$, respectively. By (V.2) and (V.3) it suffices to assume that y lies in the affine toric variety $X((WK^+)^\vee) \subset S_G$, and indeed in an orbit $O_{I,J}$ corresponding to an

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essential face of WK^+. On the other hand, the $\mathbb{G}_\beta$-orbits of $\mathbb{A}_\beta$ are indexed by $H \subset \{1, \ldots, N\}$ and are, of course, simply

$$\tilde{O}_H := \{x \in \mathbb{A}_\beta \mid x_j \neq 0 \Leftrightarrow j \in H\}.$$

Since the morphism $\mathbb{A}_\beta \to \mathbb{A}$ is induced by contraction with $\beta : \langle \alpha_i \mid i \in \Omega\rangle \to \mathbb{Q}^N_{\geqslant 0}$, it follows that $\tilde{O}_H \subset \mathbb{A}_\beta$ maps to $O_I \subset \mathbb{A}$ where

$$I = \{i \in \Omega \mid \alpha_i \cdot \beta_j = 0 \text{ for all } j \notin H\}.$$

Recall now from the proof of Theorem 5.4 that if $y \in O_{I,J}$ is unstable for an essential pair (I, J), then $J \neq \Omega$, there exists $i \notin I \cup J$ by definition of an essential pair, and $-\alpha_i^\vee : \mathbb{G}_m \to Z$ then proves to be a destabilizing cocharacter for $O_{I,J}$. To destabilize $(x, y) \in \tilde{O}_H \times_{O_I} O_{I,J}$, by the numerical criterion Lemma 5.3 it suffices to lift a multiple of $-\alpha_i^\vee$ to a 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_\beta$ such that $\lim_{t\to 0} t^\lambda x$ exists. For the necessary inequality $(\xi + m\rho) \cdot \lambda < 0$ is then automatic from $1 \ll m$.

If $t^\lambda = (t^{\ell_1}, \ldots, t^{\ell_N})$ and $x \in \tilde{O}_H$, then clearly $\lim_{t\to 0} t^\lambda x$ exists if $\ell_i \geqslant 0$ for all $i \in H$. Hence a suitable lift λ can be found if

$$-\alpha_i^\vee = \sum_{j=1}^{N} \ell_j \beta_j \tag{8.5}$$

for some $\ell_j \in \mathbb{Q}$ with $\ell_j \geqslant 0$ whenever $j \in H$.

Since $i \notin I$, there exists $j \notin H$ with $\alpha_i \cdot \beta_j \neq 0$, indeed $\alpha_i \cdot \beta_j > 0$ as $\beta_j \in \Lambda^+$. Denote by $P_i : \Lambda_\mathbb{Q} \to \Lambda_\mathbb{Q}$ the projection onto the hyperplane annihilated by α_i, that is,

$$P_i(\beta_j) = \beta_j - \frac{\alpha_i \cdot \beta_j}{2}\, \alpha_i^\vee.$$

Rearranging yields

$$-\alpha_i^\vee = \frac{2}{\alpha_i \cdot \beta_j}\, P_i(\beta_j) - \frac{2}{\alpha_i \cdot \beta_j}\, \beta_j.$$

By Lemma 8.4, $P_i(\beta_j)$ is in the support of Σ and hence can be expressed as a linear combination of the β_i with nonnegative coefficients. This establishes (8.5), completing the proof. $\square$

9. The wonderful compactification of any semisimple group

If G is semisimple and the stacky fan Σ comprises only one cone, the positive Weyl chamber $\Lambda^+_\mathbb{Q}$, equipped with the minimal cocharacters β_i along its rays, then we obtain an orbifold whose coarse moduli space is the compactification proposed by Springer [Spr06].

