

ERGODIC PROPERTIES OF SEMI-HYPERBOLIC FUNCTIONS WITH POLYNOMIAL SCHWARZIAN DERIVATIVE

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Abstract The ergodic theory and geometry of the Julia set of meromorphic functions on the complex plane with polynomial Schwarzian derivative are investigated under the condition that the function is semi-hyperbolic, i.e. the asymptotic values of the Fatou set are in attracting components and the asymptotic values in the Julia set are boundedly non-recurrent. We first show the existence, uniqueness, conservativity and ergodicity of a conformal measure m with minimal exponent h ; furthermore, we show weak metrical exactness of this measure. Then we prove the existence of a σ -finite invariant measure μ absolutely continuous with respect to m . Our main result states that μ is finite if and only if the order ρ of the function f satisfies the condition $h > 3\rho/(\rho + 1)$. When finite, this measure is shown to be metricaly exact. We also establish a version of Bowen's Formula, showing that the exponent h equals the Hausdorff dimension of the Julia set of f .

Keywords: holomorphic dynamics; Hausdorff dimension; meromorphic functions

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1. Introduction

The study of the ergodic theory and geometry of the Julia set of transcendental meromorphic functions appears to be a delicate task due to the infinite degree of these functions. For example, even the existence of conformal measures, on which the whole theory relies and which is by now completely standard in the realm of rational functions or Kleinian groups, is not known in general. By employing Nevanlinna's theory and a convenient change of the Riemannian metric, we provided a complete treatise for a very general class of hyperbolic meromorphic functions in [19, 20]. In the present paper we relax the hyperbolicity assumption and allow the Julia set to contain singularities. Clearly, one can adopt the arguments developed in the theory of rational iteration to deal with certain types of critical points. A greater challenge is to analyse the contribution of logarithmic singularities and, as we will see, this gives quite surprising results. The class of meromorphic functions with polynomial Schwarzian derivatives fits such a project best, since they do not have critical points but their inverses have finitely many logarithmic singularities.

We therefore restrict our considerations to this class of functions which, in particular, contains the tangent family; definitions and other examples are given in §2.

In the context of ergodic theory and fractal geometry, meromorphic and entire functions with logarithmic singularities have been investigated in [26–28] (see also [16] for a more complete historical outline and list of references) and, more recently, in [14]. In [26, 27] these singularities landed at poles and, in [28], they were escaping to infinity (like the trajectory of zero under the exponential function) extremely fast. In both cases the forward trajectory of images of logarithmic singularities experienced a large expansion neutralizing the contracting effect of singularities themselves. Assuming that a meromorphic map is subhyperbolic, the post-singular set is bounded, the Julia set is an entire sphere and the reference conformal measure is the Lebesgue measure, Kotus and Świątek [14] addressed the role of logarithmic singularities (algebraic singularities were also allowed).

In this paper we consider semi-hyperbolic meromorphic functions $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ with polynomial Schwarzian derivative. By ‘semi-hyperbolic’ we understand that

- (i) all the asymptotic values $a \in \mathcal{A}_f$ are finite,
- (ii) if $a \in \mathcal{F}_f$, then a belongs to an attracting component of \mathcal{F}_f and
- (iii) all the asymptotic values $a \in J_f$ have bounded orbit and are non-recurrent (cf. Definition 2.2).

Employing the full power of Nevanlinna theory, we first prove the existence of an atomless conformal measure via the Patterson–Sullivan construction. This measure is proved to be weakly metrically exact, which implies its ergodicity and conservativity. We then show the following result, in which the existence of the σ -finite measure μ is obtained by employing Martens’s general method.

Theorem 1.1. *Let f be a semi-hyperbolic meromorphic function f of polynomial Schwarzian derivative and let m be the h -conformal measure of f obtained via the Patterson–Sullivan construction. There then exists a σ -finite invariant measure μ absolutely continuous with respect to m . Moreover, the measure*

$$\mu \text{ is finite} \iff h > \frac{3\rho}{\rho + 1}$$

where $\rho = \rho(f)$ is the order of the function f . If μ is finite, then the dynamical systems (f, μ) it generates is metrically exact and, in consequence, its Rokhlin’s natural extension is K -mixing.

Notice that $3\rho/(\rho + 1) \geq 2$ if and only if the order $\rho \geq 2$. Consequently, the measure μ is most often infinite. However, in the case of the tangent family, which is just one specific example among others, this invariant measure can be finite. Curiously, finiteness of the invariant measure for the strictly pre-periodic function $z \mapsto 2\pi i e^z$ is not yet known. (Between submission and publication of the present paper, Dobbs and Skorulski [7] and Kotus and Świątek [15] showed independently that the measure is infinite.)

Let us mention that we do not assume that the Julia set is the entire sphere or that the conformal measure is the Lebesgue measure. In fact we do not assume that any conformal measure exists at all. But in the special situation when the Julia set is the entire sphere (in which case the spherical Lebesgue measure is automatically a conformal measure) and if, in addition, $h = 2 > 3\rho/(\rho + 1)$, i.e. if the order of the function $\rho < 2$, then the existence of a probability-invariant measure absolutely continuous with respect to the Lebesgue measure follows also from [14]. Indeed, in that situation our necessary and sufficient condition $\rho < 2$ coincides with the sufficient condition (Z3) from [15]. Concerning the reciprocal statement, [14] simply provides a counterexample.

The most involved part of the proof of Theorem 1.1 is to show finiteness. In the case when the measure μ is finite, the dynamical system it generates is shown to be K -mixing which, in particular, implies mixing of all orders.

We also investigate the Hausdorff dimension of the Julia set and show that this dimension coincides with h , the exponent of the conformal measure m . Notice that this holds despite the h -dimensional Hausdorff measure being shown to vanish on the Julia set.

2. The class of functions and definitions

2.1. Definitions

The reader may consult, for example, [11, 22, 23] for a detailed exposition on meromorphic functions and [2] for their dynamical aspects. We collect here the properties of interest for our concerns. The *Julia set* of a meromorphic function $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ is denoted by J_f and the *Fatou set* is denoted by \mathcal{F}_f . Note that, in contrast to [19, 20], we include here $\infty \in J_f$ since we are dealing with spherical geometry. However, $O^-(\infty) = \bigcup_{n=0}^{\infty} f^{-n}(\{\infty\})$ is a very special subset of the Julia set.

Let \mathcal{A}_f be the set of *asymptotic values* (definitions can be found, for example, in [11, p. 232] or [9, p. 270]). Note that the functions we consider do not have critical values. Therefore, \mathcal{A}_f coincides with the so-called set of *singular values* $\text{sing}(f^{-1})$. The *post-singular set* \mathcal{P}_f is the closure (in the sphere) of the set $\bigcup_{n>0} f^n(\mathcal{A}_f)$.

Concerning the singularities of a meromorphic function f , we make use of Iversen's classification (see, for example, [2]): let $a \in \text{sing}(f^{-1})$ and, for every $r > 0$, let U_r be a component of $f^{-1}(D(a, r))$ in such a way that $r_1 < r_2$ implies $U_{r_1} \subset U_{r_2}$. Then there are two possibilities:

- (a) $\bigcap_{r>0} U_r = \{c\}$ consists of one point or
- (b) $\bigcap_{r>0} U_r = \emptyset$.

In the latter case we say that our choice $r \mapsto U_r$ defines a *transcendental singularity of f^{-1} over a* . Such a singularity is called *logarithmic* if the restriction $f : U_r \rightarrow D(a, r) \setminus \{a\}$ is a universal cover for some $r > 0$. If this is the case, then the component U_r is called a *logarithmic tract*. For the functions we consider all the transcendental singularities to be logarithmic.

In case (a), the point c can be regular or a critical point $c \in \mathcal{C}_f$.

We will always denote by

$$d\sigma(z) = \frac{|dz|}{1 + |z|^2}$$

the spherical metric and by

$$|f'(z)|_\sigma = |f'(z)| \frac{1 + |z|^2}{1 + |f(z)|^2}$$

the derivative of f with respect to the spherical metric. The following direct consequence of Koebe's Distortion Theorem will be used.

Lemma 2.1. *Let $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ be a meromorphic function and suppose that $D(w, 2\delta) \subset \hat{\mathbb{C}} \setminus \mathcal{P}_f$. Then, for every $n \geq 1$, $z \in f^{-n}(w)$ and all $x, y \in D(w, \delta)$ we have that*

$$K^{-1} \leq \frac{|(f_z^{-n})'(y)|_\sigma}{|(f_z^{-n})'(x)|_\sigma} \leq K$$

for some universal constant $K \geq 1$.

Henceforth, f_z^{-n} signifies the inverse branch of f^n defined near $f^n(z)$ mapping back $f^n(z)$ to z . Another convention will be that $D(z, r)$ stands for the open spherical disc centred at z and of radius r . To indicate an open spherical r -neighbourhood of a set X we write $B(X, r)$.

2.2. Nevanlinna functions

We consider meromorphic functions $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ for which the Schwarzian derivative

$$S(f) = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 = 2P \quad (2.1)$$

is a polynomial and for which the set of asymptotic values \mathcal{A}_f does not contain ∞ . Nevanlinna [21] established that meromorphic functions with a polynomial Schwarzian derivative are exactly those functions that have only finitely many asymptotic values and no critical values (thus, these functions are sometimes called *Nevanlinna functions*). Moreover, if a Nevanlinna function has a pole, then it is of order 1. Consequently, the maps of this class are locally injective. We also mention that any solution of (2.1) is of order $\rho = p/2$, where $p = \deg(P) + 2$, and it is of normal type of its order (cf. [12]).

Standard examples are furnished by the tangent family $f(z) = \lambda \tan(z)$ for which $S(f)$ is constant. By Möbius invariance of $S(f)$, functions like

$$\frac{e^z}{\lambda e^z + e^{-z}} \quad \text{and} \quad \frac{\lambda e^z}{e^z - e^{-z}}$$

also have constant Schwarzian derivative. Examples for which $S(f)$ is a polynomial are

$$f(z) = \frac{a \operatorname{ai}(z) + b \operatorname{Bi}(z)}{c \operatorname{Ai}(z) + d \operatorname{Bi}(z)} \quad \text{with } ad - bc \neq 0, \quad (2.2)$$

where Ai and Bi are the Airy functions of the first and second kind. These are linearly independent solutions of $g'' - zg = 0$ and, in general, if g_1, g_2 are linearly independent solutions of

$$g'' + Pg = 0, \tag{2.3}$$

then $f = g_1/g_2$ is a solution of the Schwarzian equation (2.1). Conversely, every solution of (2.1) can be written (locally) as a quotient of two linearly independent solutions of the linear differential equation (2.3). The asymptotic properties of these solutions are well known due to the work of Hille [10] (see also [12]). They give a precise description of the function f near infinity. We now collect the facts that are important for our needs (more details and references are given, for example, in [20]).

First of all, there are p critical directions $\theta_1, \dots, \theta_p$ which are given by

$$\arg u + p\theta = 0 \pmod{2\pi}, \tag{2.4}$$

where u is the leading coefficient of $P(z) = uz^{p-2} + \dots$. In a sector

$$S_j = \left\{ |\arg z - \theta_j| < \frac{2\pi}{p} - \delta; |z| > R \right\},$$

where $R > 0$ is sufficiently large and $\delta > 0$, the equation (2.3) has two linear independent solutions:

$$\left. \begin{aligned} g_1(z) &= P(z)^{-1/4} \exp(iZ + o(1)), \\ g_2(z) &= P(z)^{-1/4} \exp(-iZ + o(1)); \end{aligned} \right\} \tag{2.5}$$

here

$$Z = \int_{2\text{Re}^{i\theta_j}}^z P(t)^{1/2} dt = \frac{2}{p} u^{1/2} z^{1/2} p(1 + o(1)) \quad \text{for } z \rightarrow \infty \text{ in } S_j. \tag{2.6}$$

If f is a meromorphic solution of the Schwarzian equation (2.1), then there are $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ such that

$$f(z) = \frac{ag_1(z) + bg_2(z)}{cg_1(z) + dg_2(z)}, \quad z \in S_j. \tag{2.7}$$

Observe that $f(z) \rightarrow a/c$ if $z \rightarrow \infty$ on any ray in $S_j \cap \{\arg z < \theta_j\}$ and that $f(z) \rightarrow b/d$ if $z \rightarrow \infty$ on any ray in $S_j \cap \{\arg z > \theta_j\}$. The asymptotic values of f are given by all the $a/c, b/d$ corresponding to all the sectors $S_j, j = 1, \dots, p$.

