A SOBOLEV ALGEBRA OF VOLTERRA TYPE

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Abstract

We present a family of radical convolution Banach algebras on intervals $(0, a]$ which are of Sobolev type; that is, they are defined in terms of derivatives. Among other properties, it is shown that all epimorphisms and derivations of such algebras are bounded. Also, we give examples of nontrivial concrete derivations.

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1. Introduction

Fix $n \in \mathbb{N}$. Let $\mathcal{T}_+^{(n)}(t^n)$ denote the completion of $C_c^{(\infty)}([0, \infty))$ in the norm

$$\|f\|_{\mathcal{T}_+^{(n)}(t^n)} = \int_0^\infty |f^{(n)}(t)| t^n \, dt, \quad f \in C_c^{(\infty)}([0, \infty)).$$

This space is a Banach algebra for the usual convolution product, which arises in close relationship with n-times integrated semigroups and the study of 'ill-posed' abstract Cauchy problems (see [2] and also [1] and references therein). Fractional derivative versions $\mathcal{T}_+^{(\nu)}(\phi_\nu)$ of the above algebras have been introduced in [9] with the aim of characterizing ν-times integrated semigroups by means of a suitable notion of distribution semigroups of order $\nu > 0$. In this notation, ϕ_ν is a nondecreasing function such that

$$\inf_{t>0} t^{-\nu} \phi_\nu(t) > 0.$$

In particular, if $\phi_\nu(t) \equiv t^\nu \omega(t)$ where ω is a nondecreasing (submultiplicative) weight on $(0, \infty)$, then

$$\mathcal{T}_+^{(\mu)}(t^\mu \omega) \hookrightarrow \mathcal{T}_+^{(\nu)}(t^\nu \omega) \hookrightarrow L^1(\omega), \quad \mu \geq \nu > 0.$$

As usual, we denote by $L^1(\omega)$ the space of (classes of) measurable complex functions f on $(0, \infty)$ such that $\int_0^\infty |f(x)|\omega(x)\,dx < \infty$. We also write $L^1(\mathbb{R}^+) := L^1(1)$.

It can be shown that, if ω satisfies $\rho_\omega := \lim_{t \to \infty} \omega(t)^{1/t} \neq 0$, then the convolution Banach algebra $\mathcal{T}_+^{(v)}(t^v\omega)$, $v > 0$, is *semisimple* and its Gelfand transform is equal to the Laplace transform (on the half-plane Re $z \geq -\log \rho_\omega$). This fact is well known in the case $v = 0$, that is, for $L^1(\omega)$ [4, Theorem 4.7.27(i)]. For $v > 0$ the proof will appear elsewhere.

Moreover, for $\omega \equiv 1$, the space $\mathcal{T}_+^{(v)}(t^v)$ has been studied as a semisimple Banach algebra in a series of papers including the following items: the structure of its closed ideals in [11, 13], and for a discrete version of that algebra in [12]; the properties of its Gelfand transform in [10]; and the existence of certain semigroups contained in it, which can be expressed in terms of special functions. An analysis of the applications of algebras of that type to the stability theory of operator semigroups is also in progress. In all of the above approaches the properties and applications of those algebras, and of their associated mathematical objects, are very much like those of the Banach algebra $L^1(\mathbb{R}^+)$, each one within its own setting. Thus an analoguous behavior is to be expected, at least in the semisimple case, of the weighted Banach algebras $\mathcal{T}_+^{(v)}(t^v\omega)$ by comparison with $L^1(\omega)$.

On the other hand, it is also well known that, provided ω is *radical*, that is, $\rho_\omega = 0$, then the Banach algebra $L^1(\omega)$ is radical, which is to say that $L^1(\omega)$ has no nonzero character or, equivalently, that the set of modular maximal ideals of $L^1(\omega)$ is empty; see [4, Theorem 4.7.27 (ii)]. A standard and important example of radical weight is $\omega(t) := e^{-t^2}$, $t > 0$.