Proposition 9.1. *If G is semisimple and Σ is as above, then the coarse moduli space $M_G(\Sigma)$ of $\mathcal{M}_G(\Sigma)$ is the normalization of $M_{G_{\mathrm{ad}}}(\Sigma) = \overline{G}_{\mathrm{ad}}$ in the function field of G.*

Lemma 9.2. *Suppose given a Cartesian diagram of noetherian integral schemes*

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

where X is normal, V is nonempty and open in Y, and f is finite. Then X is the normalization of Y in the function field of U.

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Proof. Let W be the normalization of Y in the function field of U. Since X is normal, f factors through $\tilde{f} : X \to W$, which is finite because f is. Since W also contains U as an open subscheme, $\tilde{f}$ is a finite birational morphism to a normal scheme, hence an isomorphism by Zariski's main theorem. □

Proof of Proposition 9.1. Take $U = G$, $V = G_{\mathrm{ad}}$, $X = M_G(\Lambda^+_{\mathbb{Q}})$, $Y = M_{G_{\mathrm{ad}}}(\Lambda^+_{\mathbb{Q}})$ in the lemma. As a geometric invariant theory quotient of a normal variety, X is itself normal. Since the morphism $\mathbb{A}_\beta \to \mathbb{A}$ is finite in this case, so is $S_{G,\beta} \to S_G$ and hence $S_{G,\beta}/\!/\mathbb{G}_\beta \to S_G/\!/\mathbb{G}_\beta$, that is, $M_G(\Lambda^+_Q) \to M_{G_{\mathrm{ad}}}(\Lambda^+_{\mathbb{Q}})$. □

10. The coarse moduli space as a spherical variety

For any reductive G and simplicial stacky fan Σ, the coarse moduli space $M(\Sigma)$ may be described as a spherical variety as follows. See Timashev [Tim11] or Pezzini [Pez10] for background on spherical varieties.

Proposition 10.1. *The coarse moduli space $M(\Sigma)$ of $\mathcal{M}_G(\Sigma)$ is the toroidal spherical embedding corresponding to the uncolored fan $w\Sigma$, where w is the longest element of W.*

Hence any $G \times G$-equivariant toroidal compactification of G with finite quotient singularities is the coarse moduli space of some $\mathcal{M}_G(\Sigma)$; for such compactifications correspond to simplicial fans whose support is the negative Weyl chamber $-u\Lambda^+_{\mathbb{Q}}$.

Proof. As the coarse moduli space of a smooth Deligne–Mumford stack, $M(\Sigma)$ is normal. It is also a scheme: indeed, it is covered by the open sets $M(\sigma)$, which are schemes since choosing a normal fan Σ_σ containing σ realizes $M(\sigma)$ as an open subset of $M(\Sigma_\sigma)$, which is quasiprojective by Theorem 8.1. Since $M(\Sigma)$ contains G as a dense open subset, it also has a dense orbit for the Borel subgroup $B \times B_- \subset G \times G$, where B_- is the Borel opposite to B. Hence it is spherical.

To prove it is toroidal, it suffices to show that every $G \times G$-orbit nontrivially intersects the *big cell*, that is, the complement of the effective divisors preserved by $B \times B_-$ but not $G \times G$. By Theorem 6.4, it suffices to prove the corresponding statement for the $G \times G \times \mathbb{G}_\beta$-action on $S^0_{G,\beta}$. For S^0_G this is proved by Vinberg [Vin95, Proposition 14], and the case of $S^0_{G,\beta}$ immediately follows, since its big cell is the fibered product of $\mathbb{A}_\beta$ with the big cell of S^0_G.

