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RESEARCH ARTICLE

Deformations of Theta Integrals and A Conjecture of Gross-Zagier

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Abstract
In this paper, we complete the proof of the conjecture of Gross and Zagier concerning algebraicity of higher Green functions at a single CM point on the product of modular curves. The new ingredient is an analogue of the incoherent Eisenstein series over a real quadratic field, which is constructed as the Doi-Naganuma theta lift of a deformed theta integral on hyperbolic 1-space.

Contents

1. Introduction

Just over half a century ago, Doi and Naganuma discovered a Hecke-equivariant lifting map from to weight k elliptic modular forms to weight (k,k) Hilbert modular forms for a real quadratic field F [DN70]. This is a special case of cyclic base change [JL70], which has now become a basic and useful tool in the theory of automorphic forms and automorphic representations. By the exceptional isogeny

$$\mathrm{O}(2,2) \sim \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{SL}_2, \tag{1.1}$$

the Doi-Naganuma lifting is also an instance of a theta lifting from SL_2 to $\mathrm{O}(2,2)$ [Kud78].

1.1. A problem posed by Gross and Zagier

In the seminal paper [GZ86], Gross and Zagier proved their formula relating the central derivative of some Rankin-Selberg L-function attached to a weight 2 level N newform f and the Néron-Tate height pairing of f-isopytic components of Heegner points in the Jacobian of the modular curve $X_0(N)$. This was extended in [GKZ87] to describe the positions of these Heegner points in the Jacobian using Fourier coefficients of modular forms. In the degenerate case $N=1$, the Gross-Zagier formula yields a beautiful factorization formula of the norm of differences of singular moduli [GZ85].

To calculate the archimedean contribution to the height pairings, one requires the automorphic Green function

$$\begin{aligned} G_s^{\Gamma_0(N)}(z_1,z_2) &:= -2\sum_{\gamma\in\Gamma_0(N)} Q_{s-1}\left(1+\frac{|z_1-\gamma z_2|^2}{2\Im(z_1)\Im(\gamma z_2)}\right),\ \Re(s)>1,\\ Q_{s-1}(t) &:= \int_0^\infty (t+\sqrt{t^2-1}\cosh(u))^{-s}\,du \end{aligned} \tag{1.2}$$

on $X_0(N)\times X_0(N)$. It is an eigenfunction with respect to the Laplacians in z_1 and z_2 with eigenvalue $s(1-s)$. The function vanishes when one of the z_i approaches the cusps and has a logarithmic singularity along the diagonal. In fact, these properties characterize it uniquely. Using Hecke operators acting on either z_1 or z_2, we can define

$$\begin{aligned} G_s^{\Gamma_0(N),m}(z_1,z_2) &:= \sum_{\gamma\in\Gamma_0(N)\backslash R_N,\ \det(\gamma)=m} G_s^{\Gamma_0(N)}(z_1,\gamma z_2)\\ &= G_s^{\Gamma_0(N)}(z_1,z_2)\mid T_{m,z_1} = G_s^{\Gamma_0(N)}(z_1,z_2)\mid T_{m,z_2}, \end{aligned} \tag{1.3}$$

where $R_N := \{\left(\begin{smallmatrix} a & b\\ Nc & d\end{smallmatrix}\right) : a,b,c,d\in\mathbb{Z}\}$. Then $G_s^{\Gamma_0(N),m}$ has a logarithmic singularity along the m-th Hecke correspondence $T_m\subset X_0(N)^2$ (see (1.2) in Chapter II of [GZ86]).

For integral parameters $s=r+1\in\mathbb{N}_{\geq 2}$, these functions are called *higher Green functions*. In Section V.1 of [GKZ87], two problems about these functions were raised. The first one was to give an interpretation of their values at Heegner points as archimedean contributions of certain higher weight height pairings. This was answered by Zhang in [Zha97] (see also [Xue10]), where the Néron-Tate height

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pairing of Heegner points is replaced by the arithmetic intersection of Heegner cycles on Kuga-Sato varieties.

The second problem dealt with the algebraicity of higher Green functions at a single CM point. Let $M^{!,\infty}_{-2r}(\Gamma_0(N))$ be the space of weakly holomorphic modular forms for $\Gamma_0(N)$ of weight $-2r < 0$ with poles only at the cusp infinity (see (2.27)). Given $f = \sum_{m \gg -\infty} c(m)q^m \in M^{!,\infty}_{-2r}(\Gamma_0(N))$, we call the following linear combination of higher Green functions

$$G^{\Gamma_0(N)}_{r+1,f}(z_1, z_2) := \sum_{m \in \mathbb{N}} c(-m)m^r G^{\Gamma_0(N),m}_{r+1}(z_1, z_2) \tag{1.4}$$

the *principal higher Green function associated to f*. Along the divisor

$$Z_f := \sum_{m \ge 1,\ c(-m) \neq 0} T_m,$$

the function $G^{\Gamma_0(N)}_{r+1,f}$ has a logarithmic singularity. Using Serre duality, this function is the same as the higher Green function defined via relations in Section V.4 of [GZ86] (see Remark 2.5). We say that it is *rational* when f has rational Fourier coefficients at the cusp infinity. Even though the theory of complex multiplication does not directly apply as in the case of automorphic Green functions, the value of a rational, principal higher Green function $G^{\Gamma_0(N)}_{r+1,f}$ at a single CM point on $X_0(N) \times X_0(N)$ should be algebraic in nature, predicted by the following conjecture (see, for example, [Mel08] and [Via11]).

Conjecture 1.1. *Suppose $f \in M^{!,\infty}_{-2r}(\Gamma_0(N))$ has rational Fourier coefficients at the cusp infinity. Then for any CM point $(z_1, z_2) \in X_0(N)^2 \backslash Z_f$ with z_j having discriminant $d_j < 0$, there exists $\alpha = \alpha(z_1, z_2) \in \overline{\mathbb{Q}} \subset \mathbb{C}$ such that*

$$G^{\Gamma_0(N)}_{r+1,f}(z_1, z_2) = |d_1 d_2|^{-r/2} \log |\alpha|.$$

Over the years, there have been a lot of partial results toward this conjecture. When $d_1 d_2$ is a perfect square, this conjecture was proved in [Zha97] conditional on the nondegeneracy of the height pairing of CM cycles. Using regularized theta liftings, an analytic proof was given in [Via11] with restrictions on N, d_j and later in full generality in [BEY21]. When $d_1 d_2$ is not a perfect square, less was known before. For $N = 1, z_1 = i$ and $r = 1$, Mellit proved the conjecture in his thesis [Mel08] using an algebraic approach. When one averages over the full Galois orbit of the CM point (z_1, z_2), the conjecture follows from [GKZ87] for r even. More partial results are available when one averages over different Galois orbits [Li22, BEY21] when $N = 1$.

Motivated by Conjecture 1.1, the first and third author, together with S. Ehlen, considered its generalization to the setting of orthogonal Shimura varieties in [BEY21]. More precisely, let V be a rational quadratic space of signature $(n, 2)$ with $n \ge 1$, and X_K be the Shimura variety associated to $\tilde{H}_{\mathrm{V}} := \mathrm{GSpin}(\mathrm{V})$ and a compact open subgroup $K \subset \tilde{H}_{\mathrm{V}}(\hat{\mathbb{Q}})$. For a nonnegative integer r and a vector-valued harmonic Maass form f of weight $1 - n/2 - 2r$, denote by Φ^r_f its regularized theta lift (see [Bru02] or equation (2.41)). This function is an eigenfunction of the Laplacian on X_K and has a logarithmic singularity along the special divisor Z_f associated to f (see (2.42)). We call it a *higher Green function* on X_K and say that it is *principal*, resp. *rational*, if f is weakly holomorphic, resp. has rational principal part Fourier coefficients. When $\mathrm{V} = M_2(\mathbb{Q})$ and $X_K = X_0(N)^2$, the function Φ^r_f becomes $G^{\Gamma_0(N)}_{r+1,f}$ (see Corollary 2.4).

For a totally real field F of degree d and an F-quadratic space $W = E$ with E/F a quadratic CM extension, suppose there is an isometric embedding

$$W_{\mathbb{Q}} := \mathrm{Res}_{F/\mathbb{Q}} W \hookrightarrow \mathrm{V}, \tag{1.5}$$

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which in particular implies that $n+2 \geq 2d$. Then we obtain a CM cycle $Z(W)$ on X_K from a torus T_W in $\tilde{H}_{\mathrm{V}}$ (see section 2.4 for details). Note that $Z(W)$ is defined over F and is the big CM cycle $Z(W, z_0^{\pm})$ in [BKY12]. We denote by $Z(W)_{\mathbb{Q}}$ the union of the F-conjugates of $Z(W)$. If F is quadratic, we write $Z(W)_{\mathbb{Q}} = Z(W) \cup Z(W)'$.

In [Li23], the second author studied the algebraicity of the *difference* of a rational, principal Φ_f^r at two CM points in $Z(W)$ and was able to verify the analogue of Conjecture 1.1 in that setting. This opens up the possibility of proving Conjecture 1.1 when one proves an algebraicity result for the averaged value $\Phi_f^r(Z(W))$. In this paper, we complete this step by proving the following result complementary to [Li23].

Theorem 1.2 (Algebraicity and factorization). *Let Φ_f^r be a rational, principal higher Green function on X_K. Suppose that $E/\mathbb{Q}$ is a biquadratic CM number field with the real quadratic subfield $F = \mathbb{Q}(\sqrt{D})$, and $Z(W) \cap Z_f = \emptyset$. Then there exists a positive integer κ and $a_1, a_2 \in F^\times$ such that*

$$\Phi_f^r(Z(W)) = \frac{1}{\kappa}\Big(\log|a_1| + \sqrt{D}\log|a_2|\Big). \tag{1.6}$$

For any prime $\mathfrak{p}$ of F, the value $\kappa^{-1}\operatorname{ord}_{\mathfrak{p}}(a_j)$ is given in (5.6). When $n = 2$, we have $a_j = 1$ for $j \equiv r \bmod 2$.

Remark 1.3. The denominator κ appears as a consequence of our matching of sections (see Propositions 4.7 and 4.11) and only depends on $Z(W)$ and r when f has integral Fourier coefficients.

Remark 1.4. Theorem 1.2 also applies to the case $r = 0$ when f has zero constant term, in which case $\Phi_f^0 = \Phi_f$ is the regularized Borcherds lift of f and we have $a_2 = 1$.

Combining Theorem 1.2 with the main result in [Li23], we deduce the algebraicity of a rational, principal higher Green function at a single CM point when $E/\mathbb{Q}$ is biquadratic – hence, Conjecture 1.1 in particular.

Theorem 1.5. *In the setting of Theorem 1.2, there exists $\kappa \in \mathbb{N}$ and Galois equivariant maps $\alpha_1, \alpha_2 : T_W(\hat{\mathbb{Q}}) \to E^{\mathrm{ab}}$ such that*

$$\Phi_f^r([z_0, h]) = \frac{1}{\kappa}\Big(\log|\alpha_1(h)| + \sqrt{D}\log|\alpha_2(h)|\Big)$$

for all $[z_0, h] \in Z(W)$. Furthermore, for $n = 2$, we can choose $\alpha_j(h) = 1$ for $j \equiv r \bmod 2$; that is, there exists a Galois-equivariant map $\alpha : T_W(\hat{\mathbb{Q}}) \to E^{\mathrm{ab}}$ such that

$$\Phi_f^r([z_0, h]) = \frac{1}{\kappa\sqrt{D}^r}\log|\alpha(h)|$$

for all $h \in T_W(\hat{\mathbb{Q}})$. In particular, Conjecture 1.1 is true.

1.2. Comparison to previous works

There has been an extensive literature on the CM-value of regularized theta lifts. When $r = 0, n = 2$ and f is weakly holomorphic, the CM-value $\Phi_f(Z(W)_{\mathbb{Q}})$ was the subject of the classical work of Gross-Zagier on singular moduli [GZ85] and generalizations by the first and third author [BY06]. More generally for arbitrary n, totally real field F and harmonic Maass form f, the value $\Phi_f(Z(W)_{\mathbb{Q}})$ is the archimedean contribution of the derivative of a Rankin-Selberg L-function involving the shadow $\xi(f)$ at $s = 0$ [BY09, BKY12, AGHMP18].

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A crucial ingredient in these works is a real-analytic Hilbert Eisenstein series E^* of parallel weight 1 over F. It is an *incoherent Eisenstein series* in the sense of the Kudla program [Kud97]. The arithmetic Siegel-Weil formula predicts that the Fourier coefficients of its derivative, $E^{*,\prime}$, are arithmetic degrees of special cycles [HY11, HY12], which are logarithms of rational numbers.

Using suitable weight 1 harmonic Maass forms in place of incoherent Eisenstein series, the first and third author, together with S. Ehlen, could prove the algebraicity result for higher Green function at a partially averaged CM cycle and deduce the Gross-Zagier conjecture for $X_K = X_0(1)^2$ when the class group of one of the imaginary quadratic fields in E is an elementary 2 group [BEY21, Theorem 1.2]. However, the factorization of the ideal generated by the algebraic numbers is not explicitly given.

Our main result in Theorem 1.2 goes far beyond these aforementioned works in an essential way by studying the regularized theta lifts at the partially averaged CM cycle $Z(W)$, which is in general only *half* of $Z(W)_{\mathbb{Q}}$ and a priori defined over F. For $r = 0$ and f weakly holomorphic, this means that $\Phi_f(Z(W))$ is the logarithm of a number in the real quadratic field F and therefore cannot be related to the Fourier coefficients of incoherent Eisenstein series!

Furthermore, this partial average is quite different, yet more natural, than the one studied in [BEY21]. Instead of using the weight 1 harmonic Maass form loc. cit., which is an elliptic modular form, we explicitly construct a Hilbert modular form $\tilde{\mathcal{I}}$, serving as a companion and substitute for the incoherent Eisenstein series, and obtain precise information concerning its Fourier coefficients. This is the main innovation of the paper and allows us to prove the exact factorization formula for the ideal generated by the algebraic numbers in the spirit of [GZ85], which was not possible in [BEY21]. Most importantly, we are able to achieve this for *arbitrary* open compact subgroup K, just as in [Li23] for the difference of two CM-values, whereas the ingredients in [BEY21] could only handle the level 1 case. This enables us to prove Theorem 1.5 for arbitrary level K, which encompasses the case in Conjecture 1.1. In that sense, this paper is the complement to [Li23], both in results and methods, for biquadratic E.

Besides the analytic approach to Conjecture 1.1, which originated from the work of Viazovska for $F = \mathbb{Q} \oplus \mathbb{Q}$ [Via11], there is also an algebraic approach in [Zha97, Mel08]. However, one must overcome serious obstacles to prove Theorem 1.5 via this approach. For $F = \mathbb{Q} \oplus \mathbb{Q}$, one needs to assume in an essential way the nondegeneracy of the restriction of the Gillet-Soulé height pairing, which is defined on Kuga-Sato varieties, to the subgroup of the Chow group spanned by CM cycles [Zha97, Theorem 5.2.2]. The nondegeneracy of this height pairing on a slightly larger subgroup is conjectured by Beilinson [Bei87] and Bloch [Blo84] (see Conjecture 1.3.1 in [Zha97]). For real quadratic F, one needs to find a substitute for the Kuga-Sato variety, construct canonical models, and define suitable cycles and arithmetic intersections such that the archimedean contribution is given by the CM-values of higher Green functions.[1] Assuming that the conjecture of Beilinson and Bloch holds in this case, one can then deduce the result in Theorem 1.5. For $n = 2$ and concrete families of CM points, it is possible verify Conjecture 1.1 by explicit constructions of cycles and calculations (see [Mel08]). In general, it is not clear at all how to construct suitable cycles, not to mention remove the nondegeneracy assumption. However, it would be very interesting to see if Theorem 1.5, which is proved via the analytic approach, can be used to prove the conjectural nondegeneracy when restricted to the above subgroup of the Chow group in [Zha97].

1.3. Proof strategy

For simplicity, we focus on the case $n = 2$, from which the general case is not hard to derive (see Section 5 for details). Applying the strategy in [Kud03] and the Rankin-Cohen operator, one can express $\Phi^r_f(Z(W)) + (-1)^r \Phi^r_f(Z(W)')$ as an F-linear combination of Fourier coefficients of the holomorphic part of $E^{*,\prime}$, which are logarithms of rational numbers. This is a standard procedure involving the Siegel-Weil formula and Stokes' Theorem (see, for example, the proof of Theorem 3.5 in [Li21]).

[1]An idea is to consider powers of the Kuga-Satake abelian scheme over an integral model of X_K, though the dimension of such an abelian scheme is 2^{n+1} and the fiber product becomes untractable quickly.

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A crucial property of the incoherent Eisenstein series is the following differential equation: (see [BKY12, Lemma 4.3])

$$2(L_1+L_2)E^{*,\prime}(g,0,\Phi^{(1,1)}) = E^*(g,0,\Phi^{(1,-1)}) + E^*(g,0,\Phi^{(-1,1)}). \tag{1.7}$$

Here, L_j are lowering operators in the j-th variable, and $\Phi^{(\epsilon_1,\epsilon_2)} = \Phi_f \otimes \Phi_\infty^{(\epsilon_1,\epsilon_2)}$ are Siegel-Weil sections in the degenerate principal series $I(0,\chi)$ with $\chi = \chi_{E/F}$ being the quadratic Hecke character of F associated to E/F (see Section 2.6 for details). In particular, $E^*(g,0,\Phi^{(\epsilon,-\epsilon)})$ is a coherent Eisenstein series of weight $(\epsilon,-\epsilon)$ for $\epsilon = \pm 1$. To prove Theorem 1.2, it suffices to understand $\Phi^r_f(Z(W)) - (-1)^r\Phi^r_f(Z(W)')$, which means we need a substitute of $E^{*,\prime}$ on the left-hand side of (1.7) such that the right-hand side is $E^*(g,0,\Phi^{(1,-1)}) - E^*(g,0,\Phi^{(-1,1)})$.

To obtain this minus sign, we apply the exceptional isogeny in (1.1) and view the coherent Eisenstein series as modular forms on the group $H_0 := \mathrm{SO}(V_0)$ for the quadratic space V_0 of signature $(2,2)$ defined in Section 3.1. Since $E/\mathbb{Q}$ is biquadratic, there is an odd character $\varrho = \varrho_f \cdot \mathrm{sgn}$ of $[F^1] = F^1\backslash\mathbb{A}_F^1$ such that $\chi = \varrho \circ \mathrm{Nm}^-$ (see Remark 2.1). By viewing ϱ as an automorphic form on $H_1 := \mathrm{SO}(V_1)$ for the quadratic space $V_1 = (F,\mathrm{Nm})$, we can consider its theta lift following the diagram

$$H_1 \xrightarrow{\theta_1} G \xrightarrow{\theta_0} H_0, \tag{1.8}$$

where $G = \mathrm{SL}_2$. The first map lifts ϱ to a weight one holomorphic cusp form $\vartheta(g',\varphi_1^-,\varrho)$ on G, which was first studied by Hecke. Here, $\varphi_1^\pm$ is a Schwartz function on $V_1(\mathbb{A})$ whose archimedean component $\varphi_\infty^\pm$ is the Schwartz function in $V_1(\mathbb{R})$ defined in (2.65). Then the second map lifts it to a coherent Eisenstein series and is an instance of the Rallis tower property ([Ral84]).[2] From this, $\theta_0\circ\theta_1$ gives us the equation

$$\begin{aligned}\mathcal{I}(g,\varphi^{(\pm1,\mp1)},\varrho) &:= \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \theta_0(g',g,\varphi_0^{(\pm1,\mp1)})\vartheta(g',\varphi_1^-,\varrho)dg' \\ &= \frac{3}{\pi}E^*(g,0,\Phi^{(\pm1,\mp1)}),\end{aligned} \tag{1.9}$$

where θ_0 is a theta kernel for the quadratic space V_0, $\varphi^{(\pm1,\mp1)} = \varphi_0^{(\pm1,\mp1)} \otimes \varphi_1^-$ is a Schwartz function on $V(\mathbb{A})$ with $V := V_0 \oplus V_1$ and $\Phi^{(\pm1,\mp1)} = F_{\varphi,\varrho}$ is the section defined in (3.37). Our first main result is Theorem 3.3, which ensures that all coherent Eisenstein series can be realized as such lifts. This is reduced to the corresponding local problem and solved in Section 3.5.

To construct $\tilde{\mathcal{I}}$, we first modify the character ϱ to the function $\tilde{\varrho}_C$ on $H_1(\mathbb{A})$ defined in (2.76). It is a preimage of ϱ under the first order invariant differential operator $t\frac{d}{dt}$ on $H_1(\mathbb{R}) \cong \mathbb{R}^\times$, and hence not a classical automorphic form on H_1. We call its lift $\vartheta(g',\varphi_1^+,\tilde{\varrho}_C)$ to G a *deformed theta integral*, since the archimedean component of $\tilde{\varrho}_C$ is essentially $\log t$ and comes from the first term in the Laurent expansion of t^s at $s=0$. This deformed theta integral was first studied in [CL20]. It satisfies the following important property (see Theorem 2.7):

$$L\vartheta(g',\varphi_1^+,\tilde{\varrho}_C) = \vartheta(g',\varphi_1^-,\varrho) + \mathrm{error}. \tag{1.10}$$

Here, L is the lowering operator on G, and error is the special value of the theta kernel on V_1.

We now define $\tilde{\mathcal{I}}(g) := \mathcal{I}(g,\varphi^{(1,1)},\tilde{\varrho}_C)$ in (4.2) using the theta kernel $\theta_0(g',g,\varphi_0^{(1,1)})$ with the archimedean component of $\varphi_0^{(1,1)}$ being the Schwartz function $\varphi_{0,\infty}^{(1,1)}$ defined in (4.1) (the integral is similar to (1.9)). A key observation is that there is an identity between the actions of the universal

[2] By a change of integration order, we can also rewrite the map as

$$G \xrightarrow{\theta} H \xrightarrow{\text{Pullback}} H_0,$$

where $H = \mathrm{SO}(V)$ contains $H_0 \times H_1$.

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enveloping algebras of H_0 and G, which gives in this special case (see (4.7) and Lemma 4.1)

$$(L_1 + L_2)\theta_0(g', g, \varphi_0^{(1,1)}) = L\theta_0(g', g, \varphi_0^{(1,-1)}) - L\theta_0(g', g, \varphi_0^{(-1,1)}), \tag{1.11}$$

where L_1, L_2, resp. L, are differential operators for the variable $g \in H_0$, resp. $g' \in G$. Putting these together, we see that $\tilde{\mathcal{I}}$ satisfies the following property (see the proof of Proposition 4.2 with $r = 0$ for details):

$$-(L_1 + L_2)\tilde{\mathcal{I}}(g) = E^*(g, 0, \Phi^{(1,-1)}) - E^*(g, 0, \Phi^{(-1,1)}) + \mathrm{error}'.$$

Up to this the term error′, which is a manifestation of the error term in (1.10), we have constructed the Hilbert modular form satisfying the desired analogue of the differential equation (1.7).

In addition to satisfying the differential equation, we still need to better understand the Fourier coefficients of $\tilde{\mathcal{I}}$ and compare them to those of $E^{*,\prime}$. This is done in Section 4.2, where we show that they are logarithms of algebraic numbers and give precise factorization information. To achieve this, we introduce a new local section with an s-variable in (3.54) and match it with the standard section involving the s-variable up to an error of $O(s^m)$ for any positive integer m. This builds upon the results in Section 3.5 and is accomplished in Theorem 3.14. These new local sections are of independent interest, as they do not come from pullback of the standard section on $H \cong \mathrm{SO}(3, 3)$. In fact, they do not even tensor together to form a global section with an s-variable.

Finally, we still need to handle the term arising from the error on the right-hand side in (1.10). This boils down to proving the rationality of a Millson theta lift, which is given in Proposition 4.9. For this, we need the Fourier expansion of such a lift computed in [ANS18], and to choose the matching section with a suitable invariance property. Proceeding essentially as in [GKZ87] or [BEY21], with $E^{*,\prime}$ replaced by its sum with $\tilde{\mathcal{I}}$, we complete the proof of Theorem 1.2.

1.4. Outlook and organization

The factorization of the algebraic numbers appearing in the Fourier coefficients of $\tilde{\mathcal{I}}$ are very closely related to the Fourier coefficients of $E^{*,\prime}$, which suggests that they should reflect the non-archimedean part of the arithmetic intersection between integral versions of Z_f and $Z(W)$ defined over the ring of integers of F. It would be very interesting to relate this arithmetic intersection to special values of derivatives of L-functions as in [BKY12] by applying and refining the results in [AGHMP18].

It would be interesting to investigate the analogues of Theorems 1.2 and 1.5 for other CM, étale $\mathbb{Q}$-algebras $E/\mathbb{Q}$. When $E/\mathbb{Q}$ has degree 4, there are four cases

1. $E/\mathbb{Q}$ is biquadratic,
2. $E/\mathbb{Q}$ is a product of imaginary quadratic fields,
3. $E/\mathbb{Q}$ is cyclic,
4. $E/\mathbb{Q}$ is a non-Galois, quartic extension.

The CM points $Z(W)$ have a moduli interpretation as abelian surfaces with CM by the reflex CM algebra $E^\#$. The present paper treats case (1). In cases (2) and (3), the reflex algebras $E^\#$ are quartic, abelian field extensions of $\mathbb{Q}$, and the CM cycle $Z(W)$ is already defined over $\mathbb{Q}$. In the last case, $E^\#/\mathbb{Q}$ is a quartic, non-Galois field, and $Z(W)$ is defined over a real quadratic field. We plan to extend the ideas and techniques in this paper to prove the analogue of Theorem 1.5 in cases (2)–(4). One difficulty that arises is that the quadratic space of signature $(3, 3)$ will have Witt rank less than 3, making it impossible to apply the Siegel-Weil formula to identify the theta integral with an Eisenstein series. Instead, one could try to add a twist to the theta integral (see [Li16]), compute its Fourier expansion and match it with that of an Eisenstein series. When $E/\mathbb{Q}$ is a field of degree greater than 4, the Hilbert Eisenstein series are over totally real fields of degree greater than 2 and hence do not arise from theta integral of elliptic modular forms. For such cases, one would need some new ideas.

In addition, there are other applications of these expected results. For cases (2) and (3), by combining the analogue of Theorem 1.2 and the idea in [Li21], we hope to obtain a non-existence result of genus 2 curves with CM Jacobian and having everywhere good reduction in certain families, generalizing the main result in [HP17]. In the last case, we expect a variation of our construction to lead to a proof of the factorization conjecture of CM-values of twisted Borcherds product in [BY07].

The paper is organized in the following way. Section 2 contains preliminaries. Most of these are standard, except for Section 2.8, which contains the adelic version of the results in [CL20]. Section 3 matches the coherent Eisenstein series with the Doi-Naganuma lift of Hecke's cusp form. Section 4 defines $\tilde{\mathcal{I}}$ and studies its various properties. Finally, we give the proofs of Theorems 1.2 and 1.5 in the last section.

2. Preliminaries

In this section, we introduce some preliminary notions, most of which are standard from the literature. The only material not easily found in the literature are in Sections 2.7 and 2.8 concerning the weight one cusp forms of Hecke in the adelic language, which are translated from the results in [CL20] in the classical language.

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a number field E, let $\mathbb{A}_E$ be its ring of adeles, $\hat{E}$ the finite adeles and $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ with $\psi = \psi_f \psi_\infty$ its usual additive character. For an algebraic group G over E, denote $[\mathrm{G}] = \mathrm{G}(E)\backslash \mathrm{G}(\mathbb{A}_E)$. As usual, let $G = \mathrm{SL}_2$ with standard Borel $B = MN \subset G$. Denote also

$$m(a) = \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right) \in M, n(b) = \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right) \in N,\ w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) \in G,$$

and

$$T(R) = \{t(a) = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) : a \in R^\times\} \subset \mathrm{GL}_2(R).$$

Throughout the paper, F will be a real quadratic field (unless stated otherwise). Let $\prime \in \mathrm{Gal}(F/\mathbb{Q})$ be the nontrivial element. It induces an automorphism of $\mathbb{A}_F, \mathbb{A}_F^\times$ and $F_p := F \otimes \mathbb{Q}_p$ for each prime $p \le \infty$, If p is a finite prime that splits in F (resp. is the infinite place), then F has two embeddings into $\mathbb{Q}_p$ (resp. $\mathbb{R}$), and F_p is a 2-dimensional vector space over $\mathbb{Q}_p$ (resp. $\mathbb{R}$). For $\lambda \in F$, let λ_1, λ_2 denote the images under those embeddings. We will also sometimes use λ to represent the pair (λ_1, λ_2) in $\mathbb{Q}_p^2$ (resp. $\mathbb{R}^2$), and λ' would represent (λ_2, λ_1). We have the incomplete Gamma function

$$\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt.$$

2.1. *Differential operators*

For a real-analytic function f on $G(\mathbb{R})$, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ acts via

$$A(f)(g) := \partial_t f(g e^{tA}) \mid_{t=0},\ A \in \mathfrak{sl}_2(\mathbb{C}). \tag{2.1}$$

We define the raising and lowering operator

$$R := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},\ L := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \tag{2.2}$$

If f is right K_∞-equivariant of weight k, then we have

$$\sqrt{v}^{-(k+2)} R(f)(g_\tau) = R_{\tau,k}(\sqrt{v}^{-k} f(g_\tau)),\ \sqrt{v}^{-(k-2)} L(f)(g_\tau) = L_{\tau,k}(\sqrt{v}^{-k} f(g_\tau)),$$

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where $R_{\tau,k}$ and $L_{\tau,k}$ are the usual raising and lowering operators given by

$$R_{\tau,k} := 2i\partial_\tau + \frac{k}{v},\ L_{\tau,k} := -2iv^2\partial_{\overline{\tau}}. \tag{2.3}$$

We say that f is holomorphic, resp. anti-holomorphic, if $L(f) = 0$, resp. $R(f) = 0$.

For $r \in \mathbb{N}_0$ and $k_1, k_2 \in \frac{1}{2}\mathbb{Z}$, define

$$\begin{aligned} Q_{r,(k_1,k_2)}(X,Y) &:= \sum_{s=0}^{r} \binom{r+k_1-1}{s}\binom{r+k_2-1}{r-s} X^{r-s}(-Y)^s, \\ \tilde{Q}_r(X,Y) &:= \frac{Q_{r,(1,1)}(X,Y)(X+Y)}{X+(-1)^r Y} \end{aligned} \tag{2.4}$$

in $\mathbb{Q}[X,Y]$. We omit (k_1,k_2) from the notation when it is $(1,1)$, in which case

$$Q_r(X,Y) = (X+Y)^r P_r\left(\frac{X-Y}{X+Y}\right),$$

with $P_r(x)$ the r-th Legendre polynomial given explicitly by

$$P_r(x) = 2^{-r}\sum_{s=0}^{r}\binom{r}{s}^2 (x-1)^{r-s}(x+1)^s = (-1)^{r_0}\sum_{s=0}^{r_0}\binom{r_0-r-1/2}{r_0-s}\binom{r-r_0-1/2}{s}x^{r-2s}, \tag{2.5}$$

where $r_0 := \lfloor r/2 \rfloor$. The second identity comes from (3.133) on page 38 of [Gou72] and direct calculation. We thank Zhiwei Sun for pointing us to this reference.

From the differential equation satisfied by P_r, we have

$$(\partial_X\partial_Y)(Q_r(X,Y)(X+Y)) = (r+1)(\partial_X+\partial_Y)Q_r(X,Y). \tag{2.6}$$

For $A \in \mathfrak{sl}_2(\mathbb{C})$, denote

$$A_1 = (A,0),\ A_2 = (0,A) \tag{2.7}$$

in $\mathfrak{sl}_2(\mathbb{C})^2$. Then we have two operators

$$\mathrm{RC}_{r,(k_1,k_2)} := (-4\pi)^{-r} Q_{r,k_1,k_2}(R_1,R_2),\ \widetilde{\mathrm{RC}}_r := (-4\pi)^{-r}\tilde{Q}_r(R_1,R_2) \tag{2.8}$$

on real-analytic functions on $G(\mathbb{R})^2$. If $f : G(\mathbb{R})^2 \to \mathbb{C}$ is holomorphic and right K_∞^2-equivariant of weight (k_1,k_2), then $\mathrm{RC}_{r,(k_1,k_2)}(f)^\Delta : G(\mathbb{R}) \to \mathbb{C}$ is holomorphic and right K_∞-equivariant of weight k_1+k_2+2r, and the operator RC is usually called the Rankin-Cohen operator. Here, $f^\Delta(g) := f(g^\Delta) = f(g,g)$ is the restriction of f to the diagonal $G(\mathbb{R}) \subset G(\mathbb{R})^2$. In fact, we have (see [BvdGHZ08, Proposition 19])

$$\mathrm{RC}_{r,(k_1,k_2)}(f)^\Delta(g_z) = (2\pi i)^{-r}\left(Q_r(\partial_{z_1},\partial_{z_2})f(g_{z_1},g_{z_2})\right)|_{z_1=z_2=z}\ .$$

For example, if $f(g_{z_1},g_{z_2}) = \mathbf{e}(m_1z_1+m_2z_2)$, then

$$\mathrm{RC}_{r,(k_1,k_2)}(f)^\Delta(g_z) = Q_{r,(k_1,k_2)}(m_1,m_2)\mathbf{e}((m_1+m_2)z). \tag{2.9}$$

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From Lemma 2.2 in [Li23], we know that there are unique constants $c_\ell^{(r;k_1,k_2)} \in \mathbb{Q}$ such that

$$(4\pi)^{-r}(R_1^r f)^\Delta = \sum_{\ell=0}^{r} c_\ell^{(r;k_1,k_2)}(4\pi)^{-r+\ell} R^{r-\ell}\mathrm{RC}_{\ell,(k_1,k_2)}(f)^\Delta \tag{2.10}$$

whenever $k_1 + k_2 + 2r < 2$.

