2

Quantum field theory

Most of the work in this book will be strictly perturbative. However it is important not to consider perturbation theory as the be-all and end-all of field theory. Rather, it must be looked on only as a systematic method of approximating a complete quantum field theory, with the errors under control. So in this chapter we will review the foundations of quantum field theory starting from the functional integral.

The purpose of this review is partly to set out the results on which the rest of the book is based. It will also introduce our notation. We will also list a number of standard field theories which will be used throughout the book. Some examples are physical theories of the real world; others are simpler theories whose only purpose will be to illustrate methods in the absence of complications.

The use of functional integration is not absolutely essential. Its use is to provide a systematic basis for the rest of our work: the functional integral gives an explicit solution of any given field theory. Our task will be to investigate a certain class of properties of the solution.

For more details the reader should consult a standard textbook on field theory. Of these, probably the most complete and up-to-date is by Itzykson & Zuber (1980); this includes a treatment of the functional integral method. Other useful references include: Bjorken & Drell (1966), Bogoliubov & Shirkov (1980), Lurié (1968), and Ramond (1981).

2.1 Scalar field theory

The simplest quantum field theory is that of a single real scalar field $\phi(x^\mu)$. The theory is defined by canonically quantizing a classical field theory. This classical theory is specified by a Lagrangian density:

$$\mathscr{L} = (\partial\phi)^2/2 - P(\phi), \tag{2.1.1}$$

from which follows the equation of motion

$$\Box\phi + P'(\phi) = 0. \tag{2.1.2}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

Here $P(\phi)$ is a function of $\phi(x)$, which we generally take to be a polynomial like $P(\phi) = m^2\phi^2/2 + g\phi^4/4!$, and $P'(\phi) = \mathrm{d}P/\mathrm{d}\phi$. (Note that we use units with $\hbar = c = 1$.)

In the Hamiltonian formulation of the same theory, we define a canonical momentum field:

$$\pi(x) \equiv \partial\mathscr{L}/\partial\dot{\phi} = \dot{\phi} = \partial\phi/\partial t, \tag{2.1.3}$$

and the Hamiltonian

$$H = \int \mathrm{d}^3x(\pi^2/2 + \vec{\nabla}\phi^2/2 + P(\phi)). \tag{2.1.4}$$

Physically, we require that a theory have a lowest energy state. If it does not then all states are unstable against decay into a lower energy state plus a collection of particles. If the function $P(\phi)$ has no minimum, then the formula (2.1.4) implies that just such a catastrophic situation exists (Baym (1960)). Thus we require the function $P(\phi)$ to be bounded below.

Quantization proceeds in the Heisenberg picture by reinterpreting $\phi(x)$ as a hermitian operator on a Hilbert space satisfying the canonical equal-time commutation relations, i.e.,

$$\left.\begin{aligned}[\pi(x), \phi(y)] &= -\mathrm{i}\delta^{(3)}(\vec{x} - \vec{y}),\\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0\end{aligned}\right\} \quad \text{if} \quad x^0 = y^0. \tag{2.1.5}$$

The Hamiltonian is still given by (2.1.4) so the equation of motion (2.1.2) follows from the Heisenberg equation of motion

$$\mathrm{i}\partial\phi/\partial t = [\phi, H]. \tag{2.1.6}$$

A solution to the theory is specified by stating what the space of states is and by giving the manner in which ϕ acts on the states. We will construct a solution by use of the functional integral. It should be noted that $\phi(x)$ is in general not a well-behaved operator, but rather it is an operator-valued distribution. Physically that means that one cannot measure $\phi(x)$ at a single point, but only averages of $\phi(x)$ over a space-time region. That is,

$$\phi_f \equiv \int \phi(x) f(x) \mathrm{d}^4x, \tag{2.1.7}$$

for any complex-valued function $f(x)$, is an operator. Now, products of distributions do not always make sense (e.g., $\delta(x)^2$). In particular, the Hamiltonian H involves products of fields at the same point. Some care is needed to define these products properly; this is, in fact, the subject of renormalization, to be treated shortly.

The following properties of the theory are standard:

(1) The theory has a Poincaré-invariant ground state $|0\rangle$, called the vacuum.

(2) The states and the action of ϕ on them can be reconstructed from the time-ordered Green's functions

$$G_N(x_1,\ldots,x_N) = \langle 0|T\phi(x_1)\ldots\phi(x_N)|0\rangle. \tag{2.1.8}$$

The T-ordering symbol means that the fields are written in order of increasing time from right to left.

(3) The Green's functions have appropriate causality properties, etc., so that they are the Green's functions of a physically sensible theory. Mathematically, these properties are summarized by the Wightman axioms (Streater & Wightman (1978)).

Bose symmetry of the ϕ-field means that the Green's functions are symmetric under interchange of any of the x's. From the equations of motion of ϕ and from the commutation relations can be derived equations of motion for the Green's functions. The simplest example is

$$\Box_y G_2(y,x) + \langle 0|TP'(\phi(y))\phi(x)|0\rangle = -i\delta^{(4)}(x-y). \tag{2.1.9}$$

For a general $(N+1)$-point Green's function, we have N δ-functions on the right:

$$\begin{aligned}\Box_y G_{N+1}(y,x_1,\ldots,x_N) &+ \langle 0|TP'(\phi(y))\phi(x_1)\ldots\phi(x_N)|0\rangle \\ &= -i\sum_{j=1}^{N}\delta^{(4)}(y-x_j)G_{N-1}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_N).\end{aligned} \tag{2.1.10}$$

This equation summarizes both the equations of motion and the commutation relations. Solving the theory for the Green's functions means in essence solving this set of coupled equations. It is in fact the Green's functions that are the easiest objects to compute. All other properties of the theory can be calculated once the Green's functions are known.

2.2 Functional-integral solution

The solution of a quantum field theory is a non-trivial problem in consistency. Only two cases are elementary: free field theory ($P = m^2\phi^2/2$), and the case of one space-time dimension, $d = 1$. The case $d = 1$ is a rather trivial field theory, for it is just the quantum mechanics of a particle with Heisenberg position operator $\phi(t)$ in a potential $P(\phi)$. (In Section 2.1, we explained the case $d = 4$. It is easy to go back and change the formulae to be valid for a general value of d.)

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

For the case of ϕ^4 theory, with

$$P(\phi) = m^2\phi^2/2 + g\phi^4/4!, \tag{2.2.1}$$

solutions are rigorously known to exist if $d = 2$ or 3 (Glimm & Jaffe (1981)). If $d > 4$ then no non-trivial solution exists (Aizenman (1981)). The case $d = 4$ is difficult; the difficulty is to perform renormalization of the ultra-violet divergences beyond perturbation theory. As we will see the theory at $d = 4$ is 'exactly renormalizable' in perturbation theory; this is the most interesting case. For the most part we will ignore the difficulties in going beyond perturbation theory. We will return to this problem in Section 7.10 when we discuss the application of the renormalization group outside of perturbation theory.

If we ignore, temporarily, the renormalization problem, then a solution for the theory can be found in terms of a functional integral. The formula for the Green's functions is written as

$$G_N(x_1, \ldots, x_N) = \mathcal{N} \int [\mathrm{d}A] \mathrm{e}^{\mathrm{i}S[A]} A(x_1) \ldots A(x_N). \tag{2.2.2}$$

(See Chapter 9 of Itzykson & Zuber (1980), or see Glimm & Jaffe (1981).) On the right-hand side of this equation $A(x)$ represents a classical field, and the integration is over the value of $A(x)$ at every space-time point. The result of the integral in (2.2.2) is the N-point Green's function for the corresponding quantum field, ϕ. In the integrand appears the classical action, which is

$$S[A] = \int \mathrm{d}^4x \mathscr{L}. \tag{2.2.3}$$

The normalization factor $\mathcal{N}$ is to give $\langle 0|0\rangle = 1$, so that

$$\mathcal{N} = \left\{ \int [\mathrm{d}A] \mathrm{e}^{\mathrm{i}S[A]} \right\}^{-1}. \tag{2.2.4}$$

Equivalent to (2.2.2) is the integral for the generating functional of Green's functions:

$$Z[J] = \mathcal{N} \int [\mathrm{d}A] \exp\left\{ \mathrm{i}S[A] + \int \mathrm{d}^4x J(x) A(x) \right\}, \tag{2.2.5}$$

where $J(x)$ is an arbitrary function. Functionally differentiating with respect to $J(x)$ gives the Green's functions, e.g.,

$$\langle 0| T\phi(x)\phi(y)|0\rangle = \frac{1}{Z[0]} \frac{\delta^2}{\delta J(x) \delta J(y)} Z[J] \bigg|_{(J=0)}. \tag{2.2.6}$$

It is somewhat delicate to make precise the definition of the integration over A. The principal steps are:

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

(1) 'Wick-rotate' time to imaginary values: $t = -i\tau$, so that space-time is Euclidean. The exponent in the integral is then:

$$-S_{\text{Eucl}}[A] = -\int d\tau\, d^3x[-\partial A^2/2 + P(A)]. \tag{2.2.7}$$

With our metric, we have $\partial A^2 = -(\partial A/\partial\tau)^2 - \vec{\nabla}A^2$. We may subtract out from $\mathscr{L}$ the minimum value of $P(A)$; this subtraction gives an overall factor in the functional integral, and it cancels between the integral and the normalization factor (2.2.4). Therefore the Euclidean action S_{Eucl} is positive definite. The factor $\exp(-S_{\text{Eucl}})$ gives much better convergence for large A and for rapidly varying A than does $\exp(iS)$ in Minkowski space.

(2) Replace space-time by a finite lattice. We may choose a cubic lattice with spacing a. Its points are then

$$x^\mu = n^\mu a.$$

where the n^μ's are integers. They are bounded to keep $\vec{x}$ inside a spatial box of volume V and to keep τ within a range $-T/2$ to $+T/2$. The integral

$$\int [dA] A(x_1) \ldots A(x_N) \exp(-S_{\text{Eucl}}[A]) \tag{2.2.8}$$

is now an absolutely convergent ordinary integral over a finite number of variables. The action S_{Eucl} is given its obvious discrete approximation.

(3) Take the continuum limit $a \to 0$, and the limits of infinite volume V and infinite time T.

(4) Analytically continue back to Minkowski space-time.

The difficulties occur at step 3. Taking the limits of infinite T and V gives divergences of exactly the sort associated with taking the thermodynamic limit of a partition function – see below. Further divergences occur when the continuum limit $a \to 0$ is taken. In addition, the canonical derivation of (2.2.2) gives an overall normalization factor which goes to infinity as $a \to 0$ or as the number of space-time points goes to infinity; this factor is absorbed by the normalization $\mathscr{N}$.