A toroidal spherical embedding is determined by its fan, which is uncolored and whose support lies in the valuation cone of its dense orbit. In the present case the dense orbit is $G = (G \times G)/G$, so the valuation cone is $-\Lambda^+_{\mathbb{Q}}$: see Timashev [Tim11, 24.9]. Since by Theorem 4.4 the closure in $M(\Sigma)$ of the maximal torus T is the toric variety $X(W\Sigma)$, it follows [Tim11, 29.7] that the desired fan is the part of $W\Sigma$ lying inside the Weyl chamber $-\Lambda^+_{\mathbb{Q}}$, which is $w\Sigma$. □

11. Relationship with the Losev–Manin space $\overline{M}_{0,\{1,1,\epsilon,\ldots,\epsilon\}}$

For the root system of type A_r, a moduli problem represented by the toric variety $X(W\Lambda^+_{\mathbb{Q}})$ of the fan of Weyl chambers has been described by Losev and Manin [LM00] (see Figure 2). It is best stated in terms of the *weighted pointed stable curves* introduced by Hassett [Has03]. These are prestable curves with positive rational weights assigned to each marked point, where weighted marked points are allowed to coincide with each other (but not with the nodes) provided that the sum of their weights does not exceed unity.

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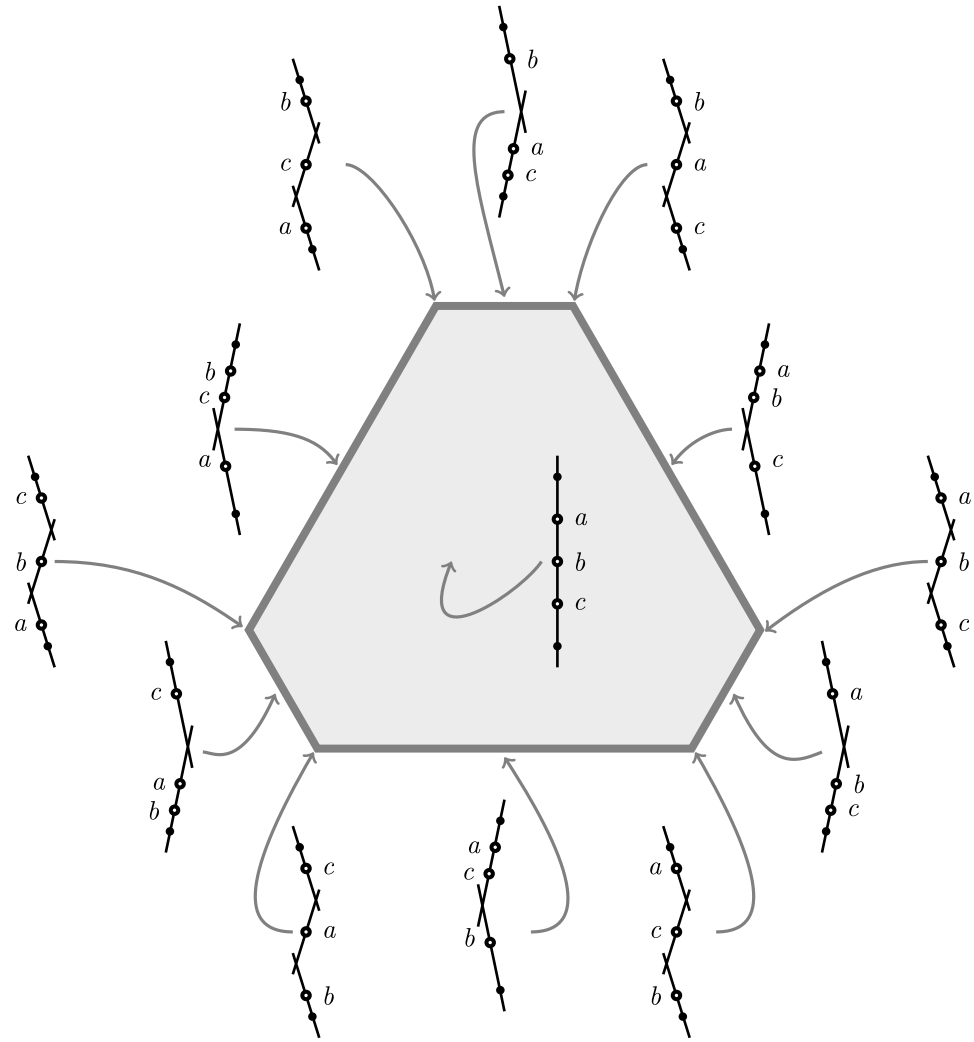