With this precise description of the asymptotic behaviour of f , one can show [20] that

$$|f'(z)| \asymp |z|^{\rho-1} |\alpha + \beta f(z) + \gamma f(z)^2| \quad \text{for } z \in S_j, \tag{2.8}$$

where $\alpha = -ab/\delta, \beta = (ad + bc)/\delta, \gamma = -cd/\delta$ and $\delta = ad - bc$.

Let us finally mention that, in the case where all asymptotic values of f are finite, $\gamma \neq 0$ and that f has (in fact, infinitely many) poles.

2.3. Semi-hyperbolic functions

Definition 2.2. A function $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ which satisfies the following three properties is called semi-hyperbolic.

- (i) $\infty \notin \mathcal{A}_f \cup \mathcal{P}_f$.
- (ii) Every asymptotic value that belongs to the Fatou set is in an attracting component.
- (iii) Asymptotic values that are in the Julia set are non-recurrent: $\mathcal{A}_f \cap \mathcal{P}_f \cap J_f = \emptyset$.

This definition implies in particular that all the asymptotic values of the function f are finite and that the post-singular set is bounded and nowhere dense in the Julia set. The natural name for this class of functions would be *boundedly non-recurrent* functions. However this may lead to confusions especially with the (quite different) non-recurrent functions studied by Rempe and van Strien [25]. However, Rempe and van Strien provide a new Mañé theorem (generalizing the work by Graczyk *et al.* [9]) which gives the expanding property on compact invariant sets. Applied to our setting and to the set \mathcal{P}_f we thus have the following useful fact.

Proposition 2.3. *If f is semi-hyperbolic, then f is expanding on $\mathcal{P}_f \cap J_f$.*

Besides the assumption that the asymptotic values and \mathcal{P}_f stay away from infinity, the conditions on the orbits of the asymptotic values imply that we deal with two types of function.

- (i) Functions having the whole set of asymptotic values in the Julia set: $\mathcal{A}_f \subset J_f$. For those functions we always have $J_f = \hat{\mathbb{C}}$.
- (ii) Some of the asymptotic values are in attracting domains and some are in the Julia set. In this case $\mathcal{F}_f \neq \emptyset$, the spherical Lebesgue measure of J_f is zero and thus, contrary to the preceding case, it cannot be used as a conformal measure.

Examples are standard. They are related to the bifurcation (or instable) part of the moduli space of a function. For the sake of completeness we provide examples of Misiurewicz type in the appendix.

From now on we fix a number $T > 0$ having the following properties:

- (T₁) $4T < |a_1 - a_2|$ for all distinct $a_1, a_2 \in \mathcal{A}_f$;
- (T₂) $B(\mathcal{P}_f, 4T) \cap \mathcal{A}_f \cap J_f = \emptyset$; and
- (T₃) $J_f \cap f^{-1}(\mathcal{A}_f \cup \mathcal{P}_f) \cap (B(\mathcal{A}_f \cup \mathcal{P}_f, 4T) \setminus (\mathcal{A}_f \cup \mathcal{P}_f)) = \emptyset$.

To every asymptotic value $a \in \mathcal{A}_f \cap J_f$ there correspond (finitely many) logarithmic tracts U_a . In the following, such a tract U_a will always be a component of $f^{-1}(D(a, T))$ and we may suppose that $U_a \cap B(\mathcal{A}_f \cup \mathcal{P}_f, 4T) \cap J_f = \emptyset$.

Because of the expanding property, we may also require that $T > 0$ is so small that $|(f^p)'|_\sigma > 2$ on $B(\mathcal{P}_f \cap J_f, T)$ for some $p \geq 1$ and that there are open neighbourhoods

Ω_1, Ω_0 of $\mathcal{P}_f \cap J_f$ such that $\bar{\Omega}_1 \subset \Omega_0 \subset B(\mathcal{P}_f, T)$ and that $g = f^p|_{\Omega_1} : \Omega_1 \rightarrow \Omega_0$ is a proper mapping. Define

$$\Omega_n = g^{-n}(\Omega_0) \quad \text{and} \quad \Gamma_n = \Omega_n \setminus \Omega_{n+1}. \tag{2.9}$$

Using the facts that repelling periodic points are dense in J_f and that J_f contains poles, one can easily prove the following.

Observation 2.4 (topological exactness of f). For every non-empty open set U intersecting J_f , there exists $n \geq 0$ such that $f^n(U) \supset \hat{\mathbb{C}} \setminus \mathcal{A}_f$. In particular, for every $r > 0$ there exists $q_r \geq 0$ such that $f^{q_r}(D(z, r)) \supset \hat{\mathbb{C}} \setminus \mathcal{A}_f$ for all $z \in J_f$.

Since \mathcal{P}_f is a closed forward-invariant set and the map $f|_{\mathcal{P}_f}$ is expanding, following the inverse trajectory of a point near \mathcal{P}_f , one can prove the following.

Observation 2.5 (repeller). The set $\mathcal{P}_f \cap J_f$ is a repeller for f ; precisely, assuming $T > 0$ to be small enough, we have

$$\bigcap_{n=0}^{\infty} f^{-n}(B(\mathcal{P}_f \cap J_f, 2T)) = \mathcal{P}_f.$$

2.4. First observations and the transfer operator

If one chooses the right metric space $(\mathbb{C}, d\sigma)$, then the ergodic theory of meromorphic functions can be well developed. This has been done in great generality and in the hyperbolic case in [19, 20]. For the functions we consider here, the right geometry is simply the spherical one (which results from (2.8); indeed, the functions satisfy the balanced growth condition of [19] with $\alpha_1 = \rho - 1$ and with $\alpha_2 = 2$, the latter meaning that one has to work with the spherical metric).

Lemma 2.6. *Let $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ be of polynomial Schwarzian derivative with $\infty \notin \mathcal{A}_f$. Then, if z belongs to a logarithmic tract $U_{2T} \subset f^{-1}(D(a, 2T))$ over an asymptotic value $a \in \mathcal{A}_f$, we have that*

$$|f'(z)|_{\sigma} \asymp (1 + |z|^{\rho+1})|f(z) - a|$$

and otherwise

$$|f'(z)|_{\sigma} \asymp 1 + |z|^{\rho+1}$$

where $\rho = \rho(f) < \infty$ is the order of f .

Proof. This follows from asymptotic description of f near infinity (in particular (2.8)) together with the fact that f has only simple poles. □

Let us consider the transfer operator with respect to the spherical geometry:

$$\mathcal{L}_t \varphi(w) = \sum_{z \in f^{-1}(w)} |f'(z)|_{\sigma}^{-t} \varphi(z) \quad \varphi \in C(J_f). \tag{2.10}$$

It follows from Lemma 2.6 that

$$\mathcal{L}_t \mathbf{1}(w) \preceq \max\{1, \text{dist}(w, \mathcal{A}_f)^{-t}\} \sum_{z \in f^{-1}(w)} (1 + |z|^{\rho+1})^{-t} \quad (2.11)$$

for every $w \in \hat{\mathbb{C}} \setminus \mathcal{A}_f$. The last sum is very well known in the theory of meromorphic functions, and a theorem of Borel [23] together with the divergence property of f established in [20, Theorem 3.2] implies that

$$\mathcal{L}_t \mathbf{1}(w) < \infty \iff t > \frac{\rho}{\rho + 1}. \quad (2.12)$$

We need the following additional properties.

Proposition 2.7. *For every $t > \rho/(\rho + 1)$ there exists a constant M_t such that*

$$\Sigma(t, w) = \sum_{z \in f^{-1}(w)} (1 + |z|^{\rho+1})^{-t} \leq M_t \quad \text{for every } w \in \hat{\mathbb{C}}.$$

The proof of this result uses parts of [19, 20] and relies heavily on Nevanlinna theory. (Good references for this are [4, 22].) Let us simply recall that $n_f(r, a)$ denotes the number of a -points of modulus at most t , that the integrated counting number $N_f(r, a)$ is defined by $dN_f(r, a) = n_f(r, a)/r$ and that $T_f(r)$ denotes the characteristic function of f .

Proof. Fix $\varepsilon = 1$ and let $A > 0$ be a constant that will be made precise later. We may suppose that the origin is not a pole of f .

Case 1 ($w \notin D(f(0), \varepsilon)$). Then we have that

$$\Sigma(t, w) \preceq C_A + \sum_{\substack{f(z)=w, \\ |z|>A}} (1 + |z|^{\rho+1})^{-t} \preceq C_A + \sum_{\substack{f(z)=w, \\ |z|>A}} |z|^{-u}$$

with $C_A = \sup_{w \in \hat{\mathbb{C}}} n_f(w, A) < \infty$ and with $u = (\rho + 1)t$. Since f is of finite order ρ , we can make the following two integrations by parts:

$$\sum_{\substack{f(z)=w, \\ |z|>A}} |z|^{-u} = -\frac{n_f(A, w)}{A^u} - u \frac{N_f(A, w)}{A^{u+1}} + u^2 \int_A^\infty \frac{N_f(s, w)}{s^{u+1}} \leq u^2 \int_A^\infty \frac{N_f(s, w)}{s^{u+1}}.$$

The First Main Theorem of Nevanlinna [19, Corollary 4.2] gives

$$N_f(r, w) \leq T_f(r) - \log[f(0), w],$$

where $[a, b]$ denotes the chordal distance on the Riemann sphere (with, in particular, $[a, b] \leq 1$ for all $a, b \in \hat{\mathbb{C}}$). Since in this first case $w \notin D(f(0), \varepsilon)$, there exists $\Theta < \infty$ such that

$$N_f(r, w) \leq T_f(r) + \Theta \quad \text{for every } w \notin D(f(0), \varepsilon).$$

Therefore,

$$\sum_{\substack{f(z)=w, \\ |z|>A}} (1 + |z|^{\rho+1})^{-t} \leq u^2 \int_A^\infty \frac{T_f(s) + \Theta}{su + 1} ds = \tilde{M}_u < \infty \quad \text{for every } w \notin D(f(0), \varepsilon).$$

All in all, there exists $M_u^{(1)} < \infty$ such that

$$\Sigma(t, w) \leq M_u^{(1)} \quad \text{for every } w \notin D(f(0), \varepsilon).$$

Case 2 ($w \in D(f(0), \varepsilon)$). We are led to find a uniform bound for

$$\sum_{\substack{f(z)=w, \\ |z|>A}} |z|^{-u}, \quad w \in D(f(0), \varepsilon).$$

Let $v \in \mathbb{C}$ be a point that is not a pole of f and such that $|f(-v) - f(0)| > 2\varepsilon$. Set $A = 3|v|$ and define the meromorphic function $g(\xi) = f(\xi - v) + v$. If $\xi = z + v$, then $f(z) = w$ is equivalent to $g(\xi) = w + v$.