2.2. *Quadratic space associated to real quadratic field*

Let $F = \mathbb{Q}(\sqrt{D})$ be a real quadratic field, which becomes a $\mathbb{Q}$-quadratic space of signature $(1,1)$ with respect to the quadratic form

$$Q_a(\lambda) := a\mathrm{Nm}(\lambda),\ \lambda \in F$$

for any $a \in \mathbb{Q}^\times$. We denote this quadratic space by V_a and identify

$$\begin{aligned} \iota_a : V_a(\mathbb{R}) = F \otimes_{\mathbb{Q}} \mathbb{R} &\cong \mathbb{R}^2 \\ (\lambda_1, \lambda_2) &\mapsto (\lambda_1, \mathrm{sgn}(a)\lambda_2)\sqrt{|a|}. \end{aligned} \tag{2.11}$$

This is an isometry, where the quadratic form on $\mathbb{R}^2$ is given by $Q((x_1, x_2)) = x_1x_2$. The special orthogonal group $H_a := \mathrm{SO}(V_a)$ satisfies

$$H_a(\mathbb{Q}) \cong F^1(\cong F^\times/\mathbb{Q}^\times),$$

where $F^1 := \{\mu \in F : \mathrm{Nm}(\mu) = 1\}$ acts on V via multiplication. Furthermore, we identify

$$H_a(\mathbb{R}) \cong \mathbb{R}^\times,$$

where $t \in \mathbb{R}^\times$ acts on $V_a(\mathbb{R}) = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^2$ via $(x_1, x_2) \mapsto (tx_1, t^{-1}x_2)$. Note that the invariant measure on $H(\mathbb{R}) \cong \mathbb{R}^\times$ is $\frac{dt}{|t|}$. We have a character $\mathrm{sgn} : H_a(\mathbb{R}) \cong \mathbb{R}^\times \to \{\pm 1\}$. Denote

$$H_a(\mathbb{R})^+ := \ker(\mathrm{sgn}) \cong \mathbb{R}_{>0}$$

its kernel, which is the connected component of $H_a(\mathbb{R})$ containing the identity. We also denote

$$H_a(\mathbb{Q})^+ := H_a(\mathbb{R})^+ \cap H_a(\mathbb{Q}) \tag{2.12}$$

which is an index 2 subgroup of $H_a(\mathbb{Q})$.

Remark 2.1. Let $\chi = \chi_{E/F}$ be a Hecke character associated to a quadratic extension E/F. Suppose $E/\mathbb{Q}$ is biquadratic. Then $\chi\ |_{\mathbb{A}^\times}$ is trivial and χ factors through the map $\mathrm{Nm}^- : \mathbb{A}_F^\times \to H_a(\mathbb{A})$; that is, there exists $\varrho = \varrho_{E/F} : H_a(\mathbb{A}) \to \{\pm 1\}$ such that

$$\varrho \circ \mathrm{Nm}^- = \chi. \tag{2.13}$$

Note that ϱ is odd if and only if E is totally imaginary. We also denote the compact subgroup

$$K_\varrho := H_a(\hat{\mathbb{Z}}) \cap \ker(\varrho). \tag{2.14}$$

Note that $H_a(\mathbb{Q})\backslash H_a(\hat{\mathbb{Q}})/K_\varrho$ is a finite set.

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2.3. The Weil representation and theta functions

Let (V, Q) be a rational quadratic space of signature (p, q), and $H_V := \mathrm{SO}(V)$. For a subfield $E \subset \mathbb{C}$, we denote $\mathcal{S}(\hat{V}; E)$, resp. $\mathcal{S}(V_p; E)$, to denote the space of Schwartz functions on $\hat{V} := V(\hat{\mathbb{Q}})$, resp. $V_p := V(\mathbb{Q}_p)$, with values in E, which is an E-vector space. We omit E from the notation if it is $\mathbb{Q}$. However, we write $\mathcal{S}(V(\mathbb{R}))$ and $\mathcal{S}(V(\mathbb{A}))$ for the space of Schwartz function on $V(\mathbb{R})$ and $V(\mathbb{A})$ valued in $\mathbb{C}$, respectively.

For a lattice[3] $L \subset V$, we denote $L^\vee \subset V$ its dual lattice, $\hat{L} := L \otimes \hat{\mathbb{Z}}$, $\hat{L}^\vee := L^\vee \otimes \hat{\mathbb{Z}}$ and

$$L_{m,\mu} := \{\lambda \in L + \mu : Q(\lambda) = m\} \tag{2.15}$$

for $m \in \mathbb{Q}, \mu \in L^\vee/L$. The finite dimensional E-subspace

$$\mathcal{S}(L; E) := \{\phi \in \mathcal{S}(\hat{V}; E) : \phi \text{ is } \hat{L}\text{-invariant with support on } \hat{L}^\vee\} \subset \mathcal{S}(\hat{V}; E), \tag{2.16}$$

is spanned by $\{\phi_{L+\mu} : \mu \in \hat{L}^\vee/\hat{L} \cong L^\vee/L\}$ with

$$\phi_{L+\mu} := \mathrm{Char}(\hat{L} + \mu) \in \mathcal{S}(\hat{V}). \tag{2.17}$$

For full sublattices $M \subset L \subset V$, it is clear that

$$\mathcal{S}(L; E) \subset \mathcal{S}(M; E) \subset \mathcal{S}(\hat{V}; E). \tag{2.18}$$

As above, we also denote $\mathcal{S}(L) := \mathcal{S}(L; \mathbb{Q})$. Furthermore, since

$$\mathcal{S}(\hat{V}) = \bigcup_{L \subset V \text{ lattice}} \mathcal{S}(L),$$

we have $\mathcal{S}(\hat{V}; E) = \mathcal{S}(\hat{V}) \otimes_{\mathbb{Q}} E$ for any subfield $E \subset \mathbb{C}$.

Suppose $V = V_1 \oplus V_2$ is a decomposition of rational quadratic spaces. For any lattice $L_i \subset V_i$, we have $\mathcal{S}(L_1 \oplus L_2; E) = \mathcal{S}(L_1; E) \otimes \mathcal{S}(L_2; E) \subset \mathcal{S}(\hat{V}_1; E) \otimes \mathcal{S}(\hat{V}_2; E)$ via the natural restriction map. This also gives us

$$\mathcal{S}(\hat{V}; E) = \mathcal{S}(\hat{V}_1; E) \otimes \mathcal{S}(\hat{V}_2; E) = \bigoplus_{L_1 \subset V_1,\ L_2 \subset V_2 \text{ lattices}} \mathcal{S}(L_1; E) \otimes \mathcal{S}(L_2; E). \tag{2.19}$$

Combining with equation (2.18), we see that for any given $\phi \in \mathcal{S}(\hat{V}; E)$, we can find a lattice $L = L_1 \oplus L_2 \subset V$ such that $L_i \subset V_i$ and

$$\phi \in \mathcal{S}(L; E) = \mathcal{S}(L_1; E) \otimes \mathcal{S}(L_2; E). \tag{2.20}$$

Let $\tilde{G}(\mathbb{A}) := \mathrm{Mp}_2(\mathbb{A})$ be the metaplectic cover of $G(\mathbb{A})$. The group $\tilde{G}(\mathbb{A}) \times H_V(\mathbb{A})$ acts on $\mathcal{S}(V(\mathbb{A}))$ via the Weil representation $\omega = \omega_{V,\psi}$ (see [Kud94, section 5] for explicit formula). For each prime $p \le \infty$, we also have the local Weil representation ω_p of $G(\mathbb{Q}_p)$ acting on $\mathcal{S}(V(\mathbb{Q}_p); \mathbb{C})$. Then $\omega_f := \otimes_{p<\infty}\omega_p$ gives a representation of $G(\hat{\mathbb{Q}})$ on $\mathcal{S}(\hat{V}; \mathbb{C})$.

For any lattice $L \subset V$, the subspace $\mathcal{S}(L; \mathbb{C})$ defined in (2.16) is a K_f-invariant subspace with $K_f := G(\hat{\mathbb{Z}})$. It has a unitary pairing $\langle , \rangle$ with the vector space $\mathbb{C}[L^\vee/L] := \oplus_{\mu \in L^\vee/L}\mathbb{C}\mathfrak{e}_\mu$ given by

$$\langle \mathfrak{e}_\mu, \phi_{\mu'} \rangle := \langle \mathfrak{e}_\mu, \phi_{\mu'} \rangle_L := \begin{cases} 1, & \mu = \mu', \\ 0, & \text{otherwise.} \end{cases} \tag{2.21}$$

[3]Lattices will be even and integral throughout the paper.

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More generally, if $L = L_1 \oplus L_2$, we have $L^\vee/L = L_1^\vee/L_1 \oplus L_2^\vee/L_2$ and $\mathbb{C}[L^\vee/L] \cong \mathbb{C}[L_1^\vee/L_1] \otimes \mathbb{C}[L_2^\vee/L_2]$. Therefore, we can extend the pairing above to

$$\langle\cdot,\cdot\rangle_{L_2} : \mathbb{C}[L^\vee/L] \times \mathcal{S}(L_2;\mathbb{C}) \to \mathbb{C}[L_1^\vee/L_1],\ (\mathfrak{v}_1 \otimes \mathfrak{v}_2, \phi) \mapsto \langle \mathfrak{v}_2, \phi\rangle \mathfrak{v}_1 \tag{2.22}$$

with $\mathfrak{v}_i \in \mathbb{C}[L_i^\vee/L_i]$ and $\phi \in \mathcal{S}(L_2;\mathbb{C})$.

With respect to the perfect pairing in (2.21), the unitary dual of ω_f is the representation ρ_L on $\mathbb{C}[L^\vee/L]$ given by

$$\begin{aligned} \rho_L(n(1))(\mathfrak{e}_\mu) &:= \mathbf{e}(Q(\mu))\mathfrak{e}_\mu, \\ \rho_L(w)(\mathfrak{e}_\mu) &:= \frac{\mathbf{e}(-(p-q)/8)}{\sqrt{|L^\vee/L|}} \sum_{\mu \in L^\vee/L} \mathbf{e}(-(\mu,\mu'))\mathfrak{e}_{\mu'}, \end{aligned} \tag{2.23}$$

where (p,q) is the signature of $V(\mathbb{R})$. This is the Weil representation on finite quadratic modules used by Borcherds in [Bor98]. If we identify $\mathcal{S}(L;\mathbb{C})$ and $\mathbb{C}[L^\vee/L]$ with $\mathbb{C}^{|L^\vee/L|}$ via the bases $\{\phi_\mu : \mu \in \hat{L}^\vee/\hat{L}\}$ and $\{\mathfrak{e}_\mu : \mu \in L^\vee/L\}$, respectively, then $\omega_f = \overline{\rho_L} = \rho_L^{-1}$. For full sublattices $M \subset L$, the following trace map

$$\mathrm{Tr}_M^L : \mathbb{C}[L^\vee/L] \to \mathbb{C}[M^\vee/M],\ \mathfrak{e}_\mu \mapsto \frac{1}{[L:M]} \sum_{h \in M^\vee/M,\ h \equiv \mu \bmod L} \mathfrak{e}_h \tag{2.24}$$

intertwines the Weil representation and is compatible with the inclusion in (2.18) in the sense that

$$\langle \mathrm{Tr}_M^L(\mathfrak{v}), \phi\rangle_M = \langle \mathfrak{e}, \phi\rangle_L \tag{2.25}$$

for any $\mathfrak{v} \in \mathbb{C}[L^\vee/L], \phi \in \mathcal{S}(L;\mathbb{C})$.

The following result will be very useful for us later.

Lemma 2.2. *For any prime p, the local Weil representation ω_p descends to $\mathcal{S}(V(\mathbb{Q}_p);\mathbb{Q}(\zeta_{p^\infty}))$ with*

$$\mathbb{Q}(\zeta_{p^\infty}) := \bigcup_{n\geq 1} \mathbb{Q}(\zeta_{p^n}) \subset \mathbb{Q}^{\mathrm{ab}} \tag{2.26}$$

the maximal abelian extension of $\mathbb{Q}$ ramified only at p.

Proof. Via $L'/L = \oplus_p L_p'/L_p$ with $L_p := L \otimes \mathbb{Z}_p$, we can write $\rho_L = \otimes_p \rho_p$ with ρ_p the Weil representation associated to the finite quadratic module L_p'/L_p and identify $\omega_p = \rho_p^{-1}$. It is well-known that any finite quadratic module can be written in the form M'/M for an even integral lattice M [Nik79]. Therefore, it suffices to prove the claim with ω_p replaced by ρ_M for an even integral lattice M with quadratic form valued in $\mathbb{Z}[1/p]$. This follows then directly from formula (2.23) and Milgram's formula [Bor98, Corollary 4.2]. □

As usual, we let $H_{k,L}(\Gamma)$ denote the space of harmonic Maass forms valued in $\mathbb{C}[L^\vee/L]$ of weight $k \in \frac{1}{2}\mathbb{Z}$ and representation ρ_L on a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ (see [BF04, section 3]). It contains the subspace $M_{k,L}^!(\Gamma)$ of vector-valued weakly holomorphic modular forms. Post-composing with Tr_M^L in (2.24) induces a map $H_{k,L}(\Gamma) \to H_{k,M}(\Gamma)$, which preserves holomorphicity and rationality of holomorphic part Fourier coefficients. If $L^\vee/L$ is trivial (resp. $\Gamma = \mathrm{SL}_2(\mathbb{Z})$), then we drop L (resp. Γ) from the above notations. Furthermore, we let

$$M_k^{!,\infty}(\Gamma) := \{f \in M_k^!(\Gamma) : f \text{ is holomorphic away from the cusp } \infty\} \tag{2.27}$$

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and denote for $f(\tau) = \sum_{m\in\mathbb{Q},\ \mu\in L^\vee/L} c(m,\mu)q^m\mathfrak{e}_\mu \in M^!_{k,L}$

$$\mathrm{prin}(f) := \sum_{m\in\mathbb{Q}_{<0},\ \mu\in L^\vee/L} c(m,\mu)q^m\mathfrak{e}_\mu \tag{2.28}$$

the principal part of *f*.

In [McG03, Theorem 4.3], McGraw extended the representation ρ_L to the metaplectic cover of GL_2. To simplify the notation, we recall it here for lattices with even rank, in which case this extension factors through GL_2. Using the short exact sequence

$$1 \to \mathrm{SL}_2 \to \mathrm{GL}_2 \stackrel{\det}{\to} \mathbb{G}_m \to 1,$$

we can identify $\mathrm{GL}_2 \cong \mathrm{SL}_2 \rtimes T$. Then ω_f extends to a $\mathbb{Q}$-linear action of $\mathrm{GL}_2(\hat{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\hat{\mathbb{Z}})$ on $\mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})$ via

$$(\omega_f(g,t(a))\phi)(x) := (\omega_f(g)\sigma_a(\phi))(x) = \sigma_a((\omega_f(t(a)^{-1}gt(a))(\phi))(x)) \tag{2.29}$$

for $g \in \mathrm{SL}_2(\hat{\mathbb{Q}}), a \in \hat{\mathbb{Q}}^\times, \phi \in \mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})$, where $\sigma_a \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ satisfies $\sigma_a(\psi_f(a')) = \psi_f(aa')$ for all $a, a' \in \hat{\mathbb{Q}}^\times$ and acts on $\mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})$ via its action on $\mathbb{Q}^{\mathrm{ab}}$. This gives us

$$\mathcal{S}(\hat{V}) = \mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})^{T(\hat{\mathbb{Z}})}. \tag{2.30}$$

For each $p < \infty$, this gives an extension of ω_p to a $\mathbb{Q}$-linear action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on $\mathcal{S}(V_p;\mathbb{Q}(\zeta_{p^\infty}))$, which satisfies

$$\mathcal{S}(V_p) = \mathcal{S}(V_p;\mathbb{Q}(\zeta_{p^\infty}))^{T(\mathbb{Z}_p)}. \tag{2.31}$$

For $\varphi \in \mathcal{S}(V(\mathbb{A}))$, we have the theta function

$$\theta_V(g,h,\varphi) := \sum_{x\in V(\mathbb{Q})} (\omega(g)\varphi)(h^{-1}x) \tag{2.32}$$

for $(g,h) \in (G\times H_V)(\mathbb{A})$ as usual. For a lattice $L \subset V$, we also denote

$$\Theta_L(\tau,h) := v^{-(p-q)/4} \sum_{\mu\in L^\vee/L} \theta_V(g_\tau,h,\phi_\mu\phi_\infty)\phi_\mu \tag{2.33}$$

the vector-valued theta function with ϕ_∞ the Gaussian.

2.4. *CM points and higher Green functions*

We follow [BKY12] and [BEY21] to recall CM points and higher Green functions. Let (V, Q) be a rational quadratic space of signature $(n,2)$, and $\tilde{H} = \tilde{H}_{\mathrm{V}} := \mathrm{GSpin}(\mathrm{V})$. For an open compact subgroup $K \subset \tilde{H}(\hat{\mathbb{Q}})$, we have the associated Shimura variety X_K, whose $\mathbb{C}$-points are given by

$$X_K(\mathbb{C}) = \tilde{H}(\mathbb{Q})\backslash(\mathbb{D}\times\tilde{H}(\hat{\mathbb{Q}})/K).$$

Here, $\mathbb{D} = \mathbb{D}^+ \sqcup \mathbb{D}^-$ is the symmetric space associated to $\mathrm{V}(\mathbb{R})$. For $m \in \mathbb{Q}$ and $\varphi \in \mathcal{S}(\hat{\mathrm{V}};\mathbb{C})$, one can define the special divisor $Z(m,\varphi)$ on X_K (see, for example, [BEY21, section 2]).

The CM points on X_K can be described as follows. For a totally real field F of degree d with real embeddings $\sigma_j, 1 \le j \le d$, denote $\alpha_j := \sigma_j(\alpha)$ for $\alpha \in F$. Suppose $\alpha_{j_0} < 0$ for some j_0 and $\alpha_j > 0$ when $j \ne j_0$. Then a CM quadratic extension E/F becomes an F-quadratic space $W = W_\alpha = E$ with respect to the quadratic form $Q_\alpha := \alpha\mathrm{Nm}_{E/F}$. Suppose there is an isometric embedding as in (1.5).

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Then the subspace $W \otimes_{F,\sigma_{j_0}} \mathbb{R} \subset \mathrm{V}(\mathbb{R})$ is a negative 2-plane and determines a point $z_0^\pm \in \mathbb{D}^\pm$ with a choice of orientation. For convenience, we denote

$$z_0 := z_0^+. \tag{2.34}$$

The group $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SO}(W))$ is contained in $\mathrm{SO(V)}$, whose preimage in $\tilde{H}_{\mathrm{V}}$ is a torus denoted by T_W. Note that $T_W(\mathbb{Q}) = E^\times/F^1$. We denote the CM cycle on X_K associated to T_W by

$$Z(W) := T_W(\mathbb{Q})\backslash(\{z_0^\pm\} \times T_W(\hat{\mathbb{Q}})/K_W) \subset X_K \tag{2.35}$$

with $K_W := K \cap T_W(\hat{\mathbb{Q}})$. It is defined over F, and its Galois conjugates are the CM cycles $Z(W(j))$ with $1 \le j \le d$, where $W(j)$ is the neighborhood F-quadratic spaces at σ_j of admissible incoherent $\mathbb{A}_F$-quadratic space $\mathbb{W}$ associated to W (see [BY11, BKY12] for details). Note that $W(j) = W_{\alpha(j)}$ for some $\alpha(j) \in F^\times$ and $\alpha(j_0) = \alpha$. For totally positive $t \in F$, we define the 'Diff' set

$$\mathrm{Diff}(W,t) := \{\mathfrak{p} : W_\mathfrak{p} \text{ does not represent } t\} \tag{2.36}$$

following [Kud97]. Note that this set is finite and odd (see [YY19, Proposition 2.7]).

When F is real quadratic (i.e. $d = 2$), for $\alpha \in F^\times$ with $\mathrm{Nm}(\alpha) < 0$, we set

$$\alpha^\vee := \alpha(2) \in F^\times. \tag{2.37}$$

Then $(\alpha^\vee)^\vee = \alpha$. Note that $\alpha^\vee$ is not necessarily the Galois conjugate of α!

Denote

$$\sigma_0 := \frac{n}{4} - \frac{1}{2}. \tag{2.38}$$

Let $L \subset \mathrm{V}$ be an even integral lattice such that K stabilizes $\hat{L}$. For $\mu \in L^\vee/L$ and $m \in \mathbb{Z} + Q(\mu)$, we write $Z(m,\mu) := Z(m,\phi_\mu)$. The automorphic Green function on $X_K\backslash Z(m,\mu)$ is defined by

$$\Phi_{m,\mu}(z,h,s) := 2\frac{\Gamma(s+\sigma_0)}{\Gamma(2s)} \sum_{\lambda \in h(L_{m,\mu})} \left(\frac{m}{Q(\lambda_{z^\perp})}\right)^{s+\sigma_0} F\left(s+\sigma_0, s-\sigma_0, 2s; \frac{m}{Q(\lambda_{z^\perp})}\right), \tag{2.39}$$

for $\mathrm{Re}(s) > \sigma_0 + 1$, where $F(a,b,c;z)$ is the Gauss hypergeometric function [AS64, Chapter 15]. At $Z(m,\mu)$, the function $\Phi_{m,\mu}$ has logarithmic singularity.

At $s = \sigma_0 + 1 + r$ with $r \in \mathbb{N}$, the function $\Phi_{m,\mu}(z,h,s)$ is called a *higher Green function*. For a harmonic Maass form $f = \sum_{m,\mu} c(m,\mu)q^{-m}\phi_\mu + O(1) \in H_{k-2r,L}$ with $k := -2\sigma_0$, define

$$\Phi_f^r(z,h) := r! \sum_{m>0,\ \mu \in L'/L} c(m,\mu)m^r \Phi_{m,\mu}(z,h,\sigma_0+1+r) \tag{2.40}$$

to be the associated higher Green function. Following from the work of Borcherds [Bor98] and generalization by Bruinier [Bru02] (also see [BEY21, Proposition 4.7]), the function Φ_f^r has the following integral representation:

$$\begin{aligned} \Phi_f^r(z,h) &= (4\pi)^{-r} \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle R_\tau^r f(\tau), \overline{\Theta_L(\tau,z,h)}\rangle d\mu(\tau) \\ &= (-4\pi)^{-r} \lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f(\tau), \overline{R_\tau^r \Theta_L(\tau,z,h)}\rangle d\mu(\tau), \end{aligned} \tag{2.41}$$

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where $\mathcal{F}_T$ is the truncated fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ at height $T > 1$ and $d\mu$ is the invariant measure given in (3.20). It has logarithmic singularity along the special divisor

$$Z_f := \sum_{m>0,\ \mu\in L^\vee/L} c(-m,\mu)Z(m,\mu) \tag{2.42}$$

on X_K. Note that $[z_0, h] \in Z(W) \cap Z_f$ if and only if

$$h(L_{m,\mu}) \cap z_0^\perp \neq \emptyset \tag{2.43}$$

for some m, μ with $c(-m,\mu) \neq 0$.

2.5. Product of modular curves as a Shimura variety

We follow and slightly modify [YY19, section 3] to express $X_0(N) \times X_0(N)$ as $\mathrm{O}(2,2)$ orthogonal Shimura variety. Consider $(\mathrm{V}, Q) = (M_2(\mathbb{Q}), \det)$, and the lattice

$$L := \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \right\} \subset \mathrm{V}$$

for any $N \in \mathbb{N}$. Then the dual lattice $L^\vee$ is given by

$$L^\vee := \left\{ \begin{pmatrix} a & b/N \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \right\} \subset \mathrm{V},$$

and $L^\vee/L \cong (\mathbb{Z}/N\mathbb{Z})^2$ is isomorphic to that of a scaled hyperbolic plane.

For $g_j \in \mathrm{SL}_2(\mathbb{Q})$ and $\Lambda \in \mathrm{V}(\mathbb{Q})$, the map

$$\Lambda \mapsto g_1 \Lambda g_2^{-1} \tag{2.44}$$

gives $\mathrm{SL}_2 \times \mathrm{SL}_2 \cong \mathrm{Spin}(\mathrm{V})$ and identifies H_V as a subgroup of $\mathrm{GL}_2 \times \mathrm{GL}_2$ [YY19, section 3.1]. Let $K(N) := K(\Gamma_0(N)) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$ be the open compact subgroup in [YY19, section 3.1] and $K := (K(N) \times K(N)) \cap H_\mathrm{V}(\hat{\mathbb{Q}})$. Then the map

$$w : \mathbb{H}^2 \to \mathbb{D}^+,\ (z_1, z_2) \mapsto \mathbb{R}\Re\left(\begin{smallmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{smallmatrix}\right) + \mathbb{R}\Im\left(\begin{smallmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{smallmatrix}\right) \tag{2.45}$$

induces an isomorphism

$$X_0(N) \times X_0(N) \cong X_K,\ (z_1, z_2) \mapsto [w(z_1, z_2), 1] \tag{2.46}$$

with X_K the Shimura variety for H_V.

Under the map (2.44), the inverse images of the discriminant kernel $\Gamma_L \subset \mathrm{SO}(L)$ are

$$\Gamma_0^\Delta(N) := \left\{ (g_1, g_2) \in \Gamma_0(N)^2 : g_1 g_2 \in \Gamma_1(N) \right\} \subset \Gamma_0(N)^2,$$

which contains $\Gamma_1(N) \times \Gamma_1(N)$ and is a normal subgroup of $\Gamma_0(N)^2$ satisfying

$$\Gamma_0(N)^2/\Gamma_0^\Delta(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times,\ \left(\left(\begin{smallmatrix} a_1 & * \\ * & * \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & * \\ * & * \end{smallmatrix}\right)\right) \mapsto a_1 a_2 \bmod N.$$

The group $\mathrm{SO}(L^\vee/L) := \mathrm{SO}(L)/\Gamma_L \cong \Gamma_0(N)^2/\Gamma_0^\Delta(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$ acts on $L^\vee/L \cong (\mathbb{Z}/N\mathbb{Z})^2$ via

$$\alpha \cdot (b, c) := (\alpha b, \alpha^{-1} c),$$

and the induced linear map on $\mathbb{C}[L^\vee/L]$ intertwines the Weil representation ρ_L.

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Now given $f \in M^!_k(\Gamma_0(N))$ for $k \in 2\mathbb{Z}$, we can lift it to a vector-valued modular form in $M^!_{k,\rho_L}$ via the following map:

$$\mathrm{vv} : M^!_k(\Gamma_0(N)) \to M^!_{k,\rho_L},\ f \mapsto \sum_{\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} (f|_k \gamma)\rho_L(\gamma)^{-1} \cdot \mathfrak{e}_0. \tag{2.47}$$

This map and its generalizations are well-studied (see, for example, [Sch09]), whose properties are summarized in the following result.

Lemma 2.3. *When $k < 0$, we have*

$$\mathrm{prin}(\mathrm{vv}(f)) = \mathrm{prin}(f)\mathfrak{e}_0. \tag{2.48}$$

for all $f \in M^{!,\infty}_k(\Gamma_0(N))$, on which space the map vv *is an isomorphism with the inverse given by*

$$g = \sum_{\mu \in L^\vee/L} g_\mu \mathfrak{e}_\mu \mapsto g_0. \tag{2.49}$$

Furthermore, it preserves the rationality of the Fourier expansion at the cusp infinity.

Proof. See Proposition 4.2 in [Sch09] and Proposition 6.12, Corollary 6.14 in [BHK+20]. □

As a consequence, we can relate the higher Green function $G^{\Gamma_0(N)}_{r+1,f}$ from the introduction to one on the Shimura variety X_K.

Corollary 2.4. *Under the isomorphism (2.46), we have*

$$G^{\Gamma_0(N)}_{r+1,f}(z_1, z_2) = -\Phi^r_{\mathrm{vv}(f)}([w(z_1, z_2), 1]) \tag{2.50}$$

with $G^{\Gamma_0(N)}_{r+1,f}$ the higher Green function defined in (1.4) for $f \in M^{!,\infty}_{-2r}(\Gamma_0(N))$ with $r > 0$.

Proof. Under (2.46), the divisor

$$Z(m, 0) = \Gamma_0(N)^2 \backslash \{(z_1, z_2) \in \mathbb{H}^2 : (\left(\begin{smallmatrix} z_1 & -z_1 z_2 \\ 1 & -z_2 \end{smallmatrix}\right), x) = 0 \text{ for some } x \in L_{m,0}\} \tag{2.51}$$

on X_K is simply the m-th Hecke correspondence T_m on $X_0(N) \times X_0(N)$. Therefore, the two sides of (2.50) have logarithmic singularity along the same divisor. Using Corollary 4.2 and Theorem 4.4 of [BEY21], we see that their difference is a smooth function in $L^2(X_0(N)^2)$ and an eigenfunction of the Laplacians in z_1 and z_2 with eigenvalue $r(1 - r) < 0$. By fixing z_2, this difference is an eigenfunction of the Laplacian on $X_0(N)$ with negative eigenvalue, which vanishes identically. This holds for any $z_2 \in X_0(N)$, and we obtain (2.50). □

Remark 2.5. Following Section V.4 of [GZ86], we call a set of integers $\{\lambda_m : m \in \mathbb{N}\}$ a *relation for* $S_{2-k}(\Gamma_0(N))$ if only finitely many λ_m are nonzero and

$$\sum_{m \geq 1} \lambda_m a_m = 0$$

for all $\sum_{m \geq 1} a_m q^m \in S_{2-k}(\Gamma_0(N))$. Since $g_0 \in S_{2-k}(\Gamma_0(N))$ for all $g \in S_{2-k,L^-}$, we have

$$\{f_P, g\} := \mathrm{CT}\left(\sum_{\mu \in L^\vee/L} f_{P,\mu} g_\mu\right) = 0$$

for $f_P = \sum_{m \geq 1} \lambda_m q^{-m} \mathfrak{e}_0$. By Serre duality [Bor99], there exists $f \in M^!_{k,L}$ with $\mathrm{prin}(f) = f_P$.

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Suppose E_1, E_2 are imaginary quadratic fields such that $E = E_1E_2$ is biquadratic containing a real quadratic field F (i.e., $E_1 \neq E_2$). Then for any CM points $z_j \in E_j$, the point $(z_1, z_2) \in \mathbb{H}^2$ is sent to $Z(W_\alpha) \cup Z(W_{\alpha^\vee}) \subset X_K$ under the isomorphism in (2.46) (see Section 3.2 in [YY19] for details).

2.6. Eisenstein series

We recall coherent and incoherent Eisenstein series for the group $G = \mathrm{SL}_2$ following [BKY12]. Let F be a totally real field of degree d and discriminant D_F, E/F be a quadratic CM extension with absolute discriminant D_E and $\chi = \chi_{E/F} = \otimes_{v\le\infty}\chi_v$ the associated Hecke character. For a standard section $\Phi \in I(s,\chi)$ with

$$I(s,\chi) := \mathrm{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(|\cdot|^s\chi) = \bigotimes_{v\le\infty} I_v(s,\chi_v),\ I_v(s,\chi_v) := \mathrm{Ind}_{B(F_v)}^{G(F_v)}(|\cdot|_v^s\chi_v), \tag{2.52}$$

we can form the Eisenstein series

$$E^*(g,s,\Phi) := \Lambda(s+1,\chi)E(g,s,\Phi),\ E(g,s,\Phi) := \sum_{\gamma\in B\backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s),$$

where $\Lambda(s,\chi)$ is the completed L-function for χ (see equation (4.6) in [BKY12]). When $\Phi = \otimes_v\Phi_v$, the Eisenstein series $E(g,s,\Phi)$ has the Fourier expansion

$$E^*(g,s,\Phi) = E_0^*(g,s,\Phi) + \sum_{t\in F^\times} E_t^*(g,s,\Phi),$$

and for $t \in F^\times$,

$$E_t^*(g,s,\Phi) = \prod_v W_{t,v}^*(g,s,\Phi_v),$$

where $W_{t,v}^*$ is the normalized local Whittaker function defined by

$$W_{t,v}^*(g_v,s,\Phi_v) := |D_E/D_F|_v^{-(s+1)/2} L(s+1,\chi_v)\int_{F_v} \Phi_v(wn(b)g_v,s)\psi_v(-tb)db. \tag{2.53}$$

For simplicity, we denote

$$W_{t,v}^*(\Phi_v) := W_{t,v}^*(1,0,\Phi_v),\ W_{t,v}^{*,\prime}(\Phi_v) := \partial_s W_{t,v}^*(1,s,\Phi_v)\mid_{s=0}. \tag{2.54}$$

We will be interested in the case when Φ is a Siegel-Weil section.

Given $\alpha \in F^\times$, we view $W_\alpha = E$ as an F-quadratic space with quadratic form $Q_\alpha(z) := \alpha z\bar{z}$. Denote ω_α the associated Weil representation. We have an SL_2-equivariant map $\lambda_\alpha : \mathcal{S}(W_\alpha(\mathbb{A}_F)) \longrightarrow I(0,\chi)$ given by

$$\lambda_\alpha(\phi)(g) = (\omega_\alpha(g)\phi)(0). \tag{2.55}$$

At each place v, there are local versions of ω_α and λ_α as well, denoted by $\omega_{\alpha,v}$ and $\lambda_{\alpha,v}$. When $\Phi = \lambda_\alpha(\phi)$, resp. $\Phi_v = \lambda_{\alpha,v}(\phi_v)$, we replace Φ, resp. Φ_v, from the notations above by ϕ, resp. ϕ_v.

Let $Z(W_\alpha)$ be the CM points on X_K as in Section 2.4 and $W_{\alpha,\mathbb{Q}} \subset \mathrm{V}$ the rational quadratic space as in (1.5). A special case of the Siegel-Weil formula (see [BKY12, Theorem 4.5]) gives us

$$\theta(g, Z(W_\alpha),\phi) = C\cdot E(g^\Delta, 0, \phi) \tag{2.56}$$

for any $\phi \in \mathcal{S}(W_{\alpha,\mathbb{Q}}(\mathbb{A})) = \mathcal{S}(W_\alpha(\mathbb{A}_F))$. Here, $C = \deg(Z(W_\alpha))/2$.