Figure 2. The moment map for the toric variety $\overline{\mathcal{M}}_{0,\{1,1,\epsilon,\epsilon,\epsilon\}} \cong \mathcal{M}_T(W\Lambda^+_{\mathbb{Q}})$, with image the permutohedron of the symmetric group S_3. The points $p_\pm = \bullet$ have weight 1; the points $a, b, c = \circ$ have weight ϵ.

Theorem 11.1 (Losev–Manin). *The toric variety associated with the A_r root system is the fine moduli space $\overline{M}_{0,\{1,1,\epsilon,\dots,\epsilon\}}$ of genus zero weighted pointed stable curves with $r+3$ marked points, two having weight 1, and the rest having small weight $\epsilon \ll 1$.*

From our point of view these toric varieties arise as the closure of the maximal torus of $G = PGL_{r+1}$ in $\mathcal{M}_G(\Lambda^+_{\mathbb{Q}})$. We may describe the functor from Losev–Manin's moduli problem to ours as follows. Let $C \to S$ be a family of weighted pointed stable curves of the type described above. Regard the marked points as sections $p_+, p_-, a_0, \dots, a_r : S \to C$. Then $C \to S$ equipped with the sections p_+ and p_- is a chain in the sense of (1.3). Let $E \to C$ be the vector bundle

$$E := \mathcal{O}(a_0) \oplus \cdots \oplus \mathcal{O}(a_r).$$

By Theorem 1.6 there is a canonical $\mathbb{G}_m$-action on C lifting to E so that the action on p_+^*E is trivial. The action on p_-^*E is then uniformly of weight -1, so the associated PGL_{r+1}-bundle is acted on trivially at $p_\pm$, hence constitutes a framed bundle chain.

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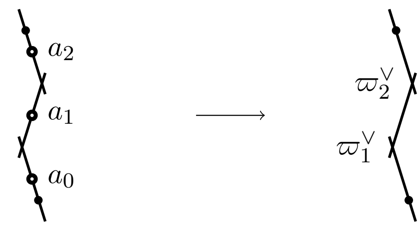
Let Σ be the stacky fan whose single cone is the positive Weyl chamber $\Lambda^+_{\mathbb{Q}}$ equipped with the fundamental coweights of PGL_{r+1}. These fundamental coweights $\varpi_i^\vee$ are the minimal cocharacters on the rays of $\Lambda^+_{\mathbb{Q}}$ and are given by $\lambda^{\varpi_i^\vee}(t) = \mathrm{diag}(t, \dots, t, 1, \dots, 1)$, where t appears i times. On the other hand, given a framed bundle of the type described above over a standard chain C_n, it is easily seen that the cocharacter at a node is $\mathrm{diag}(t^{\ell_0}, \dots, t^{\ell_r})$ where $\ell_j = -1$ if a_j lies on a component of the chain between that node and p_+, and $\ell_j = 0$ otherwise. Hence the cocharacters appearing at the nodes will be a subsequence of $w(\varpi_1^\vee), \dots, w(\varpi_r^\vee)$ for some fixed permutation $w \in W$. The framed bundle chain is therefore Σ-stable in the sense of (4.1).



This construction provides a functor from weighted pointed stable curves to framed bundle chains, yielding a morphism $\overline{M}_{0,\{1,1,\epsilon,\dots,\epsilon\}} \to \mathcal{M}_G(\Sigma)$. Since the framings respect the splitting into line bundles, this actually factors through $\mathcal{M}_T(W\Sigma)$. Indeed, by arguing as in the proof of Theorem 4.4 one shows that $\overline{M}_{0,\{1,1,\epsilon,\dots,\epsilon\}} \to \mathcal{M}_T(W\Sigma)$ is an isomorphism.