The limits of infinite volume and time are under good control. They are literally thermodynamic limits of a classical statistical mechanical system in four spatial dimensions. Recall, for example, that in ϕ^4 theory one can write

$$S[A] = g^{-1} \int d^4x(\partial\hat{A}^2/2 - m^2\hat{A}^2/2 - \hat{A}^4/4!) \tag{2.2.9}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

where $\hat{A} = g^{1/2}A$. Thus the integral $\int [dA] \exp(-S[A])$ is proportional to

$$\int [d\hat{A}] \exp\{-(1/g)S[A \to \hat{A}, g \to 1]\}. \tag{2.2.10}$$

This is the partition function of a classical system at temperature $1/g$, when the phase space is spanned by the field $\hat{A}$, and when the energy of a given configuration is

$$\hat{S}_{\text{Eucl}}[\hat{A}] = \int d^4x(-\partial\hat{A}^2/2 + m^2\hat{A}^2/2 + \hat{A}^4/4!).$$

The identity between Euclidean field theory and certain classical statistical mechanics systems has been fruitful both in working out the rigorous mathematical treatment of quantum field theory (Glimm & Jaffe (1981)) and in finding new ways to treat thermodynamic problems (Wilson & Kogut (1974)). As is particularly emphasized in Wilson's work, there is a lot of cross-fertilization between field theory and the theory of phase transitions. The methods of the renormalization group are common to both fields, and the continuum limit in field theory can be usefully regarded as a particular type of second-order phase transition.

The thermodynamic limit gives a factor $\exp(-\rho TV)$, where ρ is the ground state energy-density. This factor is clearly cancelled by $\mathcal{N}$. All the remaining divergences are associated with the continuum limit $a \to 0$. These are the divergences that form the subject of renormalization. They are called the ultra-violet (UV) divergences.

One notational change needs to be made now. In more complicated theories, there will be several fields, and the functional-integral solution of such a theory involves an integral over the values of a classical field for each quantum field. It is convenient to have a symbol for each classical field that is clearly related to the corresponding quantum field. The standard notation is to use the same symbol. Thus we change the integration variable in (2.2.2) from $A(x)$ to $\phi(x)$, with the result that

$$\langle 0|T\phi(x_1)\ldots\phi(x_N)|0\rangle = \mathcal{N}\int [d\phi] e^{iS[\phi]}\phi(x_1)\ldots\phi(x_N). \tag{2.2.11}$$

This is somewhat of an abuse of notation. However, it is usually obvious whether one is using ϕ to mean the quantum field, as on the left-hand side, or to mean the corresponding classical field, as on the right-hand side.

2.3 Renormalization

The difficult limit is the continuum limit $a \to 0$. There are divergences in this limit; this has been known from the earliest days of quantum elec-

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

trodynamics (e.g., Oppenheimer (1930)). It is possible to say that the UV divergences mean that the theory makes no physical sense, and that the subject of interacting quantum field theories is full of nonsense (Dirac (1981)). Luckily we can do better, for our ultimate aim need not be to construct a field theory literally satisfying (2.1.2)–(2.1.5). Rather, our aim is to construct a relativistic quantum theory with a local field as its basic observable. These requirements are satisfied if we construct a collection of Green's functions satisfying sensible physical properties (for example, as formulated in the Osterwalder–Schrader axioms – see Glimm & Jaffe (1981)). We may further ask that we find a theory that is close in some sense to satisfying the defining equations (2.1.2)–(2.1.5). Combining the functional integral with suitable renormalizations of the parameters of the theory satisfies these requirements.

The basic idea of renormalization comes from the observation that in one-loop graphs the divergences amount to shifts in the parameters of the action. For example, they change the mass of the particles described by $\phi(x)$ from the value m to some other effective value, which is infinite if m is finite. Renormalization is then the procedure of cancelling the divergences by adjusting the parameters in the action. To be precise, let us consider the ϕ^4 theory with

$$\mathscr{L} = (\partial A_0)^2/2 - m_0^2 A_0^2/2 - g A_0^4/4! + \Lambda_0. \tag{2.3.1}$$

The subscript zero is here used to indicate so-called bare quantities, i.e., those that appear in the Lagrangian when the $(\partial A_0)^2/2$ term has unit coefficient. (We also introduce a constant term. It will be used to cancel a UV divergence in the energy density of the vacuum.) Then we rescale the field by writing

$$A_0 = Z^{1/2} A, \tag{2.3.2}$$

so that, in terms of the 'renormalized field' A, the Lagrangian is

$$\begin{aligned} \mathscr{L} &= Z\partial A^2/2 - m_0^2 Z A^2/2 - g_0 Z^2 A^4/4! \\ &= Z\partial A^2/2 - m_B^2 A^2/2 - g_B A^4/4!. \end{aligned} \tag{2.3.3}$$

We have dropped Λ_0 from $\mathscr{L}$ since it has no effect on the Green's functions.

The Green's functions of the quantum field ϕ are now obtained by using (2.3.3) as the Lagrangian in the functional integral (2.2.2). We let Z, m_0, and g_0 be functions of the lattice spacing a, and we choose these functions (if possible) so that the Green's functions of ϕ are finite as $a \to 0$. If this can be done, then we have succeeded in constructing a continuum field theory, and it is termed 'renormalizable'. The theory may be considered close to solving

(2.1.2)–(2.1.5). This is because the theory is obtained by taking a discrete (i.e., lattice) version of the equations and then taking a somewhat odd continuum limit.

We will call m_0 the bare mass, and g_0 the bare coupling, and we will call Z the wave-function, or field-strength, renormalization. It is also common to call m_B and g_B the bare mass and coupling; but for the sake of consistency we will not do this in this book.

Another way of viewing the renormalization is to write (2.3.3) as

$$\mathscr{L} = \partial A^2/2 - m^2 A^2/2 - gA^4/4! + \delta Z \partial A^2/2 - \delta m^2 A^2/2 - \delta g A^4/4!. \tag{2.3.4}$$

We will call the first three terms the basic Lagrangian and the last three the counterterm Lagrangian. The renormalized mass m and the renormalized coupling g are finite quantities held fixed as $a \to 0$. The counterterms $\delta Z = Z - 1$, $\delta m^2 = m_B^2 - m^2$, and $\delta g = g_B - g$ are adjusted to cancel the divergences as $a \to 0$. This form of the Lagrangian is useful in doing perturbation theory; we treat $\partial A^2/2 - m^2 A^2/2$ as the free Lagrangian and the remainder as interaction. The expansion is in powers of the renormalized coupling g. The counterterms are expanded in infinite series, each term cancelling the divergences of one specific graph.

The form (2.3.4) for $\mathscr{L}$ also exhibits the fact that the theory has two independent parameters, m and g. The counterterms are functions of m, g, and of a.

We will discuss these issues in much greater depth in the succeeding chapters. For the moment it is important to grasp the basic ideas:

(1) The self-interactions of the field create, among other things, dynamical contributions to the mass of the particle, to the potential between particles, and to the coupling of the field to the single particle state. Thus the measured values of these parameters are renormalized relative to the values appearing in the Lagrangian.
(2) These contributions, or renormalizations, are infinite, in many cases. The most important theorem of renormalization theory is that they are the only infinities, in the class of theories called 'renormalizable'.
(3) The infinities are cancelled by wave-function, mass, and coupling counterterms, so that the net effect of the interactions is finite.
(4) To make quantitative the sizes of the infinities, the theory is constructed as the continuum limit of a lattice theory. The infinities appear as divergences when the lattice spacing goes to zero.

2.4 Ultra-violet regulators

In the last sections we showed how to construct field theories by defining the functional integral as the continuum limit of a lattice theory. Ultra-violet divergences appear as divergences when the lattice spacing, a, goes to zero, and are removed by renormalization counterterms. The lattice therefore is a regulator, or cut-off, for the UV divergences.

To be able to discuss the divergences quantitively and to construct a theory involving infinite renormalizations, it is necessary to use some kind of UV cut-off. Then the theory is obtained as an appropriate limit when the cut-off is removed. There are many possible ways of introducing a cut-off, of which going to a lattice is only one example. The lattice appears to be very natural when working with the functional integral. But it is cumbersome to use within perturbation theory, especially because of the loss of Poincaré invariance. There are two other very standard methods of making an ultra-violet cut-off: the Pauli–Villars method, and dimensional regularization.

The Pauli–Villars (1949) method is very traditional. In its simplest version it consists of replacing the free propagator $\mathrm{i}/(p^2 - m^2)$ in a scalar field theory by

$$\begin{aligned} S_F(p, m; M) &= \frac{\mathrm{i}}{p^2 - m^2} - \frac{\mathrm{i}}{p^2 - M^2} \\ &= \frac{\mathrm{i}}{(p^2 - m^2)} \frac{(m^2 - M^2)}{(p^2 - M^2)}. \end{aligned} \tag{2.4.1}$$

As $M \to \infty$, this approaches the original propagator. The behavior for large p has clearly been improved. Thus the degree of divergence of the Feynman graphs in the theory has been reduced. All graphs in the ϕ^4 theory, except for the one-loop self-energy are in fact made finite. In the ϕ^4 theory it is necessary to use a more general form in order to make all graphs finite:

$$\begin{aligned} S_F(p, m; M_1, M_2) &= \frac{\mathrm{i}}{(p^2 - m^2)} - \frac{\mathrm{i}}{(p^2 - M_1^2)} \frac{(m^2 - M_2^2)}{(M_1^2 - M_2^2)} \\ &\quad - \frac{\mathrm{i}}{(p^2 - M_2^2)} \frac{(m^2 - M_1^2)}{(M_2^2 - M_1^2)} \\ &= \frac{\mathrm{i}}{(p^2 - m^2)} \frac{(M_1^2 - m^2)}{(p^2 - M_1^2)} \frac{(M_2^2 - m^2)}{(p^2 - M_2^2)}. \end{aligned} \tag{2.4.2}$$

It is usually convenient to set $M_1 = M_2$.

Now the regulated propagator has extra poles at $p^2 = M^2$, or at $p^2 = M_1^2$ and $p^2 = M_2^2$. Since one of the extra poles has a residue of the opposite sign to the pole at $p^2 = m^2$, the regulated theory cannot be completely physical.

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

It is normally true that a theory with an ultra-violet cut-off has some unphysical features.

Perhaps the most convenient regulator for practical calculations is dimensional regularization. There it is observed that the UV divergences are removed by going to a low enough space-time dimension d, so d is treated as a continuous variable. In perturbation theory this can be done consistently (Wilson (1973)), as we will see when we give a full treatment of dimensional regularization in Chapter 4. However it has not been possible to make it work non-perturbatively, so it cannot at present be regarded as a fundamental method.

Since it is only the renormalized theory with no cut-off that is of true interest, the precise method of cut-off is irrelevant. In fact, all methods of ultra-violet cut-off are equivalent, at least in perturbation theory. The differences are mainly a matter of practical convenience (or of personal taste). Thus dimensional regularization is very useful for perturbation theory. But the lattice method is maybe most powerful when working beyond perturbation theory; it is possible, for example, to compute the functional integral numerically by Monte-Carlo methods (Creutz (1980, 1983), and Creutz & Moriarty (1982)).