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Suppose only the j-th real embedding of α is negative. Denote $\mathbb{1} := (1, \dots, 1)$ and $\mathbb{1}(j) := (1, \dots, -1, \dots, 1)$ with -1 at the j-th slot. The sections $\Phi(j) = \lambda_{\alpha,f}(\phi) \otimes \Phi_\infty^{\mathbb{1}(j)}$ and $\Phi = \lambda_{\alpha,f}(\phi) \otimes \Phi_\infty^{\mathbb{1}}$ are coherent and incoherent, respectively. For all $\phi \in \mathcal{S}(W_\alpha(\hat{F}); \mathbb{C})$, the Eisenstein series $E^*(g, s, \Phi(j))$ is holomorphic of weight $\mathbb{1}(j)$ at $s = 0$ and

$$E^*(\tau, \phi, \mathbb{1}(j)) := \mathrm{Nm}(v)^{-1/2} E^*(g_\tau, 0, \Phi(j)) \tag{2.57}$$

is called a *coherent* Eisenstein series. However, the Eisenstein series $E^*(g, s, \Phi)$ vanishes at $s = 0$, and its derivative

$$E^{*,\prime}(\tau, \phi) := \partial_s \mathrm{Nm}(v)^{-1/2} E^*(g_\tau, s, \Phi) \mid_{s=0} \tag{2.58}$$

is called an *incoherent* Eisenstein series, which is related to the coherent Eisenstein series via the differential equation [BKY12, Lemma 4.3]

$$2 L_{\tau_j} E^{*,\prime}(\tau, \phi) = E^*(\tau, \phi, \mathbb{1}(j)) \tag{2.59}$$

for all $1 \le j \le d$. Furthermore, it has the Fourier expansion

$$E^{*,\prime}(\tau, \phi) = \mathcal{E}(\tau, \phi) + \phi(0)\Lambda(0, \chi) \log \mathrm{Nm}(v) + \mathcal{E}^*(\tau, \phi), \tag{2.60}$$

where $\mathcal{E}^*(\tau, \phi)$ has exponential decay near the cusp infinity and

$$\mathcal{E}(\tau, \phi) = a_0(\phi) + \sum_{t \in F,\ t \gg 0} a_t(\phi) \mathbf{e}(\mathrm{Tr}(t\tau)).$$

Here, $a_0(\phi)$ is an explicit constant (see (2.24) in [YY19]) and

$$a_t(\phi) := \begin{cases} (-i)^d \tilde{W}_t(\phi) \log \mathrm{Nm}(\mathfrak{p}), & \text{if } \mathrm{Diff}(W_\alpha, t) = \{\mathfrak{p}\}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.61}$$

The coefficient $\tilde{W}_t(\phi)$ is given by (see [YY19, Proposition 2.7])

$$\tilde{W}_t(\phi) = 2^d \frac{W^{*,\prime}_{t,\mathfrak{p}}(\phi_\mathfrak{p})}{\log \mathrm{Nm}(\mathfrak{p})} \prod_{v \nmid \mathfrak{p}\infty} W^*_{t,v}(\phi_v) \in \mathbb{Q}(\phi) \tag{2.62}$$

when $\mathrm{Diff}(W_\alpha, t) = \{\mathfrak{p}\}$

2.7. Hecke's cusp form

Denote $\omega_a = \omega_{V_a,\psi}$ the Weil representation and $\theta_a := \theta_{V_a}$ the theta function as in (2.32). For a bounded, integrable function $\rho : H_a(\mathbb{Q}) \backslash H_a(\mathbb{A}) / K \to \mathbb{C}$, consider the following theta lift:

$$\vartheta_a(g, \varphi, \rho) := \int_{[H_a]} \theta_a(g, h, \varphi) \rho(h) dh. \tag{2.63}$$

The measure dh is the product measure of the local measures dh_p, where dh_p is normalized such that the maximal compact subgroup in $H_a(\mathbb{Q}_p)$ has volume 1. Such integral was first considered by Hecke in [Hec27] when $\rho = \varrho$ is an odd, continuous character – that is,

$$\varrho(h) = \mathrm{sgn}(h_\infty) \varrho_f(h_f) \tag{2.64}$$

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with ϱ_f a continuous character on $H_a(\hat{\mathbb{Q}})$ and $\varphi = \varphi_\infty^\pm \varphi_f$ with

$$\varphi_\infty^\pm(x_1, x_2) := (x_1 \pm x_2)e^{-\pi(x_1^2+x_2^2)} \in \mathcal{S}(\mathbb{R}^2). \tag{2.65}$$

Notice that $\varphi_\infty^\pm$ satisfies $\varphi_\infty^\pm(-x_1, -x_2) = -\varphi_\infty^\pm(x_1, x_2)$.

In this case, the m-th Fourier coefficient of ϑ_a is given by

$$W_m(\varphi, \varrho)(g) := \int_{[N]} \vartheta_a(ng, \varphi, \varrho)\psi(-mn)dn$$

for $m \in \mathbb{Q}$. To evaluate it, we apply the usual unfolding trick

$$\begin{aligned} W_m(\varphi, \varrho)(g) &= \int_{[N]} \int_{[H_a]} \sum_{\lambda \in V_a(\mathbb{Q})} (\omega_a(ng)\varphi)(h^{-1}\lambda)\varrho(h)dh\psi(-mn)dn \\ &= \int_{[H_a]} \sum_{\lambda \in V_{a,m}(\mathbb{Q})} (\omega_a(g)\varphi)(h^{-1}\lambda)\varrho(h)dh \\ &= \sum_{\lambda \in H_a(\mathbb{Q})\backslash V_{a,m}(\mathbb{Q})} \int_{H_{a,\lambda}(\mathbb{Q})\backslash H_a(\mathbb{A})} (\omega_a(g)\varphi)(h^{-1}\lambda)\varrho(h)dh. \end{aligned}$$

When $m = 0$, we have $\lambda = 0$ since V_a is anisotropic and $\varphi(0) = \varphi_\infty^\pm(0)\varphi_f(0) = 0$. When $Q_a(\lambda) = m \neq 0$, the group $H_{a,\lambda}$ is trivial. We can then write $g = g_\tau g_f$ with $g_\tau = n(u)m(\sqrt{v}) \in G(\mathbb{R})$ and $g_f \in G(\hat{\mathbb{Q}})$ and obtain

$$W_m(\varphi, \varrho)(g) = \sum_{\lambda \in H_a(\mathbb{Q})\backslash V_{a,m}(\mathbb{Q})} \int_{\mathbb{R}^\times} (\omega_a(g_\tau)\varphi_\infty^\pm)(t^{-1}(\iota_a(\lambda)))\frac{dt}{t} \int_{H_a(\hat{\mathbb{Q}})} \varrho_f(h)(\omega_a(g_f)\varphi_f)(h^{-1}\lambda)dh.$$

The group $H_a(\mathbb{Q})$ acts on $V_{a,m}(\mathbb{Q})$ transitively. The archimedean integral can be evaluated as

$$\begin{aligned} \int_{\mathbb{R}^\times} (\omega_a(g_\tau)\varphi_\infty^\pm)(t^{-1}(\iota_a(\lambda)))\frac{dt}{t} &= 2\mathbf{e}(mu)\frac{v}{\sqrt{|a|}} \int_0^\infty (t^{-1}\lambda_1 \pm \operatorname{sgn}(a)t\lambda_2)e^{-\pi\frac{v}{|a|}(t^{-2}\lambda_1^2+t^2\lambda_2^2)}\frac{dt}{t} \\ &= 2v\mathbf{e}(mu)\operatorname{sgn}(\lambda_1)\sqrt{|m|}\int_0^\infty (t^{-1} \pm \operatorname{sgn}(m)t)e^{-\pi v|m|(t^{-2}+t^2)}\frac{dt}{t}. \end{aligned}$$

This is 0 if $\pm\operatorname{sgn}(m) < 0$ by the change of variable $t \mapsto 1/t$ and can be otherwise evaluated using the lemma below.

Lemma 2.6. *For any $\beta > 0$, we have*

$$\int_0^\infty (t^{-1} + t)e^{-\beta\pi(t^{-2}+t^2)}\frac{dt}{t} = \frac{e^{-2\pi\beta}}{\sqrt{\beta}}.$$

Therefore, we have

$$W_m(\varphi^\pm, \varrho)((g_\tau, g_f)) = 2\sqrt{v}\mathbf{e}(mu)\mathbf{e}(|m|iv)\operatorname{sgn}(\lambda_1)\int_{H_a(\hat{\mathbb{Q}})} \varrho_f(h)(\omega_a(g_f)\varphi_f)(h^{-1}\lambda)dh, \tag{2.66}$$

when $\pm m > 0$ and $\lambda \in V_{a,m}(\mathbb{Q})$. Otherwise, it is 0. The integral in (2.66) can be evaluated locally. Notice that it always converges as $\varphi_f(h^{-1}\lambda)$ has compact support as a function of $h \in H_a(\hat{\mathbb{Q}})$.

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2.8. The Deformed theta integral

We recall the real-analytic modular form of weight one constructed in [CL20] using the notations of Section 2.7. Let ϱ be an odd, continuous character as in (2.64), and $K_\varrho \subset H_a(\hat{\mathbb{Z}})$ the open compact subgroup defined in (2.14). The intersection $H_a(\mathbb{Q})^+ \cap K_\varrho$, where $H_a(\mathbb{Q})^+ := H_a(\mathbb{Q}) \cap H_a(\mathbb{R})^+$, is a cyclic subgroup

$$\Gamma_\varrho = \langle \varepsilon_\varrho \rangle,\ \varepsilon_\varrho > 1 > \varepsilon'_\varrho > 0 \tag{2.67}$$

of the totally positive units in $\mathcal{O}$. Then we have

$$H_a(\mathbb{A}) = \bigsqcup_{\xi \in C} H_a(\mathbb{Q}) H_a(\mathbb{R})^+ K_\varrho \xi, \tag{2.68}$$

where $C \subset H_a(\hat{\mathbb{Q}})$ is a finite subset of elements representing $H_a(\mathbb{Q})^+ \backslash H_a(\hat{\mathbb{Q}})/K_\varrho$. So given $h = (h_f, h_\infty)$, we can find $\alpha \in H_a(\mathbb{Q}), t \in H_a(\mathbb{R})^+, k_1 \in K_\varrho, \xi \in C$ all depending on h such that

$$h = (\alpha k_1 \xi, \alpha t), \tag{2.69}$$

though the choice is not unique. This gives us the identification

$$H_a(\mathbb{Q}) \backslash H_a(\mathbb{A})/K_\varrho \cong \bigsqcup_{\xi \in C} \Gamma_\varrho \backslash H_a(\mathbb{R})^+ \xi \tag{2.70}$$

by sending $h = (\alpha k_1 \xi, \alpha t) \in H_a(\mathbb{A})$ as in (2.69) to $t \in H_a(\mathbb{R})^+$ in the ξ-component. Just like the decomposition (2.68), this isomorphism depends on the choice of the set of representatives C. Similarly, we have

$$H_a(\hat{\mathbb{Q}})/K_\varrho \cong \bigsqcup_{\xi \in C} \Gamma_\varrho \backslash H_a(\mathbb{Q})^+ \xi. \tag{2.71}$$

Using the Fourier coefficient W_m in (2.66) and the decomposition in (2.71), we can write

$$\vartheta_a(g, \varphi^\pm, \varrho) = \mathrm{vol}(K_\varrho) \sum_{\xi \in C} \varrho(\xi) \sum_{\beta \in \Gamma_\varrho \backslash V_a(\mathbb{Q}),\ \pm Q_a(\beta) > 0} (\omega_a(g) \varphi^{0,\pm}))(\xi^{-1}\beta) \tag{2.72}$$

for $g \in G(\hat{\mathbb{Q}}) \times B(\mathbb{R})$, where $\varphi^\pm = \varphi_f \varphi_\infty^\pm$ with $\varphi_f \in \mathcal{S}(\hat{V}_a)$ being K_ϱ-invariant and

$$\varphi^{0,\pm} = \varphi_f \varphi_\infty^{0,\pm},\ \varphi_\infty^{0,\pm}(x_1, x_2) := \mathrm{sgn}(x_1) e^{\mp 2\pi x_1 x_2}. \tag{2.73}$$

Although $\varphi_\infty^{0,\pm}$ is not a Schwartz function on $\mathbb{R}^2$, the sum above still converges absolutely. For $g \in B(\mathbb{R})$, the quantity $\omega_a(g)\varphi_\infty^{0,\pm}$ is defined with the usual formula of the Weil representation, and $\vartheta_a(g, \varphi^\pm, \varrho)$ is right $\mathrm{SO}_2(\mathbb{R})$-equivariant with weight ± 1. We also have a left $G(\mathbb{Q})$-invariant function

$$\Theta_a(g, \varphi^\pm, \varrho) := \int_{H_a(\mathbb{Q}) \backslash H_a(\hat{\mathbb{Q}})} \theta_a(g, h, \varphi^\pm) \varrho(h) dh = \mathrm{vol}(K_\varrho) \sum_{\xi \in C} \varrho(\xi) \theta_a(g, \xi, \varphi^\pm) \tag{2.74}$$

on $G(\mathbb{A})$ for $\varphi \in \mathcal{S}(V_a(\mathbb{A}))$, which is independent of the choice of C.

We now define a function $\mathrm{lg}_C : H_a(\mathbb{Q}) \backslash H_a(\mathbb{A})/K_\varrho \to [0, 1)$ by

$$\begin{aligned} \mathrm{lg}_C((\alpha k_1 \xi, \alpha t)) &:= 2 \log \varepsilon_\varrho \cdot \{\log t / \log \varepsilon_\varrho\}, \\ \{a\} &:= a - \lim_{\epsilon \to 0} \frac{1}{2}(\lfloor a + \epsilon \rfloor + \lfloor a - \epsilon \rfloor). \end{aligned} \tag{2.75}$$

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Note that $\{0\} = \frac{1}{2}$. Unlike the function considered by Hecke, lg_C cannot be written as the product of functions on $H_a(\hat{\mathbb{Q}})$ and $H_a(\mathbb{R})$. Denote

$$\tilde{\varrho}_C := \mathrm{lg}_C \cdot \varrho : H_a(\mathbb{Q})\backslash H_a(\mathbb{A})/K_\varrho \to \mathbb{C}. \tag{2.76}$$

Given $\varphi = \varphi_f \varphi_\infty \in \mathcal{S}(V_a(\mathbb{A}))$ for some K_ϱ-invariant $\varphi_f \in \mathcal{S}(\hat{V}_a; \mathbb{C})$, the deformed theta integral $\vartheta_a(g, \varphi, \tilde{\varrho}_C)$, where ϑ_a is defined in (2.63), was studied in [CL20], To describe its Fourier expansion, denote

$$\begin{aligned}\vartheta_a^*(g, \varphi_f, \varrho) &:= \int_{H_a(\mathbb{Q})^+\backslash H_a(\hat{\mathbb{Q}})} \varrho(h) \sum_{\substack{\beta \in \Gamma_\varrho\backslash V_a(\mathbb{Q}) \\ Q_a(\beta)<0}} (\omega_a(g)\varphi^{0,*})(h^{-1}\beta) dh \\ &= -\mathrm{vol}(K_\varrho)\sqrt{v} \sum_{\xi \in C} \varrho(\xi) \sum_{\substack{\beta \in \Gamma_\varrho\backslash V_a(\mathbb{Q}) \\ Q_a(\beta)<0}} (\omega_a(g_f)\varphi_f)(\xi^{-1}\beta)\mathrm{sgn}(\beta)\mathbf{e}(Q_a(\beta)\tau)\Gamma(0, 4\pi|Q_a(\beta)|v).\end{aligned}$$

for $g = (g_f, g_\tau) \in G(\mathbb{A})$ with $\varphi^{0,*} = \varphi_f \varphi_\infty^{0,*}$ and

$$\varphi_\infty^{0,*}(x_1, x_2) := -\mathrm{sgn}(x_1)e^{-2\pi x_1 x_2}\Gamma(0, 4\pi|x_1 x_2|). \tag{2.77}$$

Note that ϑ_a^* is not necessarily left-$G(\mathbb{Q})$ invariant. But it is modular after applying the lowering operator as

$$L\big(\vartheta_a^*(g, \varphi_f, \varrho)\big) = \vartheta_a(g, \varphi^-, \varrho).$$

Similarly for $\xi \in H_a(\hat{\mathbb{Q}})$, define

$$\begin{aligned}\theta_a^*(g, \xi, \varphi_f) &:= \sum_{\lambda \in V_a(\mathbb{Q})} (\omega_a(g)\varphi^*)(\xi^{-1}\lambda) = \sum_{\lambda \in V_a(\mathbb{Q})} (\omega_a(g_f)\varphi_f)(\xi^{-1}\lambda)(\omega_a(g_\tau)\varphi^*)(\lambda) \\ &= -v \sum_{\lambda \in V_a(\mathbb{Q})} (\omega_a(g_f)\varphi_f)(\xi^{-1}\lambda)\frac{\mathrm{sgn}(\lambda_1 - \mathrm{sgn}(a)\lambda_2)}{\sqrt{\pi}}\mathbf{e}(Q_a(\lambda)\tau)\Gamma\left(\frac{1}{2}, \frac{\pi v}{|a|}(\lambda_1 - \mathrm{sgn}(a)\lambda_2)^2\right).\end{aligned} \tag{2.78}$$

Here, we have employed the rapidly decaying function

$$\varphi^*(x_1, x_2) := -e^{-2\pi x_1 x_2}\mathrm{sgn}(x_1 - x_2)\Gamma\left(\frac{1}{2}, \pi(x_1 - x_2)^2\right), \tag{2.79}$$

where $B(\mathbb{R}) \subset G(\mathbb{R})$ acts via ω_a and $\mathrm{SO}_2(\mathbb{R})$ acts with weight 1. Also, we denote

$$\Theta_{a,C}^*(g, \varphi_f, \varrho) := \mathrm{vol}(K_\varrho) \sum_{\xi \in C} \varrho(\xi)\theta_a^*(g, \xi, \varphi_f), \tag{2.80}$$

which depends on the choice of C and satisfies

$$L\Theta_{a,C}^*(g, \varphi_f, \varrho) = \Theta_a(g, \varphi^-, \varrho). \tag{2.81}$$

We recall some results.

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Theorem 2.7. *Let $\tilde{\varrho}_C$ be as in (2.76) and $\varphi_f \in V_a(\hat{\mathbb{Q}})$ a right-K_ϱ invariant function. Then the integral $\vartheta_a(g, \varphi^+, \tilde{\varrho}_C)$ defines a $G(\mathbb{Q})$-invariant function in $g \in G(\mathbb{A})$ of weight 1 with respect to* $\mathrm{SO}_2(\mathbb{R})$*. Furthermore, it has the Fourier expansion*

$$\begin{aligned}\vartheta_a(g, \varphi^+, \tilde{\varrho}_C) &= \vartheta_a^*(g, \varphi_f, \varrho) + \log \varepsilon_\varrho \Theta^*_{a,C}(g, \varphi_f, \varrho)\\ &\quad + \mathrm{vol}(K_\varrho) \sum_{\substack{\xi \in C\\ \beta \in \Gamma_\varrho \backslash V_a(\mathbb{Q})\\ Q_a(\beta) > 0}} \tilde{\varrho}_C((\xi, \sqrt{|\beta/\beta'|}))(\omega_a(g)\varphi^{0,+})(\xi^{-1}\beta),\end{aligned} \tag{2.82}$$

where $\varphi^{0,+} = \varphi_f \varphi_\infty^{0,+}$ is defined in (2.73), and satisfies

$$L\vartheta_a(g, \varphi^+, \tilde{\varrho}_C) = \vartheta_a(g, \varphi^-, \varrho) + \log \varepsilon_\varrho \Theta_a(g, \varphi^-, \varrho). \tag{2.83}$$

Proof. This follows essentially from Proposition 5.5 in [CL20]. For completeness, we include a different (and slightly shorter) proof here. As in the evaluation of $\vartheta_a(g, \varphi, \varrho)$ in (2.72), we have

$$\begin{aligned}W_m(\varphi, \tilde{\varrho}_C)(g_\tau) &= \mathrm{vol}(K_\varrho)\mathbf{e}(mu)\sqrt{v} \sum_{\xi \in C} \sum_{\beta \in \Gamma_\varrho \backslash V_{a,m}(\mathbb{Q})} \varphi_f(\xi^{-1}\beta) J(\beta, v),\\ J(\beta, v) &:= 2 \log \varepsilon_\varrho \int_0^\infty \varphi_\infty^+(t^{-1} \cdot (\iota_a(\beta)\sqrt{v})) \left\{\frac{\log t}{\log \varepsilon_\varrho}\right\} \frac{dt}{t}\\ &= 2 \log \varepsilon_\varrho \mathrm{sgn}(m\beta) \int_0^\infty \varphi_\infty^{\mathrm{sgn}(m)}(\sqrt{|m|v}(t, t^{-1})) \left\{\frac{\log t}{\log \varepsilon_\varrho} + \frac{1}{2}\frac{\log|\beta/\beta'|}{\log \varepsilon_\varrho}\right\} \frac{dt}{t}.\end{aligned}$$

To verify (2.83), we start with

$$\begin{aligned}L_\tau \mathbf{e}(mu)J(\beta, v) &= \mathbf{e}(m\tau)v^2\partial_v\left(e^{2\pi mv}J(\beta, v)\right)\\ &= \mathbf{e}(m\tau)2\log \varepsilon_\varrho \mathrm{sgn}(m\beta) \int_0^\infty v^2 \partial_v e^{2\pi mv} \varphi_\infty^{\mathrm{sgn}(m)}(\sqrt{|m|v}(t, t^{-1})) \left\{\frac{\log t}{\log \varepsilon_\varrho} + \frac{1}{2}\frac{\log|\beta/\beta'|}{\log \varepsilon_\varrho}\right\} \frac{dt}{t}\\ &= v\mathbf{e}(mu)\mathrm{sgn}(m\beta)\log \varepsilon_\varrho \int_0^\infty \partial_t \varphi_\infty^{-\mathrm{sgn}(m)}(\sqrt{|m|v}(t, t^{-1})) \left\{\frac{\log t}{\log \varepsilon_\varrho} + \frac{1}{2}\frac{\log|\beta/\beta'|}{\log \varepsilon_\varrho}\right\} dt\\ &= -v\mathbf{e}(mu)\left(\mathrm{sgn}(m\beta) \int_0^\infty \varphi_\infty^{-\mathrm{sgn}(m)}(\sqrt{|m|v}(t, t^{-1}))\frac{dt}{t} - \log \varepsilon_\varrho \sum_{\varepsilon \in \Gamma_\varrho} \varphi_\infty^+(\iota_a(\beta\varepsilon))\right)\\ &= v\mathbf{e}(mu)\left(\delta_{m<0}\mathrm{sgn}(\beta) + \log \varepsilon_\varrho \sum_{\varepsilon \in \Gamma_\varrho} \varphi_\infty^+(\iota_a(\beta\varepsilon))\right).\end{aligned}$$

Substituting this into the left-hand side of (2.83) proves it.

Now to calculate the Fourier expansion, it suffices prove the claim

$$\lim_{v \to \infty} e^{2\pi mv} J(\beta, v) = \mathrm{sgn}(\beta)\, \mathrm{lg}_C((1, \sqrt{|\beta/\beta'|}). \tag{2.84}$$

For each $\beta \in \Gamma_\varrho \backslash V_a(\mathbb{Q})$, we choose the unique representative $\beta_0 \in \Gamma_\varrho \beta$ such that $|\beta_0/\beta_0'| \in [1, \varepsilon_\varrho^2)$. We can then write $J(\beta, v) = J_1(\beta_0, v) + J_2(\beta_0, v)$, where

$$\begin{aligned}J_1(\beta_0, v) &:= 2\int_0^\infty \varphi_\infty^+(t^{-1} \cdot (\iota_a(\beta_0)\sqrt{v})) \log t \frac{dt}{t},\\ J_2(\beta_0, v) &:= -2\log \varepsilon_\varrho \int_0^\infty \varphi_\infty^+(t^{-1} \cdot (\iota_a(\beta_0)\sqrt{v})) \left\lfloor \frac{\log t}{\log \varepsilon_\varrho} \right\rfloor \frac{dt}{t}.\end{aligned}$$

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For J_1, we have

$$\lim_{v\to\infty} e^{2\pi mv} J_1(\beta_0, v) = \log|\beta_0/\beta_0'|\mathrm{sgn}(m\beta_0) \lim_{v\to\infty} e^{2\pi mv} \int_0^\infty \varphi_\infty^{\mathrm{sgn}(m)}(\sqrt{|m|v}(t, t^{-1}))\frac{dt}{t}$$
$$= \log|\beta_0/\beta_0'|\mathrm{sgn}(\beta_0)\delta_{m>0}.$$

For J_2, the limit vanishes unless $|\beta_0/\beta_0'| = 1$, in which case

$$\lim_{v\to\infty} e^{2\pi mv} J_2(\beta_0, v) = 2\log\varepsilon_\varrho \mathrm{sgn}(m\beta_0) \lim_{v\to\infty} e^{2\pi mv} \int_{\varepsilon_\varrho^{-1}}^1 \varphi_\infty^{\mathrm{sgn}(m)}(\sqrt{|m|v}(t, t^{-1}))\frac{dt}{t}$$
$$= \log\varepsilon_\varrho \mathrm{sgn}(\beta_0)\delta_{m>0}.$$

Putting these together proves claim (2.84). □

Finally, we record a result as a direct consequence of Theorem 4.5 in [CL20] (see also Section 5 in [LS22]).

Proposition 2.8. *For any $\varphi_f \in \mathcal{S}(\hat{V}_1; \mathbb{C})$, there exists a real-analytic modular form $\tilde{\Theta}_{a,C}(g, \varphi^-, \varrho) = \tilde{\Theta}^+_{a,C}(g, \varphi^-, \varrho) + \tilde{\Theta}^*_{a,C}(g, \varphi^-, \varrho)$ such that $L\tilde{\Theta}_{a,C} = \Theta_a$ and $\sqrt{v}\tilde{\Theta}^+_{a,C}(g_\tau, \varphi^-, \varrho)$ is holomorphic in τ with Fourier coefficients in $\mathbb{Q}(\varphi_f)$.*

3. Doi-Naganuma lift of Hecke's cusp form

In this section, we are interested in computing the O(2, 2) theta lift of Hecke's cusp form from Section 2.7 and realize it as coherent Hilbert Eisenstein series from 2.6 over real quadratic fields. The main result of this section is the global matching Theorem 3.3, where we show that any coherent Eisenstein series can be realized as such a theta lift. This global statement follows from its local counterpart in Theorem 3.10, which is improved further in Theorem 3.14 to allow matching deformed local sections. This last result will be crucial for us in proving the factorization result in Proposition 4.7 later.

3.1. Quadratic spaces

Let $V_{\pm 1}$ be as in Section 2.2, ℓ^+, ℓ^- be isotropic lines such that $\ell^+ \oplus \ell^-$ is a hyperbolic plane and denote

$$V := V_0 \oplus V_1,\ V_0 := \ell^+ \oplus \ell^- \oplus V_{-1}. \tag{3.1}$$

We can realize

$$V_0(\mathbb{Q}) \cong \left\{\Lambda \in M_2(F) : \Lambda^t = \Lambda'\right\}$$
$$(a, b, \lambda) \mapsto \begin{pmatrix} a & \lambda \\ \lambda' & b\end{pmatrix}$$

with det as the quadratic form and furthermore write

$$V_0 = V_{00} \oplus U_D,\ V_{00} := V_0 \cap M_2(\mathbb{Q}),\ U_D := \sqrt{D}\mathbb{Q}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \cong (\mathbb{Q}, Q_D), \tag{3.2}$$

where $Q_D(x) = Dx^2$. So V has Witt rank 3 and admits the isotropic decomposition

$$V = V^+ + V^-,\ V^\pm := \ell^\pm + (V_{-1} + V_1)^\pm,\ (V_{-1} + V_1)^\pm(\mathbb{Q}) := \{(\lambda, \pm\lambda) : \lambda \in F\} \tag{3.3}$$

with $V^{\pm}$ maximal totally isotropic subspaces. For a $\mathbb{Q}$-algebra R (e.g., $R \in \{\mathbb{Q}, \mathbb{Q}_p, \mathbb{R}, \hat{\mathbb{Q}}, \mathbb{A}\}$), we will use

$$(a, b, \lambda, \mu) \in V(R),\ a, b \in R, \lambda \in R \otimes F \cong V_{-1}(R), \mu \in R \otimes F \cong V_1(R) \tag{3.4}$$

to represent elements in $V(R)$. Define elements $f_j^{\pm} \in V$ by

$$\begin{aligned} &f_1^+ := (1,0,0,0),\ f_1^- := (0,1,0,0),\ f_2^+ := (0,0,1/2,1/2),\ f_2^- := (0,0,1/2,-1/2),\\ &f_3^+ := (0,0,\sqrt{D}/2, \sqrt{D}/2),\ f_3^- := (0,0,1/(2\sqrt{D}), -1/(2\sqrt{D})). \end{aligned} \tag{3.5}$$

Then $\{f_j^{\pm} : j = 1,2,3\} \subset V^{\pm}$ is a $\mathbb{Q}$-basis of $V^{\pm}$. With respect to the ordered basis $(f_1^+, f_2^+, f_3^+, f_1^-, f_2^-, f_3^-)$, the Gram matrix of Q is $\begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$. For $i = 1, 2, 3$, the following linear transformations

$$w_i(f_j^{\pm}) := \begin{cases} f_j^{\mp}, & \text{if } i = j,\\ f_j^{\pm}, & \text{otherwise} \end{cases} \tag{3.6}$$

are easily checked to be in $\mathrm{O}(V)$. The unimodular lattice

$$V_{\mathbb{Z}} := \{(a, b, \lambda, \mu) \in V(\mathbb{Q}) \cap (\mathbb{Z}^2 \times (\mathfrak{d}^{-1})^2) : \lambda - \mu \in \mathcal{O}_F\} \subset V \tag{3.7}$$

provides V with an integral structure. Similarly for $? \in \{00, 0, 1\}$, the lattice $V_{?,\mathbb{Z}} := V_{\mathbb{Z}} \cap V_?$ in $V_?$ gives it with an integral structure.

For $? \in \{00, 0, 1, -1, \emptyset\}$, we write

$$\tilde{H}_? := \mathrm{GSpin}(V_?),\ H_? := \mathrm{SO}(V_?), \tag{3.8}$$

which are subgroups of $\tilde{H}$ and H, respectively, by acting trivially on $V_?^{\perp}$ and have the following exact sequence:

$$1 \to \mathrm{G}_m \to \tilde{H}_? \to H_? \to 1. \tag{3.9}$$

For any commutative ring R, we have explicitly

$$\iota : \mathrm{GSpin}(V_{0,\mathbb{Z}})(R) \cong \{\gamma \in \mathrm{GL}_2(\mathcal{O} \otimes_{\mathbb{Z}} R) : \det(\gamma) \in R^{\times}\}, \tag{3.10}$$

via the action of $\gamma \in \mathrm{GL}_2(\mathcal{O} \otimes_{\mathbb{Z}} R)$ on $V_{\mathbb{Z},0}(R)$

$$\Lambda \mapsto \det(\gamma)^{-1} \gamma \Lambda (\gamma')^t. \tag{3.11}$$

For any $\mathbb{Q}$-algebra R, we also have $\tilde{H}_?(R) = \mathrm{GSpin}(V_{?,\mathbb{Z}})(R)$ for $? \in \{00, 0, 1, \emptyset\}$. Therefore, through ι, we have

$$G_0 := \mathrm{Spin}(V_0) \cong G_F := \mathrm{Res}_{F/\mathbb{Q}}(G),\ G_{00} := \mathrm{Spin}(V_{00}) \cong G,\ H_{00} \cong \mathrm{PGL}_2 \tag{3.12}$$

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and will represent elements in H_0 by their preimages in G_F. Denote $T_0 := \iota^{-1}(T) \subset \tilde{H}_0$ Then the relations among these groups can be visualized in the following diagram:

$$
\begin{array}{ccccccccc}
 & & G_{00} & \hookrightarrow & G_0 & \hookrightarrow & \tilde{H}_0 & \hookleftarrow & T_0 \\
 & \overset{\cong}{\swarrow} & \big| & \overset{\cong}{\swarrow} & \big| & \swarrow & \big\uparrow & \overset{\cong}{\swarrow} & \\
\mathrm{SL}_2 & \hookrightarrow & G_F & \hookrightarrow & \mathrm{GL}_{2/F} & \hookleftarrow & T & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_{00} & \hookrightarrow & H_0 & \hookrightarrow & H_0 & &
\end{array}
\tag{3.13}
$$

Here, the horizontal and vertical arrows are natural inclusions and surjections of algebraic groups, respectively, and the diagonal arrows are induced by ι. Let $B_F \subset G_F$ be the standard parabolic subgroup, and $B_0 := \iota^{-1}(B_F) \subset G_0$. They can be visualized as

$$
\begin{array}{ccccc}
 & B_0 & \hookrightarrow & G_0 & \\
\overset{\cong}{\swarrow} & & \overset{\cong}{\swarrow} & & \\
B_F & \hookrightarrow & G_F & &
\end{array}
\tag{3.14}
$$

which gives us

$$
B_0(\mathbb{Q})\backslash G_0(\mathbb{Q}) \cong B_F(\mathbb{Q})\backslash G_F(\mathbb{Q}) = B(F)\backslash G(F) \tag{3.15}
$$

via ι. We also denote

$$
T^{\Delta} \subset T \times T_0 \subset \mathrm{GL}_2 \times \tilde{H}_0 \tag{3.16}
$$

the diagonal, which will play a crucial role in the local matching result in Section 3.5.

Now let $P \subset H$ be the Siegel parabolic stabilizing V^+, whose Levi factor is isomorphic to $\mathrm{GL}(V^+)$. Then $P_0 := P \cap H_0 \subset H$ is the subgroup stabilizing the line ℓ^+ and acting trivially on V_1. The preimage of $P_0 H_{-1} \subset H_0$ in $\tilde{H}_0$ is given by $B_0 T_0$. Combining with (3.15), we obtain

$$
(P_0 H_{-1})(\mathbb{Q})\backslash H_0(\mathbb{Q}) = (B_0 T_0)(\mathbb{Q})\backslash \tilde{H}_0(\mathbb{Q}) = B_0(\mathbb{Q})\backslash G_0(\mathbb{Q}) \cong B(F)\backslash G(F). \tag{3.17}
$$

For $\alpha \in F^\times, \beta \in F$, we then have $m(\alpha), n(\beta) \in G_0(\mathbb{Q}) \subset \tilde{H}(\mathbb{Q})$. It is easy to check that

$$
\begin{aligned}
(\omega(m(\alpha))\varphi)(a, b, \nu, \lambda) &= \varphi(a/\alpha\alpha', \alpha\alpha' b, \alpha'\nu/\alpha, \lambda), \\
(\omega(n(\beta))\varphi)(a, b, \nu, \lambda) &= \varphi(a - \beta\nu - \beta'\nu' + \beta\beta' b, \beta, \nu - \beta b, \lambda)
\end{aligned}
$$

for a Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}))$.

3.2. *Theta integral*

Let $\theta_0(g, g_1, \varphi_0)$ denote the theta function on $[G \times H_0]$ associated to $\varphi_0 \in \mathcal{S}(V_0(\mathbb{A}))$. Suppose $\varphi_{0,\infty}$ is in the polynomial Fock space $\mathbb{S}(V_0(\mathbb{R}))$ (see Section 4.1). Using $\mathbb{S}(V_0(\mathbb{R})) = \mathbb{S}(V_{00}(\mathbb{R})) \otimes \mathbb{S}(U_D(\mathbb{R}))$, we can then restrict θ_0 to the subgroup $[G \times H_{00}]$, view it as a function on $[G \times G_{00}]$, and write

$$
\theta_0(g, g_{00}, \varphi_0) = \sum_{j \in J} \theta_{00}(g, g_{00}, \varphi_{00,j}) \theta_D(g, \varphi_{D,j}), \tag{3.18}
$$

where $\varphi_0 = \sum_{j \in J} \varphi_{00,j} \varphi_{D,j}$ with $\varphi_{00,j} \in \mathcal{S}(V_{00}(\mathbb{A}))$ and $\varphi_{D,j} \in \mathcal{S}(U_D(\mathbb{A}))$.