Notice that $g(0) = f(-v) + v$. If we set $a = w + v$, then

$$|a - g(0)| = |w - f(-v)| \geq |f(-v) - f(0)| - |f(0) - w| > \varepsilon. \tag{2.13}$$

On the other hand, if $|\xi - v| \geq A$, then $|\xi| \geq A - |v| \geq 2|v|$ and

$$\frac{1}{|\xi - v|} \leq \frac{2}{|\xi|}.$$

It follows that

$$\sum_{\substack{f(z)=w, \\ |z|>A}} |z|^{-u} = \sum_{\substack{g(\xi)=a, \\ |\xi-v|>A}} |\xi - v|^{-u} \leq \sum_{\substack{g(\xi)=a, \\ |\xi|>2|v|}} \left(\frac{2}{|\xi|}\right)^u.$$

In the same way as before, we can again use the First Main Theorem of Nevanlinna theory, this time applied to the function g . Remember that by (2.13) we have $a \notin D(g(0), \varepsilon)$ whenever $w = a - v \in D(f(0), \varepsilon)$. Therefore,

$$\sum_{\substack{f(z)=w, \\ |z|>A}} |z|^{-u} \leq 2^u \sum_{\substack{g(\xi)=a, \\ |\xi|>2|v|}} |\xi|^{-u} \leq \tilde{M}_u$$

for every $a = w + v$, $w \in D(f(0), \varepsilon)$. It follows that there exists $M_u^{(2)} < \infty$ such that

$$\Sigma(t, w) \leq M_u^{(2)} \quad \text{for every } w \in D(f(0), \varepsilon).$$

All in all, we have shown that there is $M_t < \infty$ such that $\|\Sigma(t, \cdot)\|_\infty \leq M_t$. □

3. Conformal measures

Conformal measures are frequently obtained via the standard Patterson–Sullivan method. We will make use of Denker and Urbański’s $K(V)$ -method, which is explained in detail in [24, Chapter 11] (see also [16, Appendix 1]). The main reason for using this method is that, for meromorphic functions, one must check very carefully what is going on at infinity, because at this point the function is not defined. In other words, the support of a conformal measure is the non-compact set $J_f \cap \mathbb{C}$.

Definition 3.1. A probability measure m on the Julia set J_f of a meromorphic function $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ is a t -conformal measure for f if $m(\{\infty\}) = 0$ and if for every measurable set $E \subset \mathbb{C}$ for which the restriction $f|_E$ is injective, we have

$$m(f(E)) = \int_E |f'|_{\sigma}^t dm. \quad (3.1)$$

If ever the exponent t is sufficiently large that the transfer operator is well defined (in our case $t > \rho/(\rho + 1)$), then we can reformulate this definition in terms of the transfer operator: m is t -conformal if and only if $\mathcal{L}_t^* m = m$ [5]. Notice that the condition $m(\{\infty\}) = 0$ is important here. If m is a probability measure on J_f that satisfies (3.1) but has some mass $c = m(\{\infty\}) > 0$ at infinity, then we only have

$$\mathcal{L}_t^* m = m - c\delta_{\infty}.$$

If $J_f = \hat{\mathbb{C}}$, which is the case when all the asymptotic values are strictly pre-periodic, the spherical Lebesgue measure is a 2-conformal measure. Therefore, we could restrict our discussion in the following subsection to functions with non-empty Fatou set.

3.1. Existence of conformal measures

In the following $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ will again be a semi-hyperbolic meromorphic function of polynomial Schwarzian derivative.

In the $K(V)$ -method, one first restricts f to a compact invariant set that does not contain critical points or ∞ . In our case, f does not have critical points. We therefore set $V_j = D(\infty, 1/j)$, for every $j \geq 1$, and consider the compact f -invariant set

$$K_j = \bigcup_{n \geq 0} f^{-n}(\mathbb{C} \setminus V_j) = \{z \in \mathbb{C}; f^n(z) \notin V_j, n \geq 0\}.$$

Then the $K(V)$ -method provides a so-called *semi-conformal measure* m_j for $f|_{K_j}$. More precisely [16, Lemma 8.2], there exist $s_j \in [0, 2]$ and a Borel probability measure m_j on K_j such that

$$m_j(f(A)) \geq \int_A |f'|_{\sigma}^{s_j} dm_j \quad (3.2)$$

for every Borel set $A \subset \mathbb{C}$ such that $f|_A : A \rightarrow f(A)$ is one-to-one and

$$m_j(f(A)) = \int_A |f'|_{\sigma}^{s_j} dm_j \quad \text{if, in addition, } A \cap \bar{V}_j = \emptyset. \quad (3.3)$$

Passing to a subsequence if necessary, we may suppose that m_j converges weakly to some probability measure m on J_f and $s_j \rightarrow h \in [0, 2]$ as $j \rightarrow \infty$. We then have two possibilities.

Case 1 ($m(\{\infty\}) = 0$). In this case we do have an h -conformal measure for the function $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ as defined in Definition 3.1. As remarked upon right after this definition, we need a good lower bound for the exponent h in order to be able to deal with the transfer operator.

Proposition 3.2. *For a semi-hyperbolic meromorphic function f of polynomial Schwarzian derivative with $\mathcal{F}_f \neq \emptyset$ we have that*

$$2 \frac{\rho}{\rho + 1} < h.$$

Proof. Let $a \in \mathcal{A}_f \cap J_f$ and let $a' = f(a)$. The function f is expanding on $\mathcal{P}_f \cap J_f$, meaning that there exists $p > 0$ such that $|(f^p)'|_\sigma > 2$ on $B(\mathcal{P}_f \cap J_f, T)$. Again let $g = f_{\Omega_1}^p$ (see (2.9)). Notice that $g(D(g^n(a'), T)) \supset D(g^{n+1}(a'), T)$. Denote

$$A_n = D(g^n(a'), T) \setminus g_{g^n(a')}^{-1}(D(g^{n+1}(a'), T)). \tag{3.4}$$

Increasing p if necessary, we have, from the fact that $\mathcal{P}_f \cap J_f$ is compact and nowhere dense in $J_f \cap \mathbb{C}$ together with the fact that the conformal measure m has positive mass on open sets that intersect the Julia set, that

$$\inf_{n \geq 0} m(A_n) \geq c > 0.$$

If $V_n = f_a^{-1} \circ g_{a'}^{-n}(A_n)$, then

$$m(V_n) \asymp (|f'(a)|_\sigma |(g^n)'(a')|_\sigma)^{-h} m(A_n) \asymp \text{diam}(V_n)^h, \quad n \geq 1. \tag{3.5}$$

The preimages of V_n to a logarithmic tract U over a can be labelled by $U_{n,k} = f_k^{-1}(A_n)$. Let $z_{n,k} \in U_{n,k}$ be any point. Then

$$\begin{aligned} m(U_{n,k}) &\asymp |f'(z_{n,k})|_\sigma^{-h} \text{diam}(V_n)^h \\ &\asymp |z_{n,k}|^{-(\rho+1)h} |f(z_{n,k}) - a|^{-h} \text{diam}(V_n)^h \\ &\asymp |z_{n,k}|^{-(\rho+1)h}, \end{aligned} \tag{3.6}$$

where the relation

$$|z_{n,k}| \asymp (n^2 + k^2)^{1/2\rho} \tag{3.7}$$

follows from an elementary calculation based on (2.6) and (2.7). Hence,

$$1 \geq m(U) = \sum_{n,k} m(U_{n,k}) \asymp \sum_{n,k} |z_{n,k}|^{-(\rho+1)h} \asymp \sum_{n,k} (n^2 + k^2)^{-(\rho+1)h/2\rho}. \tag{3.8}$$

The assertion of the lemma now follows, since the last sum is convergent if and only if $(\rho + 1)h/2\rho > 1$. □

Case 2 ($m(\{\infty\}) = c > 0$). If this case occurs, then m is not really conformal but it still has the most important property (3.1). This implies that m has mass on every point of $O^-(\infty)$, from which it follows that m has positive mass on open sets that intersect the Julia set. Therefore, the proof of Proposition 3.2 is also valid in this case and we again have that

$$2 \frac{\rho}{\rho + 1} < h.$$

This information is sufficient to show that this second case does not occur. In order to prove this let us consider again the measures m_j . Since the associated exponents $s_j \rightarrow h$ as $j \rightarrow \infty$, there exists $j_0 \geq 1$ such that $s_j > 2\rho/(\rho + 1)$ for every $j \geq j_0$.

Lemma 3.3. *The sequence $(m_j)_j$ is tight at ∞ , i.e. for every $\varepsilon > 0$ there is $R > 0$ such that $m_j(W_R) < \varepsilon$ for every $j \geq j_0$, where $W_R = \{|z| > R\}$.*

Proof. The set W_R can be written as

$$W_R = \bigcup_{a \in \mathcal{A}_f} (U_a \cap W_R) \cup \tilde{W}_R,$$

where the union is taken over all the (finitely many) logarithmic tracts $U_a \subset f^{-1}(D(a, T))$ over the asymptotic values $a \in \mathcal{A}_f$.

Let us first consider \tilde{W}_R . From the semi-conformality of the measures m_j and, more precisely, from the inequality (3.2), one deduces, similarly to the conformal case, that

$$\int \mathcal{L}_{s_j} \varphi \, dm_j \geq \int \varphi \, dm_j \quad \text{for every integrable function } \varphi.$$

For $\varphi = \mathbf{1}_{\tilde{W}_R}$ this gives

$$m_j(\tilde{W}_R) \leq \int \mathcal{L}_{s_j} \mathbf{1}_{\tilde{W}_R} \, dm_j.$$

Since

$$s_j > 2 \frac{\rho}{\rho + 1} > \frac{\rho}{\rho + 1},$$

there exists $\gamma > 0$ such that $(\rho + 1)s_j - 2\gamma > \rho$ for every $j \geq j_0$. Then it follows from Proposition 2.7 that

$$\begin{aligned} m_j(\tilde{W}_R) &\leq \int \mathcal{L}_{s_j} \mathbf{1}_{\tilde{W}_R} \, dm_j \\ &\leq \int_{\mathbb{C}} \sum_{z \in f^{-1}(w)} \mathbf{1}_{\tilde{W}_R}(z) |z|^{-(\rho+1)s_j} \, dm_j(w) \\ &\leq \frac{1}{R^\gamma} \int_{\mathbb{C}} \sum_{z \in f^{-1}(w)} (1 + |z|)^{\gamma - (\rho+1)s_j} \, dm_j(w) \\ &\leq \frac{M}{R^\gamma} \end{aligned}$$

for every $j \geq j_0$.

Let us now figure out what happens on a logarithmic tract $U = U_a \subset f^{-1}(D(a, T))$ over $a \in \mathcal{A}_f$. Using the notation and arguments from the proof of Proposition 3.2, one has $m_j(V_n) \preceq \text{diam}(V_n)^{s_j}$ and

$$m_j(U_{n,k}) \preceq |f'(z_{n,k})|_{\sigma}^{-s_j} \text{diam}(V_n)^{s_j} \asymp |z_{n,k}|^{-(\rho+1)s_j} \asymp (n^2 + k^2)^{-(\rho+1)s_j/2\rho}.$$

Therefore,

$$m_j(U) = \sum_{n,k} m_j(U_{n,k}) \preceq \sum_{n,k} (n^2 + k^2)^{-(\rho+1)s_j/2\rho} < \infty$$

since $s_j > 2\rho/(\rho + 1)$.

The constants involved in these estimates do not depend on j . Moreover, increasing j_0 if necessary, there exists $\tau > 2\rho/(\rho + 1)$ such that $s_j \geq \tau$ for every $j \geq j_0$. Consequently, in the last sum of the previous estimation one can replace the exponent s_j by τ and it follows that $m_j(U \cap W_R) \rightarrow 0$ as $R \rightarrow \infty$ uniformly in $j \geq j_0$. \square

In conclusion, this second case does not occur. All in all we have shown that f has an h -conformal measure m . In the following, m will always refer to this conformal measure and h , or even h_f , denotes the corresponding exponent of conformality.

3.2. Additional properties

Recall the definition of the annuli Γ_n are given in (2.9). We start with the following.