Radical Banach algebras are known from the very beginning of the theory of Banach algebras, but they were not studied in depth until quite recently. The modern interest in such algebras emerges with the solution to the Kaplansky problem obtained independently by Dales and Esterle. They proved that, given an infinite compact space K, and assuming the continuum hypothesis, there always exists a discontinuous injective homomorphism $\theta \colon C(K) \to R \oplus \mathbb{C}$, for suitable commutative radical Banach algebras R. Here, $C(K)$ is the usual Banach algebra of complex continuous functions on K, and one can take as R the weighted algebra $L^1(e^{-t^2})$ or the Volterra algebra $L_*^1(0, 1)$; see [5] for a joint presentation of the Dales–Esterle theorem. This result has since been extended or complemented in several directions. For instance (always under the continuum hypothesis), algebras $L^1(\omega)$, for ω radical, are universal in the class of complex and commutative algebras with no unit which are integral domains and have the power of the continuum; see [6, Corollary 5.2]. In another direction, Esterle characterizes *all* the radical Banach algebras R for which it is possible to construct a discontinuous homomorphism $\theta \colon C(K) \to R \oplus \mathbb{C}$ (under the assumption of the continuum hypothesis again); see [7, Theorem 6.4] and [8, Theorem 5.3]. Further, these algebras form the fifth class out of a total of nine introduced in [8] as a way to classify the set of commutative radical Banach algebras. Most of the (rich set of) examples and counterexamples given in [8] are constructed from convolution algebras of ℓ^1 or L^1 type, or principal ideals thereof.

In accordance with all the above considerations, it seems sensible to investigate convolution radical Banach algebras of Sobolev type in the above setting, that is, associated with and analogous to radical algebras like $L^1(\omega)$ or $L^1_*(0, 1)$. Thus our first question is whether, similarly to the semisimple case, the Banach algebra $\mathcal{T}^{(\nu)}_+(t^\nu\omega)$, for $\nu > 0$, is radical whenever ω is a radical weight.

Somewhat disappointingly, it turns out that $\mathcal{T}^{(\nu)}_+(t^\nu\omega)$ need not be a convolution algebra if one allows ω to be a decreasing function (see Section 2). Thus, in order to find radical Banach algebras of Sobolev type, one must try some other way different from that suggested by the (continuous) inclusion $\mathcal{T}^{(\nu)}_+(t^\nu\omega) \hookrightarrow L^1(\omega)$.

Recall that the convolution Volterra algebra $L^1_*(0, 1)$ formed by all Borel measurable functions $f : (0, 1) \to \mathbb{C}$ such that $\|f\|_1 = \int_0^1 |f(t)| \, dt < \infty$, endowed with the convolution product, can be represented as the quotient $L^1_*(0, 1) \cong L^1(\mathbb{R}^+)/\mathcal{I}_1$, where $\mathcal{I}_1$ is the closed ideal

$$\mathcal{I}_1 = \{f \in L^1(\mathbb{R}^+) : f \equiv 0 \text{ a.e. on } (0, 1)\}.$$

Similarly, let us consider the quotient $\mathcal{T}^{(\nu)}_+(t^\nu)/\mathcal{I}^{(\nu)}_1$, for the closed ideal $\mathcal{I}^{(\nu)}_1 := \mathcal{T}^{(\nu)}_+(t^\nu) \cap \mathcal{I}_1$. We show in Section 2 that it is a radical Banach algebra which is indeed topologically generated by its nilpotent elements. Since there is the identification

$$f + \mathcal{I}^{(\nu)}_1 \longleftrightarrow f|_{(0,1)}, \quad \mathcal{T}^{(\nu)}_+(t^\nu)/\mathcal{I}^{(\nu)}_1 \hookrightarrow L^1_*(0, 1),$$