The Losev–Manin picture was partially extended to other classical groups by Batyrev and Blume [BB11]. The correspondence works best for the B_r root system, where the toric variety can be interpreted as a moduli space of weighted pointed stable curves with an involution. From our point of view this is because the classical group corresponding to the B_r root systems, namely SO_{2r+1}, has trivial center and hence can be embedded in PGL_{2r+1}, indeed as the fixed-point locus of an involution. This is not the case with SO_{2n} and Sp_n, leading to the complications observed in those cases by Batyrev–Blume.

12. Relationship with the Kausz space KGL_r

Kausz [Kau00] has introduced an equivariant compactification KGL_r of GL_r. It represents the moduli problem of 'bf-morphisms', also introduced by Kausz. There is presumably a faithful functor from bf-morphisms to bundle chains. However, since Kausz's moduli problem is quite technical, we prefer to study KGL_r via its alternative description by Kausz as an iterated blow-up of projective space, parallel to Vainsencher's description [Vai84] of the wonderful compactification $\overline{PGL}_r$.

Recall first Vainsencher's description. Let $X_0 = \mathbb{P}^{r^2-1}$ be the projectivization of the vector space of $r \times r$ matrices, and let A_i be the locus of matrices of rank i. Recursively define $X_i = \mathrm{Bl}(X_{i-1}, \overline{A}_i)$. Vainsencher proves that each blow-up locus, and hence each X_i, is smooth; and that moreover $X_{r-1} \cong \overline{PGL}_r$, the wonderful compactification. Furthermore, there is a recursive structure: each A_i is plainly a PGL_i-bundle over a product of Grassmannians (the projection being given by kernel and image), and its closure $\overline{A}_i$ in X_{i-1} is the associated $\overline{PGL}_i$-bundle.

Kausz's description is similar but one dimension greater. Let $Y_0 = \mathbb{P}^{r^2}$ be regarded as compactifying the vector space $\mathbb{A}^{r^2}$ of all $r \times r$ matrices, and refer to elements of the hyperplane $\mathbb{P}^{r^2-1} = \mathbb{P}^{r^2} \backslash \mathbb{A}^{r^2}$ as *matrices at infinity*. Let $B_i \subset \mathbb{A}^{r^2}$ be the locus of matrices of rank i, and let $C_i \subset \mathbb{P}^{r^2-1}$ be the locus of matrices at infinity of rank i. Recursively define $Y_i = \mathrm{Bl}(Y_{i-1}, \overline{B}_{i-1} \cup \overline{C}_i)$. Kausz proves that each blow-up locus is a disjoint union of two smooth loci, so that Y_i is also smooth; and that moreover $Y_{r-1} \cong KGL_r$, the space representing his moduli problem. Furthermore, there is a recursive structure: each B_i is a GL_i-bundle over a product of Grassmannians, and its closure in Y_i is the associated KGL_i-bundle; while each C_i is a PGL_i-bundle over a product of Grassmannians, and its closure in Y_{i-1} is the associated $\overline{PGL}_i$-bundle.

For example, KGL_2 is the blow-up of $\mathbb{P}^4$, with projective coordinates $[a, b, c, d, e]$, at the point $[0, 0, 0, 0, 1]$ and the quadric surface $ad - bc = 0 = e$.