Within perturbation theory one need not even use a cut-off. Zimmermann (1970, 1973a) has shown how to apply the renormalization procedure to the integrands rather than to the integrals for Feynman graphs. The lack of fundamental dependence on the procedure of cut-off is thereby made manifest. The application of this procedure to gauge theories, especially, is regarded by most people as cumbersome.

2.5 Equations of motion for Green's functions

We have defined a collection of Green's functions by the functional integral (2.2.2). (Implicit in the definition are a certain number of limiting procedures, as listed below (2.2.6).) This definition we will take as the basis for the rest of our work. First we must check that it in fact gives a solution of the theory. This means, in particular, that we are to derive the equations of motion (2.1.10) for the Green's functions, thus ensuring that both the operator equation of motion (2.1.2) and the commutation relations (2.1.5) hold. (For the remainder of this chapter we will not specify the details of how renormalization affects these results.)

It is convenient to work with the generating functional (2.2.5). We make the change of variable $A(x) \to A(x) + \varepsilon f(x)$, where ε is a small number, and $f(x)$ is an arbitrary function of x^μ. Since the integration measure is invariant

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

under this shift, the value of the integral is unchanged:

$$\int [\mathrm{d}A] \exp\left\{ \mathrm{i}S[A+\varepsilon f] + \int (A+\varepsilon f) J \right\} = \int [\mathrm{d}A] \exp\left\{ \mathrm{i}S[A] + \int AJ \right\}. \tag{2.5.1}$$

Picking out the terms of order ε gives

$$\int \mathrm{d}^4 y f(y) \int [\mathrm{d}A] \exp\left\{ \mathrm{i}S[A] + \int AJ \right\} \left[\mathrm{i} \frac{\delta S}{\delta A(y)} + J(y) \right] = 0, \tag{2.5.2}$$

where, as usual, we define the functional derivative

$$\frac{\delta S}{\delta A(y)} = -\Box A - \frac{\mathrm{d}P}{\mathrm{d}A}. \tag{2.5.3}$$

Since $f(y)$ is arbitrary, we get

$$\int [\mathrm{d}A] \exp\left\{ \mathrm{i}S[A] + \int AJ \right\} \left[\mathrm{i} \frac{\delta S}{\delta A(y)} + J(y) \right] = 0. \tag{2.5.4}$$

Functionally differentiating N times with respect to J, followed by setting $J = 0$, gives the equation of motion (2.1.10). For example,

$$\begin{aligned}
0 &= \mathcal{N} \frac{\delta}{\delta J(x)} [\text{left-hand side of (2.5.4)}]_{J=0} \\
&= \mathcal{N} \int [\mathrm{d}A] \mathrm{e}^{\mathrm{i}S[A]} \left[A(x) \mathrm{i} \frac{\delta S}{\delta A(y)} + \delta^{(4)}(x-y) \right] \\
&= \mathcal{N} \int [\mathrm{d}A] \mathrm{e}^{\mathrm{i}S[A]} \{ \mathrm{i}A(x)[-\Box A(y) - P'(A(y))] + \delta^{(4)}(x-y) \} \\
&= -\mathcal{N} \mathrm{i} \Box_y \int [\mathrm{d}A] \mathrm{e}^{\mathrm{i}S[A]} A(x) A(y) \\
&\quad - \mathrm{i}\mathcal{N} \int [\mathrm{d}A] \mathrm{e}^{\mathrm{i}S[A]} A(x) P'(A(y)) + \delta^{(4)}(x-y) \\
&= -\mathrm{i}\Box_y \langle 0 | T\phi(x)\phi(y) | 0 \rangle - \mathrm{i} \langle 0 | T\phi(x) P'(\phi(y)) | 0 \rangle + \delta^{(4)}(x-y) \\
&= \mathrm{i} \langle 0 | T\phi(x) \frac{\delta S}{\delta \phi(y)} | 0 \rangle + \delta^{(4)}(x-y),
\end{aligned} \tag{2.5.5}$$

which is equivalent to (2.1.9). Note that in the fourth line we have exchanged the order of integration and of differentiation for the $\Box_y$ term. We have also used the normalization condition (2.2.4). It is important that the derivative of the quantum field (next-to-last line) is *outside* the time ordering, and $\delta S/\delta\phi(y)$ in the last line is defined to be a shorthand for the combination of operators in the previous line.

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

This is somewhat paradoxical since we have the operator equation of motion:

$$0 = \frac{\delta S}{\delta \phi} = -\Box\phi - P'(\phi),$$

from which it is tempting to deduce that the Green's function $\langle 0|T\phi(x)\delta S/\delta\phi(y)|0\rangle$ should be zero. However, in view of the work above it is convenient to define this Green's function by the functional-integral formula

$$\langle 0|T\phi\frac{\delta S}{\delta\phi(y)}|0\rangle = \mathcal{N}\int[\mathrm{d}A]\mathrm{e}^{\mathrm{i}S[A]}A(x)\frac{\delta S}{\delta A(y)}.$$

Then, as we have seen, the $\Box_y$ is implicitly outside the time-ordering. Bringing it inside the time-ordering gives a commutator, so that we get the δ-function term in (2.1.9) or (2.5.5).

The momentum-space version of the equation of motion (2.1.10) is often useful. We define the momentum-space Green's functions

$$\begin{aligned}\bar{G}_N(p_1,\ldots,p_N) &= \int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_N \exp\{\mathrm{i}(p_1\cdot x_1+\cdots+p_N\cdot x_N)\}G_N(x_1,\ldots,x_N)\\ &= \bar{G}_N(p_1,\ldots,p_N)(2\pi)^4\delta^{(4)}(p_1+\cdots+p_N).\end{aligned} \tag{2.5.6}$$

The momenta p_j are to be regarded as flowing out of the Green's functions. Translation invariance of the theory implies the δ-function for momentum conservation that is explicitly factored out in the last line of (2.5.6). A convenient notation (which we will use often) is to write

$$\tilde{G}_N(p_1,\ldots,p_N) = \langle 0|T\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_N)|0\rangle. \tag{2.5.7}$$

Implicit in this formula is the definition that the integrals over x defining the momentum-space field $\tilde{\phi}(p)$ are all taken outside the time-ordering, as stated in (2.5.6). We will use a tilde over the symbol for any function to indicate the Fourier-transformed function.

Fourier transformation of the equation of motion (2.1.10) gives

$$\begin{aligned}&-q^2\langle 0|T\tilde{\phi}(q)\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_N)|0\rangle\\ &\quad+\langle 0|TP'(\tilde{\phi})(q)\tilde{\phi}(p_i)\ldots\tilde{\phi}(p_N)|0\rangle\\ &=-\mathrm{i}\sum_{j=1}^{N}\langle 0|T\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_{j-1})\tilde{\phi}(p_{j+1})\ldots\tilde{\phi}(p_N)|0\rangle(2\pi)^4\delta^{(4)}(q+p_j).\end{aligned} \tag{2.5.8}$$

2.6 Symmetries

We now turn to the consequences of symmetries. As we will see, there are many interesting problems in renormalization theory that stem from the

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

following question: If a classical field theory has certain symmetries, does the symmetry survive after quantization? Generally, it is the need for renormalization of the theory that makes this a non-trivial question.

The symmetry properties are expressed in terms of Green's functions by the Ward identities. (Historically the earliest example was found by Ward (1950) in QED.) If the symmetry is not preserved by quantization there are extra terms called anomalies. In many cases there are no anomalies, so we will derive the Ward identities in this section ignoring the subtleties that in some cases lead to anomalies. Discussion of anomalous cases is given in Chapter 13.

Consider a theory of N fields which we collectively denote by a vector $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$. Our discussion is general enough to include the case of fields with spin. We consider a symmetry group of the action $S[\boldsymbol{\phi}]$. This is a group of transformations on the classical fields

$$\boldsymbol{\phi} \to \mathbf{F}[\boldsymbol{\phi}; \boldsymbol{\omega}] \equiv \boldsymbol{\phi}', \tag{2.6.1}$$

which leaves the action invariant:

$$S[\boldsymbol{\phi}'] = S[\boldsymbol{\phi}]. \tag{2.6.2}$$

Here $\boldsymbol{\omega} = (\omega^\alpha)$ is a set of parameters of the group, which we assume here to be a Lie group, i.e., the ω's take on a continuous set of values. We let $\boldsymbol{\omega} = \mathbf{0}$ be the identity: $\mathbf{F}[\boldsymbol{\phi}; \mathbf{0}] = \boldsymbol{\phi}$. It is easiest to work with infinitesimal transformations:

$$\delta\phi_i \equiv \omega^\alpha \frac{\partial F_i}{\partial \omega^\alpha}[\boldsymbol{\phi}; \boldsymbol{\omega}]\bigg|_{\omega = 0} \equiv \omega^\alpha \delta_\alpha \phi_i. \tag{2.6.3}$$

(A summation convention on α is understood.)

In the quantum theory the symmetry is implemented as a unitary representation $U(\boldsymbol{\omega})$ of the group on the Hilbert space of states such that

$$U(\boldsymbol{\omega})\boldsymbol{\phi}U(\boldsymbol{\omega})^{-1} = F[\boldsymbol{\phi}; \boldsymbol{\omega}]. \tag{2.6.4}$$

Since the representation is unitary, we may parametrize the group so that

$$U(\boldsymbol{\omega}) = \exp(\mathrm{i}\omega^\alpha Q_\alpha), \tag{2.6.5}$$

where the generators Q_α are hermitian operators which represent the Lie algebra of the group:

$$[Q_\alpha, Q_\beta] = \mathrm{i}c_{\alpha\beta\gamma} Q_\gamma. \tag{2.6.6}$$

The normalizations are such that the structure constants $c_{\alpha\beta\gamma}$ are totally antisymmetric. The infinitesimal transformations are then given by:

$$\delta_\alpha \phi_i(x) = \mathrm{i}[Q_\alpha, \phi_i(x)]. \tag{2.6.7}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

There are a number of special cases, each with its own special features:

(1) Global internal symmetry: A finite-dimensional Lie group acts on the fields at each point of space-time, with the same transformation at each point. Thus

$$\delta_\alpha \phi_i = -\mathrm{i}(t_\alpha)_i{}^j \phi_j, \tag{2.6.8}$$

where the t_α form a hermitian matrix representation of the Lie algebra:

$$[t_\alpha, t_\beta] = \mathrm{i} c_{\alpha\beta\gamma} t_\gamma. \tag{2.6.9}$$

Single-particle states carrying this representation are annihilated by ϕ_i. The Lagrangian is invariant.