Lemma 3.4. *There exists $0 < \gamma < 1$ such that $m(\Gamma_n) \preceq \gamma^n$ for every $n \geq 0$.*

Proof. Let $\mathcal{P}_f \cap J_f \subset \bigcup_{j=1}^N D_j$, where the discs $D_j = D(x_j, 2T)$, $x_j \in \mathcal{P}_f \cap J_f$, built a Besicovitch covering of \mathcal{P}_f . We may suppose that $\Omega_0 \subset \bigcup_{j=1}^N D_j$. Fix $q \geq 1$ such that, for every $j = 1, \dots, N$,

$$m(\Gamma_q \cap D_j) \leq \eta m(\Gamma_0 \cap D_j),$$

with $\eta > 0$ some small number to be determined later. Remember that $g = f^p|_{\Omega_1}$. Clearly, all the inverse branches of g^n are well defined and of bounded distortion on every disc D_j . Let us denote these by g_*^{-n} . With this notation we can calculate, for every $n \geq 1$, that

$$\begin{aligned} m(g^{-n}(D_j \cap \Gamma_q)) &= \sum_* m(g_*^{-n}(D_j \cap \Gamma_q)) \\ &= \sum_* \frac{m(g_*^{-n}(D_j \cap \Gamma_q))}{m(g_*^{-n}(D_j \cap \Gamma_0))} m(g_*^{-n}(D_j \cap \Gamma_0)) \\ &\preceq \sum_* \frac{m(D_j \cap \Gamma_q)}{m(D_j \cap \Gamma_0)} m(g_*^{-n}(D_j \cap \Gamma_0)) \\ &\leq \eta \sum_* m(g_*^{-n}(D_j \cap \Gamma_0)) \\ &= \eta m(g^{-n}(D_j \cap \Gamma_0)). \end{aligned}$$

Summing over j and using the Besicovitch property of the covering we get that

$$m(\Gamma_{q+n}) \leq C\eta m(\Gamma_n) \quad \text{for every } n \geq 0.$$

The assertion follows, provided that η has been chosen such that $C\eta < \frac{1}{2}$. \square

In the rest of this section we denote by ν any h -conformal measure (and keep m for the conformal measure that has been constructed above). Note that, for any Borel probability measure ν on a compact metric space (X, ρ) ,

$$M_\nu(r) := \inf\{\nu(B(x, r)) : x \in \text{supp}(\nu)\} > 0$$

for every $r > 0$. Let us also prove the following.

Lemma 3.5. *For any h -conformal measure ν we have $\nu(\mathcal{P}_f \cap J_f) = 0$.*

Proof. Recall that one condition imposed on T was that, for every $z \in \mathcal{P}_f \cap J_f$ and every $n \geq 0$, there exists a holomorphic inverse branch $f_z^{-n} : D(f^n(z), 2T) \rightarrow \hat{\mathbb{C}}$ of f^n sending $f^n(z)$ to z . It then follows from the bounded distortion property (Lemma 2.1) that

$$\begin{aligned} \nu(D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1})) &\leq \nu(f_z^{-n}(D(f^n(z), T))) \\ &\leq |(f^n)'(z)|_\sigma^{-h} \nu(D(f^n(z), T)) \\ &\leq |(f^n)'(z)|_\sigma^{-h}. \end{aligned} \tag{3.9}$$

Since $\mathcal{P}_f \cap J_f$ is a nowhere-dense subset of J_f , there exists $\gamma > 0$ such that for every $y \in \mathcal{P}_f \cap J_f$ there exists $\hat{y} \in J_f$ such that

$$D_y := D(\hat{y}, \gamma) \subset D(y, K^{-2}T) \setminus \mathcal{P}_f.$$

Then

$$f_z^{-n}(D_{f^n(z)}) \subset f_z^{-n}(D(f^n(z), K^{-2}T)) \setminus \mathcal{P}_f \subset D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1}) \setminus \mathcal{P}_f \tag{3.10}$$

and

$$\nu(f_z^{-n}(D_{f^n(z)})) \geq |(f^n)'(z)|_\sigma^{-h} \nu(D_{f^n(z)}) \geq M_\nu(\gamma)|(f^n)'(z)|_\sigma^{-h}.$$

Combining this with (3.9) and (3.10), and noting that $\text{supp}(\nu) = J_f$, we get that

$$\frac{\nu(D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1}) \setminus \mathcal{P}_f)}{\nu(D(z, K^{-1}T|(f^n)'(z)|_\sigma^{-1}))} \geq M_\nu(\gamma) \quad \text{for every } n \geq 1.$$

Therefore,

$$\limsup_{r \rightarrow 0} \frac{\nu(D(z, r) \setminus \mathcal{P}_f)}{\nu(D(z, r))} \geq M_\nu(\gamma) > 0.$$

So, z is not a Lebesgue density point of ν , and therefore $\nu(\mathcal{P}_f \cap J_f) = 0$. □

3.3. Metric exactness, conservativity and ergodicity

Suppose that (X, \mathcal{F}, ν) is a probability space and that $T : X \rightarrow X$ is a measurable map such that $T(A) \in \mathcal{F}$ whenever $A \in \mathcal{F}$. The map $T : X \rightarrow X$ is said to be *weakly metrically exact* provided that $\overline{\lim}_{n \rightarrow \infty} \nu(T^n(A)) = 1$ whenever $A \in \mathcal{F}$ and $\nu(A) > 0$. The measure ν is called *conservative* if, for every Borel set A of positive measure, the ν -a.e. orbit returns infinitely many times into A . A straightforward observation concerning weak metrical exactness is as follows.

Observation 3.6. If a measurable transformation $T : X \rightarrow X$ of a probability space (X, \mathcal{F}, ν) is weakly metrically exact, then it is ergodic and conservative.

In the context of invariant measures we have the following, more involved fact, which also indicates a dynamical significance of weak metrical exactness (see, for example, [24]).

Fact 3.7. A measure-preserving transformation $T : X \rightarrow X$ of a probability space (X, \mathcal{F}, μ) is weakly metrically exact if and only if it is exact, which means that

$$\lim_{n \rightarrow \infty} \mu(T^n(A)) = 1$$

whenever $A \in \mathcal{F}$ and $\mu(A) > 0$ or, equivalently, the σ -algebra $\bigcap_{n \geq 0} T^{-n}(\mathcal{F})$ consists of sets of measure 0 and 1 only. Then Rokhlin’s natural extension $(\tilde{T}, \tilde{X}, \tilde{\mu})$ of (T, X, μ) is K -mixing.

The main result of this subsection is the following.

Theorem 3.8. *The measure m is the only regular probability measure on J_f having the property*

$$m(f(E)) = \int_E |f'|_{\sigma}^h dm \quad \text{for every measurable set } E \text{ such that } f|_E \text{ is injective.} \quad (3.11)$$

In particular, m is a unique h -conformal measure for f . The dynamical system $f : J_f \rightarrow J_f$ is weakly metrically exact with respect to m . In particular, it is ergodic and conservative.

Proof. Let

$$\mathcal{P}_f^* = \left\{ z \in J_f : \text{dist}_{\sigma}(z, \mathcal{A}_f \cup \mathcal{P}_f) > 2T \right\}.$$

By Observation 2.5,

$$J_f^* = \{ z \in J_f \setminus O^-(\infty) : \omega(z) \cap \mathcal{P}_f^* \neq \emptyset \} = J_f \setminus \bigcup_{n=0}^{\infty} f^{-n}(\mathcal{P}_f \cup \{\infty\}). \quad (3.12)$$

Take $z \in J_f^*$. Then there exists a strictly increasing sequence $(n_j = n_j(z))_{j=1}^{\infty}$ of positive integers such that

$$f^{n_j}(z) \in \mathcal{P}_f^* \setminus \{\infty\}$$

for all $j \geq 1$. Then, for every $j \geq 1$, there exists a meromorphic inverse branch $f_z^{-n_j} : D(f^{n_j}(z), 2T) \rightarrow \hat{\mathbb{C}}$ of f^{n_j} sending $f^{n_j}(z)$ to z . Let ν be a regular probability measure on J_f that satisfies (3.11). It then follows from Lemma 2.1 (the bounded distortion property) that

$$\begin{aligned} \nu(D(z, K^{-1}T|(f^{n_j})'(z)|_{\sigma}^{-1})) &\leq \nu(f_z^{-n_j}(D(f^{n_j}(z), T))) \\ &\leq |(f^{n_j})'(z)|_{\sigma}^{-h} \nu(D(f^{n_j}(z), T)) \\ &\leq |(f^{n_j})'(z)|_{\sigma}^{-h}. \end{aligned} \quad (3.13)$$

Put

$$r_j(z) = (4K)^{-1}T|(f^{n_j})'(z)|_\sigma^{-1}.$$

The above formula can then be rewritten as

$$\nu(D(z, 4r_j(z))) \preceq r_j^h(z). \tag{3.14}$$

It also follows from Lemma 2.1 that

$$\begin{aligned} \nu(D(z, r_j(z))) &\geq \nu(f_z^{-n_j}(D(f^{n_j}(z), ((4K^2)^{-1}T)))) \\ &\geq |(f^{n_j})'(z)|_\sigma^{-h} \nu(D(f^{n_j}(z), 4K^2)^{-1}T) \\ &\geq M_\nu((4K^2)^{-1}T)|(f^{n_j})'(z)|_\sigma^{-h} \\ &\asymp r_j^h(z). \end{aligned} \tag{3.15}$$

Now fix E , an arbitrary Borel set contained in J_f^* . Fix also $\varepsilon > 0$. Since the measure m is regular, for every $z \in E$ there exists $j(z) \geq 1$ such that, with $r(z) = r_{j(z)}(z)$, we will have

$$m\left(\bigcup_{z \in E} D(z, r(z))\right) \leq m(E) + \varepsilon. \tag{3.16}$$

By the $(4r)$ -Covering Theorem there now exists a countable set $\hat{E} \subset E$ such that the balls $\{D(z, r(z))\}_{z \in \hat{E}}$ are mutually disjoint and

$$\bigcup_{z \in \hat{E}} D(z, 4r(z)) \supset \bigcup_{z \in E} D(z, r(z)) \supset E.$$

Hence, using (3.14), (3.15) (with ν replaced by m) and (3.16), we get

$$\begin{aligned} \nu(E) &\leq \sum_{z \in \hat{E}} \nu(D(z, 4r(z))) \\ &\leq (4K^2/T)^h \sum_{z \in \hat{E}} r^h(z) \\ &\leq K^{2h} M_m((4K^2)^{-1}T) \sum_{z \in \hat{E}} m(D(z, r(z))) \\ &\asymp m\left(\bigcup_{z \in \hat{E}} D(z, r(z))\right) \\ &\leq m(E) + \varepsilon. \end{aligned}$$

Thus, letting $\varepsilon \searrow 0$, we get $\nu(E) \preceq m(E)$. Hence, $\nu|_{J_f^*}$ is absolutely continuous with respect to $m|_{J_f^*}$. Exchanging the roles of ν and m , we get that $m|_{J_f^*} \preceq \nu|_{J_f^*}$ and finally that $\nu|_{J_f^*}$ is equivalent to $m|_{J_f^*}$. Since, in view of Lemma 3.5,

$$m\left(\bigcup_{n=0}^\infty f^{-n}(\mathcal{P}_f)\right) = \nu\left(\bigcup_{n=0}^\infty f^{-n}(\mathcal{P}_f)\right) = 0,$$

we thus conclude that ν and m are equivalent on $J_f \setminus O^-(\infty)$. Finally, if $\nu(O^-(\infty)) > 0$, then $\nu^* = \nu|_{O^-(\infty)}$ would be a measure that satisfies (3.11) and which is without mass on $J_f \setminus O^-(\infty)$. But then we would have a contradiction, since we have just seen that m and ν^* are equivalent on $J_f \setminus O^-(\infty)$. Therefore, $\nu(O^-(\infty)) = 0$ and both measures are equivalent on the whole Julia set.