In fact, the Vainsencher and Kausz constructions are intimately related. For assigning $A \mapsto \mathbb{P}(1 \oplus A)$ gives an embedding $GL_r \to PGL_{r+1}$ restricting to an isomorphism on the standard maximal tori. It extends to a linear inclusion $Y_0 \subset X_0$ of the projective space compactifications (with $r + 1$ substituted for r in the case of X_0). It is obvious that $A_i \cap Y_0 = B_{i-1} \cup C_i$ in X_0. From a careful scrutiny of the recursive structure, one should be able to see further that $\overline{A}_i \cap Y_{i-1} = \overline{B}_{i-1} \cap \overline{C}_i$ in X_{i-1}, even scheme-theoretically, and hence, blowing up both sides, that $Y_i \subset X_i$ as the closure of GL_r. Consequently, KGL_r is nothing but the closure of GL_r in the wonderful compactification $\overline{PGL}_{r+1}$. In any case, the latter statement is proved, using the classification of toroidal group compactifications, in the thesis of Huruguen [Hur11a, 3.1.16].

Kausz's space may be realized as a moduli space of bundle chains. This is assured by Proposition 10.1, as it is a smooth toroidal equivariant compactification of GL_r. The corresponding stacky fan Σ may be described as follows. As the maximal tori of GL_r and PGL_{r+1} are identified, the cocharacter lattices of the two groups may be identified as well. Let Σ be that part of the fan of Weyl chambers of PGL_{r+1} lying in the positive Weyl chamber of GL_r, equipped with the minimal cocharacters β_i along its rays. For example, when $r = 2$, the fan Σ consists of three contiguous $60°$ sectors in the plane.

Proposition 12.1. *With notation as above,* $KGL_r \cong \mathcal{M}_{GL_r}(\Sigma)$.

Proof. Since the Weyl groups satisfy $W_{GL_r} \subset W_{PGL_{r+1}}$, the fan $W_{GL_r}\Sigma$ is precisely the fan of all Weyl chambers of PGL_r. Extension of structure group for bundle chains then determines a morphism $\mathcal{M}_{GL_r} \to \overline{PGL}_{r+1}$. This is an embedding: it separates points since a (framed) vector bundle E may be recovered from the (framed) projective bundle $\mathbb{P}(E \oplus \mathcal{O})$, and it separates tangent vectors as is easily seen from the description of the Zariski tangent spaces in Proposition 2.12. Hence $\mathcal{M}_{GL_r}$ is a subscheme of $\overline{PGL}_{r+1}$ containing GL_r as a dense open subset. It therefore coincides with KGL_r, the closure of GL_r in $\overline{PGL}_{r+1}$. $\square$

Let us spell out concretely which bundle chains are stable in the moduli problems represented by $\overline{PGL}_r$ and KGL_r. All the relevant bundles have trivial weights at $p_\pm$. So to specify a line bundle over the standard chain C_n, rather than use the notation $\mathcal{O}(b_0 \mid b_1 \mid \cdots \mid b_{n+1})$ of Remark (1.2)(b), it is now more convenient to use the multidegree notation $\mathcal{O}(d_0, \ldots, d_n)$, where $d_i = b_i - b_{i+1}$ is the degree on the ith component of the chain.

With this notation, the bundle chains over the standard chain C_n which are stable in the moduli problem represented by $\overline{PGL}_r$ are those of the form $\mathbb{P}(\mathcal{O}(v_1) \oplus \cdots \oplus \mathcal{O}(v_r))$, where each v_j is a standard basis vector in $\mathbb{Z}^{n+1}$, each of which must appear at least once. For example, the only bundle allowed over $C_0 = \mathbb{P}^1$ is $\mathbb{P}(\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1))$, the trivial bundle.

Likewise, the bundle chains over the standard chain C_n which are stable in the moduli problem represented by KGL_r are those vector bundles E such that $\mathbb{P}(\mathcal{O} \oplus E)$ is of the form stated in the last paragraph. Equivalently, $E \cong \mathcal{O}(v_1 - v_0) \oplus \cdots \oplus \mathcal{O}(v_r - v_0)$, where each v_j is a standard basis vector in $\mathbb{Z}^{n+1}$, each of which must appear at least once.