(2) Global space-time symmetry: The group effectively is a transformation on space-time; the Poincaré group and its extensions are the usual cases. For a Poincaré transformation $x^\mu \to \Lambda^\mu_\nu x^\nu + a^\mu$, we have the corresponding transformation of the fields:

$$\phi_i(x) \to \phi_j(\Lambda x + a) R(\Lambda)^j{}_i. \tag{2.6.10}$$

Here R is a finite-dimensional matrix representation of the Lorentz group (never unitary if non-trivial), acting on the spin indices of $\boldsymbol{\phi}$. The Lagrangian is not invariant. It transforms as

$$\mathscr{L}[\boldsymbol{\phi}, x] \to \mathscr{L}[\phi, \Lambda x + a],$$

so that the action $S = \int \mathrm{d}^4 x \mathscr{L}$ is invariant. Infinitesimal transformations of ϕ involve the derivative of ϕ.

(3) Global chiral symmetry: This looks like a global internal symmetry but acts differently on the left- and right-handed parts of Dirac fields (which we have yet to discuss). Anomalies are often present – see Chapter 13.

(4) Supersymmetry: This is a generalized type of symmetry where Bose and Fermi fields are related (Fayet & Ferrara (1977)). The only case that we will discuss is the BRS-invariance (Becchi, Rouet & Stora (1975)) of a gauge theory.

(5) Gauge, or local, symmetry: Any of the above symmetries may be extended to a symmetry whose parameters depend on x: $\omega = \omega(x)$. In quantum theories, these are not really implemented by unitary transformations. Their treatment is rather special. The elementary examples are general coordinate invariance in General Relativity, and gauge invariance in electromagnetism.

The basic tool for discussing symmetries is Noether's theorem, which relates them to conservation laws. This theorem in its most straightforward form applies only to symmetries of the first three types.

For a global symmetry, Noether's theorem asserts that a conserved current j_α^μ exists for each generator of a symmetry. Let the Lagrangian have the infinitesimal transformation

$$\mathscr{L} \to \mathscr{L} + \omega^\alpha \delta_\alpha \mathscr{L} = \mathscr{L} + \omega^\alpha \partial_\mu Y_\alpha^\mu, \tag{2.6.11}$$

so that the action is invariant. Define

$$j_\alpha^\mu = \sum_i \delta_\alpha \phi_i \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} - Y_\alpha^\mu. \tag{2.6.12}$$

Then the equations of motion imply conservation of j_α^μ, i.e.,

$$\partial j_\alpha^\mu / \partial x^\mu = 0. \tag{2.6.13}$$

The generators of the symmetry group are

$$Q_\alpha = \int \mathrm{d}^3 x \, j_\alpha^0. \tag{2.6.14}$$

The canonical commutation relations imply that

$$\begin{aligned} [j_\alpha^0(x), \phi_i(y)] &= -\mathrm{i}\delta_\alpha \phi_i(x) \delta^{(3)}(x - y), \quad (\text{if } x^0 = y^0), \\ [Q_\alpha, \phi_i(y)] &= -\mathrm{i}\delta_\alpha \phi_i(y), \end{aligned} \tag{2.6.15}$$

as required by (2.6.7).

We need to consider not only transformations that are symmetries of the quantum theory, but also 'broken symmetries'. There are several cases (not mutually exclusive). Let us define them, since there is a certain amount of confusion in the literature about the terminology:

(1) Explicit breaking: The classical action has a non-invariant term. If $\delta_\alpha \mathscr{L} = \partial_\mu Y_\alpha^\mu + \Delta_\alpha$ then the Noether currents are not conserved: $\partial_\mu j_\alpha^\mu = \Delta_\alpha$. An important case is where this term is small, so that it can be treated as a perturbation.

(2) Anomalous breaking: Even though the classical action is invariant, the quantum theory is not, and there is no conserved current. The classical action is important for the quantum theory, since it appears in the functional integral defining the theory. The cause of anomalous breaking is generally an ultra-violet problem: $\partial_\mu j_\alpha^\mu \neq 0$ in the UV cut-off theory, and the non-conservation does not disappear when the cut-off is removed. (Cases are conformal transformations and some chiral theories.)

(3) Spontaneous breaking: The action is invariant and the currents are conserved (in the quantum theory), but the vacuum is not invariant under the transformations.

Whether or not a symmetry is broken either spontaneously or anomalously is a dynamical question. That is, one must solve the theory, at least

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

partially, to find the answer. Frequently, perturbation theory is adequate to do this and lowest order or next-to-lowest order calculations suffice. Renormalization is an integral part of treating anomalous breaking (see Chapter 13), while renormalization-group methods are sometimes necessary in treating spontaneously broken symmetry (Coleman & Weinberg (1973)).

The case of spontaneous symmetry breaking that is not visible in perturbation theory is often termed dynamical (Jackiw & Johnson (1973), Cornwall & Norton (1973), and Gross (1976)). Anomalous breaking is sometimes called spontaneous, but this is a bad terminology, because it gives two very different phenomena the same name.

2.7 Ward identities

Ward identities express in terms of Green's functions the consequences of a symmetry (whether or not it is broken). One derivation applies the equation of motion (2.1.10) to the divergence of a Green's function of the current j^μ_α. There are two terms: one in which the current is differentiated, and one in which the θ-functions defining the time-ordered product are differentiated. Thus a Ward identity expresses not only conservation of its current but also the commutation relation (2.6.15), which is equivalent to the transformation law. The Ward identities are central to a discussion of the renormalization of a theory with symmetries, expecially if spontaneously broken.

Our derivation of Ward identities begins by making the following change of variable:

$$A_i(x) \to A_i(x) + f^\alpha(x)\delta_\alpha A_i(x) \tag{2.7.1}$$

in the functional integral for the generating functional $Z[\mathbf{J}]$. Here $\delta_\alpha A_i$ is, as before, the variation of the field A_i under a symmetry transformation, and $f^\alpha(x)$ is a set of arbitrary complex-valued functions that vanish rapidly as $x \to \infty$. We get

$$Z[\mathbf{J}] = \int [\mathrm{d}\mathbf{A}] \exp\{\mathrm{i}S[\mathbf{A} + f^\alpha \delta_\alpha \mathbf{A}] + J^i(A_i + f^\alpha \delta_\alpha A_i)\}. \tag{2.7.2}$$

(Here we assumed that the measure is invariant under the change of variables (2.7.1).) The terms in (2.7.2) that are linear in f^α give

$$\begin{aligned} 0 &= \int [\mathrm{d}\mathbf{A}] \exp\left(\mathrm{i}S + \int \mathbf{J}\cdot\mathbf{A}\right)\{-\delta S[\mathbf{A} + f^\alpha \delta_\alpha \mathbf{A}]/\delta f^\alpha(y) + \mathrm{i}J^i \delta_\alpha A_i(y)\} \\ &= \int [\mathrm{d}\mathbf{A}] \exp\left(\mathrm{i}S + \int \mathbf{J}\cdot\mathbf{A}\right)\{\partial_\mu j^\mu_{\mathrm{cl},\alpha}(y) + \mathrm{i}J^i \delta_\alpha A_i(y)\}. \end{aligned} \tag{2.7.3}$$

Here $j^\mu_{\mathrm{cl},\alpha}$ is the Noether current in the classical theory.

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

The Ward identities follow by functionally differentiating with respect to the sources $\mathbf{J}(x)$. Thus one differentiation gives

$$\frac{\partial}{\partial y^\mu}\langle 0|Tj^\mu_\alpha(y)\phi_i(x)|0\rangle = -\mathrm{i}\delta^{(4)}(x-y)\langle 0|\delta_\alpha\phi_i(y)|0\rangle, \tag{2.7.4}$$

while a double differentiation gives

$$\begin{aligned}\frac{\partial}{\partial y^\mu}&\langle 0|Tj^\mu_\alpha(y)\phi_i(w)\phi_j(x)|0\rangle \\ &= -\mathrm{i}\delta^{(4)}(w-y)\langle 0|T\delta_\alpha\phi_i(y)\phi_j(x)|0\rangle \\ &\quad -\mathrm{i}\delta^{(4)}(x-y)\langle 0|T\phi_i(w)\delta_\alpha\phi_j(y)|0\rangle.\end{aligned} \tag{2.7.5}$$

Note that, just as in our derivation of the equation of motion for Green's functions in Section 2.5, the derivative $\partial/\partial y^\mu$ is outside the time-ordering. The general case is:

$$\begin{aligned}\frac{\partial}{\partial y^\mu}&\langle 0|Tj^\mu_\alpha(y)\prod_{i=1}^{N}\phi_{n_i}|x_i)|0\rangle \\ &= -\mathrm{i}\sum_{j=1}^{N}\delta^{(4)}(y-x_j)\langle 0|T\delta_\alpha\phi_{n_j}(y)\prod_{i\neq j}\phi_{n_i}(x_i)|0\rangle.\end{aligned} \tag{2.7.6}$$

Important consequences of these Ward identities are obtained by integrating over all $\vec{y}$ (with y^0 fixed). The spatial derivatives give a surface term, which vanishes, so that we have, for example,

$$\int \mathrm{d}^3y\frac{\partial}{\partial y^0}\langle 0|Tj^0_\alpha(y)\phi_i(x)|0\rangle = -\mathrm{i}\delta(x^0-y^0)\langle 0|\delta_\alpha\phi_i(x)|0\rangle.$$

The spatial integral of j^0 is just the charge Q^0. The time derivative acts either on the charge or on the δ-functions defining the time-ordering; so we find that

$$\begin{aligned}&\langle 0|T\mathrm{d}Q_\alpha/\mathrm{d}t\,\phi_i(x)|0\rangle + \langle 0|[Q_\alpha,\phi_i(x)]|0\rangle\delta(x^0-y^0) \\ &\quad = -\mathrm{i}\delta(x^0-y^0)\langle 0|\delta_\alpha\phi_i(y)|0\rangle.\end{aligned} \tag{2.7.7}$$

In this equation and its generalizations from (2.7.6), we may choose the times of the fields $\phi_i(x_i)$ not to coincide with y^0. Therefore an arbitrary Green's function of $\mathrm{d}Q_\alpha/\mathrm{d}t$ is zero, so that the operator $\mathrm{d}Q_\alpha/\mathrm{d}t$ is zero. The remaining part of (2.7.7) therefore gives:

$$\langle 0|[Q_\alpha,\phi_i(x)]|0\rangle = -\mathrm{i}\langle 0|\delta_\alpha\phi_i(y)|0\rangle. \tag{2.7.8}$$

From (2.7.8) and its generalizations with more fields, we find that the Q_α's have the correct commutation relations with the elementary fields ϕ_i to be the generators of the symmetry group.

Finally, another specialization of (2.7.6) is to integrate it over all y^μ and to

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

drop the resulting surface term. The result is that

$$0 = \sum_{j=1}^{N} \langle 0|T\delta_\alpha \phi_{n_j}(x_j) \prod_{i \neq j} \phi_{n_i}(x_i)|0\rangle$$

$$= \delta_\alpha \langle 0|T \prod_{i=1}^{N} \phi_{n_i}(x_i)|0\rangle. \tag{2.7.9}$$

All the above equations are true for the case of a completely unbroken symmetry. The derivation breaks down at the first step if we have anomalous breaking. (In Chapter 13 we will discuss the anomaly terms that must then be inserted in the Ward identities to make them correct.) For an explicitly broken symmetry, where $\partial \cdot j = \Delta \neq 0$, we must add a term

$$\langle 0|T\Delta_\alpha(y) \prod_{i=1}^{N} \phi_{n_i}(x_i)|0\rangle \tag{2.7.10}$$

to the right-hand side of (2.7.6).