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We now define

$$I_0(h_0, \varphi_0, f) := \int_{[G]} \theta_0(g, h_0, \varphi_0) f(g) dg \tag{3.19}$$

for f a bounded, integrable function on $[G]$. Note that the measure dg is normalized so that $[G]$ has volume 1. In particular, for a right $G(\hat{\mathbb{Z}})$-invariant function ϕ on $[G]$, we have

$$\int_{[G]} \phi(g) dg = \frac{3}{\pi} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \phi(g_\tau) d\mu(\tau), \; d\mu(\tau) := \frac{dudv}{v^2}. \tag{3.20}$$

When $f(g) = \vartheta_1(g, \varphi_1, \rho)$ for a bounded, integrable function ρ on $H_1(\mathbb{Q})\backslash H_1(\mathbb{A})$, the integral I_0 above becomes

$$\begin{aligned} \mathcal{I}(h_0, \varphi, \rho) &:= \int_{[H_1]} I((h_0, h_1), \varphi)\rho(h_1) dh_1 = I_0(h_0, \varphi_0, \vartheta_1(\cdot, \varphi_1, \rho)), \\ I(h, \varphi) &:= \int_{[G]} \theta(g, h, \varphi) dg \end{aligned} \tag{3.21}$$

with $\varphi = \varphi_0 \otimes \varphi_1$.

For our purpose, $\rho = \varrho$ will be an odd, continuous character as in (2.64), and $\varphi = \varphi_f \varphi_\infty^{(\epsilon,-\epsilon)}$ for $\epsilon = \pm 1$ with

$$\begin{aligned} \varphi_\infty^{(\epsilon,-\epsilon)} &:= \varphi_{0,\infty}^{(\epsilon,-\epsilon)} \otimes \varphi_\infty^-, \\ \varphi_{0,\infty}^{(\epsilon,-\epsilon)}(a, b, \nu_1, \nu_2) &:= (\epsilon i(a+b) + (\nu_1 - \nu_2)) e^{-\pi(a^2+b^2+\nu_1^2+\nu_2^2)}, \end{aligned} \tag{3.22}$$

and $\varphi_\infty^\pm$ defined in (2.65). Here, we have identified $V(\mathbb{R}) = \mathbb{R}^2 \oplus V_1(\mathbb{R}) \oplus V_{-1}(\mathbb{R}) \cong (\mathbb{R}^2)^{\oplus 3}$ via (2.11). For any $\theta \in \mathbb{R}, \epsilon = \pm 1$, we have

$$\omega(\kappa(\theta))\varphi_\infty^{(\epsilon,-\epsilon)} = \varphi_\infty^{(\epsilon,-\epsilon)}, \; \kappa(\theta) := \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right) \in \mathrm{SO}_2(\mathbb{R}) \subset G(\mathbb{R}),$$

where ω is the Weil representation of $G(\mathbb{R})$ on $V(\mathbb{R})$. However, for $h(\theta) = (\kappa(\theta), 1), h'(\theta) = (1, \kappa(\theta)) \in H_0(\mathbb{R})$ with any $\theta \in \mathbb{R}$, it is easy to check that

$$\omega(h(\theta))\varphi_\infty^{(\epsilon,-\epsilon)} = e^{\epsilon i\theta}\varphi_\infty^{(\epsilon,-\epsilon)}, \; \omega(h'(\theta))\varphi_\infty^{(\epsilon,-\epsilon)} = e^{-\epsilon i\theta}\varphi_\infty^{(\epsilon,-\epsilon)}.$$

So $\varphi_\infty^{(\epsilon,-\epsilon)}$ is equivariant of weight $(\epsilon, -\epsilon)$ with respect to the connected component $\mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R})$ of the maximal compact of $H_0(\mathbb{R})$. Later, we will also consider the following integral

$$\mathcal{I}_f(h_0, \varphi, \varrho) := \int_{H_1(\mathbb{Q})\backslash H_1(\hat{\mathbb{Q}})} \varrho(h_1) \int_{[G]} \theta(g, (h_0, h_1), \varphi) dg dh_1, \tag{3.23}$$

which is similar to $\mathcal{I}(h_0, \varphi, \varrho)$ and well-defined as

$$\varrho(-h_1)\theta(g, (h_0, -h_1), \varphi) = \varrho(h_1)\theta(g, (h_0, h_1), \varphi)$$

for all g, h_0, h_1 and $\varphi_f \in \mathcal{S}(\hat{V})$. When $\varphi = \varphi_0 \otimes \varphi_1$, we have

$$\mathcal{I}_f(h_0, \varphi, \varrho) = I_0(h_0, \varphi_0, \Theta_1(\cdot, \varphi_1, \varrho)), \tag{3.24}$$

where Θ_a (with $a = 1$) is defined in (2.74).

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3.3. Fourier transform and Siegel-Weil formula

We follow [GQT14] to recall the Siegel-Weil formula needed for our purpose, which goes from the split orthogonal group to the symplectic group. The range we need is in the 1st term range, and was originally proved in [KR94]. Let $\varphi = \varphi_\infty \varphi_f \in \mathcal{S}(V(\mathbb{A}))$ with φ_∞ as in (3.22) above. For $(g, h) \in G(\mathbb{A}) \times H(\mathbb{A})$, we have the theta function $\theta(g, h, \varphi)$ and are interested in the value of the convergent integral $I(h, \varphi)$ defined in (3.21).

For a rational quadratic space $(V, (,)_V)$, suppose $V = U^+ + U^- + V_\circ$ with U^+, U^- complementary totally isotropic subspaces and $V_\circ = (U^+ + U^-)^\perp$. Let $W = X + Y$ denote the symplectic space of rank 2 over $\mathbb{Q}$ with the symplectic pairing $\langle , \rangle_W$. The rational vector space $\mathbb{W} := V \otimes W$ is then a symplectic space with respect to the pairing

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\mathbb{W}} := (v_1, v_2)_V \langle w_1, w_2 \rangle_W . \tag{3.25}$$

From this, we have the Fourier transform $\mathscr{F}_{U^+} : \mathcal{S}(V(\mathbb{A})) \to \mathcal{S}(((U^- \otimes W) + V_\circ)(\mathbb{A}))$ defined by

$$\mathscr{F}_{U^+}(\varphi)(\eta, v_\circ) := \int_{U^+(\mathbb{A})} \varphi(u^+, \eta_1, v_\circ)\psi((u^+, \eta_2)_V) du^+ \tag{3.26}$$

with $\eta = (\eta_1, \eta_2) \in (U^-)^2(\mathbb{A}) \cong (U^- \otimes W)(\mathbb{A})$ and $\eta_i \in U^-(\mathbb{A})$. Here, du^+ is the Usual Haar measure on $U^+(\mathbb{A}) = \mathbb{A}$. Note that we have $(u^+, \eta_1, v_\circ) \in (U^+ \otimes X + U^- \otimes X + V_\circ)(\mathbb{A}) = V(\mathbb{A})$. Note that on $\mathcal{S}((U^- \otimes W + V_\circ)(\mathbb{A}))$, the Weil representation ω acts as

$$\begin{aligned}
&(\omega(g, 1)\phi)(\eta, v_\circ) = \omega_{V_\circ}(g)\phi(\eta g, v_\circ), g \in G(\mathbb{A}), \\
&(\omega(1, a)\phi)(\eta, v_\circ) = |\det(a)|\phi(a^{-1}\eta, v_\circ), a \in \mathrm{GL}(U^+)(\mathbb{A}), \\
&(\omega(1, u)\phi)(\eta, v_\circ) = \psi(\langle u(\eta), \eta\rangle/2)\phi(\eta, v_\circ), u \in N(U^+)(\mathbb{A}) \subset \mathrm{Hom}_{\mathbb{Q}}(U^-, U^+)(\mathbb{A}),
\end{aligned} \tag{3.27}$$

which makes $\mathscr{F}_{U^+}$ an intertwining map.

For V in (3.1), we can take $U^\pm = V^\pm$ and $V_\circ$ trivial with $V^\pm$ defined in (3.3). Another possibility is to take $U^\pm = \ell^\pm$ and $V_\circ = V_1 \oplus V_{-1}$, which will be used in calculating the Fourier expansion of the theta integral I_0 in (3.19). To simplify notations, we write

$$\mathscr{F} := \mathscr{F}_{V^+}, \ \mathscr{F}_1 := \mathscr{F}_{\ell^+} \tag{3.28}$$

and use them to represent the Fourier transform at the finite and infinite places as well. For example, $\mathscr{F}_1$ is given by

$$\mathscr{F}_1(\varphi)((\eta_1, \eta_2), \nu, \lambda) = \int_{\mathbb{A}} \varphi(b, \eta_1, \nu, \lambda)\psi(b\eta_2) db \tag{3.29}$$

for $\varphi \in \mathcal{S}(V(\mathbb{A}))$. As $\mathscr{F}_1$ acts as $\mathscr{F}_1' \otimes \mathrm{id}$ on $\mathcal{S}(V(\mathbb{A})) = \mathcal{S}(V_0(\mathbb{A})) \otimes \mathcal{S}(V_1(\mathbb{A}))$, we will abuse notation and write $\mathscr{F}_1 = \mathscr{F}_1'$, which acts on $\mathcal{S}(V_0(\mathbb{A}))$.

For a place $v \le \infty$ of $\mathbb{Q}$ and corresponding local field $k = \mathbb{Q}_v$, recall we have the Siegel-Weil section

$$\begin{aligned}
\Phi_v : \mathcal{S}((V^- \otimes W)(k)) &\to I_v^H(0) \\
\phi_v &\mapsto (h \mapsto (\omega_v(h)\phi_v)(0)),
\end{aligned}$$

where $I_v^H(s) = \mathrm{Ind}_{P(k)}^{H(k)}(|\cdot|^s)$ is the degenerate principal series. The image of Φ_v is a submodule of $I_v^H(0)$ denoted by $R_v(W)$. When $v < \infty$, it is known that (see [GQT14, Proposition 5.2(ii)])

$$I_v^H(0) = R_v(W) \oplus (R_v(W) \otimes \det_H).$$

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It is clear that

$$\Phi_v(\omega(g)\phi_v) = \Phi_v(\phi_v) \tag{3.30}$$

for any $g \in G(k)$.

Given any $\phi = \otimes_v \phi_v \in \mathcal{S}((V^- \otimes W)(\mathbb{A}))$, we denote $\Phi_s(\phi) \in I^H(s)$ the standard section satisfying $\Phi_0(\phi) = \otimes_v \Phi_v(\phi_v)$. We can then form the Eisenstein series

$$E_P^H(s,\phi)(h) := \sum_{\gamma \in P(\mathbb{Q})\backslash H(\mathbb{Q})} \Phi_s(\phi)(\gamma h),$$

which has meromorphic continuation to $s \in \mathbb{C}$ and is holomorphic at $s = 0$. The regularized Siegel-Weil formula by Kudla-Rallis gives then the following equality (see [GQT14, Theorem 7.3(ii)]):

$$2I(h,\varphi) = E_P^H(0,\mathscr{F}(\varphi))(h). \tag{3.31}$$

As a special case of the proposition in Section 2 of [Mœg97], following an argument in [GPSR87], we have the following lemma.

Lemma 3.1. *For any $h \in H(\mathbb{A})$, we have*

$$E_P^H(s,\phi)(h) = \sum_{\gamma_0 \in B(F)\backslash G(F),\ \gamma_1 \in H_1(\mathbb{Q})} \Phi_s(\phi)((\gamma_0,\gamma_1)h). \tag{3.32}$$

Proof. We will show that $P(\mathbb{Q})\backslash H(\mathbb{Q}) \cong (B(F)\backslash G(F)) \times H_1(\mathbb{Q})$ with the map induced by (3.11). First, we have $P(\mathbb{Q})\backslash H(\mathbb{Q}) = (P \cap (H_0 \times H_1))(\mathbb{Q})\backslash(H_0 \times H_1)(\mathbb{Q})$. Let $H_{-1} \subset H_0$ denote the image of $\mathrm{SO}(V_{-1})$, which is isomorphic to H_1, and $P_0 := P \cap H_0$. Then $P \cap (H_0 \times H_1) = P_0 P_1^\Delta$ with $P_1^\Delta \cong H_1$ the image of the diagonal embedding of H_1 into $H_{-1} \times H_1$. From this, we obtain

$$(P \cap (H_0 \times H_1))(\mathbb{Q})\backslash(H_0 \times H_1)(\mathbb{Q}) = (P_0 P_1^\Delta)(\mathbb{Q})\backslash(H_0 \times H_1)(\mathbb{Q}) = (((P_0 H_{-1})\backslash H_0) \times H_1)(\mathbb{Q}).$$

Equation (3.17) then finishes the proof. □

Suppose $\varrho = \otimes_{p \le \infty} \varrho_p$ is an odd character of $H_1(\mathbb{A})/H_1(\mathbb{Q})$ and

$$\chi := \varrho \circ \mathrm{Nm}^- = \otimes_{v \le \infty} \chi_v \tag{3.33}$$

a totally odd character of $\mathbb{A}_F^\times / F^\times$, which can be viewed as a character on $B_0(\mathbb{A})$. Denote

$$I^{G_0}(\chi) := \mathrm{Ind}_{B_0(\mathbb{A})}^{G_0(\mathbb{A})} \chi,\ I_p^{G_0}(\chi_p) := \mathrm{Ind}_{B_0(\mathbb{Q}_p)}^{G_0(\mathbb{Q}_p)} \chi_p,\ \chi_p := \bigotimes_{v|p} \chi_v. \tag{3.34}$$

From (3.14), we see that

$$I^{G_0}(\chi) = I(0,\chi),\ I_p^{G_0}(\chi_p) = \bigotimes_{v|p} I_v(0,\chi_v) \tag{3.35}$$

with $I(s,\chi)$ and $I_v(s,\chi_v)$ defined in (2.52). Using the formula (3.31) and Lemma 3.1, we can rewrite the function $\mathcal{I}(g_0,\varphi,\varrho)$ in (3.21) as

$$2\mathcal{I}(g_0,\varphi,\varrho) = \mathrm{CT}_{s=0} \int_{[H_1]} \varrho(h_1) E_P^H(s,\mathscr{F}(\varphi))(g_0,h_1)dh_1 = E_{B_0}^{G_0}(0,F_{\varphi,\varrho})(g_0),$$

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for $g_0 \in G_0(\mathbb{A})$, where $E_{B_0}^{G_0}(s', F_{\varphi,\varrho})$ is the Eisenstein series for the standard section associated to

$$\begin{aligned} F_{\varphi,\varrho}(g_0) &:= F_{\varphi,\varrho,0}(g_0) \in \mathrm{Ind}_{B_0}^{G_0}\chi, \\ F_{\varphi,\varrho,s}(g_0) &:= \int_{H_1(\mathbb{A})} \Phi_s(\mathscr{F}(\varphi))(g_0,h_1)\varrho(h_1)dh_1. \end{aligned} \tag{3.36}$$

Note that $F_{\varphi,\varrho,s}$ is not a standard section (i.e., it depends on s when restricted to any open compact subgroup of $G_0(\hat{\mathbb{Q}})$).

If $\varrho = \otimes_{p\le\infty}\varrho_p$ and $\varphi = \otimes_{p\le\infty}\varphi_p$, then $F_{\varphi,\varrho,s}$ is a product of local integrals.

$$F_{\varphi_p,\varrho_p,s}(g_{0,p}) := \int_{H_1(\mathbb{Q}_p)} \Phi_s(\mathscr{F}(\varphi_p))(g_{0,p},h_1)\varrho(h_1)dh_1,\ F_{\varphi_p,\varrho_p} := F_{\varphi_p,\varrho_p,0} \in I_{0,p}(\chi_p). \tag{3.37}$$

Recall that dh_1 is normalized so that the maximal compact subgroup of $H_1(\mathbb{Q}_p)$ has volume 1. We have explicitly

$$F_{\varphi_p,\varrho_p}(g) = \int_{H_1(\mathbb{Q}_p)\times F_p\times\mathbb{Q}_p} \varphi_p((g,h_1)^{-1}(x,0,\lambda,\lambda))\varrho_p(h_1)dx\, d\lambda\, dh_1 \tag{3.38}$$

with $d\lambda$ the self-dual measure on F_p such that $\int_{\mathcal{O}_{F_p}} d\lambda = |D|_p^{1/2}$. From this, we see that

$$\sigma_a(|D|_p^{1/2}F_{\varphi_p,\varrho_p}(g)) = |D|_p^{1/2}F_{\sigma_a(\varphi_p),\varrho_p}(g),\ F_{\varphi_p,\varrho_p}(t_0g) = F_{\varphi_p,\varrho_p}(g) \tag{3.39}$$

for all $a \in \mathbb{Z}_p^\times$ and $t_0 \in T_0(\mathbb{Z}_p)$. At all but finitely many cases, the function $F_{\varphi_p,\varrho_p,s}$ is given explicitly as follows.

Lemma 3.2. *Suppose p is unramified in E and φ_p is the characteristic function of the maximal lattice $V_\mathbb{Z}\otimes\mathbb{Z}_p \subset V_p$. Then*

$$F_{\varphi_p,\varrho_p,s}(g_p) = (1-p^{-2-2s})\prod_{v|p} L(1+s,\chi_v) \tag{3.40}$$

for all $g_p \in G_0(\mathbb{Z}_p)$.

Proof. Since φ_p is $G_0(\mathbb{Z}_p)$-invariant, we can suppose $g_p = 1$.

If p is inert in F, then

$$\Phi_s(\mathscr{F}(\varphi_p))(1,h_1) = \Phi_0(\mathscr{F}(\varphi_p))(1,h_1) = 1 = \varrho(h_1)$$

for all $h_1 \in H_1(\mathbb{Q}_p) = H_1(\mathbb{Z}_p) = \mathcal{O}_{F_p}^1 \subset \mathcal{O}_{F_p}^\times$, and $F_{\varphi_p,\varrho_p,s}(g_p) = \int_{H_1(\mathbb{Z}_p)} dh_1 = 1$.

If p is split in F, we have $F_p \cong \mathbb{Q}_p^2$, $H_1(\mathbb{Q}_p) = \{(\alpha,\alpha^{-1}) \in F_p : \alpha \in \mathbb{Q}_p^\times\} \cong \mathbb{Q}_p^\times$ and $\chi_v = \chi_{v'}$ is a character of $\mathbb{Q}_p^\times$. Straightforward (though involved) calculations show that

$$\Phi_s(\mathscr{F}(\varphi_p))(1,h_1) = \Phi_0(\mathscr{F}(\varphi_p))(1,h_1)\min\{|h_1|_v,|h_1|_{v'}\}^s.$$

For $h_1 = (\alpha,\alpha^{-1})$ with $o(\alpha) = m$, we have

$$\begin{aligned} \Phi_0(\mathscr{F}(\varphi_p))(1,h_1) &= \int_{\mathbb{Q}_p^3} \mathrm{Char}(\mathbb{Z}_p^6)(a,0,\lambda_1,\lambda_2,\alpha^{-1}\lambda_1,\alpha\lambda_2)da d\lambda_1 d\lambda_2 \\ &= \int_{p^{\max\{0,m\}}\mathbb{Z}_p} d\lambda_1 \int_{p^{\max\{0,-m\}}\mathbb{Z}_p} d\lambda_2 = p^{-|m|}. \end{aligned} \tag{3.41}$$

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Since p is unramified in E, we have $\varrho_p((\alpha,\alpha^{-1})) = \epsilon^{o(\alpha)}$ with $\epsilon := \varrho_p((p,p^{-1})) = \chi_v(p) = \chi_{v'}(p)$. Putting these together then gives us

$$\begin{aligned} F_{\varphi_p,\varrho_p,v,s}(1) &= \int_{\mathbb{Q}_p^\times} \min\{|\alpha|_p, |\alpha^{-1}|_p\}^{1+s}\epsilon^{o(\alpha)}|\alpha|_p^s d^\times\alpha \\ &= \sum_{m\in\mathbb{Z}} \epsilon^m p^{-|m|(1+s)} = L(1+s,\chi_v)L(1+s,\chi_{v'})(1-p^{-2-2s}). \end{aligned} \tag{3.42}$$

This finishes the proof. □

3.4. Matching global sections

The function $\mathcal{I}(g_0, \varphi^{(k,k')}, \varrho)$ is a Hilbert modular form of weight (k,k'). We want to suitably choose ϱ and φ_f and compare this function to a coherent Eisenstein series.

Let $\chi = \chi_{E/F}$ be a Hecke character associated to a quadratic extension E/F with $E/\mathbb{Q}$ biquadratic, and $\varrho : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$ the character satisfying (2.13), whose kernel in $H_1(\hat{\mathbb{Z}})$ is denoted by K_ϱ. Let $\alpha \in F^\times$, W_α be the same as in Section 2.6. For our purpose, we will choose $\phi_\infty^{(k,k')} \in \mathcal{S}(W_\alpha(F\otimes\mathbb{R}))$ to be eigenfunctions of $K_\infty = \mathrm{SL}_2(\mathbb{R})^2$ with weight (k,k') and normalized to have

$$\phi_\infty^{(k,k')}(0) = 1.$$

The matching result we will prove is the following.

Theorem 3.3. *For $\alpha \in F^\times$ with $\mathrm{Nm}(\alpha) < 0$, given any $\phi_f \in \mathcal{S}(\hat{W}_\alpha)$, there exists $\varphi_f \in \mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})$ such that $\omega_f(-1)\varphi_f = -\varphi_f$ for $-1 \in H_1(\hat{\mathbb{Q}})$, it is invariant with respect to the compact subgroup $G(\hat{\mathbb{Z}})T^\Delta(\hat{\mathbb{Z}})K_\varrho \subset G(\mathbb{A}) \times H(\mathbb{A})$ and satisfies*

$$\frac{\pi}{3}F_{\varphi,\varrho} = 2\Lambda(1,\chi)\lambda_\alpha(\phi) \in I(0,\chi). \tag{3.43}$$

Here, $\varphi = \varphi_f\varphi_\infty^{(\epsilon,-\epsilon)}$ with $\epsilon := \mathrm{sgn}(\alpha_1) = -\mathrm{sgn}(\alpha_2)$ and $\varphi_\infty^{(\pm1,\mp1)}$ defined in (3.22), and $\phi = \phi_f\phi_\infty^{(\epsilon,-\epsilon)}$. In particular, we have the equality

$$\frac{\pi}{3}\mathcal{I}(g,\varphi,\varrho) = E^*(g,\phi). \tag{3.44}$$

Remark 3.4. The constants $\Lambda(0,\chi) = \Lambda(1,\chi) = \frac{\sqrt{D_E/D}}{\pi^2}L(1,\chi)$ and $\sqrt{D_E}$ are in $\mathbb{Q}^\times$.

Remark 3.5. For $L \subset W_\alpha(\hat{\mathbb{Q}})$ a lattice and $\mu \in L^\vee/L$, suppose $\varphi_\mu \in \mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})$ satisfies (3.43) with $\phi_f = \phi_{L+\mu}$. Then it is easy to see that

$$\sum_{\mu\in L^\vee/L} \mathcal{I}(g_\tau^\Delta, \varphi_\mu, \varrho)\mathfrak{e}_\mu : \mathbb{H} \to \mathbb{C}[L^\vee/L]$$

is a (non-holomorphic) vector-valued modular form of weight 0 on $\mathrm{SL}_2(\mathbb{Z})$ with representation $\overline{\rho_L}$.

Remark 3.6. If we decompose $V_0 = U \oplus U^\perp$ with $U = \ell^+ + \ell^-$ the hyperbolic plane, then it is easy to see that $T_0 \subset \mathrm{SO}(U) \subset H_0$. Therefore, for any $\varphi \in \mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})^{T^\Delta(\hat{\mathbb{Z}})}$, we can write it as

$$\varphi = \sum_{j\in J} \varphi_{U,j} \otimes \varphi_{U^\perp,j}$$

such that $\varphi_{U,j} \in \mathcal{S}(\hat{U};\mathbb{Q}^{\mathrm{ab}})^{T^\Delta(\hat{\mathbb{Z}})}$ and $\varphi_{U^\perp,j} \in \mathcal{S}(\hat{U}^\perp;\mathbb{Q}^{\mathrm{ab}})^{T^\Delta(\hat{\mathbb{Z}})}$ for all $j \in J$. This in particular implies that $\varphi_{U^\perp,j}$ is $T(\hat{\mathbb{Z}})$-invariant (i.e., it is $\mathbb{Q}$-valued by (2.30)).

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Proof of Theorem 3.3. Suppose $\phi = \otimes_{v\le\infty}\phi_v$. By Theorem 3.10, there exists $\varphi_p \in \mathcal{S}(V_p;\mathbb{Q}(\zeta_{p^\infty}))$ invariant with respect to $G(\mathbb{Z}_p)T^\Delta(\mathbb{Z}_p)$ and satisfying (3.52). Furthermore, φ_p is the characteristic function of the maximal lattice in V_p for all but finitely many p. Therefore, $\varphi_f := \bigotimes_{p<\infty}\varphi_p$ is in $\mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})^{(G\cdot T^\Delta)(\hat{\mathbb{Z}})}$ and satisfies

$$F_{\varphi_f,\varrho_f} = \zeta(2)^{-1}L(1,\chi)\sqrt{D_E/D}\lambda_\alpha(\phi_f) = 6\Lambda(1,\chi)\lambda_\alpha(\phi_f).$$

Since $\varrho_f(-1) = \mathrm{sgn}(-1) = -1$, the function $\omega_f(-1)\varphi_f$ with $-1 \in H_1(\hat{\mathbb{Q}})$ also satisfies these conditions, and we can replace φ_f by $(\varphi_f - \omega_f(-1)\varphi_f)/2$ so that $\omega_f(-1)\varphi_f = -\varphi_f$. Furthermore, we have $F_{\omega_f(h)\varphi_f,\varrho_f} = F_{\varphi_f,\varrho_f}$ for all $h \in K_\varrho$, and can therefore average over K_ϱ to ensure that φ_f is K_ϱ-invariant.

To prove (3.44), it suffices to check that $F_{\varphi_\infty^{(\epsilon,-\epsilon)},\varrho_\infty}(g) = \pi^{-1}\lambda_\alpha(\phi_\infty^{(\epsilon,-\epsilon)})(g)$ for $g = (g_{\tau_1}, g_{\tau_2})$. Using

$$\begin{aligned}
&\Phi_\infty(\mathcal{F}(\varphi_\infty^{(\epsilon,-\epsilon)}))(g,t)\\
&\quad= \mathcal{F}(\omega(g,t)\varphi_\infty^{(\epsilon,-\epsilon)})(0) = \int_{\mathbb{R}^3}(\omega(g,t)\varphi_\infty^{(\epsilon,-\epsilon)})(a,0,\lambda_1,\lambda_2,\lambda_1,\lambda_2)da d\lambda_1 d\lambda_2\\
&\quad= \int_{\mathbb{R}^3}\varphi_\infty^{(\epsilon,-\epsilon)}\left(\frac{a-\lambda_1u_1-\lambda_2u_2}{\sqrt{v_1v_2}},0,v_1\lambda_1/\sqrt{v_1v_2},v_2\lambda_2/\sqrt{v_1v_2},t^{-1}\lambda_1,t\lambda_2\right)da d\lambda_1 d\lambda_2\\
&\quad= \int_{\mathbb{R}^2}(v_1\lambda_1 - v_2\lambda_2)(t^{-1}\lambda_1 - t\lambda_2)e^{-\frac{\pi}{v_1v_2}((v_1\lambda_1-v_2\lambda_2)^2+v_1v_2((t^{-1}\lambda_1+t\lambda_2)^2))}d\lambda_1 d\lambda_2\\
&\quad= \int_{\mathbb{R}^2}x\frac{2x+(t^{-1}v_2-tv_1)y}{v_1t+v_2t^{-1}}e^{-\frac{\pi}{v_1v_2}(x^2+v_1v_2y^2))}\frac{dxdy}{v_1t+v_2t^{-1}}\\
&\quad= \pi^{-1}2(v_1v_2)^{3/2}\frac{t^2}{(v_1t^2+v_2)^2},
\end{aligned}$$

where we have used the change of variable $x = v_1\lambda_1 - v_2\lambda_2$, $y = t^{-1}\lambda_1 + t\lambda_2$, we obtain

$$F_{\varphi_\infty^{(\epsilon,-\epsilon)},\varrho_\infty}(g) = \int_0^\infty \Phi_\infty(\mathcal{F}(\varphi_\infty))(g,t)\frac{dt}{t} = \pi^{-1}2(v_1v_2)^{3/2}\int_0^\infty\frac{tdt}{(v_1t^2+v_2)^2} = \pi^{-1}\sqrt{v_1v_2}.$$

However, we have

$$\lambda_\alpha(\phi_\infty^{(\epsilon,-\epsilon)})(g) = \sqrt{v_1v_2}.$$

This finishes the proof. □

The requirement that φ_f in Theorem 3.3 is invariant with respect to $T^\Delta(\hat{\mathbb{Z}})$ will be important to deduce important rationality results in Section 4.3. We give a taste of such results in the following lemma.

Lemma 3.7. *If $\varphi_0 \in \mathcal{S}(V_0;\mathbb{Q}^{\mathrm{ab}})$ is invariant with respect to $T^\Delta(\hat{\mathbb{Z}}) \subset (\mathrm{GL}_2\times H_0)(\hat{\mathbb{Z}})$, then $\mathcal{F}_1(\varphi_0) \in \mathcal{S}(((\ell^-\otimes W)+V_{-1})(\hat{\mathbb{Q}});\mathbb{Q}^{\mathrm{ab}})$ satisfies*

$$\sigma_a(\mathcal{F}_1(\varphi_0)((\eta_1,\eta_2),\nu)) = \mathcal{F}_1(\varphi_0)((a^{-1}\eta_1,\eta_2),\nu) \tag{3.45}$$

for any $\sigma_a \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ associated to $a \in \hat{\mathbb{Z}}^\times$ as in section 2.3. In particular, we have

$$\mathcal{F}_1(\varphi_0)((0,r),\nu) \in \mathbb{Q} \tag{3.46}$$

for all $r \in \hat{\mathbb{Q}}$, $\nu \in \hat{F}$.

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Proof. Using the expression for $\mathscr{F}_1$ in (3.29), we can write

$$\begin{aligned}\sigma_a(\mathscr{F}_1(\varphi_0)((\eta_1,\eta_2),\nu)) &= \sigma_a\left(\int_{\hat{\mathbb{Q}}}\varphi_0(b,\eta_1,\nu)\psi_f(b\eta_2)db\right) = \int_{\hat{\mathbb{Q}}}\sigma_a(\varphi_0(b,\eta_1,\nu))\psi_f(ab\eta_2)db\\ &= \int_{\hat{\mathbb{Q}}}\omega((t(a),1))(\varphi_0)(b,\eta_1,\nu)\psi_f(ab\eta_2)db = \int_{\hat{\mathbb{Q}}}\omega((1,\iota(t(a^{-1}))))(\varphi_0)(b,\eta_1,\nu)\psi_f(ab\eta_2)db\\ &= \int_{\hat{\mathbb{Q}}}\varphi_0(ab,a^{-1}\eta_1,\nu)\psi_f(ab\eta_2)db = \mathscr{F}_1(\varphi_0)((a^{-1}\eta_1,\eta_2),\nu).\end{aligned}$$

For the second step, we moved σ_a inside the integral as φ_0 is a Schwartz function and the integral is a finite sum. The third and fourth steps used (2.29) and the invariance of φ_0 under $(t,\iota(t)) \in T^\Delta(\hat{\mathbb{Z}})$, respectively. Equation (3.46) now follows from (3.45) via (2.29). □

3.5. Matching local sections I

The goal of this section is to prove Theorem 3.10, the non-archimedean local counterpart of the matching result 3.3. For this purpose, we fix a prime $p < \infty$ throughout this section. The main input to Theorem 3.10 is the following surjectivity result.

Proposition 3.8. *Let ϱ_p and χ_p be as in (3.33). Then the following map*

$$\begin{aligned}\beta : \mathcal{S}(V_p;\mathbb{C})^{G(\mathbb{Z}_p)} \subset \mathcal{S}(V_p;\mathbb{C}) &\longrightarrow I_p^{G_0}(\chi_p)\\ \varphi &\longmapsto F_{\varphi,\varrho_p}\end{aligned} \tag{3.47}$$

is surjective. Furthermore, if $\Phi \in I_p^{G_0}(\chi_p)$ is valued in $\mathbb{Q}(\zeta_{p^\infty})$, then there exists $\varphi \in \mathcal{S}(V_p;\mathbb{Q}(\zeta_{p^\infty}))^{G(\mathbb{Z}_p)}$ satisfying $\beta(\varphi) = \Phi$. Here, $\mathbb{Q}(\zeta_{p^\infty}) \subset \mathbb{Q}^{\mathrm{ab}}$ is the subfield defined in (2.26).

Proof. Using (3.35), we can suppose $\Phi = \otimes_{v|p}\Phi_v$ with $\Phi_v \in I_v(0,\chi_v)$. Since $F_{\omega(g)\varphi,\varrho_p} = F_{\varphi,\varrho_p}$ for all $g \in G(\mathbb{Z}_p)$ and $\varphi \in \mathcal{S}(V_p;\mathbb{C})$, it suffices to prove the surjectivity of β on $\mathcal{S}(V_p;\mathbb{C})$. To do this, we will use the m-th Fourier coefficient of $\Phi_v \in I_v(0,\chi_v)$ for $m \in F_v$, which is defined by

$$W_m(\Phi_v) := \int_{F_v}\Phi_v(wn(b))\psi_v(-mb)db \tag{3.48}$$

with ψ_v an additive character of F_v. For $m = (m_v)_{v|p} \in F_p$ and $\varphi \in \mathcal{S}(V_p;\mathbb{C})$, we denote

$$\begin{aligned}W_m(\varphi) &:= \prod_{v|p} W_{m_v}((F_{\varphi,\varrho_p})_v)\\ &= \int_{F_p\times H_1(\mathbb{Q}_p)}(\omega_p(wn(b),h_1)\mathscr{F}(\varphi))(0)\psi_p(-mb)\varrho_p(h_1)dh_1\,db\end{aligned} \tag{3.49}$$

with $\psi_p := \prod_{v|p}\psi_v$. Now, the $G(F_v)$-module $I_v(0,\chi_v)$ can be written as

$$I_v(0,\chi_v) = \oplus_{\alpha\in F_v^\times/\mathrm{Nm}(E_v^\times)}R(W_\alpha)$$

with $R(W_\alpha)$ the image of $\lambda_{\alpha,v}$ and irreducible. So $I_v(0,\chi_v)$ is irreducible if and only if χ_v is trivial. Otherwise, it is the direct sum of two irreducible submodules. Furthermore, for $\Phi \in R(W_\alpha)$, the coefficient $W_m(\Phi)$ is zero unless $m/\alpha \in \mathrm{N}_{E_v/F_v}E_v^\times$. We then have two cases to consider, depending on whether $\chi_v = \chi_{v'}$ is trivial or not.