Passing to the proof of weak metrical exactness of f with respect to the measure m , suppose that $E \subset J_f$ and

$$\limsup_{n \rightarrow \infty} \sup \{m(f^n(E) \cap D(y, K^{-2}T))/m(D(y, K^{-2}T)) : y \in \mathcal{P}_f^*\} = 1. \tag{3.17}$$

We shall show that

$$\limsup_{n \rightarrow \infty} m(f^n(E)) = 1. \tag{3.18}$$

By virtue of Observation 2.4 there exists $q \geq 0$ such that

$$f^q(D(y, K^{-2}T)) \supset \hat{\mathbb{C}} \setminus \mathcal{A}_f$$

for all $y \in J_f$. Clearly, by conformality of m , for every $\varepsilon > 0$ there then exists $\delta > 0$ such that if $y \in J_f$, $G \subset D(y, K^{-2}T)$, and if $m(G)/m(D(y, K^{-2}T)) \geq 1 - \delta$, then $m(f^q(G)) \geq 1 - \varepsilon$. Combining this with (3.17) yields (3.18). In order to obtain the weak metrical exactness of m , suppose by contradiction that $E \subset J_f$ and $\limsup_{n \rightarrow \infty} m(f^n(E)) < 1$. By (3.17) and (3.18), this implies that

$$2\kappa := \liminf_{n \rightarrow \infty} \inf \{m(D(y, K^{-2}T) \setminus f^n(E))/m(D(y, K^{-2}T)) : y \in \mathcal{P}_f^*\} > 0.$$

So, for all $n \geq 1$ large enough, say $n \geq p$,

$$\inf \{m(D(y, K^{-2}T) \setminus f^n(E))/m(D(y, K^{-2}T)) : y \in \mathcal{P}_f^*\} \geq \kappa > 0.$$

Fix $z \in E \cap J_f^*$. We shall show that z is not a Lebesgue density point for the measure m . Let $n_j = n_j(z) \geq p$, $j \geq 1$, have the same meaning as in the first part of the proof. Then

$$f_z^{-n_j}(D(f^{n_j}(z), K^{-2}T) \setminus f^{n_j}(E)) \subset D(z, K^{-1}T|(f^{n_j})'(z)|_\sigma^{-1}) \setminus E \tag{3.19}$$

and

$$\begin{aligned} m(f_z^{-n_j}(D(f^{n_j}(z), K^{-2}T) \setminus f^{n_j}(E))) &\geq K^{-h} |(f^{n_j})'(z)|_\sigma^{-h} m(D(f^{n_j}(z), K^{-2}T) \setminus f^{n_j}(E)) \\ &\geq \kappa K^{-h} |(f^{n_j})'(z)|_\sigma^{-h} m(D(f^{n_j}(z), K^{-2}T)) \\ &\geq \kappa K^{-h} M_m(K^{-2}T) |(f^{n_j})'(z)|_\sigma^{-h}. \end{aligned}$$

Combining this along with (3.19) and (3.13), we get that

$$\frac{m(D(z, K^{-1}T|(f^{n_j})'(z)|_\sigma^{-1}) \setminus E)}{m(D(z, K^{-1}T|(f^{n_j})'(z)|_\sigma^{-1}))} \geq \kappa K^{-2h} M_m(K^{-2}T).$$

Therefore,

$$\lim_{r \rightarrow 0} \frac{m(D(z, r) \setminus E)}{m(D(z, r))} \geq \kappa K^{-2h} M_m(K^{-2}T) > 0.$$

So, z is not a Lebesgue density point for m . Thus $m(E \cap J_f^*) = 0$. Since $m(J_f^*) = 1$ (see Lemma 3.5 and (3.12)), we finally obtain that $m(E) = 0$. The weak metrical exactness of f with respect to m is established. Ergodicity and conservativity follow from Observation 3.6. Since ν (introduced in the first part of the proof) is equivalent to m , the equality $\nu = m$ follows from ergodicity of m . This completes the proof. \square

4. Invariant measures

We now consider $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ a semi-hyperbolic meromorphic function f of polynomial Schwarzian derivative and investigate invariant measures equivalent to the conformal measure m obtained in the previous section. In particular, in the course of this section we prove Theorem 1.1.

4.1. Existence of σ -finite invariant measures.

Since we have already established conservativity of the conformal measure m , we can use the method of Martens [18] (see also [16] for a description of this method) in order to obtain the following.

Proposition 4.1. *Let f be a semi-hyperbolic meromorphic function f of polynomial Schwarzian derivative and let m be the conservative h -conformal measure of f with $m(\mathcal{P}_f) = 0$. Then there exists μ a σ -finite invariant measure absolutely continuous with respect to m .*

Proof. Using a Whitney decomposition of $\mathbb{C} \setminus (\mathcal{A}_f \cup \mathcal{P}_f)$ it is easy to construct a countable partition $\{A_n; n \geq 0\}$ of $X = J_f \setminus (\{\infty\} \cup \mathcal{A}_f \cup \mathcal{P}_f)$ such that for every $n, m \geq 0$ there exists $k \geq 0$ such that

$$m(f^{-k}(A_m) \cap A_n) > 0.$$

Since m has no mass on $J_f \setminus X$ and since m is conservative, Martens's result [18] applies and gives the σ -finite invariant measure absolutely continuous with respect to m . Notice that for every Borel set $A \subset X$ we have that

$$\mu(A) = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n m(f^{-k}(A))}{\sum_{k=0}^n m(f^{-k}(A_0))}. \quad (4.1)$$

For the choice of the set A_0 there is much freedom. In particular, we will use the fact that $A_0 \subset X$ is such that all the inverses of the iterates of f are well defined and have bounded distortion. \square

Let $\Delta = \hat{\mathbb{C}} \setminus B(\mathcal{A}_f \cup \mathcal{P}_f, T)$.

Lemma 4.2. *There is $K > 1$ such that $1/K \leq \varphi < K$ on Δ , where $\varphi = d\mu/dm$.*

Proof. Let $z \in \Delta$. From (4.1) it follows that

$$\varphi(z) = \lim_{r \rightarrow 0} \frac{\mu(D(z, r))}{m(D(z, r))} \asymp \lim_{r \rightarrow 0} \frac{1}{m(D(z, r))} \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \mathcal{L}_h^k \mathbf{1}(z) m(D(z, r))}{\sum_{k=0}^n \mathcal{L}_h^k \mathbf{1}(z_0) m(A_0)},$$

where $z_0 \in A_0$ is any point. Now, if $z_1, z_2 \in \Delta$ are any two points, then they can be joined by a chain of at most $N = N(\Delta)$ spherical discs of radius T . On each of these discs all the inverse branches of every iterate of f are well defined and have distortion bounded by some universal constant. Therefore,

$$\mathcal{L}_h^k \mathbf{1}(z) \asymp \mathcal{L}_h^k \mathbf{1}(z_0) \quad \text{for every } k \geq 0.$$

The lemma is proved. □

This simple observation on the density h has several important applications, starting with the following.

Proposition 4.3. $\mu(B(\mathcal{A}_f \cap J_f, T)) < \infty$.

Proof. It suffices to show that $\mu(D(a, T)) < \infty$, $a \in \mathcal{A}_f \cap J_f$. The measure μ being invariant, we have $\mu(D(a, T)) = \mu(f^{-1}(D(a, T)))$. By the choice of the constant $T > 0$ (see (T₃)), we obtain

$$f^{-1}(D(a, T)) \cap B(\mathcal{A}_f \cup \mathcal{P}_f, T) \cap J_f = \emptyset.$$

It therefore follows from Lemma 4.2 that

$$\mu(D(a, T)) = \mu(f^{-1}(D(a, T))) \asymp m(f^{-1}(D(a, T))) < \infty.$$

□

4.2. When is the σ -finite invariant measure finite?

To our great surprise it turns out that finiteness of the invariant measure μ does depend on the order of the function.

Theorem 4.4. *Let f be a semi-hyperbolic meromorphic function of polynomial Schwarzian derivative and let m be the (unique) h_f -conformal measure of f . Then there is a finite f -invariant measure μ absolutely continuous with respect to m if and only if $h > 3\rho/(\rho + 1)$.*

Consequently, the invariant measure μ can be finite in the particular case of the tangent family and also for the examples of (2.2) that involve the Airy functions. Notice that $3\rho/(\rho + 1) \geq 2$ as soon as the order $\rho = \frac{1}{2} \deg(P) + 1 \geq 2$.

Corollary 4.5. *A semi-hyperbolic solution f of the polynomial Schwarzian equation $S(f) = 2P$ can have a finite invariant measure absolutely continuous with respect to the h_f -conformal measure if and only if $\deg(P) = 0$ or $\deg(P) = 1$.*

We now prove Theorem 4.4 in several steps and again use the notation given in (2.9). In the following let us consider f , a semi-hyperbolic meromorphic function of polynomial Schwarzian derivative, and let m again be the h_f -conformal measure of f .

Lemma 4.6. *If $h_f \leq 3\rho/(\rho + 1)$, then there is no finite invariant measure absolutely continuous with respect to m .*

Proof. Suppose to the contrary that such a finite invariant measure μ exists. Remember that $g = f^p|_{\Omega_1} : \Omega_1 \rightarrow \Omega_0$. Let $a \in \mathcal{A}_f \cap J_f$, and set $a'' = f^p(a)$ and $D'' = D(a'', T)$. By invariance of μ we have that

$$\mu(\Omega_n) = \mu(f^{-p}(\Omega_n)) \geq \mu(f_a^{-p}(\Omega_n \cap D'')) + \mu(\Omega_{n+1}), \quad n \geq 0.$$

If we define $W_n = f_a^{-p}(\Omega_n \cap D'')$, then we get inductively that

$$\mu(\Omega_0) \geq \sum_{n \geq 0} \mu(W_n).$$

Since $|(f^p)'|_\sigma$ is bounded on $B(\{a\} \cup (\mathcal{P}_f) \cap J_f, 2T)$, there exists $L > 1$ such that $W_n \supset D(a, L^{-n})$ for every $n \geq 1$. Therefore,

$$\mu(\Omega_0) \geq \sum_{n \geq 0} \mu(D(a, L^{-n})) \geq \sum_{n \geq 0} \mu(f^{-1}(D(a, L^{-n})) \cap U_a)$$

with U_a a logarithmic tract over the asymptotic value a . But, on U_a , μ is equivalent to the conformal measure m (Lemma 4.2) and, with the same calculations that lead to (3.8), we get that

$$\sum_{n \geq 0} m(f^{-1}(D(a, L^{-n})) \cap U_a) \asymp \sum_{n \geq 0} \left(\sum_{N \geq n} \sum_k (N^2 + k^2)^{-(\rho+1)/(2\rho)h} \right), \tag{4.2}$$

which is finite if and only if $h > 3\rho/(\rho + 1)$. □

It remains to investigate the case when $h > 3\rho/(\rho + 1)$. In order to do so we write

$$f^{-p}(\Gamma_n) = \Gamma_{n+1} \cup W_n \cup S_n, \tag{4.3}$$

where

$$W_n = \bigcup_{a \in \mathcal{A}_f} W_n^a \quad \text{with } W_n^a = f_a^{-p}(D''_a \cap \Gamma_n), D''_a = D(f^p(a), T)$$

and where S_n is the remaining set. The measure μ being f -invariant, the sequence $(\mu(\Gamma_n))_n$ is decreasing. We need the following additional property.

Lemma 4.7. *For the σ -finite invariant measure μ we have that $\lim_{n \rightarrow \infty} \mu(\Gamma_n) = 0$.*

Proof. Let $l = \lim_{n \rightarrow \infty} \mu(\Gamma_n)$. From (4.3) it follows inductively that

$$\mu(\Gamma_0) = l + \sum_{n=0}^{\infty} (\mu(W_n) + \mu(S_n)).$$

It is therefore natural to consider the set $B = \bigcup_{n=0}^{\infty} (W_n \cup S_n)$. Define $\Gamma_{\infty} = \Gamma_0 \cup B$ and let f_{∞} be the induced map, i.e. the first return map, of f^p on the set Γ_{∞} . Since μ is conservative, the conditional measure $\mu_{\infty} = \mu/\mu(\Gamma_{\infty})$ is f_{∞} -invariant [1]. Hence,

$$\mu_{\infty}(B) = \mu_{\infty}(f_{\infty}^{-1}(\Gamma_0)) = \mu_{\infty}(\Gamma_0),$$

which implies that $\mu(B) = \mu(\Gamma_0)$. But this is only possible if $l = 0$. □

The last step of the proof of Theorem 4.4 is the following.