In the case of a spontaneously broken theory the basic Ward identities (2.7.4)–(2.7.6) remain true – we still have an exact symmetry. But the integrated Ward identities (2.7.7) and (2.7.9) are no longer true. Equation (2.7.9) must be false if the vacuum is not invariant, and the derivation fails because the surface term is not zero. This is caused by the existence of zero-mass particles. These Nambu–Goldstone bosons (Goldstone, Salam & Weinberg (1962)) are characteristic of theories with a spontaneously broken symmetry.

2.8 Perturbation theory

As an example, consider again ϕ^4 theory, with classical Lagrangian

$$\mathscr{L} = Z(\partial A)^2/2 - m_B^2 A^2/2 - g_B A^4/4!. \tag{2.8.1}$$

We will expand the Green's functions in powers of the renormalized coupling g, for small g. To expand the functional-integral formula (2.2.2) in powers of g, we write

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1, \tag{2.8.2}$$

where $\mathscr{L}_0$ is the free Lagrangian:

$$\mathscr{L}_0 = (\partial A)^2/2 - m^2 A^2/2, \tag{2.8.3}$$

and $\mathscr{L}_1$ is the interaction Lagrangian:

$$\mathscr{L}_1 = -gA^4/4! + (Z-1)(\partial A)^2/2 - (m_B^2 - m^2)A^2/2 - (g_B - g)A^4/4!. \tag{2.8.4}$$

We will expand the renormalization counterterms, $Z-1$, $m_B^2 - m^2$, and

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

$g_B - g$, in powers of g, so that all of the terms in $\mathscr{L}_I$ have at least one power of g. The series expansion of the Green's functions is then obtained from (2.2.2) as:

$$G_N(x_1, x_2, \ldots, x_N) = \frac{\sum_{n=0}^{\infty} (i^n/n!) \int [dA] A(x_1) \cdots A(x_N) \left[\int d^4y \mathscr{L}_I(y) \right]^n \exp(iS_0[A])}{\sum_{n=0}^{\infty} (i^n/n!) \int [dA] \left[\int d^4y \mathscr{L}_I(y) \right]^n \exp(iS_0[A])}. \quad (2.8.5)$$

Here

$$S_0[A] = \int d^4y \mathscr{L}_0 = \int d^4y (\partial A^2/2 - m^2 A^2/2)$$

is the free action.

Each of the terms in the series is a Green's function in the free-field theory (aside from a common normalization), so (2.8.5) is equivalent to the Gell-Mann–Low (1951) formula:

$$G_N(x_1, \ldots, x_N) = \frac{\sum_{n=0}^{\infty} (i^n/n!) \left(\prod_{j=1}^{n} \int d^4y_j \right) \langle 0 | T \phi_F(x_1) \ldots \phi_F(x_N) \prod_{j=1}^{n} \mathscr{L}_I(y_i) | 0 \rangle}{\sum_{n=0}^{\infty} (i^n/n!) \left(\prod_{j=1}^{n} \int d^4y_j \right) \langle 0 | T \prod_{j=1}^{n} \mathscr{L}_I(y_j) | 0 \rangle}. \quad (2.8.6)$$

Here ϕ_F is a free quantum field of mass m. It is the field generated from the free Lagrangian $\mathscr{L}_0 = (\partial \phi_F)^2/2 - m^2 \phi_F^2/2$. Then $\mathscr{L}_I$ is the quantum interaction Lagrangian, $\mathscr{L} - \mathscr{L}_0$, which is a function of the free field ϕ_F.

To compute the integrals in (2.8.5) it suffices to compute the generating functional of free-field Green's functions:

$$Z_0[J] \equiv \frac{\int [dA] \exp\left(iS_0[A] + \int JA \right)}{\int [dA] \exp(iS_0[A])}. \quad (2.8.7)$$

This is done by completing the square, i.e., by making the following change of variable:

$$A(x) \to A(x) + \int d^4y G_F(x - y) J(y). \quad (2.8.8)$$

Here, $G_F(x)$ is the Feynman propagator satisfying

$$(\Box + m^2) G_F(x) = -i\delta^{(4)}(x), \quad (2.8.9)$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

and a boundary condition that, after rotating to Euclidean space by $x^0 = -i\tau$, $G_F(x) \to 0$ as $x \to \infty$. Thus

$$G_F(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k^2 - m^2 + i\varepsilon}. \tag{2.8.10}$$

The result is that

$$Z_0[J] = \exp\left\{\tfrac{1}{2}\int d^4x\, d^4y\, J(x) G_F(x-y) J(y)\right\}. \tag{2.8.11}$$

Green's functions of free fields are obtained by differentiating with respect to J; for example

$$\begin{aligned}\langle 0|T\phi(x)\phi(y)|0\rangle &= \frac{\delta^2 Z}{\delta J(x)\delta J(y)}\bigg|_{J=0} \\ &= G_F(x-y).\end{aligned} \tag{2.8.12}$$

We can now derive the well-known Feynman rules for the interacting theory from (2.8.6). These can be given either in momentum or coordinate space. In either case the Green's function G_N is written as a sum over all possible topologically distinct Feynman graphs. Each graph Γ consists of a number of vertices joined by lines. It has N 'external vertices', one for each $\phi(x_i)$, with one line attached, and some number, n, of interaction vertices. The interaction vertices are of several types, corresponding to the terms in the interaction Lagrangian (2.8.4). The vertex for the A^4 interaction has four lines attached and the vertices for the ∂A^2 and A^2 interactions have two lines attached. The value of the graph, denoted $I(\Gamma)$, is the integral over the position y_j of the n interaction vertices. The integrand is a product of factors:

(1) $G_F(w-z)$ for each line, where w and z are the positions of the vertices at its end.
(2) A combinatorial factor $1/S(\Gamma)$.
(3) $-ig_B$ for each A^4 interaction.
(4) $-i(m_B^2 - m^2)$ for each A^2 interaction.
(5) $-i(Z-1)\partial^2/\partial w^\mu \partial w_\mu$ for each $(\partial A)^2$ interaction: the derivatives with respect to w act on one of the propagators attached to the vertex.

For each Feynman graph a number of equal contributions arise in expanding (2.8.5). If Γ has no symmetries and if it has no counterterm vertices, then this number is $n!(4!)^n$ so that the explicit $n!$ in (2.8.5) and the $4!$ in each interaction are cancelled. Graphs with symmetries have a number of contributions smaller by a factor of the symmetry number $S(\Gamma)$. (For example, the self-energy graph Fig. 2.8.1 has $S = 6$.) The combinatorial factor is then the inverse of $S(\Gamma)$.

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

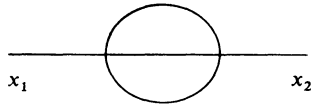

Fig. 2.8.1. A graph with symmetry factor $S = 6$.

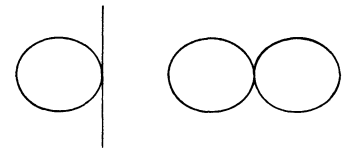

Fig. 2.8.2. A graph with a vacuum bubble.

The denominator of (2.8.5) is the sum of all graphs with no external lines. The result is to cancel all graphs in the numerator that have disconnected vacuum bubbles (like Fig. 2.8.2.).

In momentum space each line is assigned a (directed) momentum k. The Feynman rules are:

(1) A factor $i/[(2\pi)^4(k^2 - m^2 + i\varepsilon)]$ for a line with momentum k.
(2) A factor $(2\pi)^4$ times a momentum conservation δ-function for each vertex (external or interaction).
(3) An integral over the momentum of every line.
(4) A combinatorial factor $1/S(\Gamma)$.
(5) $-ig_B$ for each A^4 interaction.
(6) $-i(m_B^2 - m^2)$ for each A^2 interaction.
(7) $i(Z-1)p^2$ for each $(\partial A)^2$ interaction, where p is the momentum flowing on one of the propagators attached to the vertex.

The perturbation series in (2.8.5) need not be convergent, but only asymptotic. Let $G_{N,n}$ be the sum up to order g^n of the perturbation series for the Green's function G_N. Then it is asymptotic to G_N if for any n the error satisfies

$$|G_N - G_{N,n}| = O(g^{n+1}) \tag{2.8.13}$$

as $g \to 0$. In general, perturbation theory is asymptotic but not convergent. This is rigorously known (Glimm & Jaffe (1981)) for the ϕ^4 theory in the cases that the space-time dimension is $d = 0, 1, 2, 3$. ($d = 0$ is the case of the ordinary integral

$$\int dx \exp(-m^2x^2/2 - gx^4/4!),$$

while $d = 1$ is the quantum mechanics of the anharmonic oscillator.) Physically, the reason for non-convergence is that when $g < 0$ the energy is unbounded below and so the vacuum-state continued from $g > 0$ is unstable. (Dyson (1952) first observed this phenomenon in quantum electrodynamics.)

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

In later chapters we will assume (2.8.13). When we compute large-momentum behavior, it will be important to understand the maximum possible validity and accuracy of the calculations if the perturbation theory is asymptotic but not convergent.

2.9 Spontaneously broken symmetry

Consider the ϕ^4 interaction. If m^2 is positive and g is small, we have a theory of particles of mass m slightly perturbed by the interaction. This interaction is basically a repulsive δ-function potential, as can be seen by examining the Hamiltonian in the non-relativistic approximation. There is a symmetry $\phi \to -\phi$.

But if m^2 is negative this interpretation is incorrect. The true situation can be discovered by noticing that the functional integral (in Euclidean space) is dominated by classical fields with the lowest Euclidean action, which is

$$S_{\text{Eucl}}[A] = \int d^4x[-(\partial A)^2/2 + m^2A^2/2 + gA^4/4!]. \tag{2.9.1}$$

(Remember that $(\partial A)^2 = -(\partial A/\partial\tau)^2 - \vec{\nabla}A^2$ is negative.) If $m^2 > 0$, then the minimum action field is $A = 0$. But, if $m^2 < 0$, then there are two minima; these are constant fields with $P'(A) = 0$, i.e., $A = A_+ \equiv \sqrt{(-6m^2/g)}$ and $A = A_- \equiv -\sqrt{(-6m^2/g)}$.