When $\chi_v = \chi_{v'}$ is trivial, Lemma 3.12 gives us φ such that $W_m(\varphi) \neq 0$ for some $m \in F_p^\times$. So for $v \mid p$, the restriction of $\mathrm{im}(\beta) \subset I_p^{G_0}(\chi_p)$ to $G(F_v)$ gives a nonzero section in $I_v(0,\chi_v)$ and generates a nontrivial, irreducible sub $G(F_v)$-module. As $I_v(0,\chi_v)$ is irreducible, the map β is

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surjective. When $\chi_v = \chi_{v'}$ is nontrivial, we again apply Lemma 3.12 to obtain a submodule $R \subset I_v(0,\chi_v) = R(W_{\alpha_0}) \oplus R(W_{\alpha_1})$ from $\mathrm{im}(\beta)$ such that $\pi_i(R)$ is nontrivial with $\pi_i : I_v(0,\chi_v) \to R(W_{\alpha_i})$ the projection. As $R(W_{\alpha_i})$ is irreducible, we have $\pi_i(R) = \pi_i(I_v(0,\chi_v))$. Consider $R_i := \ker \pi_i \cap R$ as a submodule of the irreducible module $\ker \pi_i$. As $R(W_{\alpha_0})$ and $R(W_{\alpha_1})$ are not isomorphic [KR92, Proposition 3.4], R_i cannot be trivial for both $i = 0, 1$, otherwise, $R \cong \pi_i(R) = R(W_{\alpha_i})$. Thus, $R_i = \ker \pi_i \subset R$ for an i, which implies $R = I_v(0,\chi_v)$ and proves surjectivity.

When $\Phi = \otimes_{v|p}\Phi_v$ has value in $\mathbb{Q}(\zeta_{p^\infty})$, we apply the surjectivity of β and the discussion in Section 2.3 to choose $\varphi_j \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))^{G(\mathbb{Z}_p)}$ and $c_j \in \mathbb{C}$ such that

$$\varphi := \sum_{j=1}^{J} c_j \varphi_j \in \mathcal{S}(V_p; \mathbb{C})$$

satisfies $\beta(\varphi) = \Phi$ and J is minimal. Therefore, $F_{\varphi,\varrho_p} = \sum_{j=1}^{J} c_j F_{\varphi_j,\varrho_p}$ is valued in $\mathbb{Q}(\zeta_{p^\infty})$. By the minimality of J, the section F_{φ_j,ϱ_p} is not identically zero for all j. Therefore, the set $\{1, c_1, \cdots, c_J\} \subset \mathbb{C}$ is linearly dependent over $\mathbb{Q}(\zeta_{p^\infty})$. The minimality of J then implies that $J = 1$ and $c_1 \in \mathbb{Q}(\zeta_{p^\infty})$, and hence, $\varphi \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))^{G(\mathbb{Z}_p)}$. □

Using this proposition, we can match any continuous function on $G_0(\mathbb{Z}_p)$ via the map β. Furthermore, we can incorporate Galois action to obtain the following result.

Proposition 3.9. *In the setting of Proposition 3.8, given any continuous function $\Phi : G_0(\mathbb{Z}_p) \to \mathbb{C}$ satisfying*

$$\Phi(m(a)n(b)k) = \chi(a)\Phi(k), \tag{3.50}$$

for all $m(a), n(b) \in B_0(\mathbb{Z}_p), k \in G_0(\mathbb{Z}_p)$, there exists $\varphi \in \mathcal{S}(V_p; \mathbb{C})^{G(\mathbb{Z}_p)}$ such that $F_{\varphi,\varrho_p}(g) = \Phi(g)$ for all $g \in G_0(\mathbb{Z}_p)$. Furthermore, if Φ takes values in $\mathbb{Q}(\zeta_{p^\infty})$ and satisfies

$$\sigma_a(|D|_p^{-1/2}\Phi(t_0^{-1} g t_0)) = |D|_p^{-1/2}\Phi(g), \tag{3.51}$$

with $t_0 = \iota(t(a)) \in \tilde{H}_0(\mathbb{Z}_p), t(a) = \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right) \in T \subset \mathrm{GL}_2(\mathbb{Z}_p)$ for all $a \in \mathbb{Z}_p^\times$ and $g \in G_0(\mathbb{Z}_p)$, then $\varphi \in \mathcal{S}(V; \mathbb{Q}(\zeta_{p^\infty}))^{G(\mathbb{Z}_p)}$ can be chosen to be $T^\Delta(\mathbb{Z}_p)$-invariant.

Proof. A continuous function Φ on $G_0(\mathbb{Z}_p)$ satisfying (3.50) can be uniquely extended to a section $\tilde{\Phi} \in I_p^{G_0}(\chi_p)$ by setting

$$\tilde{\Phi}(g) := \chi_p(a)\Phi(k)$$

with $g = m(a)n(b)k$ the Iwasawa decomposition of g. Therefore, the first claim is a direct consequence of Proposition 3.8.

For the second claim, we take any $\varphi \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))^{G(\mathbb{Z}_p)}$ and observe that

$$\begin{aligned}\frac{F_{\omega_p(t,t_0)\varphi_p,\varrho_p}(g)}{|D|_p^{1/2}} = \frac{F_{\omega_p(t)\varphi_p,\varrho_p}(g t_0)}{|D|_p^{1/2}} = \frac{F_{\sigma_a(\varphi_p),\varrho_p}(t^{-1} g t_0)}{|D|_p^{1/2}} &= \sigma_a\left(\frac{F_{\varphi_p,\varrho_p}(t^{-1} g t)}{|D|_p^{1/2}}\right) \\ &= \sigma_a(|D|_p^{-1/2}\Phi(t_0^{-1} g t_0)) = |D|_p^{-1/2}\Phi(g)\end{aligned}$$

for any $(t, t_0) \in T^\Delta(\mathbb{Z}_p)$ with $t = t(a), t_0 = \iota(t(a))$ and $g \in G_0(\mathbb{Z}_p)$. Here, we used (3.39) for the first line. By averaging φ over $T^\Delta(\mathbb{Z}_p)$, we can suppose that it is $T^\Delta(\mathbb{Z}_p)$-invariant. This finishes the proof. □

We are now ready to state and prove the local matching result. This is just the Kudla matching principle [Kud03] in some sense.

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Theorem 3.10. *For any $\phi_v \in \mathcal{S}(W_\alpha(F_v))$ with $v \mid p$, there exists $\varphi_p \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))^{(G\cdot T^\Delta)(\mathbb{Z}_p)}$ such that*

$$F_{\varphi_p,\varrho_p} = (1 - p^{-2})|D/D_E|_p^{1/2} \prod_{v|p} L(1, \chi_v)\lambda_{\alpha,v}(\phi_v). \tag{3.52}$$

In addition, if p is unramified in E and co-prime to α, and ϕ_v is the characteristic function of the maximal lattice in $W_\alpha(F_v)$, then we can choose φ_p to be the characteristic function of the maximal lattice in V_p.

Proof. Suppose p and α are co-prime, $E_p/\mathbb{Q}_p$ is unramified and $\phi_v = \mathrm{Char}(\mathcal{O}_{E_v}), \varphi_p = \mathrm{Char}(\mathcal{O}_{F_p} \times \mathbb{Z}_p^2 \times \mathcal{O}_{F_p})$. Then it is easy to check that F_{φ_p,ϱ_p} and $\prod_{v|p} \lambda_{\alpha,v}(\phi_v)$ are both right $G(\mathcal{O}_{F_p})$-invariant. Since they are both in $\prod_{v|p} I(0, \chi_v)$, we only need to check that

$$F_{\varphi_p,\varrho_p}(1) = (1 - p^{-2}) \prod_{v|p} L(1, \chi_v)\lambda_{\alpha,v}(\phi_v)(1)$$

by the Iwasawa decomposition of $G(F_p)$. This is given precisely by Lemma 3.2 and proves (3.52) for all but finitely many places.

When ϕ_v is $\mathbb{Q}$-valued, we can use (2.29) to check that

$$\begin{aligned}\sigma_a\left(\prod_{v|p} \lambda_{\alpha,v}(\phi_v)(t^{-1}gt)\right) &= \prod_{v|p} \sigma_a\Big(\omega_{\alpha,v}(t^{-1}gt)(\phi_v)(0)\Big) = \prod_{v|p}\big(\omega_{\alpha,v}(g)(\sigma_a(\phi_v))(0)\big)\\ &= \prod_{v|p}\big(\omega_{\alpha,v}(g)(\phi_v)(0)\big) = \prod_{v|p} \lambda_{\alpha,v}(\phi_v)(g)\end{aligned}$$

for any $(t, t_0) \in T^\Delta(\mathbb{Z}_p)$ with $t = t(a), t_0 = \iota(t(a))$ and $g \in G_0(\mathbb{Z}_p)$. Proposition 3.9 combined with Remark 3.4 then completes the proof. □

Finally, we record the two local calculation lemmas used in proving Proposition 3.8.

Lemma 3.11. *Suppose $F_p/\mathbb{Q}_p$ is non-split with valuation ring $\mathcal{O}_p$, uniformizer ϖ, residue field size q, and a nontrivial additive character ψ. For a character ϱ of $H_1(\mathbb{Q}_p) = F_p^1 \subset \mathcal{O}_p^\times$, let*

$$n(\psi) := \min\{n : \psi(\varpi^n\mathcal{O}_p) = 1\},\ n(\varrho) := \min\{n \geq 0 : \varrho(K_n) = 1\}$$

be the conductors of ψ and ϱ, respectively, where $K_n := F_p^1 \cap (1 + \varpi^n\mathcal{O}_p)$. Then

$$\int_{F_p^1} \varrho(x)\psi(mx)dx \neq 0$$

for some $m \in \mathcal{O}_p^\times$ if and only if $n(\varrho) \leq n(\psi)$.

Proof. Let

$$f(m) = \begin{cases} \int_{F_p^1} \varrho(x)\psi(mx)dx & \text{if } m \in \mathcal{O}_p^\times \\ 0 & \text{otherwise.}\end{cases}$$

Then $f \in \mathcal{S}(F_p)$ and its Fourier transformation with respect to ψ is

$$\begin{aligned}\hat{f}(m) &= \int_{F_p} f(n)\psi(-nm)dn = \int_{F_p^1} \varrho(x) \int_{\mathcal{O}_p^\times} \psi(n(x-m))dndx\\ &= \int_{F_p^1} \varrho(x)(\mathrm{Char}(m + \varpi^{n(\psi)}\mathcal{O}_p)(x) - q^{-1}\,\mathrm{Char}(m + \varpi^{n(\psi)-1}\mathcal{O}_p)(x))dx.\end{aligned}$$

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First, assume that there is some $h_0 \in F_p^1$ such that $h_0 - m \in \varpi^{n(\psi)}\mathcal{O}_p$. Then

$$\begin{aligned}\hat{f}(m) &= \varrho(h_0)\left(\int_{K_{n(\psi)}} \varrho(x)dx - q^{-1}\int_{K_{n(\psi)-1}} \varrho(x)dx\right)\\ &= \begin{cases} 0 & \text{if } n(\varrho) > n(\psi),\\ \varrho(h_0)\mathrm{vol}(K_{n(\psi)}) & \text{if } n(\varrho) = n(\psi),\\ \varrho(h_0)(\mathrm{vol}(K_{n(\psi)}) - q^{-1}K_{n(\psi)-1}) & \text{if } n(\varrho) < n(\psi).\end{cases}\end{aligned}$$

Next, we assume that there no $h_0 \in F_p^1$ such that $h_0 - m \in \varpi^{n(\psi)}\mathcal{O}_p$ but some $h_0 \in F_p^1$ with $h_0 - m \in \varpi^{n(\psi)-1}\mathcal{O}_p$. Then

$$\hat{f}(m) = \begin{cases} 0 & \text{if } n(\varrho) \geq n(\psi),\\ -q^{-1}\varrho(h_0)\mathrm{vol}(K_{n(\psi)-1}) & \text{if } n(\varrho) < n(\psi).\end{cases}$$

Finally, if there is no $h_0 \in F_p^1$ with $h_0 - m \in \varpi^{n(\psi)-1}\mathcal{O}_p$, then $\hat{f}(m) = 0$. Now the lemma is clear. □

Lemma 3.12. *When χ_v is trivial, there exists $\phi \in \mathcal{S}(V_p)$ such that F_{ϕ,ϱ_p} is nontrivial. When χ_v is nontrivial, then for any $\epsilon = (\epsilon_v)_{v|p}$ with $\epsilon_v = \pm 1$, there exists $\phi^\epsilon \in \mathcal{S}(V_p)$ and $m^\epsilon \in F_p^\times$ such that $W_{m^\epsilon}(\phi^\epsilon) \neq 0$ and $m^\epsilon = (m_v^{\epsilon_v})_{v|p}$ with $\chi_v(m_v^{\epsilon_v}) = \epsilon_v$.*

Proof. When χ_v is trivial, the character ϱ_p of $H_1(\mathbb{Q}_p)$ is also trivial. Suppose ϕ is the characteristic function of the maximal lattice in V_p; then the integral in (3.38) is positive at $g = 1$, which means F_{ϕ,ϱ_p} is nontrivial.

Suppose now that χ_v, hence ϱ_p, is nontrivial. We can suppose that $n(\psi) = 0$. When $\phi = \phi_0 \otimes \phi_1$ with $\phi_i \in \mathcal{S}(V_{i,p})$, we can apply (3.38) to write

$$W_m(\phi) = \int_{F_p\times H_1(\mathbb{Q}_p)\times F_p\times\mathbb{Q}_p} \phi_0((wn(b))^{-1}\cdot\left(\begin{smallmatrix} x & \\ \lambda & \end{smallmatrix}\right)\lambda'0)\phi_1(h_1^{-1}\lambda)\psi_p(-mb)\varrho_p(h_1)dx\,d\lambda\,dh_1\,db.$$

We first assume that p is non-split and use the notation in Lemma 3.11. In this case, there is a unique place v of F above p, and $\epsilon = \pm 1$. For $n \geq \max\{n(\varrho)+1, 1\}$ and $\beta \in \mathcal{O}_p^\times$, let $\phi_1 = \phi_{1,\beta} = \mathrm{Char}(\beta + p^n\mathcal{O}_p)$ and

$$\phi_0 = \mathrm{Char}\begin{pmatrix} \mathbb{Z}_p & \varpi^n\mathcal{O}_p \\ \varpi^n\mathcal{O}_p & 1+p^n\mathbb{Z}_p\end{pmatrix}.$$

Then

$$\begin{aligned}W_m(\phi_0\otimes\phi_1) &= \int_{F_p\times F_p^1\times F_p\times\mathbb{Q}_p} \phi_0\begin{pmatrix} b\lambda+b'\lambda'+bb'x & -\lambda'-bx \\ -\lambda-b'x & x\end{pmatrix}\phi_1(h^{-1}\lambda)\varrho(h)\psi(-mb)dx\,d\lambda\,dh\,db\\ &= \int_{F_p^1} c(h)\varrho(h)dh,\end{aligned}$$

where

$$\begin{aligned}c(h) &:= \int_{\beta h+\varpi^n\mathcal{O}_p}\int_{-\lambda'+\varpi^n\mathcal{O}_{\mathfrak{p}}}\int_{1+p^n\mathbb{Z}_p} \psi(-mb)dx\,db\,d\lambda\\ &= p^{-n(1+f)}\,\mathrm{Char}(\varpi^{-n}\mathcal{O}_p)(m)\int_{\beta h+\varpi^n\mathcal{O}_p}\psi(m\lambda')d\lambda = C\,\mathrm{Char}(\varpi^{-n}\mathcal{O}_p)(m)\psi(m\beta' h^{-1})\end{aligned}$$

for some nonzero constant C. Here, $f = 1$ or 2 depending on whether $F/\mathbb{Q}$ is ramified or inert at p. Using $\varrho(h) = \varrho(h^{-1})$, we have

$$W_m(\phi_0 \otimes \phi_1) = C\,\mathrm{Char}(\varpi^{-n}\mathcal{O}_p)(m) \int_{F_p^1} \varrho(h)\psi(m\beta' h)dh. \tag{3.53}$$

If $F_p/\mathbb{Q}_p$ is inert, then χ_p is ramified and nontrivial when restricted to $\mathcal{O}_p^\times$. By Lemma 3.11, we can find $m_0 \in \varpi^{-n}\mathcal{O}_p^\times$ such that

$$W_{m_0/\beta'}(\phi_0 \otimes \phi_{1,\beta}) = C \int_{F_p^1} \varrho(h)\psi(m_0 h)dh \neq 0$$

as $n(\psi(\varpi^{-n}\cdot)) = n \geq n(\varrho)$. We can choose $\beta = \beta^\pm$ such that $\chi(m_0/(\beta^\pm)') = \pm 1$. Then taking $\phi^\pm = \phi_0 \otimes \phi_{1,\beta^\pm}$ proves the Lemma. If F_p/Q_p is ramified, then E_v/F_v is inert and $\chi_v(\varpi) = -1, \chi_v\mid_{\mathcal{O}_p^\times} = 1$. Again by Lemma 3.11, we can find $m_j \in \varpi^{-n+j}\mathcal{O}_p^\times$ for $j = 0, 1$ such that

$$W_{m_j}(\phi_0 \otimes \phi_{1,1}) = C \int_{F_p^1} \varrho(h)\psi(m_j h)dh \neq 0$$

as $n(\psi(m_j\cdot)) = n - j \geq n(\varrho)$. Therefore, $\phi^\pm = \phi_0 \otimes \phi_{1,1}$ satisfies the Lemma.

Finally, we come to the case when $p = v_1 v_2$ splits and $\eta := \chi_{v_1} = \chi_{v_2}$ is nontrivial. In this case, $F_p = F_{v_1} \times F_{v_2} = \mathbb{Q}_p^2$ and $\eta = \varrho_p$ is a character of $\mathbb{Q}_p^\times \cong H_1(\mathbb{Q}_p)$. For $m \in F_p$, we write $m = (m_1, m_2)$ with $m_j \in \mathbb{Q}_p$ and

$$V_{0,p} \cong M_2(\mathbb{Q}_p)$$
$$\left(\begin{smallmatrix} a & \lambda \\ \lambda' & b \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & \lambda_1 \\ \lambda_2 & b \end{smallmatrix}\right).$$

So we take $\phi_1 = \phi_{1,1} \otimes \phi_{1,2}$ with $\phi_{1,j} \in \mathcal{S}(\mathbb{Q}_p)$ and $\phi_0 \in \mathcal{S}(M_2(\mathbb{Q}_p))$. Simple calculation gives us

$$W_m(\phi_0 \otimes \phi_1) = \int_{\mathbb{Q}_p^2 \times \mathbb{Q}_p^\times \times \mathbb{Q}_p^2 \times \mathbb{Q}_p} \phi_{1,1}(h^{-1}\lambda_1)\phi_{1,2}(h\lambda_2)\phi_0\left(\begin{smallmatrix} b_1\lambda_1 + b_2\lambda_2 + b_1 b_2 x & -\lambda_2 - b_1 x \\ -\lambda_1 - b_2 x & x \end{smallmatrix}\right)$$
$$\cdot\, \eta(h)\psi(-m_1 b_1 - m_2 b_2)dx\, d\lambda_1\, d\lambda_2\, d^\times h\, db_1\, db_2.$$

Taking $\phi_{1,j} = \mathrm{Char}(1 + p^n\mathbb{Z}_p)$, $n \geq \max\{1, n(\eta)\}$, and

$$\phi_0 = \mathrm{Char}(\left(\begin{smallmatrix} \mathbb{Z}_p & p^n\mathbb{Z}_p \\ p^n\mathbb{Z}_p & 1 + p^n\mathbb{Z}_p \end{smallmatrix}\right)),$$

the same calculation as above gives

$$W_m(\phi_1 \otimes \phi_0) = C \int_{\mathbb{Q}_p^\times} \eta(x)\psi(m_2 x + m_1 x^{-1})\,\mathrm{Char}(p^{-n}\mathbb{Z}_p^2)(m_1, m_2)\,\mathrm{Char}(p^{-n}\mathbb{Z}_p^2)(m_1 x^{-1}, m_2 x)\frac{dx}{|x|}$$

for some nonzero constant C.

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When η is ramified, we take $n = n(\rho)$, $m_1 = p^l$ with $l \geq n$ and $m_2 = m_0 p^{-n} \in p^{-n}\mathbb{Z}_p^\times$ and obtain (write $o(x) = \mathrm{ord}_p x$)

$$\begin{aligned}
W_m(\phi_1 \otimes \phi_0) &= C \int_{0 \leq o(x) \leq n+l} \eta(x)\psi(m_2 x)\psi(m_1 x^{-1}) \frac{dx}{|x|} \\
&= C \sum_{0 \leq i \leq n} \int_{p^i \mathbb{Z}_p^\times} \eta(x)\psi(m_2 x) \frac{dx}{|x|} + C \sum_{1 \leq i \leq l} \int_{p^{-n-i}\mathbb{Z}_p^\times} \eta(x)^{-1}\psi(m_1 x) \frac{dx}{|x|} \\
&= C \left[\eta(m_0)^{-1} \int_{\mathbb{Z}_p^\times} \eta(x)\psi(p^{-n}x)dx + \eta(p)^{n+l} \int_{\mathbb{Z}_p^\times} \eta^{-1}(x)\psi(p^{-n}x)dx \right] \\
&\neq 0,
\end{aligned}$$

for some m_2.

When η is unramified, we have $n(\eta) = 0$ and take $n = 1$. For $m_j \in p^{-1}\mathbb{Z}_p^\times$, $j = 1, 2$, we have

$$W_{(m_1,m_2)}(\phi_0 \otimes \phi_1) = C \int_{\mathbb{Z}_p^\times} \psi(m_2 x + m_1 x^{-1})dx.$$

If we sum this over $m_j \in p^{-1}\mathbb{Z}_p^\times$, then the result is nonzero. So there exists $m_j \in p^{-1}\mathbb{Z}_p^\times$ such that $W_{(m_1,m_2)}(\phi_0 \otimes \phi_1) \neq 0$. For $m_j \in \mathbb{Z}_p^\times$, $j = 1, 2$, we have

$$\begin{aligned}
W_{(m_1,m_2)}(\phi_0 \otimes \phi_1) &= C \left[\int_{\mathbb{Z}_p^\times} dx + \eta(p) \int_{p\mathbb{Z}_p^\times} \psi(m_1 x^{-1}) \frac{dx}{|x|} + \eta(p)^{-1} \int_{p^{-1}\mathbb{Z}_p^\times} \psi(m_2 x) \frac{dx}{|x|} \right] \\
&= C(1 - p^{-1} + p^{-1} + p^{-1}) \neq 0,
\end{aligned}$$

as $\eta(p) = -1$. Replacing ϕ_1 by $\phi_1' = \mathrm{Char}(1 + p^n\mathbb{Z}_p, p + p^n\mathbb{Z}_p)$, the same calculation gives

$$W_{(m_1,m_2)}(\phi_0 \otimes \phi_1') \neq 0$$

when $m_j \in \mathbb{Z}_p^\times$ and $m_{3-j} \in p^{-1}\mathbb{Z}_p^\times$. This completes the proof. □

3.6. Matching local sections II

In order to give the factorization result, we also need a matching result involving the following local sections with the s parameter. When $p = vv'$ splits in F, we define a slightly modified section

$$F_{\varphi_p,\varrho_p,v,s}(g_p) := \int_{H_1(\mathbb{Q}_p)} \Phi_0(\mathscr{F}(\varphi_p))(g_p, h_1)\varrho(h_1)|h_1|_v^s dh_1. \tag{3.54}$$

This function on $G_0(\mathbb{Q}_p)$ depends on the choice of $v \mid p$ and has the following property.

Lemma 3.13. *When $|s| < 1$, the integral in (3.54) converges absolutely and defines a rational function in p^s defined over $\mathbb{Q}(\zeta_{p^\infty})(\varphi_p)$. Furthermore, when restricted to the first (resp. second) components of $G_0(\mathbb{Q}_p) \cong G(F_p) \cong G(F_v) \times G(F_{v'})$, it defines a section in $I(s, \chi_v)$ (resp. $I(-s, \chi_{v'})$).*

Proof. For the first claim, one can suppose $\omega(g_p)\mathscr{F}_1(\varphi_p)$ is the characteristic function of

$$C_1 \times (p^{a_1}\mathbb{Z}_p + r_1) \times (p^{a_2}\mathbb{Z}_p + r_2) \times (p^{b_1}\mathbb{Z}_p + t_1) \times (p^{b_2}\mathbb{Z}_p + t_2),$$

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with $C_1 \subset \mathbb{Q}_p^2$ a compact subset and $a_j, b_j \in \mathbb{Z}, r_j, t_j \in \mathbb{Q}_p$. As in Lemma 3.2, the integral defining $F_{\varphi_p,\varrho_p,v,s}(g_p)$ is given by

$$F_{\varphi_p,\varrho_p,v,s}(g_p) = \int_{\mathbb{Q}_p^\times}\int_{\mathbb{Q}_p^4}(\omega(g_p)\mathscr{F}_1(\varphi_p))(0,0,\lambda_1,\lambda_2,\alpha\lambda_1,\alpha^{-1}\lambda_2)d\lambda_1 d\lambda_2 \varrho((\alpha,\alpha^{-1}))|\alpha|_p^s d^\times\alpha.$$

Suppose $(0,0) \in C_1$; otherwise, the integral vanishes identically. When $|\alpha|_p \geq p^N$ for N sufficiently large, we have $|\alpha^{-1}\lambda_2|_p$ very small for all $\lambda_2 \in p^{a_2}\mathbb{Z}_p + r_2$. Therefore, when $t_2 \notin p^{b_2}\mathbb{Z}_p$ or $r_1 \notin p^{a_1}\mathbb{Z}_p$, the integral over those α with $|\alpha|_p \geq p^N$ is zero. When $t_2 \in p^{b_2}\mathbb{Z}_p$ and $r_1 \in p^{a_1}\mathbb{Z}_p$, we have

$$\begin{aligned}
&\int_{|\alpha|_p\geq p^N}\int_{\mathbb{Q}_p^4}(\omega(g_p)\mathscr{F}_1(\varphi_p))(0,0,\lambda_1,\lambda_2,\alpha\lambda_1,\alpha^{-1}\lambda_2)d\lambda_1 d\lambda_2\varrho((\alpha,\alpha^{-1}))|\alpha|_p^s d^\times\alpha\\
&= \mathrm{vol}(C_1)\sum_{n\geq N}\varrho((p,p^{-1}))^{-n}p^{ns}\int_{\mathbb{Z}_p^\times}\varrho((a,a^{-1}))\\
&\quad\times \mathrm{vol}(p^{a_1}\mathbb{Z}_p \cap p^n(p^{b_1}+a^{-1}t_1))\mathrm{vol}((p^{a_2}\mathbb{Z}_p+r_2)\cap p^{-n+b_2}\mathbb{Z}_p)d^\times a\\
&= \mathrm{vol}(C_1)\mathrm{vol}(p^{a_2}\mathbb{Z}_p+r_2)\mathrm{vol}(p^{b_1}+a^{-1}t_1)\int_{\mathbb{Z}_p^\times}\varrho((a,a^{-1}))d^\times a\sum_{n\geq N}(\varrho((p,p^{-1}))p^{-1+s})^n,
\end{aligned}$$

which converges when $|s| < 1$ and defines a rational function in p^s. The same argument takes care of the case when $|\alpha|_p$ is sufficiently small. This proves the first claim.

For the second claim, it is clear from the definition that $F_{\varphi_p,\varrho_p,v,s}$ is locally constant as φ_p is a Schwartz function. For the transformation property, we have

$$(\omega(m(\alpha)g,h)\mathscr{F}(\varphi_p))(0) = |\alpha\alpha'|_v(\omega(g,h(\alpha'/\alpha))\mathscr{F}(\varphi))(0) \tag{3.55}$$

for $\alpha = (\alpha_1,\alpha_2) \in F_p^\times = (\mathbb{Q}_p^\times)^2$. A change of variable plus $\varrho(\alpha/\alpha') = \chi_v(\alpha)\chi_{v'}(\alpha)$ and $|\alpha|_v = |\alpha_1|_p, |\alpha'|_v = |\alpha_2|_p$ then finishes the proof. □

Now, we will extend the matching result in Theorem 3.10 to standard sections.

Theorem 3.14. *In the setting of Theorem 3.10, suppose $p = vv'$ splits and let $\lambda_{\alpha,v,s}(\phi_v) \in I_v(s,\chi_v)$ denote the standard section associated to $\lambda_{\alpha,v}(\phi_v) \in I_v(0,\chi_v)$ for $\phi_v \in \mathcal{S}(W_\alpha(F_v))$. For any $r \in \mathbb{N}$, there exists $\varphi_p \in \mathcal{S}(V_p;\mathbb{Q}(\zeta_{p^\infty}))^{(G\cdot T^\Delta)(\mathbb{Z}_p)}$ such that*

$$F_{\varphi_p,\varrho_p,v,s}(g) = \mathcal{L}(s)\lambda_{\alpha,v,s}(\phi_v)(g_v)\lambda_{\alpha,v,-s}(\phi_{v'})(g_{v'}) + O(s^r). \tag{3.56}$$

for all $g = (g_v,g_{v'}) \in G_0(\mathbb{Q}_p)$, where $\mathcal{L}(s) := (1-p^{-2})|D_E|_p^{-1}L(1+s,\chi_v)L(1-s,\chi_{v'})$

Remark 3.15. If ϱ_p is unramified and φ_p is the characteristic function of the maximal lattice in V_p, then

$$F_{\varphi_p,\varrho_p,v,s}(1) = L(1+s,\chi_v)L(1-s,\chi_{v'})(1-p^{-2}) \tag{3.57}$$

by a similar calculation as in Lemma 3.2, and $F_{\varphi_p,\varrho_p,v,s}/(L(1+s,\chi_v)L(1-s,\chi_{v'})(1-p^{-2}))$ is already the standard section $\lambda_{\alpha,v,s}(\phi_v)(g_v)\lambda_{\alpha,v,-s}(\phi_{v'})(g_{v'})$.

Proof. For $\varphi \in \mathcal{S}(V_p;\mathbb{C})$ and $g \in G_0(\mathbb{Q}_p)$, we write

$$\begin{aligned}
F_{\varphi,\varrho_p,v,s}(g) &= \sum_{n\geq 0}\frac{a_n(\varphi,g)}{n!}(-\log p\cdot s)^n,\\
a_n(\varphi,g) &:= (-\log p)^{-n}\partial_s^n\Big(F_{\varphi,\varrho_p,v,s}(g)\Big)\Big|_{s=0} = \int_{H_1(\mathbb{Q}_p)}\Phi_0(\mathscr{F}(\varphi_p))(g_p,h_1)\varrho(h_1)\,\mathrm{ord}_v(h_1)^n dh_1.
\end{aligned}$$

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It is easy to check from definition that $F_{\varphi,\varrho_p,v,s} : G_0(\mathbb{Z}_p) \to \mathbb{C}$ satisfies (3.50), and hence, so does the function $a_n(\varphi, g)$ for all $n \geq 0$. Now we define

$$\varphi^{(n)} := \frac{1}{(-2)^n n!}(\omega((p,p^{-1})^2) - 1)^n \varphi \in \mathcal{S}(V_p; \mathbb{C})$$

with $(p,p^{-1}) \in H_1(\mathbb{Q}_p)$. An easy induction shows that $F_{\varphi^{(n)},\varrho_p,v,s} = \frac{1}{n!}(\frac{1-p^{2s}}{2})^n F_{\varphi,\varrho_p,v,s}$, which implies

$$a_{n'}(\varphi^{(n)}, g) = \begin{cases} 0 & \text{if } n' < n, \\ F_{\varphi,\varrho_p} & \text{if } n' = n. \end{cases} \tag{3.58}$$

When $\varphi \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))$ is $(G \cdot T^\Delta)(\mathbb{Z}_p)$-invariant, so is the function $\varphi^{(n)} \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))$. Furthermore, the function $a_n(\varphi, \cdot) : G_0(\mathbb{Z}_p) \to \mathbb{Q}(\zeta_{p^\infty})$ satisfies conditions (3.50) and (3.51). By Proposition 3.9, there exists $\varphi_n \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))^{(G\cdot T^\Delta)(\mathbb{Z}_p)}$ such that

$$F_{\varphi_n,\varrho_p}(k) = a_n(\varphi, k) \tag{3.59}$$

for all $k \in G_0(\mathbb{Z}_p)$.

Now we prove the theorem by induction on r. The case $r = 1$ is just the content of Theorem 3.10. Note that $|D|_p = 1$ when p splits in F. Now suppose we have φ satisfying (3.56) for some $r \geq 1$. As

$$\Phi_s := \lambda_{\alpha,v,s}(\phi_v)(g_v)\lambda_{\alpha,v,-s}(\phi_{v'})(g_{v'}) \in I(s, \chi_v)I(-s, \chi_{v'})$$

is a standard section, it satisfies $\Phi_s(k) = \Phi_0(k)$ when $k \in G_0(\mathbb{Z}_p)$. So in that case, we have

$$F_{\varphi,\varrho_p,v,s}(k) - \mathcal{L}(s)\Phi_s(k) = (a_r(\varphi,k) - c_r\Phi_0(k))\frac{(-\log p \cdot s)^r}{r!} + O(s^{r+1}),$$

with $c_r := (-\log p)^{-r}\partial_s^r \mathcal{L}(s)\mid_{s=0}$ rational. If we set

$$\tilde{\varphi} := \varphi - \varphi_r^{(r)} - c_r\varphi^{(r)} \in \mathcal{S}(V_p, \mathbb{Q}(\zeta_{p^\infty}))^{(G\cdot T^\Delta)(\mathbb{Z}_p)},$$

then equations (3.58) and (3.59) give us

$$\begin{aligned} &F_{\tilde{\varphi},\varrho_p,v,s}(k) - \mathcal{L}(s)\Phi_s(k) \\ &= \Big(a_r(\varphi,k) - F_{\varphi_r,\varrho_p}(k) + c_rF_{\varphi,\varrho_p}(k) - c_r\Phi_0(k)\Big)\frac{(-\log p\cdot s)^r}{r!} + O(s^{r+1}) = O(s^{r+1}). \end{aligned}$$

So $\tilde{\varphi}$ satisfies the claim for $r + 1$. This completes the proof. □

Now, we can state a consequence of the matching result in Theorem 3.14.