Lemma 4.8. *If $h_f > 3\rho/(\rho + 1)$, then the measure μ is finite.*

Proof. We have to show that $\mu(\Omega_0) < \infty$. Since $\lim_{n \rightarrow \infty} \mu(\Gamma_n) = 0$, it follows by induction that

$$\mu(\Omega_0) = \sum_{N=0}^{\infty} \mu(\Gamma_N) = \sum_{N=0}^{\infty} \left(\sum_{n=N}^{\infty} \mu(W_n) + \mu(S_n) \right).$$

Let us first consider the term corresponding to S_n .

Again choose a Besicovitch covering of Ω_0 by discs $D_j = D(x_j, 2T)$, $x_j \in \mathcal{P}_f \cap J_f$. Let D be one of these discs and denote by f_*^{-p} the inverse branches of f^p defined on D such that

$$S_n = \bigcup_{D \in \{D_j\}} \bigcup_{*} f_*^{-p}(\Gamma_n \cap D) \quad \text{for every } n \geq 0.$$

Since there exists $c > 0$ for which the sets $S_n \subset \Delta = \hat{\mathbb{C}} \setminus B(\mathcal{A}_f \cup \mathcal{P}_f, cT)$, we have $\mu(S_n) \asymp m(S_n)$ (Lemma 4.2). Therefore, we can make the following estimation:

$$\mu\left(\bigcup_{*} f_*^{-p}(\Gamma_n \cap D)\right) \asymp \sum_{*} m(f_*^{-p}(\Gamma_n \cap D)) \asymp \sum_{*} |(f^p)'(z_*)|_{\sigma}^{-h} m(\Gamma_n \cap D),$$

where, for every $*$, z_* is any fixed point in $f_*^{-p}(D)$. Since $D \cap B(\mathcal{A}_f, T) = \emptyset$, it follows from Lemma 2.6 together with Proposition 2.7 that

$$\mu\left(\bigcup_{*} f_*^{-p}(\Gamma_n \cap D)\right) \leq \sum_{*} (1 + |z|^{\rho+1})^{-h} m(\Gamma_n \cap D) \leq m(\Gamma_n \cap D).$$

Summing now over the discs of the Besicovitch covering and using the exponential decay of the m -mass of the sets Γ_n given in Lemma 3.4, we finally get

$$\mu(S_n) \leq m(\Gamma_n) \leq \gamma^n,$$

and thus

$$\sum_{N=0}^{\infty} \sum_{n \geq N}^{\infty} \mu(S_n) < \infty.$$

It now suffices to obtain the corresponding statements for the sets W_n . Notice again that

$$\mu(W_n) = \mu(f^{-1}(W_n)) \asymp m(f^{-1}(W_n)).$$

The set $f^{-1}(W_n)$ contains a subset that lies in parabolic tracts and a remaining set, say S'_n . The m -mass of the latter can be estimated exactly as for S_n . It therefore suffices to see what happens in just one tract, U_a , and to estimate the mass of $U_a \cap f^{-1}(W_n)$. Clearly, there exists $c > 0$ such that $W_n \subset D(a, c2^{-n})$. We can therefore conclude, just as in (4.2), that

$$\sum_{N=0}^{\infty} \sum_{n \geq N}^{\infty} \mu(W_n) < \infty$$

if and only if $h > 3\rho/(\rho + 1)$. □

5. Bowen’s Formula, Hausdorff dimension and Hausdorff measures

We start with the following fact concerning the h -dimensional Hausdorff measure \mathcal{H}^h on J_f .

Proposition 5.1. *If $h < 2$, then the h -dimensional Hausdorff measure of J_f vanishes: $\mathcal{H}^h(J_f) = 0$. If $h = 2$, then $J_f = \hat{\mathbb{C}}$. In either case $\text{HD}(J_f) \leq h$.*

Proof. Fix an arbitrary

$$z \in J_f \setminus \bigcup_{n=0}^{\infty} f^{-n}(\mathcal{A}_f \cup \{\infty\}).$$

Then there exists an increasing unbounded sequence $(n_j)_{j=1}^{\infty}$ such that for every $j \geq 1$ there exists a meromorphic inverse branch $f_z^{-n_j} : D(f^{n_j}(z), 2T) \rightarrow \hat{\mathbb{C}}$ sending $f^{n_j}(z)$ to z . Then $f_z^{-n_j}(D(f^{n_j}(z), T)) \subset D(z, KT|(f^{n_j})'(z)|^{-1})$, and therefore

$$\begin{aligned} m(D(z, KT|(f^{n_j})'(z)|^{-1})) &\geq K^{-h}|(f^{n_j})'(z)|_{\sigma}^{-h} m(D(f^{n_j}(z), T)) \\ &\geq M_m(T)(K^2T)^{-h}(KT|(f^{n_j})'(z)|_{\sigma}^{-1})^h. \end{aligned}$$

Hence,

$$\limsup_{r \rightarrow 0} \frac{m(D(z, r))}{r^h} \geq \liminf_{j \rightarrow \infty} \frac{m(D(z, KT|(f^{n_j})'(z)|_{\sigma}^{-1}))}{(KT|(f^{n_j})'(z)|_{\sigma}^{-1})^h} \geq M_m(T)(K^2T)^{-h} > 0.$$

Thus,

$$\mathcal{H}^h|_{J_f} \leq Cm \tag{5.1}$$

with some universal constant $C > 0$. Proceeding further, suppose first that $h < 2$. Recall that $W_R = \{z \in \hat{\mathbb{C}} : |z| > R\}$. It follows from (3.6) that

$$m(W_R) \geq R^{2\rho - (\rho+1)h}. \tag{5.2}$$

Due to conservativity and ergodicity of the measure m , there exists a Borel set

$$Y \subset J_f \setminus \bigcup_{n \geq 0} f^{-n}(\infty)$$

such that $m(Y) = 1$ and $\infty \in \omega(z)$ for all $z \in Y$. Fix one $z \in Y$. There then exists an unbounded increasing sequence $(n_j)_{j=1}^\infty$ such that

$$\lim_{j \rightarrow \infty} |(f^{n_j})(z)| = +\infty \quad \text{and} \quad |(f^{n_j})(z)| \geq 4T^{-1} \tag{5.3}$$

for all $j \geq 1$. Therefore, there exist meromorphic inverse branches

$$f_z^{-n_j} : W_{|(f^{n_j})(z)|} \rightarrow \hat{\mathbb{C}}$$

sending $f^{n_j}(z)$ to z . Set $r_j = 2K|(f^{n_j})'(z)|_\sigma^{-1}|f^{n_j}(z)|^{-1}$. Looking at (5.2), we get

$$\begin{aligned} \frac{m(D(z, r_j))}{r_j^h} &\geq \frac{m(f_z^{-n_j}(D(f^{n_j})(z), 2|(f^{n_j})(z)|^{-1}))}{r_j^h} \\ &\geq \frac{K^{-h}|(f^{n_j})'(z)|_\sigma^{-h}m(D(f^{n_j})(z), 2|(f^{n_j})(z)|^{-1})}{K^h|(f^{n_j})'(z)|_\sigma^{-h}|(f^{n_j})(z)|^{-h}} \\ &\geq K^{-2h}|(f^{n_j})(z)|^h m(W_{|(f^{n_j})(z)|}) \\ &\geq K^{-2h}|(f^{n_j})(z)|^h |(f^{n_j})(z)|^{2\rho - (\rho+1)h} \\ &= K^{-2h}|(f^{n_j})(z)|^{\rho(2-h)}. \end{aligned}$$

Since $2 - h > 0$, we therefore conclude from this and (5.3) that

$$\limsup_{r \rightarrow 0} \frac{m(D(z, r))}{r^h} \geq \lim_{j \rightarrow \infty} \frac{m(D(z, r_j))}{r_j^h} \geq \lim_{j \rightarrow \infty} K^{-2h}|(f^{n_j})(z)|^{\rho(2-h)} = +\infty.$$

Thus, $\mathcal{H}^h(Y) = 0$. Since $\mathcal{H}^h(J_f \setminus Y) = 0$ by (5.1), we thus have $\mathcal{H}^h(J_f) = 0$. The case when $h < 2$ is complete.

If $h = 2$, then, for the sequence $(n_j)_{j=1}^\infty$ from the beginning of the proof, we will have $m(D(z, r_j)) \asymp r_j^2$, which implies that m and l_s , the spherical Lebesgue measure on $\hat{\mathbb{C}}$, are equivalent. So, $l_s(J_f) > 0$. Now, if $J_f \neq \hat{\mathbb{C}}$, then J_f would be nowhere dense in $\hat{\mathbb{C}}$ and, in the same way as Lemma 3.5, making use of the Lebesgue Density Theorem, we may prove that $l_s(J_f) = 0$. This contradiction finishes the proof. \square

Although $\mathcal{H}^h(J_f) = 0$ (if $h < 2$), we shall, however, show that $h = \text{HD}(J_f)$. The proof will use the induced (first return) map we now describe. Let

$$X = J_f \setminus \left(B(\mathcal{P}_f, T) \cup \bigcup_{a \in \mathcal{A}_f} f_a^{-1}(D(f(a), T)) \right). \tag{5.4}$$

Let $f_* : X \rightarrow X$ be the first return map of f on X . That is,

$$f_*(x) = f^{\tau(x)}(x),$$

where $\tau(x) = \min\{n \geq 1 : f^n(x) \in X\}$. Since $f : J_f \rightarrow J_f$ is conservative with respect to the measure μ (see Theorem 3.8), the map f_* is well defined on the complement of a set of μ measure zero; in fact, as it is easy to see, it is well defined on the complement of $\bigcup_{n \geq 0} f^{-n}(\mathcal{P}_f)$, which is of measure zero by Lemma 3.5 and by formula (4.1). Since the Radon–Nikodým derivative $d\mu/dm$ is uniformly bounded from above on X , $\mu(X) < +\infty$. For every $x \in X$ define

$$f'_*(x) = (f^{\tau(x)})'(x) \quad \text{and} \quad |f'_*(x)|_\sigma = |(f^{\tau(x)})'(x)|_\sigma.$$

We shall prove the following.

Lemma 5.2. $\beta := \inf\{|f'_*(z)|_\sigma : z \in X\} > 0$ and there exists $k \geq 1$ so large that $|(f_*^k)'(z)|_\sigma \geq 2$ for all $z \in X$.

Proof. In the course of the proof of this lemma, Q denotes an appropriately large positive constant.

Suppose first that $z \in X \cap U_a$, where U_a is a logarithmic tract over some $a \in \mathcal{A}_f \cap J_f$ such that $f(U_a) = f_a^{-1}(D(f(a), T))$. Let $n \geq 0$ be the least integer such that $f^{n+1}(z) \notin D(\mathcal{P}_f, T)$. Then

$$\begin{aligned} |f'_*(z)|_\sigma &= |(f^{n+1})'(z)|_\sigma \\ &\geq Q^{-1}|f(z) - a|(1 + |z|^{\rho+1})|(f^n)'(z)|_\sigma \\ &\geq Q^{-2}(1 + |z|^{\rho+1}) \\ &\geq Q^{-2} \end{aligned} \tag{5.5}$$

and

$$|f'_*(z)|_\sigma \geq 2Q^2 \quad \text{if, in addition, } |z| \geq R. \tag{5.6}$$

For all other $z \in X$, Lemma 2.6 implies that

$$|f'_*(z)|_\sigma \geq Q^{-1} \geq Q^{-2}. \tag{5.7}$$

If, in addition, $|z| > R$ with $R > 0$ large enough, then

$$|f'_*(z)|_\sigma \geq 2Q^2. \tag{5.8}$$

The first part of our lemma is thus proved. We shall now demonstrate the following.