We choose to impose the boundary condition $A(x) \to A_+$ as $x \to \infty$ in the functional integral. (The condition $A \to A_-$ gives equivalent physics, because of the $A \to -A$ symmetry of the action.) Then field configurations with A close to A_+ will dominate. We may understand this by observing that field configurations with large regions where A is not close to A_+ or A_- will give small contributions to the Euclidean functional integral (2.2.8) because their action S_{Eucl} is so big. Indeed, a constant field with A not equal to A_+ or A_- has infinitely more action than one with $A = A_+$ or A_-, and its contribution to the integral is zero. One's first inclination then is that the only configurations that contribute have $A \to A_+$ or $A \to A_-$ as $x \to \infty$. However, other configurations contribute, because there are many of them – one has to integrate over all possible fluctuations. However, one can argue – even rigorously (Glimm & Jaffe (1981) – that in general A will be close to A_+ or to A_-. A typical configuration of the classical field $A(x)$ will be close to one of these values over almost all of space-time.

Given our choice of boundary condition $A \to A_+$, even more is true: a typical configuration is close to $A = A_+$ almost everywhere, rather than to either A_- or A_+. The reason is that if it had a large region with $A(x)$ close to

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

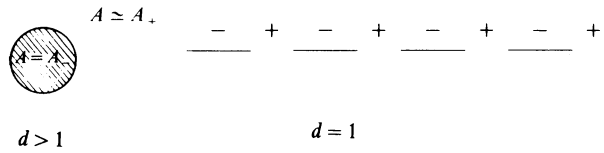

Fig. 2.9.1. Illustrating transitions between regions with fields close to different minima of the potential

A_- (Fig. 2.9.1), then there would be a contribution to the action proportional to the size of the boundary between the regions of positive $A(x)$ and of negative $A(x)$. Only if the space-time dimension is $d = 1$ will we have a finite contribution from the boundary. This special case is quantum mechanics of a particle in a potential with two wells. The particle can tunnel between the two wells.

In the case we have discussed, of a discrete rather than of a continuous symmetry, the argument that A is close to A_+ almost everywhere for the important configurations is correct in all space-time dimensions greater than one. The quantum field therefore has a vacuum expectation value close to A_+:

$$\langle 0|\phi(x)|0\rangle = \mathcal{N}\int [\mathrm{d}A]A(x)\mathrm{e}^{-S[A]} \simeq A_+ .$$

In the case of a continuous symmetry, there is a continuous series of minima of the potential. A field configuration can interpolate between different minima without going over a big hump in the potential. The only penalty comes from the gradient terms in the action. This suppresses configurations that do not stay close to one minimum, but only in more than two space-time dimensions. In one space-time dimension there is no spontaneous breaking of a continuous symmetry (Mermin & Wagner (1966), Hohenberg (1967), and Coleman (1973)).

Perturbation theory can be considered as a saddle point expansion about the minimum of the action. We write $A(x) = A'(x) + v$ where $v = A_+$. Now we treat $A'(x)$ as the independent variable. We have

$$\mathcal{L} = (\partial A')^2/2 - M^2A'^2/2 - gvA'^3/3! - gA'^4/4! + C. \qquad (2.9.2)$$

Here $C \equiv -m^2v^2/2 - gv^4/4!$, and $M^2 \equiv gv^2/2 + m^2 = -2m^2 > 0$. We now have a theory of particles of mass M with both an A'^3 and an A'^4 interaction. The symmetry is hidden; its only obvious manifestation is in the relation between the A'^3 coupling and the mass and A'^4 terms:

$$gv = M(3g)^{1/2}. \qquad (2.9.3)$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

We will show later that renormalization counterterms are correctly given by continuing in m^2 from positive m^2. The vacuum expectation value of ϕ has corrections which can be computed in perturbation theory

$$\begin{aligned}\langle 0|\phi|0\rangle &= v + \langle 0|\phi'|0\rangle \\ &= v + O(g^{1/2}).\end{aligned} \tag{2.9.4}$$

Exactly similar methods can be applied if there is a continuous symmetry. Then the Goldstone theorem tells us that there will be a massless scalar particle for each broken generator.

2.10 Fermions

The field theories obtained by functional integration as in Section 2.2 are all theories of bosons. This follows from the symmetry of the Green's functions under exchange of fields (e.g., $\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\phi(y)\phi(x)|0\rangle$). In turn, this symmetry property follows from the functional-integral formula (2.2.2) because the integration variables (the values of the classical field $A(x)$) commute with each other.

To get a theory with quantized fields, it is necessary to define something like an integral over anticommuting variables. A rather small number of properties of integration are needed to derive the equations of motion for Green's functions. Requiring these properties determines the integration operation uniquely (Itzykson & Zuber (1980)).

As an example, consider the following Lagrangian for a free Dirac field:

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - M)\psi. \tag{2.10.1}$$

Here ψ is a four-dimensional column vector and $\bar{\psi}$ a row vector, while $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. The generating functional of Green's functions is written as:

$$Z[\eta, \bar{\eta}] = \mathscr{N} \int [d\psi\, d\bar{\psi}] \exp\left(i\int \mathscr{L} + \int \bar{\eta}\psi + \int \bar{\psi}\eta\right). \tag{2.10.2}$$

The fields and the sources $\eta(x)$ and $\bar{\eta}(x)$ take their values in the fermionic sector of a Grassmann algebra. In the lattice approximation the definition of the integration in (2.10.2) is really algebraic (Itzykson & Zuber (1980)). Green's functions are defined by differentiating with respect to the sources.

One important difference between ordinary integration and Grassmann integration will be important in treating gauge theories. The simplest case of this difference is in the integral over two variables x and $\bar{x}$ of $\exp(i\bar{x}ax)$, where a is a real number. For ordinary real variables the integral is

$$\int dx\, d\bar{x} e^{i\bar{x}ax} = 2\pi \int d\bar{x}\delta(\bar{x}a) = 2\pi/a. \tag{2.10.3}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

For Grassmann variables we get

$$\int \mathrm{d}x\,\mathrm{d}\bar{x}\,\mathrm{e}^{\mathrm{i}\bar{x}ax} = \mathrm{i}a. \tag{2.10.4}$$

The overall normalization is irrelevant for our applications, for it is always cancelled by the overall normalization factor in the functional integral. What matters is that the a-dependence of (2.10.4) is inverse to that in (2.10.3).

2.11 Gauge theories

A gauge symmetry is an invariance under a group G where the group transformation is different at each space-time point. The earliest examples were General Relativity (where G is $GL(4)$, the group of linear transformations of the coordinate system), and electrodynamics (where G is the group of phase rotations). Yang & Mills (1954) and Shaw (1955) generalized the idea to a general group. Beg & Sirlin (1982) and Buras (1981) explain some of the uses of gauge theories as theories of physics.

Let G be a simple group and let a matter field ψ transform as

$$\psi(x) \to \exp(-\mathrm{i}g\omega^{\alpha}(x)t_{\alpha})\psi(x) = U(\omega(x))^{-1}\psi(x). \tag{2.11.1}$$

The field ψ is a column vector of components, and the hermitian matrices t_{α} form a representation of the group, with structure constants $c_{\alpha\beta\gamma}$ defined by

$$[t_{\alpha}, t_{\beta}] = \mathrm{i}c_{\alpha\beta\gamma}t_{\gamma}. \tag{2.11.2}$$

The matrices $U(\omega)$ form a representation of the group.

In order that the action be gauge invariant, we need a covariant derivative:

$$D_{\mu}\psi \equiv (\partial_{\mu} + \mathrm{i}g\mathsf{A}_{\mu})\psi. \tag{2.11.3}$$

Here we have introduced the gauge potential A_{μ}. It is a vector under Lorentz transformation. As far as its gauge symmetry properties are concerned, it can be written as a matrix A_{μ} or in terms of components A_{μ}^{α}:

$$\mathsf{A}_{\mu} = \sum_{\alpha} A_{\mu}^{\alpha} t_{\alpha}. \tag{2.11.4}$$

It transforms under the gauge group as:

$$\mathsf{A}_{\mu}(x) \to U(\omega(x))^{-1}[\mathsf{A}_{\mu}(x) - \mathrm{i}g^{-1}\partial_{\mu}]U(\omega(x)). \tag{2.11.5}$$

To build an action, we need the field-strength tensor

$$\begin{aligned} \mathsf{F}_{\mu\nu} &= \partial_{\mu}\mathsf{A}_{\nu} - \partial_{\nu}\mathsf{A}_{\mu} + \mathrm{i}g[\mathsf{A}_{\mu}, \mathsf{A}_{\nu}], \\ F_{\mu\nu}^{\alpha} &= \partial_{\mu}A_{\nu}^{\alpha} - \partial_{\nu}A_{\mu}^{\alpha} - gc_{\alpha\beta\gamma}A_{\mu}^{\beta}A_{\nu}^{\gamma}, \end{aligned} \tag{2.11.6}$$

which transforms as $\mathsf{F} \to U^{-1}\mathsf{F}U$.

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

A gauge-invariant Lagrangian with spin $\frac{1}{2}$ matter fields is

$$\begin{aligned}\mathscr{L}_{\text{inv}} &= -(F^{\alpha}_{\mu\nu})^2/4 + \bar{\psi}(\mathrm{i}\not{D} - M)\psi \\ &= -\operatorname{tr} \mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu}/2 + \bar{\psi}(\mathrm{i}\not{D} - M)\psi,\end{aligned} \tag{2.11.7}$$

where we assumed the conventional normalization of the t_α's, viz., $\operatorname{tr} t_\alpha t_\beta = \delta_{\alpha\beta}/2$. In an exactly similar way, an action can be set up using scalar fields. If there are matter fields in several irreducible representations a term for each is needed in $\mathscr{L}$. The transformation (2.11.5) ensures that the coupling g is the same for all matter fields if the group is non-abelian.

The form of the infinitesimal transformations is needed:

$$\begin{aligned}\delta_\omega \psi &= -\mathrm{i}g\omega^\alpha t_\alpha \psi, \\ \delta_\omega \bar{\psi} &= \mathrm{i}g\omega^\alpha \bar{\psi} t_\alpha, \\ \delta_\omega A^\alpha_\mu &= \partial_\mu \omega^\alpha + g c_{\alpha\beta\gamma}\omega^\beta A^\gamma_\mu, \\ \delta_\omega F^\alpha_{\mu\nu} &= g c_{\alpha\beta\gamma}\omega^\beta F^\gamma_{\mu\nu}.\end{aligned} \tag{2.11.8}$$

If the group is not simple, then it is the product of several simple groups, e.g., $SU(2) \otimes SU(2)$. For each there is a gauge field and an independent coupling.

2.12 Quantizing gauge theories

A gauge theory such as the one defined by the Lagrangian (2.11.7) can be solved by the functional integral. Thus, as an example, we can write for the fermion propagator.