Proposition 3.16. *For matching sections $\varphi_p \in \mathcal{S}(V_p; \mathbb{Q}(\zeta_{p^\infty}))^{G(\mathbb{Z}_p)}$ and $\{\phi_v \in \mathcal{S}(W_\alpha(F_v)) : v \mid p\}$ as in Theorem 3.14 with $r = 2$, we have*

$$\prod_{v|p} W^*_{t,v}(\phi_v) = \sum_{n\in\mathbb{Z}}\int_{H_1(\mathbb{Q}_p)} \varrho_p(h)\mathscr{F}_1(\varphi_p)((0,p^n), t'/p^n, -h^{-1}t/p^n)dh \tag{3.60}$$

for all $t \in F_p^\times$. Furthermore, when $p = vv'$ is a split prime and $W^*_{t,v}(\phi_v) = 0$, then we have

$$\frac{W^{*,\prime}_{t,v}(\phi_v)W^*_{t,v'}(\phi_{v'})}{\log p} = \sum_{n\in\mathbb{Z}}\int_{H_1(\mathbb{Q}_p)} \varrho_p(h)\mathscr{F}_1(\varphi_p)((0,p^n), t'/p^n, -h^{-1}t/p^n)o_v(h)dh. \tag{3.61}$$

Remark 3.17. Since $o_{v'}(h) = -o_v(h)$ for all $h \in H_1(\mathbb{Q}_p)$, the left-hand side of (3.61) gets a minus sign if $o_v(h)$ is replaced by $o_{v'}(h)$ on the right-hand side.

Proof. To prove (3.60). we apply the definition of $W^*_{t,v}$ in (2.54) and Theorem 3.10 to obtain

$$\begin{aligned}\prod_{v|p} W^*_{t,v}(\phi_v) &= |D/D_E|_p^{1/2} \prod_{v|p} L(1,\chi_v) \int_{F_v} \lambda_{\alpha,v}(\phi_v)(wn(b_v))\psi_v(-tb_v)db_v \\ &= (1-p^{-2})^{-1} \int_{F_p} F_{\varphi_p,\varrho_p}(wn(b))\psi_p(-\operatorname{Tr}(tb))db \\ &= (1-p^{-2})^{-1} \int_{H_1(\mathbb{Q}_p)} \varrho(h_1) \int_{F_p} \mathcal{F}(\omega_p((wn(b),h_1))\varphi_p)(0)\psi_p(-\operatorname{Tr}(tb))db dh_1.\end{aligned}$$

Now the right-hand side of (3.60) can be rewritten as

$$\begin{aligned}\text{Right-hand side of (3.60)} &= \int_{H_1(\mathbb{Q}_p)} \varrho_p(h_1) \sum_{n\in\mathbb{Z}} \mathcal{F}_1(\omega_p(h_1)\varphi_p)((0,p^n),t'/p^n,-t/p^n)dh_1 \\ &= \int_{H_1(\mathbb{Q}_p)} \varrho_p(h_1) \sum_{n\in\mathbb{Z}} (1-p^{-1})^{-1} \int_{p^n\mathbb{Z}_p^\times} \mathcal{F}_1(\omega_p(h_1)\varphi_p)((0,u),t'/u,-t/u)d^\times u dh_1 \\ &= (1-p^{-1})^{-1} \int_{H_1(\mathbb{Q}_p)} \varrho_p(h_1) \int_{\mathbb{Q}_p^\times} \mathcal{F}_1(\omega_p(h_1)\varphi_p)((0,u),t'/u,-t/u)d^\times u dh_1.\end{aligned}$$

For the second line, we have used $\omega(m(a))\varphi_p = \varphi_p$ for all $a \in \mathbb{Z}_p^\times$ since φ_p is $G(\mathbb{Z}_p)$-invariant. Equation (3.60) now follows from applying Proposition 3.18 to $\varphi = \omega_p(h_1)\varphi_p$.

We now prove (3.61). Let $\Phi_s \in I(s,\chi_v)I(-s,\chi_{v'})$ be the standard sections extending $\lambda_{\alpha,v}(\phi_v)\lambda_{\alpha,v'}(\phi_{v'})$. By Theorem 3.14 with $r = 2$, we have

$$F_{\varphi_p,\varrho_p,v,s} = (1-p^{-2})|D_E|_p^{-1} L(1,\chi_v)L(1,\chi_{v'})\lambda_{\alpha,v,s}(\phi_v)\lambda_{\alpha,v',-s}(\phi_{v'}) + O(s^2).$$

Therefore, using $W^*_{t,v}(\phi_v) = 0$, we have

$$\begin{aligned}W^{*,\prime}_{t,v}(\phi_v)W^*_{t,v'}(\phi_{v'}) &= \partial_s\Big(W^*_{t,v}(1,s,\phi_v)W^*_{t,v'}(1,-s,\phi_{v'})\Big)\Big|_{s=0} \\ &= |D_E|_p^{-1} L(1,\chi_v)L(1,\chi_{v'})\partial_s\left(\int_{F_p} \Phi_s((wn(b_v),wn(b_{v'})))\psi_v(-tb_v)\psi_{v'}(-tb_{v'})db_v db_{v'}\right)\Big|_{s=0} \\ &= (1-p^{-2})^{-1}\partial_s\left(\int_{F_p} F_{\varphi_p,\varrho_p,v,s}(wn(b))\psi_p(-\operatorname{Tr}(tb))db\right)\Big|_{s=0} \\ &= (1-p^{-2})^{-1}\partial_s\left(\int_{H_1(\mathbb{Q}_p)} \varrho(h_1)|h_1|_v^s \int_{F_p} \mathcal{F}(\omega_p((wn(b),h_1))\varphi_p)(0)\psi_p(-\operatorname{Tr}(tb))db dh_1\right)\Big|_{s=0} \\ &= \log p (1-p^{-2})^{-1} \int_{H_1(\mathbb{Q}_p)} \varrho(h_1)\operatorname{ord}_v(h_1) \int_{F_p} \mathcal{F}(\omega_p((wn(b),h_1))\varphi_p)(0)\psi_p(-\operatorname{Tr}(tb))db dh_1.\end{aligned}$$

Applying Proposition 3.18 and continuing as in the second half of the proof of (3.60) then proves (3.61). □

We end this section with the following technical result used in the proof of the previous proposition.

Proposition 3.18. *For any $\varphi \in \mathcal{S}(V_p;\mathbb{C})^{G(\mathbb{Z}_p)}$ and $t \in F_p^\times$, we have*

$$(1+1/p)^{-1}\int_{F_p}(\omega_p((wn(\beta))\mathscr{F}(\varphi))(0)\psi_p(-\mathrm{Tr}(t\beta))d\beta = \int_{\mathbb{Q}_p^\times}\mathscr{F}_1(\varphi)((0,u),t'/u,-t/u)d^\times u. \tag{3.62}$$

Proof. Since the left-hand side is essentially the Fourier transform of $(\omega(wn(\beta))\mathscr{F}(\varphi_p))(0)$ as a function of $\beta \in F_p$, it suffices to calculate the inverse Fourier transform of the right-hand side, though we need to be careful about the singularity of right-hand side when $t \notin F_p^\times$. To take care of this, we define

$$G_\epsilon(t,\varphi) := \begin{cases}\int_{\mathbb{Q}_p^\times}\mathscr{F}_1(\varphi)((0,u),t'/u,-t/u)d^\times u, & \text{if } |t| > \epsilon,\\ 0, & \text{otherwise.}\end{cases} \tag{3.63}$$

for $\epsilon > 0$ and $t \in F_p$. Note that $|t| := \min\{|t_1|,|t_2|\}$ when $t = (t_1,t_2) \in F_p = \mathbb{Q}_p^2$, Given any fixed $t \in F_p^\times$, the limit $\lim_{\epsilon\to 0} G_\epsilon(t,\varphi)$ exists and is the right-hand side of (3.62). Also for any fixed $\epsilon > 0$, the function $G_\epsilon(t,\varphi)$ is a Schwartz function on F_p. Its inverse Fourier transform is given by

$$\begin{aligned}\hat{G}_\epsilon(\beta,\varphi) &:= \int_{F_p} G_\epsilon(t,\varphi)\psi_p(\mathrm{Tr}(t'\beta'))dt\\ &= \int_{F_p\setminus D_\epsilon}\int_{\mathbb{Q}_p^\times}\mathscr{F}_1(\varphi)((0,u),t'/u,-t/u)d^\times u\psi_p(\mathrm{Tr}(t'\beta'))dt\\ &= \int_{\mathbb{Q}_p^\times}\int_{F_p\setminus D_{\epsilon/|u|}}\mathscr{F}_1(\varphi)((0,u),\tilde{t}',-\tilde{t})\psi_p(\mathrm{Tr}(\tilde{t}'u\beta'))|u|^2 d\tilde{t}d^\times u,\end{aligned}$$

where $D_\epsilon \subset F_p$ is the ϵ-neighborhood of 0. Note that

$$\mathscr{F}_1(\varphi)((0,u),\tilde{t}',-\tilde{t})\psi_p(\mathrm{Tr}(\tilde{t}'u\beta')) = \mathscr{F}_1(\varphi_\beta)((0,u),-\tilde{t},-\tilde{t}),$$

where $\varphi_\beta := \omega(w_2 n(\beta))\varphi$ and $w_i \in H(\mathbb{Q})$ is defined in (3.6). Therefore, $\hat{G}_\epsilon(\beta,\varphi)$ is given by

$$\begin{aligned}\hat{G}_\epsilon(\beta,\varphi) &= \int_{\mathbb{Q}_p^\times}\int_{F_p\setminus D_{\epsilon/|u|}}\mathscr{F}_1(\varphi_\beta)((0,u),-\tilde{t},-\tilde{t})|u|^2 d\tilde{t}d^\times u\\ &= \int_{\mathbb{Q}_p^\times}\left(\mathscr{F}(\varphi_\beta)((0,u),0) - \int_{D_{\epsilon/|u|}}\phi_\beta((0,u),s,s)ds\right)|u|^2 d^\times u\end{aligned}$$

with $s = -\tilde{t}$ and $ds = d\tilde{t}$. Using the $G(\mathbb{Z}_p)$-invariance of φ, we have

$$\mathscr{F}(\varphi_\beta)((0,p^n u),0) = \mathscr{F}(\varphi_\beta)((0,p^n),0) = \mathscr{F}(\varphi_\beta)(a,0)$$

for all $u \in \mathbb{Z}_p^\times, n \in \mathbb{Z}$ and $a \in (p^n\mathbb{Z}_p)^2 - (p^{n+1}\mathbb{Z}_p)^2$. Applying this, we can evaluate the first part as

$$\begin{aligned}&\int_{\mathbb{Q}_p^\times}\mathscr{F}(\varphi_\beta)((0,u),0)|u|^2 d^\times u = \sum_{n\in\mathbb{Z}}\int_{\mathbb{Z}_p^\times}\mathscr{F}(\varphi_\beta)((0,p^n u),0)|p^n u|^2 d^\times u\\ &= \sum_{n\in\mathbb{Z}}\mathscr{F}(\varphi_\beta)((0,p^n),0)p^{-2n}(1-1/p) = (1+1/p)^{-1}\sum_{n\in\mathbb{Z}}\mathscr{F}(\varphi_\beta)((0,p^n),0)\int_{(p^n\mathbb{Z}_p)^2-(p^{n+1}\mathbb{Z}_p)^2}da\\ &= (1+1/p)^{-1}\sum_{n\in\mathbb{Z}}\int_{(p^n\mathbb{Z}_p)^2-(p^{n+1}\mathbb{Z}_p)^2}\mathscr{F}(\varphi_\beta)(a,0)da = (1+1/p)^{-1}\int_{\mathbb{Q}_p^2}\mathscr{F}(\varphi_\beta)(a,0)da\\ &= (1+1/p)^{-1}\mathscr{F}(\omega(w_1)\varphi_\beta)(0) = (1+1/p)^{-1}(\omega(wn(\beta))\mathscr{F}(\varphi))(0).\end{aligned}$$

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Then for any fixed $t \in F_p^\times$, we have

$$\begin{aligned} G_\epsilon(t,\varphi) &= \int_{F_p} G_\epsilon(\beta,\varphi)\psi_p(-\mathrm{Tr}(\beta t))d\beta \\ &= (1+1/p)^{-1}\int_{F_p}((\omega(wn(\beta))\mathscr{F})(\varphi))(0)\psi_p(-\mathrm{Tr}(\beta t))d\beta - E_\epsilon(\varphi), \\ E_\epsilon(\varphi) &:= \int_{F_p}\psi_p(-\mathrm{Tr}(\beta t))\int_{\mathbb{Q}_p^\times}|u|^2\int_{D_{\epsilon/|u|}}\mathscr{F}_1(\varphi)((0,u),-s',s)\psi_p(-\mathrm{Tr}(su\beta))dsd^\times ud\beta. \end{aligned}$$

Since $\mathscr{F}_1(\varphi)$ is a Schwartz function, we can replace the domain $\mathbb{Q}_p^\times \times D_{\epsilon/|u|}$ with a compact open subset independent of β and interchange the order of integration to compute the integral over β first, which yields

$$E_\epsilon(\varphi) = \int_{\mathbb{Q}_p^\times}|u|^2\int_{D_{\epsilon/|u|}}\int_{F_p}\psi_p(-\mathrm{Tr}(\beta(t+su)))d\beta\mathscr{F}_1(\varphi)((0,u),-s',s)dsd^\times u.$$

When ϵ is sufficiently small, we have $t \notin D_\epsilon$ and $t+su \neq 0$, in which case $E_\epsilon = 0$. This finishes the proof. □

4. Doi-Naganuma lift of the deformed theta integral

In this section, we will define and study the properties of the function $\tilde{\mathcal{I}}$ discussed in the introduction. In particular, we will calculate its Fourier coefficients and images under lowering differential operators. The actions of differential operators follow from those on the theta kernel, which are given in Section 4.1. The Fourier expansion computations are carried out in Section 4.2, with the main result being Proposition 4.7. Section 4.3 contains rationality results about theta lifts that will be needed to handle the error term mentioned in Section 1.3.

Choose $\varphi^{(1,1)} := \varphi_f\varphi_\infty^{(1,1)} \in \mathcal{S}(V(\mathbb{A}))^{K_\varrho}$ with $\varphi_\infty^{(1,1)} := \varphi_{0,\infty}^{(1,1)} \otimes \varphi_\infty^+ \in \mathcal{S}(V_0(\mathbb{R})) \otimes \mathcal{S}(V_1(\mathbb{R}))$ and

$$\varphi_{0,\infty}^{(1,1)}(a,b,\nu,\nu') := -i(a-b+i(\nu+\nu'))e^{-\pi(a^2+b^2+\nu^2+(\nu')^2)} \in \mathcal{S}(V_0(\mathbb{R})). \tag{4.1}$$

For $\tilde{\varrho}_C$ as in (2.76), we can define

$$\tilde{\mathcal{I}}(g_0) := \mathcal{I}(g_0,\varphi^{(1,1)},\tilde{\varrho}_C) = \int_{[H_1]}\int_{[G]}\theta(g,(g_0,h_1),\varphi^{(1,1)})dg\tilde{\varrho}_C(h_1)dh_1. \tag{4.2}$$

We will now analyze various properties of this integral.

4.1. Lowering operator action

To calculate the action of differential operators on $\tilde{\mathcal{I}}$, it suffices to understand the effect on φ_0 via the Weil representation, which can be done in the Fock model. For this, we follow the appendices in [FM06] and [Li22] (see also [KM90]).

We identify $(V_0(\mathbb{R}),Q) = (M_2(\mathbb{R}),\det) \cong \mathbb{R}^{2,2}$ with the basis

$$v_1 := \frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}, v_2 := \frac{1}{\sqrt{2}}\begin{pmatrix}0&-1\\1&0\end{pmatrix}, v_3 := \frac{1}{\sqrt{2}}\begin{pmatrix}-1&0\\0&1\end{pmatrix}, v_4 := \frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}, \tag{4.3}$$

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which identifies $\mathcal{S}(\mathbb{R}^{2,2}) \cong \mathcal{S}(V_0(\mathbb{R}))$ and gives us

$$\begin{pmatrix} a & v_1 \\ v_2 & b \end{pmatrix} = \frac{a+b}{\sqrt{2}}v_1 + \frac{-a+b}{\sqrt{2}}v_3 + \frac{-v_1+v_2}{\sqrt{2}}v_2 + \frac{v_1+v_2}{\sqrt{2}}v_4. \tag{4.4}$$

The *polynomial Fock space* is the subspace $\mathbb{S}(\mathbb{R}^{2,2}) \subset \mathcal{S}(\mathbb{R}^{2,2})$ spanned by functions of the form $\prod_{1\le j\le 4} D_j^{r_j}\varphi^\circ$ for $r_j \in \mathbb{N}_0$, where $\varphi^\circ \in \mathcal{S}(\mathbb{R}^{2,2})$ is the Gaussian

$$\varphi^\circ(x_1,x_2,x_3,x_4) := e^{-\pi(x_1^2+x_2^2+x_3^2+x_4^2)}$$

and D_r are operators on $\mathcal{S}(\mathbb{R}^{2,2})$ defined by

$$D_r := \partial_{x_r} - 2\pi x_r,\ 1 \le r \le 4. \tag{4.5}$$

There is an isomorphism $\iota : \mathbb{S}(\mathbb{R}^{2,2}) \to \mathscr{P}(\mathbb{C}^4) = \mathbb{C}[\mathfrak{z}_1,\mathfrak{z}_2,\mathfrak{z}_3,\mathfrak{z}_4]$ such that $\iota(\varphi^\circ) = 1$, D_r acts as $(-1)^{\lfloor (r-1)/2\rfloor} i\mathfrak{z}_r$. We now set

$$\mathfrak{v} := \mathfrak{z}_1 + i\mathfrak{z}_2,\ \mathfrak{w} := \mathfrak{z}_3 - i\mathfrak{z}_4. \tag{4.6}$$

Then using (4.4), the Schwartz functions $\varphi_{0,\infty}^{(\epsilon,\epsilon')} \in \mathcal{S}(V_0(\mathbb{R}))$ in (3.22) and (4.1) become

$$\begin{aligned} \iota(\varphi_{0,\infty}^{(1,-1)}) &= i\sqrt{2}\iota((x_1+ix_2)\varphi^\circ) = -\frac{i\sqrt{2}}{4\pi}\iota((D_1+iD_2)\varphi^\circ) = \frac{\sqrt{2}}{4\pi}\mathfrak{v}, \\ \iota(\varphi_{0,\infty}^{(-1,1)}) &= -\frac{\sqrt{2}}{4\pi}\overline{\mathfrak{v}},\ \iota(\varphi_{0,\infty}^{(1,1)}) = -\frac{\sqrt{2}}{4\pi}\mathfrak{w}. \end{aligned} \tag{4.7}$$

Let $(W,\langle,\rangle)$ be the $\mathbb{R}$-symplectic space of dimension 2, and $\mathbb{W} := V_0(\mathbb{R}) \otimes W$ the symplectic space with the skew-symmetric form $(,) \otimes \langle,\rangle$. The Lie algebra $\mathfrak{sp}(\mathbb{W}\otimes\mathbb{C})$ acts on $\mathcal{S}(V_0)$ through the infinitesimal action induced by ω, which we also denote by ω. In $\mathfrak{sp}(\mathbb{W}\otimes\mathbb{C})$, we have the subalgebra $\mathfrak{sp}(W\otimes\mathbb{C}) \times \mathfrak{o}(V_0\otimes\mathbb{C})$. Through ι, the elements $L, R \in \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sp}(W\otimes\mathbb{C})$ defined in (4.11) act on $\mathbb{C}[\mathfrak{z}_1,\mathfrak{z}_2,\mathfrak{z}_3,\mathfrak{z}_4]$ as (see [FM06, Lemma A.2])

$$\omega(L) = -8\pi\partial_{\mathfrak{v}}\partial_{\overline{\mathfrak{v}}} + \frac{1}{8\pi}\mathfrak{w}\overline{\mathfrak{w}},\ \omega(R) = -8\pi\partial_{\mathfrak{w}}\partial_{\overline{\mathfrak{w}}} + \frac{1}{8\pi}\mathfrak{v}\overline{\mathfrak{v}}. \tag{4.8}$$

Using the isomorphism

$$\begin{aligned} \mathfrak{sl}_2(\mathbb{C})^2 &\to \mathfrak{o}(V_0\otimes\mathbb{C}) \\ (A,B) &\mapsto (v \mapsto Av + vB^t), \end{aligned}$$

we see that the elements $L_1 = (L,0), L_2 = (0,L_2), R_1 = (R,0), R_2 = (0,R)$ in $\mathfrak{sl}_2(\mathbb{C})^2$ act on $\mathbb{C}[\mathfrak{z}_1,\mathfrak{z}_2,\mathfrak{z}_3,\mathfrak{z}_4]$ through ι as (see [FM06, Lemma A.1])

$$\begin{aligned} \omega(L_1) &= 8\pi\partial_{\mathfrak{v}}\partial_{\mathfrak{w}} - \frac{1}{8\pi}\overline{\mathfrak{v}\mathfrak{w}},\ \omega(R_1) = 8\pi\partial_{\overline{\mathfrak{v}}}\partial_{\overline{\mathfrak{w}}} - \frac{1}{8\pi}\mathfrak{v}\mathfrak{w}, \\ \omega(L_2) &= 8\pi\partial_{\overline{\mathfrak{v}}}\partial_{\mathfrak{w}} - \frac{1}{8\pi}\mathfrak{v}\overline{\mathfrak{w}},\ \omega(R_2) = 8\pi\partial_{\mathfrak{v}}\partial_{\overline{\mathfrak{w}}} - \frac{1}{8\pi}\overline{\mathfrak{v}}\mathfrak{w}. \end{aligned} \tag{4.9}$$

For convenience, we slightly abuse notation and write L, R, L_j, R_j for their corresponding actions on $\mathscr{P}(\mathbb{C}^4)$.

When we consider the decomposition $V_0 = V_{00} \oplus U_D$ in (3.2), the map ι induces $\mathbb{S}(V_{00}(\mathbb{R})) \cong \mathscr{P}(\mathbb{C}^3) = \mathbb{C}[\mathfrak{z}_1, \mathfrak{z}_3, \mathfrak{z}_4]$ and $\mathbb{S}(U_D(\mathbb{R})) \cong \mathscr{P}(\mathbb{C}) = \mathbb{C}[\mathfrak{z}_2]$. For $a, b, c \in \mathbb{N}_0$, we also define $\varphi_{00,\infty}^{(a,b)} \in \mathbb{S}(V_{00}(\mathbb{R}))$ and $\varphi_{D,\infty}^{c} \in \mathbb{S}(\mathbb{R})$ by

$$\varphi_{00,\infty}^{(a,b)} := \left(-\frac{\sqrt{2}}{4\pi}\right)^{a+b} \iota^{-1}(\mathfrak{z}_1^a \mathfrak{w}^b),\ \varphi_{D,\infty}^{(c)} := \left(-\frac{\sqrt{2}i}{4\pi}\right)^{c} \iota^{-1}(\mathfrak{z}_2^c). \tag{4.10}$$

For $r \in \mathbb{N}_0$, we have the operators $\mathrm{RC}_r, \widetilde{\mathrm{RC}}_r$ defined in (2.8) that also act on $\mathscr{P}(\mathbb{C}^4)$. They are related by the following lemma.

Lemma 4.1. *In the notations above, we have*

$$\begin{aligned}(L_1 + L_2)\mathrm{RC}_r(\mathfrak{w}) &= -L(\widetilde{\mathrm{RC}}_r(\mathfrak{v} + (-1)^r\overline{\mathfrak{v}})),\\ (\widetilde{\mathrm{RC}}_r - (R_1 + R_2)^r)(\mathfrak{v}) &= (-8\pi)^{1-2r} 2^r R(\mathfrak{w}^r p_r(\mathfrak{v}, \overline{\mathfrak{v}})),\\ (\widetilde{\mathrm{RC}}_r - (R_1 + R_2)^r)(\overline{\mathfrak{v}}) &= (-8\pi)^{1-2r} 2^r R(\mathfrak{w}^r p_r(\overline{\mathfrak{v}}, \mathfrak{v})),\end{aligned} \tag{4.11}$$

where $p_r(X, Y) := -(\tilde{Q}_r(X, Y) - (X + Y)^r)/Y \in \mathbb{Q}[X, Y]$ *for all* $r \in \mathbb{N}_0$.

Proof. It is easy to check that

$$\begin{aligned}\mathrm{RC}_r(\mathfrak{w}) &= (-8\pi)^{-2r} 2^r \mathfrak{w}^{r+1} Q_r(\mathfrak{v}, \overline{\mathfrak{v}}),\\ \widetilde{\mathrm{RC}}_r(\mathfrak{v}) &= (-8\pi)^{-2r} 2^r \mathfrak{w}^r \tilde{Q}_r(\mathfrak{v}, \overline{\mathfrak{v}})\mathfrak{v},\\ \widetilde{\mathrm{RC}}_r(\overline{\mathfrak{v}}) &= (-8\pi)^{-2r} 2^r \mathfrak{w}^r \tilde{Q}_r(\mathfrak{v}, \overline{\mathfrak{v}})\overline{\mathfrak{v}}.\end{aligned} \tag{4.12}$$

This leads directly to the second equation in (4.11) from the definition. To prove the first equation, it is enough to verify

$$(L_1 + L_2)\mathfrak{w}^{r+1} Q_r(\mathfrak{v}, \overline{\mathfrak{v}}) = -L(Q_r(\mathfrak{v}, \overline{\mathfrak{v}})(\mathfrak{v} + \overline{\mathfrak{v}})\mathfrak{w}^r),$$

which follows from (2.6). □

Proposition 4.2. *Let* $\phi_f \in \mathcal{S}(W_\alpha(\hat{F}))$ *and* $\varphi_f \in \mathcal{S}(\hat{V}; \mathbb{Q}^{\mathrm{ab}})$ *be matching sections as in Theorem 3.3 and denote* $\epsilon := \mathrm{sgn}(\alpha_1) = -\mathrm{sgn}(\alpha_2)$. *Then for* $\tilde{\mathcal{I}}$ *defined in (4.2), we have*

$$\begin{aligned}\frac{\pi}{3}(L_1 + L_2)\mathrm{RC}_r\tilde{\mathcal{I}}(g) = &-(-4\pi)^{-r}(R_1 + R_2)^r(E^*(g, \phi^{(1,-1)}) - (-1)^r E^*(g, \phi^{(-1,1)}))\\ &- 2\log\varepsilon_\varrho \frac{\pi}{3}\widetilde{\mathrm{RC}}_r\mathcal{I}_f(g, \varphi^{(1,-1)} - (-1)^r\varphi^{(-1,1)}, \varrho),\end{aligned} \tag{4.13}$$

where $\mathcal{I}_f$ *is defined in (3.23) and* $\varphi^{(\pm 1,\mp 1)} = \varphi_f \otimes \varphi_\infty^{(\pm 1,\mp 1)} \in \mathcal{S}(V(\mathbb{A}))$ *is defined in (3.22).*

Proof. Suppose $\varphi_f = \varphi_{0,f} \otimes \varphi_{1,f}$ and denote $\varphi_1^{\pm} = \varphi_{1,f}\varphi_\infty^{\pm}, \varphi_0^{(k,k')} = \varphi_{0,f}\varphi_{0,\infty}^{(k,k')}$. Then

$$\begin{aligned}(L_1 + L_2)\mathrm{RC}_r(\tilde{\mathcal{I}}(g)) &= \int_{[G]} (L_1 + L_2)\mathrm{RC}_r(\theta_0(g', g, \varphi_0^{(1,1)}))\vartheta_1(g', \varphi_1^+, \tilde{\varrho}_C)dg'\\ &= \int_{[G]} L(\widetilde{\mathrm{RC}}_r(\theta_0(g', g, \varphi_0^{(1,-1)}) - (-1)^r\theta_0(g', g, \varphi_0^{(-1,1)})))\vartheta_1(g', \varphi_1^+, \tilde{\varrho}_C)dg'\end{aligned}$$

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by Lemma 4.1 and (4.7). Now applying Stokes' theorem and Theorem 2.7 gives us

$$\int_{[G]} L(\widetilde{\mathrm{RC}}_r(\theta_0(g',g,\varphi_0^{\epsilon,-\epsilon})))\vartheta_1(g',\varphi_1^+,\tilde{\varrho}_C)dg'$$
$$= -\int_{[G]} \widetilde{\mathrm{RC}}_r(\theta_0(g',g,\varphi_0^{\epsilon,-\epsilon}))L\vartheta_1(g',\varphi_1^+,\tilde{\varrho}_C)dg'$$
$$= -\int_{[G]} \widetilde{\mathrm{RC}}_r(\theta_0(g',g,\varphi_0^{\epsilon,-\epsilon}))\big(\vartheta_1(g',\varphi_1^-,\varrho) + 2\log\varepsilon_\varrho\Theta_1(g',\varphi_1^-,\varrho)\big)dg'$$

with $\epsilon = \pm 1$. Since $R\vartheta_1(g',\varphi_1^-,\varrho) = 0$, we can apply Stokes' theorem, Lemma 4.1 and Theorem 3.3 to obtain

$$\int_{[G]} \widetilde{\mathrm{RC}}_r(\theta_0(g',g,\varphi_0^{\epsilon,-\epsilon}))\vartheta_1(g',\varphi_1^-,\varrho)dg'$$
$$= (-4\pi)^{-r}(R_1+R_2)^r \int_{[G]} (\theta_0(g',g,\varphi_0^{\epsilon,-\epsilon}))\vartheta_1(g',\varphi_1^-,\varrho)dg'$$
$$= (-4\pi)^{-r}(R_1+R_2)^r\mathcal{I}(g,\varphi^{(\epsilon,-\epsilon)},\varrho) = \frac{3}{\pi}(-4\pi)^{-r}(R_1+R_2)^r E^*(g,\phi^{(\epsilon,-\epsilon)}).$$

Putting these together finishes the proof. □

To understand the first term on the right-hand side of (4.13), recall the decomposition for $\theta_0(g',g^\Delta,\varphi_0)$ in (3.18) when $\varphi_{0,\infty} \in \mathbb{S}(V_0)$. This allows us to define

$$(\mathrm{RC}'^{,\Delta}_{r',(k_1,k_2)}\theta_0)(g',g_{00}^\Delta,\varphi_0) := (-4\pi)^{-r'}Q_{r',(k_1,k_2)}(R_1',R_2')\big(\theta_{00}(g_1',g_{00},\varphi_{00})\theta_D(g_2',\varphi_D)\big)\,|_{g_1'=g_2'=g'} \tag{4.14}$$

for $\varphi_0 = \varphi_{00}\otimes\varphi_D$ with $\varphi_{00} \in \mathcal{S}(V_{00}(\mathbb{A})), \varphi_D \in \mathcal{S}(U_D(\mathbb{A}))$, and R_j' the raising operator on g_j'. In the Fock model, R_1', resp. R_2', acts on $\mathbb{C}[\mathfrak{z}_1,\mathfrak{z}_3,\mathfrak{z}_4]$, resp. $\mathbb{C}[\mathfrak{z}_2]$, as

$$\omega(R_1') = -8\pi\partial_{\mathfrak{w}}\partial_{\overline{\mathfrak{w}}} + \frac{1}{8\pi}\mathfrak{z}_1^2,\ \omega(R_2') = \frac{1}{8\pi}\mathfrak{z}_2^2. \tag{4.15}$$

This definition also extends by linearity to all $\varphi_0 \in \mathcal{S}(V_0(\mathbb{A}))$ satisfying $\varphi_{0,\infty} \in \mathbb{S}(V_0(\mathbb{R}))$. We now record the following lemma.

Lemma 4.3. *For $r \in \mathbb{N}_0$, denote $r_0 := \lfloor r/2 \rfloor$. Then*

$$\begin{aligned}(\widetilde{\mathrm{RC}}_r\theta_0)(g',g_{00}^\Delta,\varphi_0^{(1,-1)} + (-1)^r\varphi_0^{(-1,1)})\\ = -2^{2r_0-r+1}(\mathrm{RC}'^{,\Delta}_{r_0,(-r+1/2,r-2r_0+1/2)}\theta_0)(g',g_{00}^\Delta,\varphi_{0,f}(\varphi_{00,\infty}^{(1,r)}\otimes\varphi_{D,\infty}^{r-2r_0}))\end{aligned} \tag{4.16}$$

Proof. Suppose $\varphi_{0,f} = \varphi_{00,f}\otimes\varphi_{D,f}$. Then equations in (4.7) imply

$$(\widetilde{\mathrm{RC}}_r\theta_0)(g',g_{00}^\Delta,\varphi_0^{(1,-1)} - (-1)^r\varphi_0^{(-1,1)}) = \frac{\sqrt{2}}{4\pi}\theta_0(g',g_{00}^\Delta,\varphi_{0,f}\,\iota^{-1}\widetilde{\mathrm{RC}}_r(\mathfrak{v}+(-1)^r\overline{\mathfrak{v}})).$$

From (4.12) and the definition of P_r in (2.5), we have

$$\begin{aligned}\widetilde{\mathrm{RC}}_r(\mathfrak{v}+(-1)^r\overline{\mathfrak{v}})) &= (-4\pi)^{-2r}2^{-r}\mathfrak{w}^r\tilde{Q}_r(\mathfrak{v},\overline{\mathfrak{v}})(\mathfrak{v}+(-1)^r\overline{\mathfrak{v}}) = (-4\pi)^{-2r}2^{-r}\mathfrak{w}^r Q_r(\mathfrak{v},\overline{\mathfrak{v}})(\mathfrak{v}+\overline{\mathfrak{v}})\\ &= (-4\pi)^{-2r}2\mathfrak{w}^r\mathfrak{z}_1^{r+1}P_r(i\mathfrak{z}_2/\mathfrak{z}_1)\\ &= (-4\pi)^{-2r}2\mathfrak{w}^r\mathfrak{z}_1(-1)^{r_0}\sum_{s=0}^{r_0}\binom{r_0-r-1/2}{r_0-s}\binom{r-r_0-1/2}{s}\mathfrak{z}_1^{2s}(i\mathfrak{z}_2)^{r-2s}\\ &= (-4\pi)^{r_0-2r}2i^r(-2)^{r_0}Q_{r_0,(-r+1/2,r-2r_0+1/2)}(R_1',R_2')\mathfrak{z}_1\mathfrak{w}^r\mathfrak{z}_2^{r-2r_0}.\end{aligned}$$

Substituting the definition (4.10) finishes the proof. □

The following technical lemma concerns a change of regularized integrals and follows from the proof of Lemma 5.4.3 in [Li22].