Claim 5.3. *There exists $l = l(R) \geq 1$ such that*

$$|(f_*^n)'(z)|_\sigma \geq 2Q^2$$

for all $n \geq l$ and all $z \in D(0, R) \cap X$.

Proof of claim. Suppose to the contrary that there exist an increasing sequence $n_j \rightarrow \infty$ and a sequence $z_j \in X \cap \bar{D}(0, R)$ such that

$$|(f_*^{n_j})'(z_j)|_\sigma < 2Q^2 \tag{5.9}$$

for all $j \geq 1$. Since $f_*^{n_j}(z_j) \in X$, there exists a unique meromorphic inverse branch

$$f_{z_j}^{-N_j} : D(f_*^{n_j}(z_j, 2T)) \rightarrow \hat{\mathbb{C}}$$

of f^{N_j} , sending $f_*^{n_j}(z_j)$ to z_j , where

$$N_j = \tau(z_j) + \tau(f_*(z_j)) + \dots + \tau(f_*^{n_j-1}(z_j)).$$

It then follows from Lemma 2.1 and (5.9) that

$$f_{z_j}^{-N_j}(D(f_*^{n_j}(z_j, T))) \supset D(z_j, (2KQ^2)^{-1}T)$$

or, equivalently,

$$f^{N_j}(D(z_j, (2KQ^2)^{-1}T)) \subset D(f_*^{n_j}(z_j), T).$$

Passing to a subsequence, we may assume without loss of generality that the sequence $(z_j)_1^\infty$ converges to a point $z \in J_f \cap \bar{D}(0, R)$ and $|z_j - z| < (4KQ^2)^{-1}T$ for all $j \geq 1$. Since

$$D(f_*^{n_j}(z_j, T)) \cap B(\mathcal{P}_f, T) = \emptyset,$$

it follows from Montel's Theorem that the family

$$\{f^{N_j}|_{D(z, (4KQ^2)^{-1}T)}\}_{j=1}^\infty$$

is normal, contrary to the fact that $z \in J_f$. The claim is proved. □

Let $k = 2l$. If $|f_*^j(z)| \geq R$ for all $j = 1, 2, \dots, l$, then by (5.7)–(5.6), we get

$$|(f_*^k)'(z)|_\sigma \geq (2Q^2)^l Q^{-2l} = 2^l \geq 2.$$

If $|f_*^j(z)| < R$ for some $0 \leq j \leq l$, let j be minimal with this property. It then follows, from (5.8), (5.6) and the claim, that

$$|(f_*^k)'(z)|_\sigma = |(f_*^j)'(z)|_\sigma |(f_*^{k-j})'(f^j(z))|_\sigma \geq |(f_*^{k-j})'(f^j(z))|_\sigma \geq 2Q^2 \geq 2.$$

This completes the proof. □

Now, we shall prove the following lemma.

Lemma 5.4. *The function $z \mapsto \log |f'_*(z)|_\sigma$ is integrable on X with respect to μ_X , the conditional measure on X induced by μ . In addition,*

$$\chi := \int \log |f'_*|_\sigma d\mu_X > 0.$$

Proof. Since the Radon–Nikodým derivative $d\mu/dm$ is uniformly bounded on X , it suffices to demonstrate that the function $z \mapsto \log |f'_*(z)|_\sigma$ is integrable on X with respect to the measure m ($\chi > 0$ follows immediately from Lemma 5.2). For every $a \in \mathcal{A}_f \cap J_f$ let $A_n(a)$, $n \geq 0$, be the annuli defined by formula (3.4). Set

$$A_n = \bigcup_{a \in \mathcal{A}_f} A_n(a).$$

Partition $X \setminus f(A_0)$ by disjoint Borel sets X_n , $n \geq 0$, such that $D(x_n, 2 \operatorname{diam}(X_n)) \cap (\mathcal{A}_f \cup \mathcal{P}_f) = \emptyset$ with some $x_n \in X_n$. Then, by Lemma 2.6 and Proposition 2.7, we get that

$$\begin{aligned} & \int_{X \cap f^{-1}(X \setminus f(A_0))} |\log |f'_*|_\sigma| dm \\ &= \sum_{n=1}^\infty \int_{X \cap f^{-1}(X_n)} |\log |f'_*|_\sigma| dm \\ &\asymp \sum_{n=1}^\infty m(X_n) \sum_{z \in X \cap f^{-1}(w_n)} |f'_*(z)|_\sigma^{-h} |\log |f'_*(z)|_\sigma| \\ &\asymp \sum_{n=1}^\infty m(X_n) \sum_{z \in X \cap f^{-1}(w_n)} (1 + |z|^{\rho+1})^{-h} |\log(1 + |z|^{\rho+1}) + O(1)| \\ &\preccurlyeq \sum_{n=1}^\infty m(X_n) \sum_{z \in X \cap f^{-1}(w_n)} (1 + |z|^{\rho+1})^{-t} \\ &\leq M_t \sum_{n=1}^\infty m(X_n) \leq M_t < +\infty, \end{aligned} \tag{5.10}$$

where w_n is an arbitrary point in X_n and t is a fixed number in $(\rho/(\rho + 1), h)$. Now, following notation from Proposition 3.2, for every $a \in \mathcal{A}_f$ and every $n \geq 0$, set

$$\begin{aligned} \Gamma_a &= f^{-1}(f(a)) \setminus (\mathcal{A}_f \cup \mathcal{P}_f), \\ Y_n(a) &= \bigcup_{b \in \Gamma} f_b^{-1} \circ g_{f(a)}^{-n}(A_n(a)) \cup \bigcup_{b \in f^{-1}(a)} f_b^{-1} \circ f_a^{-1} \circ g_{f(a)}^{-n}(A_n(a)) \end{aligned}$$

and

$$Y_a = \bigcup_{n=0}^\infty Y_n(a).$$

Keep $t \in (\rho/(\rho + 1), h)$. Again, by virtue of Lemma 2.6 and Proposition 2.7, and also Lemma 3.4, we get that

$$\begin{aligned} & \int_{Y_a} |\log |f'_*|_\sigma| dm \\ &= \sum_{n=0}^\infty \int_{Y_n(a)} |\log |f'_*|_\sigma| dm \end{aligned}$$

$$\begin{aligned}
 & \asymp \sum_{n=0}^{\infty} \left(\sum_{b \in \Gamma} m(f_b^{-1} \circ g_{f(a)}^{-n}(A_n(a))) |\log |f'(b)| |(g^n)'(f(a))| + O(1) \right) \\
 & \quad + \sum_{b \in f^{-1}(a)} m(f_b^{-1} \circ f_a^{-1} \circ g_{f(a)}^{-n}(A_n(a))) |\log |f'(b)| |(g^n)'(f(a))| + O(1) \Big) \\
 & \preceq \sum_{n=0}^{\infty} \sum_{b \in \Gamma} (1 + |b|^{\rho+1})^{-h} \gamma^n |\log(1 + |b|^{\rho+1}) + \log |(g^n)'(f(a))| + O(1)| \\
 & \quad + \sum_{n=0}^{\infty} \sum_{b \in f^{-1}(a)} (1 + |b|^{\rho+1})^{-h} \gamma^n |\log(1 + |b|^{\rho+1}) + \log |(g^n)'(f(a))| + O(1)| \\
 & \leq \sum_{n=0}^{\infty} \gamma^n \sum_{b \in \Gamma} (1 + |b|^{\rho+1})^{-h} \gamma^n |\log(1 + |b|^{\rho+1}) + O(n)| \\
 & \quad + \sum_{n=0}^{\infty} \gamma^n \sum_{b \in f^{-1}(a)} (1 + |b|^{\rho+1})^{-h} \gamma^n |\log(1 + |b|^{\rho+1}) + O(n)| \\
 & \preceq \sum_{n=0}^{\infty} \gamma^n \sum_{b \in \Gamma \cup f^{-1}(a)} (1 + |b|^{\rho+1})^{-h} |\log(1 + |b|^{\rho+1})| \\
 & \quad + \sum_{n=0}^{\infty} n \gamma^n \sum_{b \in \Gamma \cup f^{-1}(a)} (1 + |b|^{\rho+1})^{-h} \\
 & \preceq \sum_{n=0}^{\infty} \gamma^n \sum_{b \in \Gamma \cup f^{-1}(a)} (1 + |b|^{\rho+1})^{-t} + M_h \sum_{n=0}^{\infty} n \gamma^n \\
 & < +\infty.
 \end{aligned}$$

Hence,

$$\int_{\bigcup_{a \in \mathcal{A}_f} Y_a} |\log |f'_*|_{\sigma}| dm < +\infty. \tag{5.11}$$

Finally, for every $a \in \mathcal{A}_f$, let

$$U_a = \bigcup_{n,k \geq 1} U_{n,k}(a).$$

In view of (3.6) and (3.7) we get that

$$\begin{aligned}
 \int_{U_a} |\log |f'_*|_{\sigma}| dm &= \sum_{n,k \geq 1} \int_{U_{n,k}(a)} |\log |f'_*|_{\sigma}| dm \\
 &\asymp \sum_{n,k \geq 1} m(U_{n,k}(a)) |\log(|z_{n,k}|^{\rho+1} |f(z_{n,k}) - a| |f(z_{n,k}) - a|^{-1})| \\
 &\asymp \sum_{n,k \geq 1} |z_{n,k}|^{-h(\rho+1)} |\log(|z_{n,k}|^{\rho+1})| \\
 &\asymp \sum_{n,k \geq 1} (n^2 + k^2)^{-h(\rho+1)/2\rho} \log(n^2 + k^2)
 \end{aligned}$$

$$\begin{aligned} &\asymp \sum_{n,k \geq 1} (n^2 + k^2)^{-t(\rho+1)/2\rho} \\ &< +\infty. \end{aligned}$$

Hence,

$$\int_{\bigcup_{a \in \mathcal{A}_f} U_a} |\log |f'_*|_\sigma| \, dm < +\infty.$$

Adding this equation to (5.10) and (5.11), we conclude that

$$\int_X |\log |f'_*|_\sigma| \, dm < +\infty,$$

and the proof is complete. □

The main result of this section is the following.

Theorem 5.5. *It holds that $\text{HD}(J_f) = h$.*

Proof. In view of Proposition 5.1 it suffices to show that $\text{HD}(J_f) \geq h$. Let $X \subset J_f$ be the set defined by (5.4) and let $f_* : X \rightarrow X$ be the corresponding induced map. By virtue of Lemmas 5.4 and 5.2 and Birkhoff’s Ergodic Theorem, there exists a Borel set $\hat{X} \subset X$ such that $\mu(\hat{X}) = 1$ and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |(f_*^n)'(z)|_\sigma = \chi > 0$$

for all $z \in \hat{X}$. In particular,

$$\lim_{n \rightarrow \infty} \frac{\log |(f_*^{k(n+1)})'(z)|_\sigma}{\log |(f_*^{kn})'(z)|_\sigma} = 1, \tag{5.12}$$

where $k \geq 1$ comes from Lemma 5.2. For every $z \in \hat{X}$ and every $n \geq 0$, define

$$r_n(z) = (2K)^{-1} |(f_*^{kn})'(z)|_\sigma^{-1}.$$

Fix $\varepsilon \in (0, h)$. By virtue of (5.12), for every $z \in \hat{X}$ we have

$$\frac{r_n(z)}{r_{n+1}(z)} \leq r_n(z)^{-\varepsilon/2} \tag{5.13}$$

for all $n \geq 1$ large enough. It follows from Lemma 2.1 and conformality of m that

$$\begin{aligned} m(D(z, r_n)) &\leq m(f_*^{-N_{kn}(z)}(D(f_*^{kn}(z), \frac{1}{2}T))) \\ &\leq K^h |(f_*^{N_{kn}(z)})'(z)|_\sigma^{-h} m(D(f_*^{kn}(z), \frac{1}{2}T)) \\ &\leq K^h |(f_*^{kn})'(z)|_\sigma^{-h} \\ &= (2K^2T)^h r_n^h. \end{aligned} \tag{5.14}$$

Now, keeping $z \in \hat{X}$, take an arbitrary radius $r \in (0, (2K)^{-1}T)$. Since the sequence $(r_n)_0^\infty$ is strictly decreasing, there exists a unique $n \geq 0$ such that $r_{n+1} \leq r < r_n$. In view of (5.14) and (5.13), we get that

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{m(D(z, r))}{r^{h-\varepsilon}} &\leq \lim_{n \rightarrow \infty} \frac{m(D(z, r_n))}{r_{n+1}^{h-\varepsilon}} \\ &= \lim_{n \rightarrow \infty} \left(\frac{m(D(z, r_n))}{r_n^{h-\varepsilon}} \left(\frac{r_n}{r_{n+1}} \right)^{h-\varepsilon} \right) \\ &\leq \lim_{n \rightarrow \infty} (r_n^\varepsilon r_n^{-\varepsilon/2}) \\ &= \lim_{n \rightarrow \infty} r_n^{\varepsilon/2} \\ &= 0. \end{aligned}$$

Since $m(\hat{X}) > 0$, we therefore conclude that

$$\mathcal{H}^{h-\varepsilon}(J_f) \geq \mathcal{H}^{h-\varepsilon}(\hat{X}) = +\infty.$$

Thus, $\text{HD}(J_f) \geq h - \varepsilon$ and, eventually, letting $\varepsilon \searrow 0$, we get $\text{HD}(J_f) \geq h$. This completes the proof. □

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Appendix A. Examples of semi-hyperbolic functions

The aim of this appendix is to explain briefly some examples of semi-hyperbolic functions in any given holomorphic family of Nevanlinna functions. In order to do so we first make some general comments.