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \mathscr{N}_{GI}\int [\mathrm{d}A][\mathrm{d}\psi \mathrm{d}\bar{\psi}]\psi(x)\bar{\psi}(y)\exp\left(\mathrm{i}\int \mathscr{L}_{\text{inv}}\right). \tag{2.12.1}$$

In fact, a lattice approximation to the functional integral forms the basis of Monte-Carlo calculations (Creutz (1983) and Creutz & Moriarty (1982)). The only trouble with (2.12.1) is that it is exactly zero. To see this we observe that, given any field configuration, we can make a gauge transformation on it, as in (2.11.1) and (2.11.5). The new field configuration has the same action as the old field. Thus the only dependence on the gauge transformation is in the explicit $\psi(x)\bar{\psi}(y)$. Now the gauge transformation is independent at each space-time point. So (2.12.1) contains a factor

$$\begin{aligned}&\left(\prod_{\text{points } z}\int \mathrm{d}U(z)\right)U^{-1}(x) \otimes U(y) \\ &\quad = (\text{number})\left(\int \mathrm{d}U(x)U(x)^{-1}\right)\left(\int \mathrm{d}U(y)U(y)\right),\end{aligned} \tag{2.12.2}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

which is zero. (Note that the propagator is a matrix in the representation space of the gauge group.)

The vanishing of (2.12.1) is not a fundamental problem, for we may choose only to work with Green's functions of gauge-invariant operators (e.g., $\bar{\psi}\psi$, $F^{\alpha}_{\mu\nu}F^{\alpha\mu\nu}$, the Wilson loop (Wilson (1974) and Kogut (1983))). But the vanishing is a disaster for formulating perturbation theory; for among the basic objects needed to write the Feynman rules are the propagators for the elementary fields. An elegant solution to this problem was given by Faddeev & Popov (1967). The integral over all gauge fields is written as the product of the integral over fields satisfying some given gauge condition (such as $\partial\cdot A^{\alpha}=0$) and of the integral over all gauge transformations. Any field configuration can be obtained by gauge transforming some configuration that satisfies the gauge condition. For a gauge-invariant Green's function, the integral over gauge transformations amounts to an overall factor which cancels an inverse factor in the normalization. So the integral over gauge transformations can be consistently omitted.

The new integral over fields with the gauge-fixing condition imposed also provides a solution to the theory. But the gauge-variant Green's functions like (2.12.1) no longer vanish. It is necessary, moreover, to find the correct measure for the integral; this was the key point of the work of Faddeev and Popov.

These authors also constructed a slightly different formulation; it is this formulation that is most often used, and that we will review now. A detailed treatment and further references are to be found in Itzykson & Zuber (1980). Here we will merely summarize the argument and derive the Ward identities in the form that we will use them.

We will consider gauge conditions of the form $F_{\alpha}[A,x]=f_{\alpha}(x)$. There is one condition at each point of space-time and for each generator of the group. The functional F_{α} might be $\partial\cdot A^{\alpha}$, for example. The functions $f_{\alpha}(x)$ are any real valued functions of x.

Faddeev and Popov write an arbitrary Green's function as

$$\langle 0|TX|0\rangle = \mathcal{N}_{GI}\int[\mathrm{d}A][\mathrm{d}\psi][\mathrm{d}\bar{\psi}]X\exp(\mathrm{i}S_{\mathrm{inv}})\Delta[A]\prod_{x,\alpha}\delta(F_{\alpha}-f_{\alpha}). \tag{2.12.3}$$

Here S_{inv} is the gauge invariant action, and X is any product of fields. The factor $\Delta[A]$ is a Jacobian that arises in transforming variables to the set of fields that satisfy the gauge condition plus the set of gauge transformations. The key result is that $\Delta[A]$ is a determinant, so that it can be written as

$$\Delta = \int[\mathrm{d}c_{\alpha}][\mathrm{d}\bar{c}_{\alpha}]\exp(\mathrm{i}\mathscr{L}_{\mathrm{gc}}). \tag{2.12.4}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

Here c_α and $\bar{c}_\alpha$ are anticommuting scalar fields, called the Faddeev–Popov ghosts. The so-called gauge-compensating Lagrangian is

$$\mathscr{L}_{\mathrm{gc}} = \bar{c}^\alpha \delta_c F_\alpha[A, x], \tag{2.12.5}$$

where $\delta_c F_\alpha$ is the infinitesimal transformation of F_α with ω replaced by $\mathbf{c}$. For the case $F_\alpha = \partial \cdot A^\alpha$

$$\mathscr{L}_{\mathrm{gc}} = \partial^\mu \bar{c}(\partial_\mu c_\alpha + g c_{\alpha\beta\gamma} c_\beta A_\mu^\gamma) + \text{divergence}. \tag{2.12.6}$$

We treat c and $\bar{c}$ as independent fields. They are not genuine physical fields, as they do not obey the usual spin-statistics theorem.

A convenient form of solution to the theory is obtained by averaging over all f_α's, with weight $\exp(-\xi^{-1}\int f_\alpha^2/2)$. This leaves gauge-invariant Green's functions unaltered, and gives the following formula:

$$\langle 0|TX|0\rangle = \mathscr{N}\int[\text{d fields}]X\mathrm{e}^{\mathrm{i}S} \tag{2.12.7}$$

with a different normalization. The integral is over all fields $(A, \psi, \bar{\psi}, c, \bar{c})$. The action S contains three terms:

$$S = \int \mathrm{d}^4x(\mathscr{L}_{\mathrm{inv}} + \mathscr{L}_{\mathrm{gf}} + \mathscr{L}_{\mathrm{gc}}). \tag{2.12.8}$$

We have already defined the gauge-invariant Lagrangian by (2.11.7) and $\mathscr{L}_{\mathrm{gc}}$ by (2.12.5). The gauge-fixing term is

$$\mathscr{L}_{\mathrm{gf}} = -\mathrm{i}F_\alpha^2/(2\xi), \tag{2.12.9}$$

where ξ is an arbitrarily chosen parameter. (If desired, it may be absorbed into a redefinition of F_α.)

The advantage of the form (2.12.8) is that Green's functions of the elementary fields are defined as in a simple non-gauge theory. For a gauge-invariant observable X the equations (2.12.3) and (2.12.7) define the same objects as

$$\langle 0|TX|0\rangle = \mathscr{N}_{GI}\int[\mathrm{d}A\mathrm{d}\psi\mathrm{d}\bar{\psi}]X\exp(\mathrm{i}S_{\mathrm{inv}}). \tag{2.12.10}$$

If X is gauge variant, then all the definitions give different results, and (2.12.7) depends on ξ. Quantities that depend on the choice of a gauge fixing are called gauge dependent, of course. We see that gauge invariance of the operators in a Green's function implies gauge independence.

It is important to distinguish the concepts of gauge invariance and gauge independence. Gauge invariance is a property of a classical quantity and is invariance under gauge transformations. Gauge independence is a property of a quantum quantity when quantization is done by fixing the gauge. It is

independence of the method of gauge fixing. Gauge invariance implies gauge independence, but only if the gauge fixing is done properly.

Gauge theories such as (2.11.7) have a dimensionless coupling if space-time is four dimensional. General results, which we will treat in later chapters, imply that the theories need renormalization. However these same results imply that many more counterterms *may* be needed than are available by renormalizing (2.11.7). In Chapter 12 we will prove that the extra couplings are absent. The tools needed are the Ward identities for the gauge symmetry. These we will prove in the next section. It is also necessary to prove that the unphysical degrees of freedom represented, for example, by the ghost fields c_α and $\bar{c}_\alpha$ do not enter unitarity relations. This proof also needs the gauge properties exhibited in the Ward identities (see Itzykson & Zuber (1980)).

2.13 BRS invariance and Slavnov–Taylor identities

After gauge fixing, the gauge invariance of a gauge theory is no longer manifest in the functional-integral solution. Slavnov (1972) and Taylor (1971) were the first to derive the generalized Ward identities that carry the consequences of gauge invariance. Their derivation was very much simplified by Becchi, Rouet & Stora (1975) through the discovery of what is now called the BRS symmetry of the action (2.12.8).

BRS symmetry is in fact a supersymmetry, that is, its transformations involve parameters that take their values in a Grassmann algebra. Let $\delta\lambda$ be a fermionic Grassmann variable. Then the BRS transformation of a matter or a gauge field is defined to be a gauge transformation with $\omega^\alpha = c^\alpha \delta\lambda$. Thus

$$\begin{aligned}
\delta_{\mathrm{BRS}}\psi &= -\mathrm{i}g(c^\alpha\delta\lambda)t_\alpha\psi = \mathrm{i}gt_\alpha c^\alpha\psi\delta\lambda,\\
\delta_{\mathrm{BRS}}\bar{\psi} &= \mathrm{i}g\bar{\psi}t_\alpha c^\alpha\delta\lambda,\\
\delta_{\mathrm{BRS}}A^\alpha_\mu &= (\partial_\mu c^\alpha + g c_{\alpha\beta\gamma}c^\beta A^\gamma_\mu)\delta\lambda.
\end{aligned} \tag{2.13.1}$$

Observe that $\delta\lambda$ is fermionic, so it anticommutes with fermion fields (c, $\bar{c}$, ψ, $\bar{\psi}$). The ghost fields transform as

$$\begin{aligned}
\delta_{\mathrm{BRS}}c^\alpha &= -\tfrac{1}{2}g c_{\alpha\beta\gamma}c^\beta c^\gamma\delta\lambda,\\
\delta_{\mathrm{BRS}}\bar{c}_\alpha &= F_\alpha\delta\lambda/\xi.
\end{aligned} \tag{2.13.2}$$

(Note that c^α and $\bar{c}_\alpha$ are not related by hermitian conjugation, contrary to appearances.) Since $\mathscr{L}_{\mathrm{inv}}$ is invariant under gauge transformations, it is BRS invariant. Hence

$$\begin{aligned}
\delta_{\mathrm{BRS}}\mathscr{L} &= \delta_{\mathrm{BRS}}(\mathscr{L}_{\mathrm{gf}} + \mathscr{L}_{\mathrm{gc}})\\
&= -(1/\xi)F_\alpha\delta_{c\delta\lambda}F_\alpha[A;x] - (1/\xi)F_\alpha\delta\lambda\delta_c F_\alpha - \bar{c}_\alpha\delta_{\mathrm{BRS}}(\delta_c F_\alpha)\\
&= 0.
\end{aligned} \tag{2.13.3}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

In the second line we used $\mathscr{L}_{gc} = -\bar{c}_\alpha \delta_c F_\alpha$, while to prove the last line zero we anticommuted $\delta\lambda$ and c in the first two terms of the second line. In addition we used the nilpotence of the BRS transformation:

$$\left(\frac{\delta_{BRS}}{\delta\lambda}\right)^2 (\psi \text{ or } \bar{\psi} \text{ or } A_\mu^\alpha \text{ or } c^\alpha) = 0,$$

$$\left(\frac{\delta_{BRS}}{\delta\lambda}\right)^3 \bar{c}_\alpha = 0, \tag{2.13.4}$$

which follows from anticommutativity of the c's.