Lemma 4.4. *Given $\varphi_{i,f}\in\mathcal{S}(\hat{V}_i)$ with $i=0,1$, let $\Gamma\subset\mathrm{PSL}_2(\mathbb{Z})\subset G_{00}(\mathbb{Q})$ be a congruence subgroup that acts trivially on $\varphi_{0,f}$. For any $a\ge 1, b\ge 0$ and $f\in M^!_{-2b}(\Gamma)$, we have*

$$\begin{aligned}\int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} y^b f(z)\int_{[\mathrm{SL}_2]}\theta_1(g,\xi,\varphi_1^-)\theta_0(g,g_z^\Delta,\varphi_{0,f}(\varphi_{00,\infty}^{(a,b)}\otimes\varphi_{D,\infty}^{b-a+1}))dg\,d\mu(z)\\ =\int_{[\mathrm{SL}_2]}^{\mathrm{reg}}\theta_1(g,\xi,\varphi_1^-)\int_{\Gamma\backslash\mathbb{H}}^{\mathrm{reg}} y^b f(z)\theta_0(g,g_z^\Delta,\varphi_{0,f}(\varphi_{00,\infty}^{(a,b)}\otimes\varphi_{D,\infty}^{b-a+1}))d\mu(z)dg.\end{aligned}$$

4.2. Fourier expansion of $\tilde{\mathcal{I}}$

To evaluate the Fourier expansion of $\tilde{\mathcal{I}}$ in (4.2), we change to a mixed model of the Weil representation using the partial Fourier transform $\mathscr{F}_1$ defined in (3.28).

Throughout the section, we write

$$g_0=(g_{z_1},g_{z_2})\in G_0(\mathbb{R}) \tag{4.17}$$

for $(z_1,z_2)\in\mathbb{H}^2$ with $z_i=x_i+iy_i$ and $g_\tau\in G(\mathbb{R})$ with $\boldsymbol{\tau}=\boldsymbol{u}+i\boldsymbol{v}\in\mathbb{H}$. Then (3.27) implies

$$\begin{aligned}\omega_{0,\infty}(g_0)\mathscr{F}_1(\varphi_{0,\infty})((0,r),\nu) &= \sqrt{y_1y_2}\,\mathbf{e}(r(x_2\nu+x_1\nu'))\mathscr{F}_1(\varphi_{0,\infty})((0,r\sqrt{y_1y_2}),\nu\sqrt{y_2/y_1}),\\ \omega_{0,\infty}(g_{\boldsymbol{\tau}})\mathscr{F}_1(\varphi_{0,\infty})((0,r),\nu) &= \sqrt{\boldsymbol{v}}\,\mathbf{e}(-\boldsymbol{u}\nu\nu')\mathscr{F}_1(\varphi_{0,\infty})((0,r/\sqrt{\boldsymbol{v}}),\sqrt{\boldsymbol{v}}\nu).\end{aligned} \tag{4.18}$$

Also when $\varphi_{0,\infty}=\varphi_{0,\infty}^{(k,k')}$ with $k,k'=\pm1$ as given in (3.22) and (4.1), we have

$$\mathscr{F}_1(\varphi_{0,\infty}^{(k,k')})((0,r),\nu)=\varphi_{0,\infty}^{(k,k')}(ir,0,\nu)e^{-2\pi r^2} \tag{4.19}$$

with $r\in\mathbb{R}, \nu=(\nu_1,\nu_2)\in\mathbb{R}^2$. After applying Poisson summation and unfolding, we can rewrite the theta kernel $\theta_0(g,g_0,\varphi_0)$ as

$$\begin{aligned}\theta_0(g,g_0,\varphi_0) &= \sum_{\lambda\in V_0(\mathbb{Q})}\omega_0(g,g_0)\varphi_0(\lambda)=\sum_{\substack{\nu\in V_{-1}(\mathbb{Q})\\ \eta\in\mathbb{Q}^2}}\omega_{-1}(g)\omega_0(g_0)\mathscr{F}_1(\varphi_0)(\eta g,\nu)\\ &= \sum_{\nu\in V_{-1}(\mathbb{Q})}\omega_{-1}(g)\omega_0(g_0)\left(\mathscr{F}_1(\varphi_0)((0,0),\nu)+\sum_{\substack{r\in\mathbb{Q}^\times\\ \gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}}\mathscr{F}_1(\varphi_0)((0,r)\gamma g,\nu)\right).\end{aligned}$$

For a bounded, integrable function f on $[G]$ such that $\theta_0(g, g_0, \varphi_0) f(g)$ is right $\mathrm{SL}_2(\hat{\mathbb{Z}})\mathrm{SO}_2(\mathbb{R})$-invariant, we have

$$I_0(g_0, \varphi_0, f) = \int_{[G]} \theta_0(g, g_0, \varphi_0) f(g) dg = \frac{3}{\pi} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \theta_0(g_\tau, g_0, \varphi_0) f(g_\tau) d\mu(\boldsymbol{\tau}),$$

which can be written as $I_0(g_0, \varphi_0, f) = I_0^0(g_0, \varphi_0, f) + I_0^+(g_0, \varphi_0, f)$ with (see, for example, equation (4.2) in [Kud16])

$$\begin{aligned}
I_0^0(g_0, \varphi_0, f) &:= \frac{3}{\pi} \sum_{\nu \in V_{-1}(\mathbb{Q})} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \omega_{-1}(g_\tau)\omega_0(g_0)\mathscr{F}_1(\varphi_0)((0,0),\nu) f(g_\tau) d\mu(\boldsymbol{\tau}),\\
I_0^+(g_0, \varphi_0, f) &= \frac{3}{\pi} \sum_{\substack{\nu \in V_{-1}(\mathbb{Q}),\ r \in \mathbb{Q}^\times \\ \gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(g_{\gamma\tau})\omega_{-1}(g_{\gamma\tau})\omega_0(g_0)\mathscr{F}_1(\varphi_0)((0,r)g_{\gamma\tau}, \nu) d\mu(\boldsymbol{\tau})\\
&= \sum_{\nu \in V_{-1}(\mathbb{Q}),\ r \in \mathbb{Q}^\times} \mathscr{F}_1(\varphi_{0,f})((0,r),\nu)\mathfrak{F}_{r,\nu}(\varphi_{0,\infty})(z_1, z_2, f)\\
\mathfrak{F}_{r,\nu}(\varphi)(z_1, z_2, f) &:= \frac{3}{\pi} \int_{\Gamma_\infty\backslash\mathbb{H}} (\omega_0(g_\tau, g_0)\mathscr{F}_1(\varphi))((0,r),\nu) f(g_\tau) d\mu(\boldsymbol{\tau}).
\end{aligned} \tag{4.20}$$

Using the $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariance of $\theta_0(g, g_0, \varphi_0) f(g)$, we can rewrite for any $N \in \mathbb{N}$

$$\mathfrak{F}_{r,\nu}(\varphi)(z_1, z_2, f) := N^{-1}\frac{3}{\pi} \int_{\Gamma_\infty^N\backslash\mathbb{H}} (\omega_0(g_\tau, g_0)\mathscr{F}_1(\varphi))((0,r),\nu) f(g_\tau) d\mu(\boldsymbol{\tau}) \tag{4.21}$$

with $\Gamma_\infty^N := \{n(Nb) : b \in \mathbb{Z}\} \subset \Gamma_\infty$.

For our purpose, we are interested in the case when $f(g) = \vartheta_1(g, \varphi_1, \rho)$ with $\rho \in \{\varrho, \tilde{\varrho}_C\}, \varphi_1 = \varphi_1^\pm$ and $\varphi_0 = \varphi_0^{(k,k')}$ for $k, k' = \pm 1$. In that case, we have $I_0(g_0, \varphi_0, f) = \mathcal{I}(g_0, \varphi, \rho)$ with $\varphi = \varphi_0 \otimes \varphi_1$, and denote

$$\mathcal{I}^+(g_0, \varphi, \rho) := I_0^+(g_0, \varphi_0, \vartheta(\cdot, \varphi_1, \rho)),\ \mathcal{I}^0(g_0, \varphi, \rho) := I_0^0(g_0, \varphi_0, \vartheta(\cdot, \varphi_1, \rho)). \tag{4.22}$$

This can be extended by $\mathbb{Q}$-linearity to all $\varphi = \varphi_f \varphi_\infty \in \mathcal{S}(V(\mathbb{A}))$ with $\varphi_f \in \mathcal{S}(\hat{V})$ and $\varphi_\infty \in \mathbb{S}(V(\mathbb{R}))$.

The constant term I_0^0 in the Fourier expansion of $I_0(g_0, \varphi_0, f)$ is independent of x_1, x_2 and can be evaluated by the change of model in section 3.3. For our purpose, we will state a decay result needed to prove Theorem 5.1

Lemma 4.5. *Suppose there is $s \in \mathbb{R}$ such that $|f(g_\tau)| \ll v^s$ for all τ in the usual fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Then*

$$\lim_{y\to\infty} y^{-c}\left(\partial_{y_1}^a \partial_{y_2}^b \frac{I_0^0(g_0, \varphi_0, f)}{\sqrt{y_1 y_2}}\right)|_{y_1=y_2=y} = 0$$

for any $a, b, c \in \mathbb{N}_0$ satisfying $a + b + c \geq 1$. When $a = b = c = 0$, the limit exists.

Remark 4.6. It is easy to check that $f(g) = \vartheta(g, \varphi_1, \tilde{\varrho}_C)$ fulfills the condition in the lemma.

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Proof. Let $\mathcal{F}$ denote the fundamental domain. Then we can use (4.18) to obtain

$$\frac{I_0^0(g_0,\varphi_0,f)}{\sqrt{y_1y_2}} = \frac{3}{\pi}\sum_{\nu\in F}\mathscr{F}_1(\varphi_{0,f})((0,0),\nu)\left(\nu\sqrt{y_2/y_1}+\nu'\sqrt{y_1/y_2}\right)$$
$$\times\int_{\mathcal{F}}\mathbf{e}(-\boldsymbol{u}\nu\nu')e^{-\pi\boldsymbol{v}(\nu^2y_2^2+(\nu')^2y_1^2)/(y_1y_2)}f(g_\tau)\frac{d\boldsymbol{u}\,d\boldsymbol{v}}{\boldsymbol{v}}.$$

Since $\mathscr{F}_1(\varphi_{0,f})$ is a Schwartz function, we can suppose the sum over $\lambda\in F$ is replaced by a sum over the translate of a lattice. For the integral on the second line, we can trivially estimate it by

$$\int_{\sqrt{3}/2}^{\infty}e^{-\pi\boldsymbol{v}(\nu^2y_2^2+(\nu')^2y_1^2)/(y_1y_2)}\boldsymbol{v}^s\frac{d\boldsymbol{v}}{\boldsymbol{v}}.$$

From this, we see that $|y^{-1}I_0^0((g_z,g_z),\varphi_0,f)|$ is bounded independent of y, and the second claim holds. This also gives the first claim for $a=b=0$ and $c\geq 1$. The other cases follow from first applying $\partial_{y_1}^a\partial_{y_2}^b$ to $\frac{I_0^0(g_0,\varphi_0,f)}{\sqrt{y_1y_2}}$ and then conducting the same estimate. □

We will now evaluate the non-constant term $\mathcal{I}^+$. Let ϱ and $\tilde{\varrho}_C$ be as in (2.76), and $K_\varrho\subset H_1(\hat{\mathbb{Z}}),\Gamma_\varrho\subset H_1^+(\mathbb{Q})$ be as in (2.14) and (2.67), respectively. For $f(g)=\vartheta_1(g,\varphi_1^-,\varrho)$ and $\varphi_{0,\infty}=\varphi_{0,\infty}^{(\pm1,\mp1)}$ defined in (3.22), we can apply (4.18), (4.19) and (4.21) to obtain

$$\mathfrak{F}_{r,\nu}(\varphi_{0,\infty}^{(\pm1,\mp1)})(z_1,z_2,f)=\operatorname{sgn}(\nu)2\sqrt{y_1y_2}\mathbf{e}(r((x_2\mp iy_2)\nu+(x_1\pm iy_1)\nu'))$$
$$\times\operatorname{vol}(K_\varrho)\sum_{\xi\in C}\varrho(\xi)\sum_{\substack{\beta\in\Gamma_\varrho\backslash F^\times\\ \beta\beta'=\nu\nu'<0}}\operatorname{sgn}(\beta)\varphi_{1,f}(\xi^{-1}\beta)$$

if $\mp r\nu>0$, and zero otherwise. After the change of variable $t=r\nu'$, we have

$$\mathcal{I}^+(g_0,\varphi^{(1,-1)},\varrho)=\frac{3}{\pi}\sqrt{y_1y_2}\sum_{t\in F^\times,\ t_1>0>t_2}c_t(\varphi_f,\varrho)\mathbf{e}(t_1z_1+t_2\overline{z_2}),$$
$$\mathcal{I}^+(g_0,\varphi^{(-1,1)},\varrho)=\frac{3}{\pi}\sqrt{y_1y_2}\sum_{t\in F^\times,\ t_2>0>t_1}c_t(\varphi_f,\varrho)\mathbf{e}(t_1\overline{z_1}+t_2z_2),$$
$$c_t(\varphi_f,\varrho):=2\operatorname{vol}(K_\varrho)\sum_{\xi\in C}\varrho(\xi)\sum_{r\in\mathbb{Q}^\times}\sum_{\substack{\beta\in\Gamma_\varrho\backslash F^\times\\ \operatorname{Nm}(\beta)=\operatorname{Nm}(t)/r^2<0}}\operatorname{sgn}(\beta_1t_2/r)\mathscr{F}_1(\varphi_f)((0,r),t/r,\xi^{-1}\beta).$$

In the case of $\mathcal{I}^+(g_0,\varphi,\tilde{\varrho}_C)$, we have the following result.

Proposition 4.7. *Given $\phi_f\in\mathcal{S}(\hat{W}_\alpha)$, let $\varphi_f\in\mathcal{S}(\hat{V})^{G(\hat{\mathbb{Z}})T^\Delta(\hat{\mathbb{Z}})K_\varrho}$ be a matching section as in Theorem 3.3. For $\tilde{\varrho}_C$ as in (2.76), we have*

$$\frac{\mathcal{I}^+(g_0,\varphi_f\varphi_\infty^{(1,1)},\tilde{\varrho}_C)}{\sqrt{y_1y_2}}=\frac{3}{\pi}\sum_{t\in F^\times,\ t\gg0}\tilde{c}_t(\varphi_f,\varrho)\mathbf{e}(t_1z_1+t_2z_2)+\sum_{t\in F^\times}e_t(\varphi_f;y_1,y_2)\mathbf{e}(t_1z_1+t_2z_2),\tag{4.23}$$

where $\tilde{c}_t(\varphi_f,\varrho)\in\mathbb{C}$ and $e_t(\varphi_f;\cdot):\mathbb{R}_{>0}^2\to\mathbb{C}$ are given in (4.27) and (4.28) below. There is a constant $M\in\mathbb{N}$ such that $\tilde{c}_t(\varphi_f,\varrho)=0$ when $t\notin M^{-1}\mathcal{O}$.

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Let

$$S_C := \{v \text{ prime in } F : \mathrm{ord}_v(\xi) \neq 0 \text{ for some } \xi \in C\} \tag{4.24}$$

be a finite set of primes, and then there exists $\kappa \in \mathbb{N}$ *and* $\beta(t, \phi_f) \in F^\times$ *such that* $\tilde{c}_t(\varphi_f, \varrho) = -\frac{2}{\kappa} \log|\beta(t, \phi_f)/\beta(t, \phi_f)'|$ *and*

$$\kappa^{-1} \mathrm{ord}_{\mathfrak{p}}(\beta(t, \phi_f)/\beta(t, \phi_f)') = \begin{cases} \tilde{W}_t(\phi_f), & \text{if } \mathrm{Diff}(W_\alpha, t) = \{\mathfrak{p}\}, \\ -\tilde{W}_t(\phi_f), & \text{if } \mathrm{Diff}(W_\alpha, t) = \{\mathfrak{p}'\}, \\ 0, & \text{otherwise,} \end{cases} \tag{4.25}$$

when $\mathrm{Diff}(W_\alpha, t) \subset S_C^c$ *with* $\tilde{W}_t$ *defined in (2.62). Furthermore, the function* e_t *satisfies*

$$\lim_{y \to \infty} y^{-c} \left(\partial_{y_1}^a \partial_{y_2}^b e_t(y_1, y_2) \right) |_{y_1 = y_2 = y} = 0 \tag{4.26}$$

for all $a, b, c \in \mathbb{N}_0$.

Remark 4.8. Note that we have

$$\tilde{\mathcal{I}}(g_0) = \mathcal{I}(g_0, \varphi^{(1,1)}, \tilde{\varrho}_C) = \mathcal{I}^0(g_0, \varphi^{(1,1)}, \tilde{\varrho}_C) + \mathcal{I}^+(g_0, \varphi^{(1,1)}, \tilde{\varrho}_C)$$

from (4.2) and (4.22).

Proof. Suppose $\varphi_f = \varphi_{0,f} \otimes \varphi_{1,f}$, as the general case follows by linearity. We first prove (4.23). Using (4.20), it is enough to evaluate $\mathfrak{F}_{r,\nu}(\varphi_{0,\infty}^{(1,1)})(z_1, z_2, \vartheta_1(\cdot, \varphi_1^+, \tilde{\varrho}_C))$. If we set $t := r\nu'$, then we have by (4.18), (4.19), and Theorem 2.7

$$\mathfrak{F}_{r,\nu'}(\varphi_{0,\infty}^{(1,1)})(z_1, z_2, \vartheta_1(\cdot, \varphi_1^+, \tilde{\varrho}_C)) = \left(e_{r,\nu'}(\varphi_{1,f}; y_1, y_2) + \tilde{c}_{r,\nu'}(\varphi_{1,f}, \varrho) \right) \sqrt{y_1 y_2} \mathbf{e}(t_1 z_1 + t_2 z_2))$$

with $\tilde{c}_{r,\nu'}(\varphi, \varrho)$ and $e_{r,\nu}(\varphi; y, y')$ given by

$$\tilde{c}_{r,\nu'}(\varphi, \varrho) := \begin{cases} 4\mathrm{vol}(K_\varrho) \displaystyle\sum_{\substack{\beta \in \Gamma_\varrho \backslash F^\times \\ \beta\beta' = \nu\nu' > 0 \\ \xi \in C}} \varrho(\xi) \mathrm{sgn}(r\beta) \varphi(\xi^{-1}\beta) \log(\varepsilon_\varrho) \left\{ \log_{\varepsilon_\varrho} \sqrt{|\beta/\beta'|} \right\}, & t \gg 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$e_{r,\nu}(\varphi; y_1, y_2) := -\frac{\mathrm{vol}(K_\varrho)}{\sqrt{y_1 y_2}} \sum_{\substack{\beta \in \Gamma_\varrho \backslash F^\times \\ \beta\beta' = \nu\nu' \\ \xi \in C}} \varrho(\xi) \varphi(\xi^{-1}\beta) \left(\delta_{\nu\nu' < 0} \mathrm{sgn}(\beta) e_{r,\nu}^*(y_1, y_2) + \frac{\log \varepsilon_\varrho}{\sqrt{\pi}} e_{r,\nu,\beta}^\dagger(y_1, y_2) \right).$$

Here, we have set

$$e_{r,\nu}^*(y_1, y_2) := \int_0^\infty \Gamma(0, 4\pi|\nu\nu'|\mathbf{v}) K_{r,\nu}(\mathbf{v}, y_1, y_2) \frac{d\mathbf{v}}{\mathbf{v}},$$

$$e_{r,\nu,\beta}^\dagger(y_1, y_2) := \int_0^\infty \sum_{\epsilon \in \Gamma_1} \mathrm{sgn}(\beta\epsilon - \beta'\epsilon') \Gamma(1/2, \pi(\beta\epsilon - \beta'\epsilon')^2 \mathbf{v}) \sqrt{\mathbf{v}} K_{r,\nu}(\mathbf{v}, y_1, y_2) \frac{d\mathbf{v}}{\mathbf{v}},$$

$$K_{r,\nu}(\mathbf{v}, y_1, y_2) := e^{-\pi\left(\frac{A}{\sqrt{\mathbf{v}}} - B\sqrt{\mathbf{v}}\right)^2} \left(\frac{A}{\sqrt{\mathbf{v}}} + B\sqrt{\mathbf{v}} \right), \ A := r\sqrt{y_1 y_2}, \ B := \frac{\nu y_1 + \nu' y_2}{\sqrt{y_1 y_2}}.$$

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So if we set

$$\tilde{c}_t(\varphi_f, \varrho) := \sum_{r \in \mathbb{Q}^\times} \mathscr{F}_1(\varphi_{0,f})((0,r), t'/r)\tilde{c}_{r,t'/r}(\varphi_{1,f}, \varrho), \tag{4.27}$$

$$e_t(\varphi_f; y_1, y_2) := \sum_{r \in \mathbb{Q}^\times} \mathscr{F}_1(\varphi_{0,f})((0,r), t'/r)e_{r,t'/r}(\varphi_{1,f}; y_1, y_2), \tag{4.28}$$

then equation (4.23) holds by (4.20). Since φ_f is a Schwartz function, the sum defining $\tilde{c}_t$ is finite and equals to 0 when $t \notin M^{-1}\mathcal{O}$ for some $M \in \mathbb{N}$ depending only on φ_f.

To prove (4.25), notice that

$$\tilde{c}_t(\varphi_f, \varrho) = 4\mathrm{vol}(K_\varrho) \sum_{r \in \mathbb{Q}^\times} \sum_{\substack{\beta \in \Gamma_\varrho \backslash F^\times \\ 1 \le |\beta/\beta'| \le \varepsilon_\varrho^2 \\ \beta\beta' = tt'/r^2 > 0 \\ \xi \in C}} \varrho(\xi)\mathscr{F}_1(\varphi_f)((0,r), t'/r, \xi^{-1}\beta)\mathrm{sgn}(r\beta) \log\sqrt{|\beta/\beta'|}.$$

By Theorem 3.10 and Lemma 3.7, there exists $c \in \mathbb{N}$ such that $2c\mathscr{F}_1(\varphi_f)((0,r), \nu, \lambda) \in \mathbb{Z}$ for all $r \in \hat{\mathbb{Q}}$ and $\nu, \lambda \in \hat{F}$. Then we can write

$$\tilde{c}_t(\varphi_f, \varrho) = -\frac{2}{\kappa} \log\left|\frac{\beta(t, \varphi_f)}{\beta(t, \varphi_f)'}\right|,$$
$$\beta(t, \varphi_f) := \prod_{r \in \mathbb{Q}^\times} \prod_{\substack{\beta \in \Gamma_\varrho \backslash F^\times \\ 1 \le |\beta/\beta'| \le \varepsilon_\varrho^2 \\ \beta\beta' = tt'/r^2 > 0 \\ \xi \in C}} (r\beta)^{-\mathrm{vol}(K_\varrho)\varrho(\xi)\kappa\mathscr{F}_1(\varphi_f)((0,r), t'/r, \xi^{-1}\beta)\mathrm{sgn}(r\beta)}.$$

For any split rational prime $p = \mathfrak{p}\mathfrak{p}'$ with any $\mathfrak{p} \notin S_C$, we have

$$\begin{aligned}
&\kappa^{-1} \mathrm{ord}_{\mathfrak{p}}\, \beta(t, \varphi_f) \\
&= -\mathrm{vol}(K_\varrho) \sum_{r \in \mathbb{Q}^\times} \sum_{\substack{\beta \in \Gamma_\varrho \backslash F^\times \\ \beta\beta' = tt'/r^2 > 0 \\ \xi \in C}} \varrho(\xi)\mathscr{F}_1(\varphi_f)((0,r), t'/r, \xi^{-1}\beta)\mathrm{sgn}(r\beta)\, \mathrm{ord}_{\mathfrak{p}}(\xi^{-1}r\beta) \\
&= -\mathrm{vol}(K_\varrho) \sum_{r \in \mathbb{Q}^\times} \sum_{\substack{h \in \Gamma_\varrho \backslash H_1(\mathbb{Q})^+ \\ \xi \in C}} \varrho(\xi)\mathscr{F}_1(\varphi_f - \omega_f(-1)\varphi_f) \\
&\quad \times ((0,r), t'/r, \xi^{-1}h^{-1}t/r)\mathrm{sgn}(h^{-1}t)\, \mathrm{ord}_{\mathfrak{p}}(\xi^{-1}h^{-1}t) \\
&= 2\sum_{r \in \mathbb{Q}^\times} \int_{H_1(\hat{\mathbb{Q}})} \varrho(h_1)\mathscr{F}_1(\varphi_f)((0,r), t'/r, -h_1^{-1}t/r)\, \mathrm{ord}_{\mathfrak{p}}(h_1^{-1}t')dh_1
\end{aligned}$$

since $t \gg 0$, $\mathrm{ord}_{\mathfrak{p}}(\beta) = \mathrm{ord}_{\mathfrak{p}}(\xi^{-1}\beta)$ and $\varrho_f(h) = \mathrm{sgn}(h)$, where $h_1 = -\xi h \in H_1(\hat{\mathbb{Q}})$ and $\varrho(h_1) = -\varrho(\xi)$. The last step follows from (2.71).

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Notice that the quantity above factors as the following product of sums of local integrals

$$\begin{aligned}
&\kappa^{-1}\operatorname{ord}_{\mathfrak{p}}\beta(t,\varphi_f)\\
&= 2\prod_{\ell<\infty,\ \ell\neq p}\left(\sum_{n\in\mathbb{Z}}\int_{H_1(\mathbb{Q}_\ell)}\varrho_\ell(h_{1,\ell})\mathscr{F}_1(\varphi_{f,\ell})((0,\ell^n),t'/\ell^n,-h_{1,\ell}^{-1}t/\ell^n)dh_{1,\ell}\right)\\
&\quad\times\left(\sum_{n\in\mathbb{Z}}\int_{H_1(\mathbb{Q}_p)}\varrho_p(h_{1,p})\mathscr{F}_1(\varphi_{f,p})((0,p^n),t'/p^n,-h_{1,p}^{-1}t/p^n)\operatorname{ord}_{\mathfrak{p}}(h_{1,p}^{-1}t')dh_{1,p}\right).
\end{aligned}$$

Applying (3.60) turns the first line on the right-hand side into $2\prod_{v<\infty,\ v\nmid p}W^*_{t,v}(\phi_v)$. If this is nonzero, then $\mathrm{Diff}(W_\alpha,t)$ is either $\{\mathfrak{p}\}$ or $\{\mathfrak{p}'\}$ as it has odd size. If $\mathrm{Diff}(W_\alpha,t)=\{\mathfrak{p}\}$, then $W^*_{t,\mathfrak{p}}(\phi_{\mathfrak{p}})=0$ and Proposition 3.16 tell us that the second line on the right-hand side becomes

$$\frac{W^{*,\prime}_{t,\mathfrak{p}}(\phi_{\mathfrak{p}})W^*_{t,\mathfrak{p}'}(\phi_{\mathfrak{p}''})}{\log p}+W^*_{t,\mathfrak{p}}(\phi_{\mathfrak{p}})W^*_{t,\mathfrak{p}'}(\phi_{\mathfrak{p}'})\operatorname{ord}_{\mathfrak{p}}(t')=\frac{W^{*,\prime}_{t,\mathfrak{p}}(\phi_{\mathfrak{p}})W^*_{t,\mathfrak{p}'}(\phi_{\mathfrak{p}'})}{\log p}$$

as $\operatorname{ord}_{\mathfrak{p}}(h_{1,p}^{-1}t')=\operatorname{ord}_{\mathfrak{p}}(h_{1,p}^{-1})+\operatorname{ord}_{\mathfrak{p}}(t')$. This gives us

$$\kappa^{-1}\operatorname{ord}_{\mathfrak{p}}\beta(t,\varphi_f)=\tilde{W}_t(\phi_f)/2.$$

Repeating the above argument together with Remark 3.17, we obtain $\kappa^{-1}\operatorname{ord}_{\mathfrak{p}'}\beta(t,\varphi_f)=-\tilde{W}_t(\phi_f)/2$. Putting this together gives us (4.25), where the case with $\mathrm{Diff}(W_\alpha,t)=\{\mathfrak{p}'\}$ is obtained similarly.

Now we will prove (4.26). Since φ has compact support, the summation over ξ and β in $e_{r,\nu}$ and the summation over r in (4.28) are finite sums, it suffices to establish (4.26) with e_t replaced by $e^*_{r,\nu}$ and $e^\dagger_{r,\nu,\beta}$ with $r>0$. For any fixed $C,\epsilon>0$, $s\in\mathbb{R}$ and $a,b,c\in\mathbb{N}_0$, we have

$$\begin{aligned}
\left|y^c\partial_y^a\partial_{y'}^b\int_0^{A^{1-\epsilon}}e^{-C\mathbf{v}}\mathbf{v}^sK_{r,\nu}(\mathbf{v},y,y')\frac{d\mathbf{v}}{\mathbf{v}}\right|&\ll\int_0^{A^{1-\epsilon}}K_{r,\nu}(\mathbf{v},y,y')\frac{d\mathbf{v}}{\mathbf{v}}\ll e^{-A^{\epsilon/2}},\\
\left|y^c\partial_y^a\partial_{y'}^b\int_{A^{1-\epsilon}}^\infty e^{-C\mathbf{v}}\mathbf{v}^sK_{r,\nu}(\mathbf{v},y,y')\frac{d\mathbf{v}}{\mathbf{v}}\right|&\ll\int_{A^{1-\epsilon}}^\infty e^{-C\mathbf{v}/2}d\mathbf{v}\ll e^{-A^{1/2-\epsilon/2}}
\end{aligned}$$

when B is in a compact subset of $\mathbb{R}$ and $A>0$ is sufficiently large. Furthermore, it is easy to see that there exists $C>0$ such that

$$|\Gamma(0,4\pi|\nu\nu'|\mathbf{v})|\ll\mathbf{v}^{-1}e^{-C\mathbf{v}},\ \left|\sum_{\varepsilon\in\Gamma_1}\operatorname{sgn}(\beta\varepsilon-\beta'\varepsilon')\Gamma(1/2,\pi(\beta\varepsilon-\beta'\varepsilon')^2\mathbf{v})\right|\ll\mathbf{v}^{-1/2}e^{-C\mathbf{v}}$$

for all $\mathbf{v}>0$. Combining these then proves (4.26). □

4.3. *Rationality of theta lifts*

Recall that the rational quadratic space V_α is the restriction of scalars of the F-quadratic space W_α. The following result shows that the Millson theta lift preserves rationality.

Proposition 4.9. *Let* $f=\sum_{\mu\in L^\vee/L}f_\mu\mathfrak{e}_\mu\in M^!_{-2r,\rho_L}$ *be weakly holomorphic for some* $r\in\mathbb{N}$ *and lattice* $L\subset V_\alpha$. *For* $\varphi^{(1,r)}=\varphi^{(1,r)}_{00,\infty}\varphi_f$ *with* $\varphi_f\in\mathcal{S}(\hat{V}_{00};\mathbb{Q}^{\mathrm{ab}})^{T^\Delta(\hat{\mathbb{Z}})}$, *let* $\Gamma(N)\subset\mathrm{SL}_2(\mathbb{Z})$ *be a congruence*

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subgroup contained in $\ker(\rho_L)$ *that fixes* φ_f. *The following regularized integral*[4]

$$I^M(\tau, f_\mu, \varphi_f) := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]} \int_{\Gamma(N)\backslash\mathbb{H}}^{\mathrm{reg}} y^r f_\mu(z) \theta_{00}(g_\tau, h_z, \varphi^{(1,r)}) d\mu(z) \tag{4.29}$$

defines a weakly holomorphic modular form of weight $-r + 1/2 < 0$. *Suppose f has rational Fourier coefficients at the cusp* ∞*, so does* $I^M(\tau, f_\mu, \varphi_f)$ *for all* $\mu \in L^\vee/L$.

Remark 4.10. When $r = 0$ and f has vanishing constant term, the same proof shows that the weakly holomorphic modular form $I^M(\tau, f_\mu, \varphi_f)$ has rational Fourier coefficients up to algebraic multiples of weight $1/2$ unary theta series.

Proof. We will use the Fourier expansion of Millson theta lift calculated in Theorem 5.1 of [ANS18], which we now recall. Fix an orientation on $V_{00}(\mathbb{R})$. For an isotropic line $\ell \subset V_{00}$, let $G_{00,\ell} \subset G_{00}$ be its stabilizer and $\gamma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_\ell^{-1} G_{00,\ell} \gamma_\ell = G_{00,\ell_\infty}$ with $\ell_\infty = \mathbb{Q}v_\infty$ and $v_\infty := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Denote $c_\ell(m, \mu)$ the m-th Fourier coefficient of $f_\mu \mid_{-2r} \gamma_\ell$. If $x \in V_{00}(\mathbb{Q})$ satisfies $\sqrt{-Q(x)} = d \in \mathbb{Q}_{>0}$, then $x^\perp$ is a hyperbolic plane spanned by two isotropic lines ℓ_x and ℓ_{-x} such that $x, \gamma_{\ell_x} v_\infty, \gamma_{\ell_{-x}} v_\infty$ is positively oriented. We can then define $r_x \in \mathbb{Q}$ by

$$\gamma_{\ell_x}^{-1} \cdot x = -d \begin{pmatrix} 2r_x & 1 \\ 1 & 0 \end{pmatrix}.$$

Suppose $r \geq 1$. From [ANS18, Theorem 5.1], we know that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] \cdot I^M$ is weakly holomorphic of weight $-r + 1/2 < 0$ with principal part given by[5]

$$\sum_{d>0} \frac{\mathbf{e}(-d^2\tau)}{2d^{1+r}} \sum_{\substack{x \in \Gamma_L \backslash V_{00,-d^2}(\mathbb{Q}) \\ w \in \mathbb{Q}_{<0}}} w^k \varphi_f(x) (c_{\ell_x}(w, \mu) \mathbf{e}(r_x w) + (-1)^{r+1} c_{\ell_{-x}}(w, \mu) \mathbf{e}(r_{-x} w)) \in \mathbb{Q}^{\mathrm{ab}}.$$

Note that the inner sum above vanishes for d sufficiently large by Proposition 4.7 in [BF04], and I^M is uniquely determined by its principal part because its weight is negative. Now we can enlarge N such that $Nr_{\pm x} w \in \mathbb{Z}$ whenever $c_{\ell_{\pm x}}(w, \mu) \neq 0$. Then given a prime $p \nmid N$, for an element $x \in \Gamma_L \backslash V_{00,-d^2}(\mathbb{Q})$ to have a representative $\tilde{x} \in V_{00}$ such that both $t(p) \cdot \tilde{x}$ and $\tilde{x}$ are both p-integral is equivalent to finding a p-integral representative $\left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)$ with $p \nmid A$. Note that the set

$$S_d(\varphi) := \{x \in \Gamma_L \backslash V_{00,-d^2}(\mathbb{Q}) : \varphi(x) \neq 0\}$$

is a finite set for any $\varphi \in \mathcal{S}(\hat{V}_{00})$.

For any $\sigma_a \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ with $a \in \hat{\mathbb{Z}}^\times$, we have $t(a) \in T(\hat{\mathbb{Z}}) \subset \mathrm{GL}_2(\hat{\mathbb{Z}})$. Choose an odd prime $p \nmid N$ such that $a \equiv p \bmod N$ and every $x \in S_d(\varphi_f)$ has a p-integral representative $\tilde{x} \in V_{00}(\mathbb{Q})$ such that $t(p) \cdot \tilde{x}$ is p-integral. Let $\tilde{S}_d(\varphi_f)$ be such a set of representatives.