First of all, two functions f_0, f_1 are called *topologically equivalent* if there exist homeomorphisms $\psi : \mathbb{C} \rightarrow \mathbb{C}$ and $\phi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $\phi \circ f_0 = f_1 \circ \psi$ (and we then write $f_0 \simeq_{\text{top}} f_1$). Then, for a given function $f_0 : \mathbb{C} \rightarrow \hat{\mathbb{C}}$, define the space

$$\mathcal{M}(f_0) = \{f : \mathbb{C} \rightarrow \hat{\mathbb{C}}; f \simeq_{\text{top}} f_0\}.$$

Eremenko and Lyubich [8] showed (for entire functions of class \mathcal{S} but this also works for Nevanlinna functions; see also [6]) that this space is a holomorphic family (parametrized by the asymptotic plus two extra values). Now, Nevanlinna’s topological characterization implies that all (or none) of the functions $f \in \mathcal{M}(f_0)$ have polynomial Schwarzian derivative. As usual, the space $\mathcal{M} = \mathcal{M}(f_0)$ splits into two parts:

$$\mathcal{M} = \mathcal{M}^{\text{stable}} \cup \mathcal{M}^{\text{bif}},$$

where $\mathcal{M}^{\text{stable}}$ is the set of J -stable functions and \mathcal{M}^{bif} is the bifurcation locus (or unstable set); we refer the reader to [13] for a detailed discussion.

Fix in the following an arbitrary Nevanlinna function f_0 , consider the associated space $\mathcal{M} = \mathcal{M}(f_0)$ and set $\mathcal{A}_{f_0} = \{a_1, \dots, a_d\}$, where $a_i \neq a_j$ for $i \neq j$. We first construct a particular stable function.

Lemma A 1. *There exists $f_1 \in \mathcal{M}$ such that each of the asymptotic values $\mathcal{A}_{f_1} = \{b_1, \dots, b_d\}$ is in a different attracting component.*

Proof. For $j = 1, \dots, d$, set $D_j = \mathbb{D}(a_j, T)$ and let U_j be a logarithmic tract over a_j , i.e. a component of $f^{-1}(D_j)$. From the discussion on the critical directions in § 2 it follows that there exist $R > 0$ and critical directions θ_j such that the sector $S_j = \{|\arg z - \theta_j| < (2\pi/p) - \delta; |z| > R\}$ has a non-empty intersection with the tract U_j .

Let $\phi: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a quasiconformal map fixing ∞ and such that every $a'_j = \phi(a_j)$ has argument θ_j and $|a'_j| \geq 2$. It follows from the Ahlfors–Bers Measurable Mapping Theorem that there is a second quasiconformal map $\psi: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ also fixing ∞ and such that

$$g = \phi \circ f_0 \circ \psi: \mathbb{C} \rightarrow \hat{\mathbb{C}}$$

is holomorphic. By construction, $\mathcal{A}_g = \{a'_1, \dots, a'_d\}$. On the other hand, we know that g is again a Nevanlinna function. Therefore, the discussion on the critical directions in § 2 also applies to g . Replacing if necessary g by $e^{i\alpha}g$ for some angle α , it follows that f_0 and g do have the same critical directions. We denote by V_j the component of $g^{-1}(D(a'_j, T))$ that intersects $\psi^{-1}(U_j)$.

Fix now

$$\beta = \frac{\pi}{p} \in \left(0, \frac{2\pi}{p}\right)$$

and let $r > 1$. The function

$$f_1 = re^{i\beta}g$$

has asymptotic values $b_j = re^{i\beta}a'_j$ and, if r has been chosen to be sufficiently large, then

$$\overline{D(b_j, T)} \subset V_j.$$

This implies that the logarithmic tracts V_j belong to attracting components of the Fatou set. \square

In the same way, it is possible to construct a function $f_2 \in \mathcal{M}(f_0)$ such the first $d - 1$ asymptotic values behave like b_1, \dots, b_{d-1} but the last one is in the same attracting component as one of the asymptotic values b_1, \dots, b_{d-1} . Therefore, f_2 is also a stable map but is not in the same component of $\mathcal{M}^{\text{stable}}$ as f_1 . We showed the following.

Lemma A 2. *The bifurcation locus $\mathcal{M}^{\text{bif}} \neq \emptyset$.*

In order to get a semi-hyperbolic example in \mathcal{M} , we choose a particular point $g \in \mathcal{M}^{\text{bif}}$. First, we join f_1 to f_2 by a path in \mathcal{M} as follows: suppose that a'_1 is the closest neighbour of a'_d in the sense that the difference of the arguments $L = |\arg a'_1 - \arg a'_d|$ is minimal. We may suppose that $L = \arg a'_1 - \arg a'_d$. Then, for $\lambda \in [0, L]$, let f_λ be constructed

precisely like f_1 , except that the first quasiconformal map ϕ will be replaced by ϕ_λ such that again $a'_j = \phi_\lambda(a_j)$ for $j = 1, \dots, d-1$ but with

$$\phi_\lambda(a_d) = |a'_d|^{-\lambda/L} |a'_1|^{2\lambda/L} e^{i\lambda} a'_d.$$

Then there exists $\lambda \in (0, L)$ such that $f_\lambda \in \mathcal{M}^{\text{bif}}$. More precisely, $d-1$ asymptotic values of f_λ are in attracting stable domains, whereas the last one has unstable dynamics under perturbation of f_λ in \mathcal{M} . Using the now standard normal family argument [17, Proposition 2.2] (a detailed exposition can be found in [3]), we get the following.

Proposition A 3. *Any open neighbourhood of f_λ in \mathcal{M} contains a Nevanlinna function g having $d-1$ asymptotic values in attracting components and one more asymptotic value which eventually is mapped into a repelling cycle. This (Misiurewicz-type) function is, in particular, semi-hyperbolic.*

References

1. J. AARONSON, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, Volume 50 (American Mathematical Society, Providence, RI, 1997).
2. W. BERGWELER, Iteration of meromorphic functions, *Bull. Am. Math. Soc.* **29**(2) (1993), 151–188.
3. F. BERTELOOT AND V. MAYER, *Rudiments de dynamique holomorphe*, Cours Spécialisés, Volume 7 (Société Mathématique de France, Paris, 2001).
4. W. CHERRY AND Z. YE, *Nevanlinna's theory of value distribution*, Springer Monographs in Mathematics (Springer, 2001).
5. M. DENKER AND M. URBAŃSKI, On the existence of conformal measures, *Trans. Am. Math. Soc.* **328** (1991), 563–587.
6. R. L. DEVANEY AND L. KEEN, Dynamics of meromorphic maps: maps with polynomial Schwarzian derivative, *Annales Scient. Éc. Norm. Sup.* **22**(1) (1989), 55–79.
7. N. DOBBS AND B. SKORULSKI, Non-existence of absolutely continuous invariant probabilities for exponential maps, *Fund. Math.* **198**(3) (2008), 283–287.
8. A. EREMENKO AND M. LYUBICH, Dynamical properties of some classes of entire functions, *Annales Inst. Fourier* **42**(4) (1992), 1–32.
9. J. GRACZYK, J. KOTUS AND G. SWIATEK, Non-recurrent meromorphic functions, *Fund. Math.* **182** (2004), 269–281.
10. E. HILLE, *On the zeros of the functions of the parabolic cylinder*, Arkiv for Matematik, Astronomi och Fysik, Volume 18 (no. 26) (Kungliga Vetenskapsakademien, Stockholm, 1924).
11. E. HILLE, *Analytic function theory*, Volume II (Ginn, Boston, MA, 1962).
12. E. HILLE, *Ordinary differential equations in the complex domain* (Dover, New York, 1997).
13. L. KEEN AND J. KOTUS, Dynamics of the family $\lambda \tan(z)$, *Conform. Geom. Dyn.* **1** (1997), 28–57.
14. J. KOTUS AND G. SWIATEK, Invariant measures for meromorphic Misiurewicz maps, *Math. Proc. Camb. Phil. Soc.* **145** (2008), 685–697.
15. J. KOTUS AND G. SWIATEK, No finite invariant density for Misiurewicz exponential maps, *C. R. Acad. Sci. Paris Sér. I* **346** (2008), 559–562.
16. J. KOTUS AND M. URBAŃSKI, Fractal measures and ergodic theory of transcendental meromorphic functions, in *Transcendental Dynamics and Complex Analysis*, London Mathematical Society Lecture Notes Series, Volume 348 (Cambridge University Press, 2008).

17. M. YU. LYUBICH, The dynamics of rational transforms: the topological picture, *Russ. Math. Surv.* **41**(4) (1986) 43–117.
18. M. MARTENS, The existence of σ -finite invariant measures: applications to real one-dimensional dynamics, ArXiv eprint, math/9201300.
19. V. MAYER AND M. URBAŃSKI, Geometric thermodynamical formalism and real analyticity for meromorphic functions of finite order, *Ergod. Theory Dynam. Syst.* **28** (2008), 915–946.
20. V. MAYER AND M. URBAŃSKI, *Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order*, Memoirs of the American Mathematical Society, Volume 203 (American Mathematical Society, Providence, RI, 2010).
21. R. NEVANLINNA, Über Riemannsche Flächen mit endlich vielen Windungspunkten, *Acta Math.* **58** (1932), 295–373.
22. R. NEVANLINNA, *Eindeutige analytische Funktionen* (Springer, 1953).
23. R. NEVANLINNA, *Analytic functions* (Springer, 1970).
24. F. PRZYTYCKI AND M. URBAŃSKI, *Conformal fractals: ergodic theory methods*, in press (Cambridge University Press; available at www.math.unt.edu/~urbanski/book1.html).
25. L. REMPE AND S. VAN STRIEN, Absence of line fields and Mañé’s Theorem for non-recurrent transcendental functions, eprint arXiv:0802.0666 (2008).
26. B. SKORULSKI, Non-ergodic maps in the tangent family, *Indagationes Math.* **14** (2003), 103–118.
27. B. SKORULSKI, Metric properties of the Julia set of some meromorphic functions with an asymptotic value eventually mapped onto a pole, *Math. Proc. Camb. Phil. Soc.* **138** (2005), 117–138.
28. M. URBAŃSKI AND A. ZDUNIK, Geometry and ergodic theory of non-hyperbolic exponential maps, *Trans. Am. Math. Soc.* **359** (2007), 3973–3997.