By applying the Noether theorem we find a conserved current:

$$j^\mu_{BRS} = g\bar{\psi}\gamma^\mu t_\alpha \psi c^\alpha - F^{\alpha\mu\nu} D_\nu c^\alpha - (1/\xi)\partial\cdot A^\alpha D^\mu c^\alpha$$
$$-\tfrac{1}{2} g(\partial^\mu \bar{c}_\alpha) c^\beta c^\gamma c_{\alpha\beta\gamma}. \tag{2.13.5}$$

Although the BRS transformations involve Grassmann-valued parameters, the derivation of Ward identities given in Section 2.7 goes through unchanged. For our purposes, we only need the integrated Ward identity (2.7.9). A case of (2.7.9) applied to BRS invariance is called a Slavnov–Taylor identity. A simple example is

$$\begin{aligned} 0 &= \delta_{BRS} \langle 0 | T A^\alpha_\mu(x) c_\beta(y) | 0 \rangle / \delta\lambda \\ &= -\langle 0 | T(\partial_\mu c_\alpha + g c_{\alpha\delta\gamma} c^\delta A^\gamma) \bar{c}_\beta(y) | 0 \rangle \\ &\quad + (1/\xi) \langle 0 | T A^\alpha_\mu(x) \partial\cdot A^\beta(y) | 0 \rangle . \end{aligned} \tag{2.13.6}$$

We have defined the notation δ_{BRS} (quantity)$/\delta\lambda$ to mean that the $\delta\lambda$ in the BRS variation is commuted or anticommuted to the right and then deleted.

The most used cases of the Slavnov–Taylor identities are:

$$\begin{aligned} 0 &= \delta_{BRS} \langle 0 | T X \bar{c}_\alpha(x) | 0 \rangle / \delta\lambda \\ &= -\langle 0 | T(\delta_{BRS} X / \delta\lambda) \bar{c}_\alpha | 0 \rangle + (1/\xi) \langle 0 | T X \partial\cdot A^\alpha | 0 \rangle . \end{aligned} \tag{2.13.7}$$

Here X is a product of fields with total ghost number zero.

We will also need equations of motion. Let:

$$\begin{aligned} \mathscr{L}_{\bar{\psi}} &= \frac{\partial \mathscr{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathscr{L}}{\partial \partial_\mu \bar{\psi}} = (\mathrm{i}\not{D} - M)\psi, \\ \mathscr{L}_{\psi} &= \bar{\psi}(-\mathrm{i}\overleftarrow{\not{D}} - M), \\ \mathscr{L}_{A^\alpha_\mu} &= -D_\nu F^{\alpha\mu\nu} - g\bar{\psi}\gamma^\mu t_\alpha \psi + (1/\xi)\partial^\mu \partial\cdot A^\alpha + g c_{\alpha\beta\gamma}(\partial^\mu \bar{c}^\beta) c_\gamma, \\ \mathscr{L}_{\bar{c}_\alpha} &= -\partial_\mu D^\mu c^\alpha, \\ \mathscr{L}_{c^\alpha} &= -\Box \bar{c}_\alpha - g c_{\alpha\beta\gamma}(\partial_\mu \bar{c}_\beta) A^{\gamma\mu}. \end{aligned} \tag{2.13.8}$$

Then each of these is zero. Furthermore, for ϕ equal to any field in the theory we have:

$$\langle 0 | T \mathscr{L}_\phi(x) X | 0 \rangle = \mathrm{i} \langle 0 | T \delta X / \delta\phi(x) | 0 \rangle, \tag{2.13.9}$$

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

where time derivatives in $\mathscr{L}_\phi$ are taken outside the time-ordering (as usual – see Section 2.5).

2.14 Feynman rules for gauge theories

Feynman rules are given in Fig. 2.14.1 for the Lagrangian of (2.11.7), with the gauge-fixing term $F_\alpha = \partial \cdot A^\alpha$. Note that these agree with the figures but not the equations of Marciano & Pagels (1978). They are the rules for quantum chromodynamics (QCD), the theory of strong interactions, if the gauge group is $SU(3)$. The fermions are the quarks and consist of several triplets of $SU(3)$, each with its own mass term. (In the conventional terminology, the gauge field is called the gluon field and the gauge symmetry is called the color symmetry of strong interactions. Each irreducible representation in the quark field is called a flavor, and has a label: u, d, s, c, b, etc.)

The same Lagrangian also describes quantum electrodynamics (QED) if we change the gauge group to $U(1)$. In that case there is but one gauge field (the photon) and, since the group is abelian, the three- and four-point self-

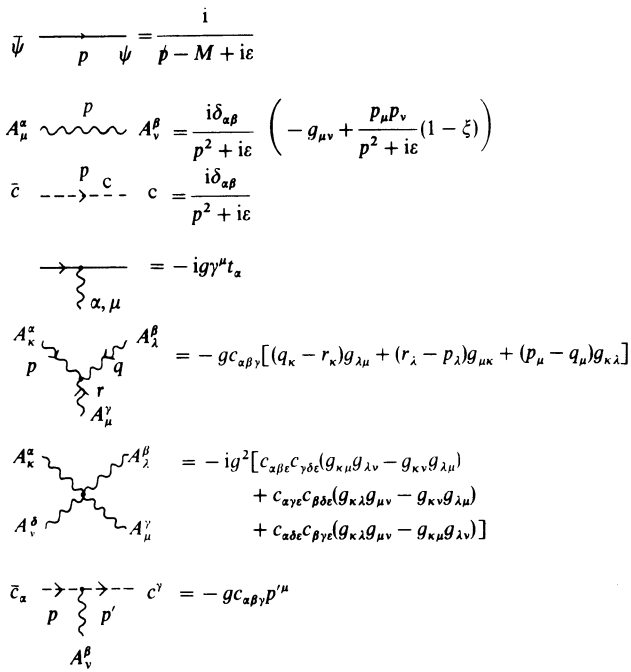

Fig. 2.14.1. Feynman rules for the gauge theory defined by the Lagrangian (2.11.7).

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

interactions of the gauge field vanish. Moreover, with the gauge fixing term $F[A] = \partial \cdot A$ there is no coupling to the ghost fields, so we may drop them from consideration. The transformations on the matter fields are simple phase rotations: $\psi \to e^{-iq\omega}\psi$ where q is the charge of the field (negative for the electron field). The transformation of the gauge field is $A_\mu \to A_\mu + \partial_\mu \omega$.

2.15 Other symmetries of (2.11.7)

The Lagrangian (2.11.7) is gauge invariant. After gauge fixing we get (2.12.8), which is not gauge invariant, but which has BRS symmetry. The action (2.12.8) also is invariant under global gauge transformations – those with constant ω – because we chose the gauge fixing not to break this symmetry.

There are also what in strong interactions are called flavor symmetries. These are transformations that act identically on every member of an irreducible representation of the gauge group. In this case they give conservation of the number of each of the different flavors of quark. Other flavor symmetries include the discrete symmetries of parity and time-reversal invariance.

Charge-conjugation is also an invariance of (2.12.8), and its action on the ghost fields is rather interesting. Let us define ε_α by the parity of the representation matrices under transposition:

$$t_\alpha = \varepsilon_\alpha t_\alpha^{\mathrm{T}} (\sum). \tag{2.15.1}$$

In this and the following equations, the symbol $(\sum)$ means that the summation convention on repeated indices is suspended. The fermion and gauge fields transform as usual:

$$\begin{aligned} A_\alpha^\mu &\to -A_\alpha^\mu \varepsilon_\alpha (\sum), \\ \psi &\to \eta_c (i\gamma^0\gamma^2)(\bar{\psi})^{\mathrm{T}}, \end{aligned} \tag{2.15.2}$$

where η_c is a real matrix such that

$$t_\alpha^{\mathrm{T}} = \eta_c^{\mathrm{T}} t_\alpha \eta_c. \tag{2.15.3}$$

The ghosts transform as:

$$c^\alpha \to -\varepsilon^\alpha c^\alpha, \quad \bar{c}^\alpha \to -\varepsilon^\alpha \bar{c}^\alpha. \tag{2.15.4}$$

Consider the $c^\alpha - \bar{c}^\beta - A_\mu^\gamma$ and the $A_\lambda^\alpha - A_\mu^\beta - A_\nu^\gamma$ Green's functions. They are invariant under global gauge transformations so only two couplings of the gauge indices are possible: $c_{\alpha\beta\gamma}$ which is antisymmetric, and a symmetric coupling which we can call $d_{\alpha\beta\gamma}$. Charge-conjugation invariance prohibits the symmetric coupling.

https://doi.org/10.1017/9781009401807.002 Published online by Cambridge University Press

2.16 Model field theories

Although the concepts of quantum field theory are very general, we have reviewed them by examining mainly two specific models. The first was the theory of a real scalar field (2.1.1), mostly with the ϕ^4 interaction (2.2.1). The second was a gauge theory (2.11.7) with matter described by some Dirac fields. It should be clear that the general principles apply to any Lagrangian $\mathscr{L}$. A field theory is specified by listing its elementary fields and giving a formula for $\mathscr{L}$. It is solved by a functional-integral representation of its Green's functions.

The aim of physics is to describe the real world. To the extent that a field-theoretic description is the correct one, the fundamental problem in physics is to find the correct field theory. In fact the Lagrangian (2.11.7) appears to do this for strong and electromagnetic interactions if the gauge group and matter fields are correctly chosen. Weak interactions can be included by the Weinberg–Salam theory, and many speculations have been made about extensions (see the proceedings of most recent conferences on high-energy physics).

Our aim in this book would be badly served by only treating real theories. One reason is that we wish to develop techniques and concepts applicable to any field theory, for example not only to the many Grand Unified Theories currently under discussion (see Langacker (1981) and Ross (1981) for reviews), but also to the theories to be invented in the future. As is usual in the subject, we will make use of field theories that are more properly called models. The ϕ^4 theory is an obvious case. Another important reason for using models is to be able to discuss particular aspects of the methods without having other complications to clutter up the presentation.

Particular models will be introduced as needed. Some will recur often, such as the ϕ^4 theory and the simple gauge theory (2.11.7).

Another frequently used model is the ϕ^3 interaction of a real scalar field:

$$\mathscr{L} = (\partial\phi)^2/2 - m^2\phi^2/2 - g\phi^3/3!. \tag{2.16.1}$$

This is much more unphysical than the other models. It is not even completely consistent. Because of the ϕ^3 interaction, the energy is not bounded below. This is manifest in the classical theory and true in the quantum theory (Baym (1960)). Hence any state must catastrophically decay. But the perturbation theory is well-defined, and somewhat simpler than for the ϕ^4 theory. So it proves very convenient to use the ϕ^3 model in treating the elements of the theory of renormalization within perturbation theory.

Another way of constructing models is to change the dimension, d, of space-time from its physical value 4. One motivation for this is that the renormalization problem becomes easier as d is reduced; the degree of divergence of a Feynman graph decreases. As we will see in Chapter 4, it is both useful and possible to treat d as a continuous variable, for the purpose of computing the values of terms in the perturbation expansion.