Denote $\tilde{x}' := t(p) \cdot \tilde{x}$ for $\tilde{x} \in \tilde{S}_d(\varphi_f)$, which is p-integral and satisfies $\ell_{\pm\tilde{x}'} = t(p) \cdot \ell_{\pm\tilde{x}}$ and

$$\gamma_{\ell_{\pm\tilde{x}'}} \equiv t(p) \gamma_{\ell_{\pm\tilde{x}}} t(p)^{-1} \equiv t(a) \gamma_{\ell_{\pm\tilde{x}}} t(a)^{-1} \bmod N, \ r_{\pm\tilde{x}'} w - p r_{\pm\tilde{x}} w \in \mathbb{Z}$$

when $c_{\ell_{\tilde{x}}}(w)$ or $c_{\ell_{-\tilde{x}}}(w)$ is nonzero. By equation (2.29), $\Gamma(N) \subset \ker(\rho_L)$ and the fact that f has rational Fourier coefficients, we then have

$$\sigma_a(f \mid_{-2r} \gamma_{\ell_{\pm\tilde{x}}}) = \rho_L(t(a) \gamma_{\ell_{\pm\tilde{x}}}) f = \rho_L(t(a) \gamma_{\ell_{\pm\tilde{x}}} t(a)^{-1}) f = \rho_L(\gamma_{\ell_{\pm\tilde{x}'}}) f = f \mid_{-2r} \gamma_{\ell_{\pm\tilde{x}'}}.$$

[4]The regularization is the same as in [BF04] or [ANS18].

[5]Up to a sign $(-1)^{r+1}$ depending on the orientation of $V_{00}(\mathbb{R})$.

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These imply that

$$\sigma_a(c_{\ell_{\pm\tilde{x}}}(w,\mu)) = c_{\ell_{\pm\tilde{x}'}}(w,\mu),\ \sigma_a(\mathbf{e}(r_{\tilde{x}}w)) = \mathbf{e}(pr_{\tilde{x}}w) = \mathbf{e}(r_{\tilde{x}'}w) \tag{4.30}$$

for all $d > 0$, $\tilde{x} \in \tilde{S}_d(\varphi_f)$ and $w \in \mathbb{Q}_{<0}$. Finally, we have $\varphi_f(\tilde{x}) = \varphi_f(t(p)^{-1}\tilde{x}')$, which implies

$$\varphi_f(\tilde{x})) = \varphi_f(t(a)^{-1}\tilde{x}') = \omega(\iota(t(a)))\varphi_f)(\tilde{x}') \tag{4.31}$$

since $\varphi_f \in \mathcal{S}(L)$ and p is co-prime to the level of L. Here, ι is the map defined in (3.10). The map $\tilde{x} \mapsto \tilde{x}'$ then gives a bijection between $S_d(\varphi_f)$ and $S_d(\omega(\iota(t(a)))\varphi_f)$. From this, we then obtain

$$\sigma_a(I^M(\tau, f_\mu, \varphi_f)) = I^M(\tau, f_\mu, \omega(\iota(t(a)))\sigma_a(\varphi_f)) = I^M(\tau, f_\mu, \omega(t(a), \iota(t(a)))\varphi_f). \tag{4.32}$$

As $(t(a), \iota(t(a))) \in T^\Delta(\hat{\mathbb{Z}})$, and φ_f is $T^\Delta(\hat{\mathbb{Z}})$-invariant, the modular form $I^M(\tau, f_\mu, \varphi_f)$ has rational Fourier coefficients. □

From Propositions 2.8 and 4.9, we can deduce the following result.

Proposition 4.11. *Let $r \in \mathbb{N}_0$ and $f \in M^!_{-2r,\rho_L}$ as in Proposition 4.9. For all $\mu \in L^\vee/L$ and congruence subgroup $\Gamma(N) \subset \ker(\rho_L)$ fixing $\varphi_\mu \in \mathcal{S}(\hat{V};\mathbb{Q}^{\mathrm{ab}})^{(G\cdot T^\Delta)(\hat{\mathbb{Z}})}$, which is a matching section of ϕ_μ as in Theorem 3.3, the regularized integral*

$$c_\mu(f) := \sqrt{D}^{-r}\frac{\pi}{3}\frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma(N)]}\int^{\mathrm{reg}}_{\Gamma(N)\backslash\mathbb{H}} v^r f_\mu(\tau)\widetilde{\mathrm{RC}}_r\mathcal{I}_f(g^\Delta_\tau, \varrho, \varphi^{(1,-1)}_\mu - (-1)^r\varphi^{(-1,1)}_\mu)d\mu(\tau) \tag{4.33}$$

is a rational number.

Proof. Since φ_μ is $G(\hat{\mathbb{Z}})$-invariant, we can rewrite the constant $c_\mu := c_\mu(f)$ as

$$\sqrt{D}^r c_\mu = \int^{\mathrm{reg}}_{\Gamma_L\backslash\mathbb{H}} v^r f_\mu(\tau)\lim_{T'\to\infty}\int_{\mathcal{F}_{T'}}\int_{H_1(\mathbb{Q})\backslash H_1(\hat{\mathbb{Q}})}\theta(g_{\tau'}, (g^\Delta, h_1), \varphi^r_\mu)\varrho(h_1)dh_1 d\mu(\tau')d\mu(\tau)$$

with $\varphi^r_\mu := \widetilde{\mathrm{RC}}_r(\varphi^{(1,-1)}_\mu - (-1)^r\varphi^{(-1,1)}_\mu) \in \mathcal{S}(V(\mathbb{A}))$. Using Lemma 4.4, we can switch the regularized integral in g with the limit in T'. Then by the rational decomposition $V = V_{00} \oplus U_D \oplus V_1$, we can write

$$\varphi_\mu = \sum_{j\in J}\varphi_{00,\mu,j}\otimes\varphi_{D,\mu,j}\otimes\varphi_{1,\mu,j},$$

with $\varphi_{00,\mu,j} \in \mathcal{S}(\hat{V}_{00};\mathbb{Q}^{\mathrm{ab}})^{T^\Delta(\hat{\mathbb{Z}})}$, $\varphi_{1,\mu,j} \in \mathcal{S}(\hat{V}_1)$ and $\varphi_{D,\mu,j} \in \mathcal{S}(\hat{U}_D)$. The constant c_μ can then be rewritten as

$$c_\mu = \sum_{j\in J} c_{\mu,j},$$

where $c_{\mu,j}$ is defined by

$$c_{\mu,j} := \sqrt{D}^{-r}\mathrm{vol}(K_\varrho)\lim_{T'\to\infty}\int_{\mathcal{F}_{T'}}\Theta_1(g_{\tau'}, \varphi^-_{1,\mu,j}, \varrho)$$
$$\times\int^{\mathrm{reg}}_{\Gamma(L)\backslash\mathbb{H}} v^r f_\mu(\tau)\theta_0(g_{\tau'}, g^\Delta, \widetilde{\mathrm{RC}}_r(\varphi^{(1,-1)}_{0,\mu,j} - (-1)^r\varphi^{(-1,1)}_{0,\mu,j}))d\mu(\tau)d\mu(\tau').$$

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Now with Lemma 4.3, we obtain

$$\frac{c_{\mu,j}}{2^{2r_0-r+1}} = \mathrm{vol}(K_\varrho) \lim_{T'\to\infty} \int_{\mathcal{F}_{T'}} (v')^{-1/2}\Theta_1(g_{\tau'}, \varphi^-_{1,\mu,j}, \varrho)G_{\mu,j}(\tau')d\mu(\tau'),$$

where $G_{\mu,j}$ is a weakly holomorphic modular form of weight -1 defined by

$$G_{\mu,j}(\tau') := \sqrt{v'}\sqrt{D}^{-3r}\mathrm{RC}'^{,\Delta}_{r_0,(-r+1/2,r-2r_0+1/2)}\Big(\theta_D(g_{\tau'_2}, \varphi^{r-2r_0}_{D,\mu,j})I^M(\tau'_1, f_\mu, \varphi_{00,\mu,j})\Big)\,|_{\tau'_1=\tau'_2=\tau'}$$

and has rational Fourier coefficient at the cusp ∞ by Proposition 4.11. As φ_μ is $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariant, the function $(v')^{-1/2}\sum_{j\in J}\Theta_1(g_{\tau'}, \varphi^-_{1,\mu,j}, \varrho)G_{\mu,j}(\tau')$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant in τ'. Applying Proposition 2.8 and Stokes' Theorem, we then have

$$\begin{aligned}\frac{c_\mu}{2^{2r_0-r+1}} &= \mathrm{vol}(K_\varrho) \lim_{T'\to\infty} \int_{\mathcal{F}_{T'}} \sum_{j\in J} L_{\tau'}\Big(\sqrt{v'}\tilde{\Theta}_{1,C}(g_{\tau'}, \varphi^-_{1,\mu,j}, \varrho)\Big)G_{\mu,j}(\tau', j)d\mu(\tau')\\ &= -\mathrm{vol}(K_\varrho)\sum_{j\in J}\mathrm{CT}\Big(\sqrt{v'}\tilde{\Theta}^+_{1,C}(g_{\tau'}, \varphi^-_{1,\mu,j}, \varrho)G_{\mu,j}(\tau', j)\Big) \in \mathbb{Q}.\end{aligned}$$

This finishes the proof. □

5. Proofs of theorems

In this section, we will prove Theorem 1.2. First, we state and prove the case for $\mathrm{O}(2,2)$.

Theorem 5.1. *Let E be a biquadratic CM number field with real quadratic subfield F. Let $W = W_\alpha$ be an F-quadratic space and $W_{\alpha^\vee}$ its neighborhood quadratic space as in Section 2.4. Suppose $\alpha_1 < 0 < \alpha_2$. For $r \in \mathbb{N}_0$ and a lattice $L \subset W_{\mathbb{Q}}$, suppose*

$$f = \sum_{m\in\mathbb{Q},\ \mu\in L^\vee/L} c(m,\mu)q^m\mathfrak{e}_\mu \in M^!_{-2r,\rho_L}$$

is a weakly holomorphic modular form with rational Fourier coefficients. Furthermore, suppose it has vanishing constant term when $r = 0$. Then there exists $\kappa, M \in \mathbb{N}$ depending on f such that

$$\begin{aligned}&\kappa\Big(\Phi^r_f(Z(W_\alpha)) - (-1)^r\Phi^r_f(Z(W_{\alpha^\vee}))\Big)\\ &= -\frac{\deg(Z(W))}{\Lambda(0,\chi)}\sum_{m>0,\ \mu\in L^\vee/L} c(-m,\mu)m^r \sum_{\substack{\lambda\in F^\times\cap M^{-1}\mathcal{O}\\ \lambda\gg 0\\ \mathrm{Tr}(\lambda)=m}} P_r\Big(\frac{\lambda-\lambda'}{m}\Big)\log\Big|\frac{\beta(\lambda,\phi_\mu)}{\beta(\lambda,\phi_\mu)'}\Big|,\end{aligned} \tag{5.1}$$

where $\beta(t,\phi_f) \in F^\times$ is nonzero which has the property

$$\kappa^{-1}\,\mathrm{ord}_{\mathfrak{p}}(\beta(t,\phi_f)/\beta(t,\phi_f)') = \begin{cases}\tilde{W}_t(\phi_f), & \text{if } \mathrm{Diff}(W,t) = \{\mathfrak{p}\},\\ -\tilde{W}_t(\phi_f), & \text{if } \mathrm{Diff}(W,t) = \{\mathfrak{p}'\},\\ 0, & \text{otherwise}.\end{cases} \tag{5.2}$$

Proof. By the Siegel-Weil formula in (2.56), we have

$$\Phi^r_f(Z(W)) = \frac{C'}{(-4\pi)^r}\cdot\int^{\mathrm{reg}}_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} v^r\sum_{\mu\in L^\vee/L} f_\mu(\tau)R^r_\tau E^*(g^\Delta_\tau, \phi_\mu^{(\mathrm{sgn}(\alpha),-\mathrm{sgn}(\alpha))})d\mu(\tau)$$

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with $C' := \deg(Z(W))/(2\Lambda(0,\chi)) \in \mathbb{Q}^\times$. For each $\mu \in L^\vee/L$, let $\varphi_\mu \in \mathcal{S}(\hat{V})^{(G\cdot T^\Delta)(\hat{\mathbb{Z}})}$ be a matching section of ϕ_μ as in Theorem 3.3. Then we can apply Proposition 4.2 to obtain

$$\begin{aligned}
&\Phi_f^r(Z(W_{\alpha^\vee})) - (-1)^r\Phi_f^r(Z(W_\alpha)) \\
&\quad = \frac{C'}{(-4\pi)^r}\cdot\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}}\sum_{\mu\in L^\vee/L} f_\mu(\tau)R_\tau^r\Big(E^*(g_\tau^\Delta,\phi_\mu^{(1,-1)}) - (-1)^r E^*(g_\tau^\Delta,\phi_\mu^{(-1,1)})\Big)d\mu(\tau) \\
&\quad = -C'\cdot\frac{\pi}{3}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}}\sum_{\mu\in L^\vee/L} f_\mu(\tau)L_\tau(\mathrm{RC}_r\tilde{\mathcal{I}})(g_\tau^\Delta,\tilde{\varrho}_C,\varphi_\mu^{(1,1)})d\mu(\tau) \\
&\qquad + \frac{C'2\log\varepsilon_\varrho}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma(N)]}\sum_{\mu\in L^\vee/L}\frac{\pi}{3}\int_{\Gamma(N)\backslash\mathbb{H}}^{\mathrm{reg}} f_\mu(\tau)\widetilde{\mathrm{RC}}_r\mathcal{I}_f(g_\tau^\Delta,\varrho,\varphi_\mu^{(1,-1)} - (-1)^r\varphi_\mu^{(-1,1)})d\mu(\tau) \\
&\quad = -C'\cdot\frac{\pi}{3}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}}\sum_{\mu\in L^\vee/L} f_\mu(\tau)L_\tau(\mathrm{RC}_r\tilde{\mathcal{I}})(g_\tau^\Delta,\tilde{\varrho}_C,\varphi_\mu^{(1,1)})d\mu(\tau) + 2C'\sqrt{D}^r\log\varepsilon_\varrho\sum_{\mu\in L^\vee/L} c_\mu(f).
\end{aligned}$$

By Proposition 4.11, we know that $c_\mu(f) \in \mathbb{Q}$ for all $\mu \in L^\vee/L$. For the other term, we can apply Stokes' theorem to obtain

$$\begin{aligned}
&C'\cdot\frac{\pi}{3}\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{\mu\in L^\vee/L} f_\mu(\tau)L_\tau(\mathrm{RC}_r\tilde{\mathcal{I}})(g_\tau^\Delta,\tilde{\varrho}_C,\varphi_\mu^{(1,1)})d\mu(\tau) \\
&\quad = C'\cdot\frac{\pi}{3}\lim_{v\to\infty}\int_0^1\sum_{\mu\in L^\vee/L} f_\mu(\tau)(\mathrm{RC}_r\tilde{\mathcal{I}})(g_\tau^\Delta,\tilde{\varrho}_C,\varphi_\mu^{(1,1)})du \\
&\quad = C'\sum_{m>0,\ \mu\in L^\vee/L} c(-m,\mu)\sum_{\lambda\in F^\times,\ \lambda\gg 0,\ \mathrm{Tr}(\lambda)=m} m^r P_r\left(\frac{\lambda-\lambda'}{m}\right)\tilde{c}_\lambda(\varphi_\mu,\varrho).
\end{aligned}$$

For the last step, we have applied Remark 4.8 to replace $\tilde{\mathcal{I}}$ with $\mathcal{I}^0 + \mathcal{I}^+$, used Lemma 4.5 to see that $\mathcal{I}^0$ contribute nothing, and substitute in the Fourier coefficients of $\mathrm{RC}_r\mathcal{I}^+$ in terms of $\tilde{c}_\lambda(\varphi_\mu,\varrho)$, the λ-th Fourier coefficient of $\tilde{\mathcal{I}}(g_\tau^\Delta,\varphi_\mu^{(1,1)},\tilde{\varrho}_C)$. Note that $m^r P_r((\lambda-\lambda')/m)$ appears by (2.9) and is a rational multiple of $\sqrt{D}^r$. As the sum above is finite, we can choose C such that S_C^c contains $\mathrm{Diff}(W,t)$ for all the t that appears in this sum. Finally, the knowledge about the factorization of these coefficients in Proposition 4.7 finishes the proof. □

Corollary 5.2. *In the setting of Theorem 5.1, suppose that Z_f does not intersect with $Z(W)$ when $r = 0$. Then there exists $\kappa \in \mathbb{N}$ and $\gamma(\lambda,\phi_\mu) \in F^\times$ such that*

$$\kappa\Phi_f^r(Z(W)) = -2\frac{\deg(Z(W))}{\Lambda(0,\chi)}\sum_{m>0,\ \mu\in L^\vee/L} c(-m,\mu)m^r\sum_{\substack{\lambda\in F^\times\cap M^{-1}\mathcal{O}\\ \lambda\gg 0\\ \mathrm{Tr}(\lambda)=m}} P_r\left(\frac{\lambda-\lambda'}{m}\right)\log|\gamma(\lambda,\phi_\mu)| \tag{5.3}$$

and

$$\kappa^{-1}\mathrm{ord}_{\mathfrak{p}}(\gamma(t,\phi_f)) = \begin{cases}\tilde{W}_t(\phi_f), & \textit{if } \mathrm{Diff}(W,t) = \{\mathfrak{p}\},\\ 0, & \textit{otherwise}.\end{cases} \tag{5.4}$$

Remark 5.3. The constant $\frac{\deg(Z(W))}{\Lambda(0,\chi)}$ can be explicitly given when $X_K = X_0(1)^2$ (see [Li21, Remark 3.6]).

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Proof. By Theorem 5.10 in [BEY21], we have

$$\begin{aligned}&(\Phi^r_f(Z(W_\alpha)) + (-1)^r\Phi^r_f(Z(W_{\alpha^\vee})))\\&= \frac{\deg(Z(W))}{\Lambda(0,\chi)}\sum_{m>0,\ \mu\in L^\vee/L} c(-m,\mu)m^r \sum_{\substack{\lambda\in F^\times\\ \lambda\gg 0\\ \mathrm{Tr}(\lambda)=m}} P_r\left(\frac{\lambda-\lambda'}{m}\right)a_\lambda(\phi_\mu),\end{aligned}$$

where $a_t(\phi_\mu)$ is t-th the Fourier coefficient of the holomorphic part of the incoherent Eisenstein series, and given in (2.61). Adding this to (5.1) and applying (5.2) finishes the proof. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Write $\mathrm{V} = \mathrm{V}_\circ \oplus W_{\mathbb{Q}}$ and $L_W := L \cap W_{\mathbb{Q}}$, $L_\circ := L \cap \mathrm{V}_\circ$. Then $L_\circ \oplus L_W \subset L$ is a full sublattice, and we can write

$$\langle f(\tau), \overline{\Theta_L(\tau, Z(W))}\rangle_L = \langle \tilde{f}(\tau,\tau), \overline{\Theta_{L_W}(\tau, Z(W))}\rangle_{L_W}$$

with $\tilde{f}(\tau_1,\tau_2) := \langle \mathrm{Tr}^L_{L_\circ\oplus L_W}(f(\tau_1)), \overline{\Theta_{L_\circ}(\tau_2)}\rangle_{L_\circ}$. Using (2.10), we have

$$\begin{aligned}(4\pi)^{-r}\langle R^r_\tau f(\tau), \overline{\Theta_L(\tau, Z(W))}\rangle_L &= (4\pi)^{-r}(R^r_{\tau_1}\langle f(\tau_1), \overline{\Theta_L(\tau, Z(W))}\rangle_L \mid_{\tau_1=\tau}\\ &= \langle (4\pi)^{-r}(R^r_{\tau_1}(\tilde{f})^\Delta, \overline{\Theta_{L_W}(\tau, Z(W))}\rangle_{L_W}\\ &= \sum_{\ell=0}^{r} c_\ell^{(r;k_1,k_2)}(4\pi)^{-r+\ell}\langle R^{r-\ell}_\tau \tilde{f}_\ell, \overline{\Theta_{L_W}(\tau, Z(W))}\rangle_{L_W},\end{aligned}$$

where $k_1 = -2r + 1 - \frac{n}{2}$, $k_2 = \frac{n}{2} - 1$ and

$$\tilde{f}_\ell := \mathrm{RC}_{\ell,(k_1,k_2)}(\tilde{f})^\Delta \in M^!_{-2r+2\ell, L_W}$$

has rational Fourier coefficients. Therefore, we have

$$\Phi^r_f(Z(W)) = \sum_{\ell=0}^{r} c_\ell^{(r;k_1,k_2)}\Phi^{r-\ell}_{\tilde{f}_\ell}(Z(W)). \tag{5.5}$$

If f has the Fourier expansion

$$f(\tau) = \sum_{\nu\in L^\vee/L,\ n\in\mathbb{Z}+Q(\nu)} c(n,\nu)q^n\mathfrak{e}_\nu,$$

then the (m,μ)-th Fourier coefficient of $\tilde{f}_\ell$, denoted by $c_\ell(m,\mu)$, can be expressed as

$$c_\ell(m,\mu) = \sum_{\lambda_\circ\in L_\circ^\vee} Q_{\ell,(k_1,k_2)}(m - Q(\lambda_\circ), Q(\lambda_\circ))c(m - Q(\lambda_\circ), (\lambda_\circ,\mu)),$$

with $Q_{\ell,(k_1,k_2)}(X,Y) \in \mathbb{Q}[X,Y]$ defined in (2.4). In particular, when $Z(W) \cap Z_f = \emptyset$, we have $c_r(0,\mu) = 0$ for all $0 \le \ell \le r$ and $\mu \in L^\vee/L$ as $c(-Q(\lambda_\circ), (\lambda_\circ,\mu)) = 0$ for all $\lambda_\circ \in L_\circ^\vee$ and $\mu \in L_W^\vee/L_W$ by (2.43).

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By Corollary 5.2, we can write

$$\kappa\Phi_f^r(Z(W)) = -2\frac{\deg(Z(W))}{\Lambda(0,\chi)}\sum_{\ell=0}^{r} c_\ell^{(r;k_1,k_2)} \sum_{m>0,\ \mu\in L_W^\vee/L_W} c_\ell(-m,\mu)m^{r-\ell}$$
$$\times \sum_{\substack{\lambda\in F^\times\cap M^{-1}\mathcal{O}\\ \lambda\gg 0\\ \mathrm{Tr}(\lambda)=m}} P_{r-\ell}\left(\frac{\lambda-\lambda'}{m}\right)\log|\gamma_\ell(\lambda,\phi_\mu)|$$

with $\gamma_\ell(\lambda,\phi_\mu)\in F^\times$ having factorization as in (5.4) independent of ℓ, and express $\Phi_f^r(Z(W))$ as in (1.6) such that

$$\kappa^{-1}\operatorname{ord}_{\mathfrak{p}}(a_j) = -2\frac{\deg(Z(W))}{\Lambda(1,\chi)} \sum_{\substack{m>0\\ \mu\in L_W^\vee/L_W\\ \lambda_\circ\in L_\circ^\vee}} c(-m-Q(\lambda_\circ),(\lambda_\circ,\mu))$$
$$\times \sum_{\substack{0\le\ell\le r\\ r-\ell\equiv j \bmod 2}} c_\ell^{(r;k_1,k_2)} Q_{\ell,(k_1,k_2)}(-m-Q(\lambda_\circ),Q(\lambda_\circ))$$
$$\times \sum_{\substack{\lambda\in F^\times\cap M^{-1}\mathcal{O}\\ \lambda\gg 0\\ \mathrm{Tr}(\lambda)=m\\ \mathrm{Diff}(W,\lambda)=\{\mathfrak{p}\}}} \frac{1}{\sqrt{D}^{\,j \bmod 2}} P_{r-\ell}\left(\frac{\lambda-\lambda'}{m}\right)\tilde{W}_\lambda(\phi_\mu) \tag{5.6}$$

for all prime $\mathfrak{p}$ of F. □

Finally, we prove Theorem 1.5.

Proof of Theorem 1.5. By the main result in [Li23], we have $\tilde{\kappa}\in\mathbb{N}$ and Galois equivariant maps $\tilde{\alpha}_j : T_W(\hat{\mathbb{Q}}) \to E^{\mathrm{ab}}$ satisfying

$$\Phi_f^r([z_0,h]) - \Phi_f^r([z_0,h']) = \frac{1}{\tilde{\kappa}}\left(\log\left|\frac{\tilde{\alpha}_1(h)}{\tilde{\alpha}_1(h')}\right| + \sqrt{D}\log\left|\frac{\tilde{\alpha}_2(h)}{\tilde{\alpha}_2(h')}\right|\right) \tag{5.7}$$

for all $h,h'\in T_W(\hat{\mathbb{Q}})$. Furthermore, when $n=2$, we have $\tilde{\alpha}_j = 1$ for $j\equiv r \bmod 2$. Setting $\alpha_j(h) := a_j\prod_{[z_0,h']\in Z(W)\setminus[z_0,h]}\frac{\tilde{\alpha}_j(h)}{\tilde{\alpha}_j(h')}$ and $\kappa := \tilde{\kappa}|Z(W)|$ and applying equations (5.7) and (1.6) proves the first two claims. Combining with Corollary 2.4, we see that Conjecture 1.1 holds. □

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References

[AGHMP18] F. Andreatta, E. Z. Goren, B. Howard and K. M. Pera, 'Faltings heights of abelian varieties with complex multiplication', *Ann. of Math. (2)* **187**(2) (2018), 391–531.

[ANS18] C. Alfes-Neumann and M. Schwagenscheidt, 'On a theta lift related to the Shintani lift', *Adv. Math.* **328** (2018) 858–889.

[AS64] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (National Bureau of Standards Applied Mathematics Series) vol. 55 (For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, DC, 1964).

[Bei87] A. A. Beilinson, 'Height pairing between algebraic cycles', in *K-theory, Arithmetic and Geometry (Moscow, 1984–1986)* vol. 1289 (Lecture Notes in Math.) (Springer, Berlin, 1987), 1–25.

[BEY21] J. H. Bruinier, S. Ehlen and T. Yang, 'CM values of higher automorphic Green functions for orthogonal groups', *Invent. Math.* **225**(3) (2021), 693–785.

[BF04] J. H. Bruinier and J. Funke, 'On two geometric theta lifts', *Duke Math. J.* **125**(1) (2004), 45–90.

[BHK+20] J. H. Bruinier, B. Howard, S. S. Kudla, M. Rapoport and T. Yang, 'Modularity of generating series of divisors on unitary Shimura varieties', *Astérisque* **421** (2020), 7–125.

[BKY12] J. H. Bruinier, S. S. Kudla and T. Yang, 'Special values of Green functions at big CM points', *Int. Math. Res. Not. IMRN* **9** (2012), 1917–1967.

[Blo84] S. Bloch, 'Height pairings for algebraic cycles', in *Proceedings of the Luminy Conference on Algebraic K-theory (Luminy, 1983)* vol. 34 (1984), 119–145.

[Bor98] R. E. Borcherds, 'Automorphic forms with singularities on Grassmannians', *Invent. Math.* **132**(3) (1998), 491–562.

[Bor99] R. E. Borcherds. 'The Gross-Kohnen-Zagier theorem in higher dimensions', *Duke Math. J.* **97**(2) (1999), 219–233.

[Bru02] J. H. Bruinier, *Borcherds Products on O(2, l) and Chern Classes of Heegner Divisors* (Lecture Notes in Mathematics) vol. 1780 (Springer-Verlag, Berlin, 2002).

[BvdGHZ08] J. H. Bruinier, G. van der Geer, G. Harder and D. Zagier, *The 1-2-3 of Modular Forms* (Universitext. Springer-Verlag, Berlin, 2008) Lectures from the Summer School on Modular Forms and Their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[BY06] J. H. Bruinier and T. Yang, 'CM-values of Hilbert modular functions', *Invent. Math.* **163**(2) (2006), 229–288.

[BY07] J. H. Bruinier and T. Yang, 'Twisted Borcherds products on Hilbert modular surfaces and their CM values', *Amer. J. Math.* **129**(3) (2007), 807–841.

[BY09] J. H. Bruinier and T. Yang, 'Faltings heights of CM cycles and derivatives of L-functions', *Invent. Math.* **177**(3) (2009), 631–681.

[BY11] J. H. Bruinier and T. Yang, 'CM values of automorphic Green functions on orthogonal groups over totally real fields', in *Arithmetic Geometry and Automorphic Forms* (Adv. Lect. Math. (ALM)) vol. 19 (Int. Press, Somerville, MA, 2011), 1–54.

[CL20] P. Charollois and Y. Li, 'Harmonic Maass forms associated to real quadratic fields', *J. Eur. Math. Soc. (JEMS)* **22**(4) (2020), 1115–1148.

[DN70] K. Doi and H. Naganuma, 'On the functional equation of certain Dirichlet series', *Invent. Math.* **9** (1969/70), 1–14.

[FM06] J. Funke and J. Millson, 'Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms', *Amer. J. Math.* **128**(4) (2006), 899–948.

[GKZ87] B. Gross, W. Kohnen and D. Zagier, 'Heegner points and derivatives of L-series. II', *Math. Ann.* **278**(1–4) (1987), 497–562.

[Gou72] H. W. Gould, *Combinatorial Identities* (Henry W. Gould, Morgantown, WV, 1972). Astandardized set of tables listing 500 binomial coefficient summations.

[GPSR87] S. Gelbart, I. Piatetski-Shapiro and S. Rallis, *Explicit Constructions of Automorphic L-functions* (Lecture Notes in Mathematics) vol. 1254 (Springer-Verlag, Berlin, 1987).

[GQT14] W. T. Gan, Y. Qiu and S. Takeda, 'The regularized Siegel-Weil formula (the second term identity) and the Rallis inner product formula', *Invent. Math.* **198**(3) (2014), 739–831.

[GZ85] B. H. Gross and D. B. Zagier, 'On singular moduli', *J. Reine Angew. Math.* **355** (1985), 191–220.

[GZ86] B. H. Gross and D. B. Zagier, 'Heegner points and derivatives of L-series', *Invent. Math.* **84**(2) (1986), 225–320.

[Hec27] E. Hecke, 'Zur Theorie der elliptischen Modulfunktionen', *Math. Ann.* **97**(1) (1927), 210–242.

[HP17] P. Habegger and F. Pazuki, 'Bad reduction of genus 2 curves with CM jacobian varieties', *Compos. Math.* **153**(12) (2017), 2534–2576.

[HY11] B. Howard and T. Yang, 'Singular moduli refined', in *Arithmetic Geometry and Automorphic Forms* (Adv. Lect. Math. (ALM)) vol. 19 (Int. Press, Somerville, MA, 2011), 367–406.

[HY12] B. Howard and T. Yang, *Intersections of Hirzebruch-Zagier Divisors and CM Cycles* (Lecture Notes in Mathematics) vol. 2041 (Springer, Heidelberg, 2012).

[JL70] H. Jacquet and R. P. Langlands, *Automorphic Forms on GL(2)* (Lecture Notes in Mathematics) vol. 114 (Springer-Verlag, Berlin-New York, 1970).

[KM90] S. S. Kudla and J. J. Millson, 'Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables', *Inst. Hautes Études Sci. Publ. Math.* **71** (1990), 121–172.

[KR92] S. S. Kudla and S. Rallis, 'Ramified degenerate principal series representations for $\mathrm{Sp}(n)$', *Israel J. Math.* **78**(2–3) (1992), 209–256.

[KR94] S. S. Kudla and S. Rallis, 'A regularized Siegel-Weil formula: the first term identity', *Ann. of Math. (2)* **140**(1) (1994), 1–80.

[Kud78] S. S. Kudla, 'Theta-functions and Hilbert modular forms', *Nagoya Math. J.* **69** (1978), 97–106.

[Kud94] S. S. Kudla, 'Splitting metaplectic covers of dual reductive pairs', *Israel J. Math.* **87**(1–3) (1994), 361–401.

[Kud97] S. S. Kudla, 'Central derivatives of Eisenstein series and height pairings', *Ann. of Math. (2)* **146**(3) (1997), 545–646.

[Kud03] S. S. Kudla, 'Integrals of Borcherds forms', *Compos. Math.* **137**(3) (2003), 293–349.

[Kud16] S. S. Kudla, 'Another product for a Borcherds form', in *Advances in the Theory of Automorphic Forms and Their L-functions* (Contemp. Math.) vol. 664 (Amer. Math. Soc., Providence, RI, 2016), 261–294.

[Li16] Y. Li, 'Real-dihedral harmonic Maass forms and CM-values of Hilbert modular functions', *Compos. Math.* **152**(6) (2016), 1159–1197.

[Li21] Y. Li, 'Singular units and isogenies between CM elliptic curves', *Compos. Math.* **157**(5) (2021), 1022–1035.

[Li22] Y. Li, 'Average CM-values of higher green's function and factorization', *Amer J. Math.* **144**(5) (2022), 1241–1298.

[Li23] Y. Li, 'Algebraicity of higher Green functions at a CM point', *Invent. Math.* **234**(1) (2023), 375–418.

[LS22] Y. Li and M. Schwagenscheidt, 'Mock modular forms with integral Fourier coefficients', *Adv. Math.* **399** (2022), Paper No. 108264, 30 pp.

[McG03] W. J. McGraw, 'The rationality of vector valued modular forms associated with the Weil representation', *Math. Ann.* **326**(1) (2003), 105–122.

[Mel08] A. Mellit, 'Higher Green's functions for modular forms', Preprint, 2008, arXiv:0804.3184.

[Mœg97] C. Mœglin, 'Non nullité de certains relêvements par séries théta', *J. Lie Theory* **7**(2) (1997), 201–229.

[Nik79] V. V. Nikulin, 'Integer symmetric bilinear forms and some of their geometric applications', *Izv. Akad. Nauk SSSR Ser. Mat.* **43**(1) (1979), 111–177, 238.

[Ral84] S. Rallis, 'On the Howe duality conjecture', *Compos. Math.* **51**(3) (1984), 333–399.

[Sch09] N. R. Scheithauer, 'The Weil representation of $SL_2(\mathbb{Z})$ and some applications', *Int. Math. Res. Not. IMRN* **8** (2009), 1488–1545.

[Via11] M. Viazovska, 'CM values of higher Green's functions', Preprint, 2011. arXiv:1110.4654.

[Xue10] H. Xue, 'Gross-Kohnen-Zagier theorem for higher weight forms', *Math. Res. Lett.* **17**(3) (2010), 573–586.

[YY19] T. Yang and H. Yin, 'Difference of modular functions and their CM value factorization', *Trans. Amer. Math. Soc.* **371**(5) (2019), 3451–3482.

[Zha97] S. Zhang, 'Heights of Heegner cycles and derivatives of L-series', *Invent. Math.* **130**(1) (1997), 99–152.