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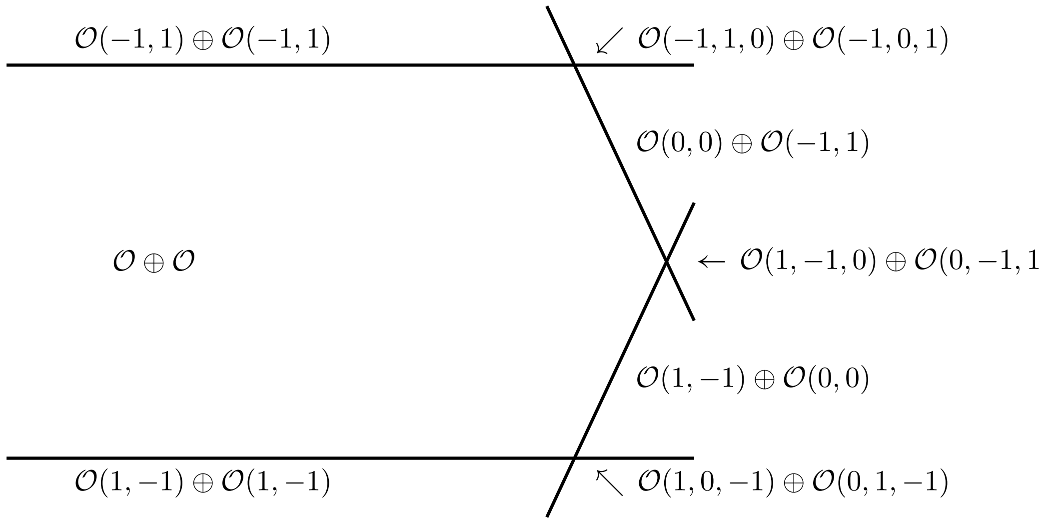

Figure 3. The orbits of the $GL_2 \times GL_2$-action on KGL_2, labeled with the isomorphism classes of their corresponding bundle chains, using the multidegree notation indicated in the text.

For example, KGL_2, has 8 $GL_2 \times GL_2$-orbits: one of codimension 0, four of codimension 1, and three of codimension 2. Their incidences, and the isomorphism classes of the vector bundles appearing in the corresponding bundle chains, are shown in Figure 3.

From the fan perspective, the choice of KGL_r as a compactification of GL_r seems somewhat arbitrary. There are many other possible fans covering the Weyl chamber of GL_r leading to equally good compactifications. For example, when $r = 2$, the fan consisting of two contiguous $90°$ sectors in the plane leads to the compactification $[(\overline{SL}_2 \times \mathbb{P}^1)/\mu_2]$.

On the other hand, several of the alternative descriptions of $\overline{PGL}_r$ given by the second author [Tha99] carry over neatly to KGL_r by tracing through the closures of the relevant loci, starting from $GL_r \subset PGL_{r+1}$. For example, let $\mathbb{G}_m$ act by scalar multiplication on an r-dimensional vector space V and thus on the Grassmannian $X = \mathrm{Gr}_r(V \oplus V^*)$. Then $\overline{PGL}_r$ is the Chow quotient of X by $\mathbb{G}_m$; and likewise, KGL_r is the Chow quotient of $\mathbb{P}^1 \times X$ by $\mathbb{G}_m$. Or again, if $\mathrm{ev} : \overline{M}_{0,2}(X, r) \to X^2$ denotes evaluation on the moduli space of stable maps, then $\overline{PGL}_r = \mathrm{ev}^{-1}(V, V^*)$; and likewise, KGL_r is a fiber of evaluation on the so-called 'graph space'. That is to say, if $\mathrm{Ev} : \overline{M}_{0,2}(\mathbb{P}^1 \times X, 1 \times r) \to (\mathbb{P}^1 \times X)^2$ again denotes evaluation, then $KGL_r = \mathrm{Ev}^{-1}([1,0] \times V, [0,1] \times V^*)$. From this perspective, the choice of KGL_r seems less arbitrary.

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