a natural question in this respect is which elements of $L^1_*(0, 1)$ correspond to the classes $f + \mathcal{I}^{(\nu)}_1$ ($f \in \mathcal{T}^{(\nu)}_+(t^\nu)$). A complete answer to that problem is given in Section 3, for *integer* $\nu = n$. Namely, the quotient algebra $\mathcal{T}^{(n)}_+(t^n)/\mathcal{I}^{(n)}_1$ coincides with the space $\mathcal{V}^{(n)}(0, 1)$ formed by all functions $f : (0, 1] \longrightarrow \mathbb{C}$ for which there exist $f, f', \ldots, f^{(n-1)}$ on $(0, 1]$, $f^{(n-1)}$ is absolutely continuous on $(0, 1]$, and $\int_0^1 |f^{(n)}(x)| x^n \, dx < \infty$. Moreover, the quotient norm in $\mathcal{T}^{(n)}_+(t^n)/\mathcal{I}^{(\nu)}_1$ is equivalent to the norm

$$\|f\|_{\mathcal{V}^{(n)}(0,1)} := \int_0^1 |f^{(n)}(x)| x^n \, dx + \max_{0 \le i \le n-1} |f^{(i)}(1)|, \quad f \in \mathcal{V}^{(n)}(0, 1).$$

A consequence of the above equivalence is that the space $\mathcal{V}^{(n)}(0, 1)$ is a radical Banach algebra for the convolution and the above norm (Corollary 8). Thus $\mathcal{V}^{(n)}(0, 1)$ is a generalization of the Volterra algebra formed by (higher-order) absolutely continuous functions on $(0, 1)$. The representation of the elements $f + \mathcal{I}^{(\nu)}_1$ for $f \in \mathcal{T}^{(\nu)}_+(t^\nu)$ and general fractional ν is rather more difficult than in the integer case $\nu \in \mathbb{N}$, and in fact it remains unsolved. We briefly discuss the reason for that at the end of Section 3.

In Section 4, on the basis of results obtained in [13], we show that all closed ideals of $\mathcal{V}^{(n)}(0, 1)$ are standard, and then it follows from the main theorem of [15] that all epimorphisms onto $\mathcal{V}^{(n)}(0, 1)$ and all derivations from $\mathcal{V}^{(n)}(0, 1)$ into itself are bounded. However, we have not been able to find a complete characterization of the

set of all such derivations. The paper ends with a result, Corollary 16, in which a fairly large class of concrete derivations of $\mathcal{V}^{(n)}(0, 1)$ is given.

As the reader will have already noticed, by a Banach algebra we understand a Banach space endowed with a (jointly) continuous multiplication (so the algebra norm need not be submultiplicative with constant one).

Throughout the paper, we use the variable constant convention, in which C denotes a constant which may not be the same from line to line. The constant is frequently written with subindexes to emphasize that it depends on some parameters or functions.

2. Quotient radical Sobolev algebras

Let ω be a submultiplicative continuous weight on $(0, \infty)$. For $\nu > 0$, the Banach space $\mathcal{T}_+^{(\nu)}(t^\nu \omega)$ is defined as the completion of $C_c^{(\infty)}[0, \infty)$ in the norm

$$\int_0^\infty |W^\nu f(x)| x^\nu \omega(x) \, dx, \quad f \in C_c^{(\infty)}[0, \infty),$$

where $W^\nu f$ denotes the (fractional) Weyl derivative of f of order ν, defined as

$$W^\nu f(x) = \frac{(-1)^m}{\Gamma(m - \nu)} \frac{d^m}{dx^m} \int_x^\infty (t - x)^{m-\nu-1} f(t) \, dt,$$

where $m := [\nu] + 1$ with $[\nu]$ the integer part of ν. When ν itself is integer, say $\nu = n$, then $W^n f = (-1)^n (d/dx)^n f$.

As pointed out in the introductory section to this paper, if ω is nondecreasing then the space $\mathcal{T}_+^{(\nu)}(t^\nu \omega)$ is in fact a Banach algebra for the usual convolution on $(0, \infty)$, and a subalgebra of $L^1(\omega)$; moreover, it is semisimple if $\lim_{t \to \infty} \omega(t)^{1/t} \neq 0$. More details and properties of the Weyl fractional derivative and algebras $\mathcal{T}_+^{(\nu)}(t^\nu \omega)$ can be found in [9, 17].

Here, and in analogy to the $L^1(\omega)$ case, we would like $\mathcal{T}_+^{(\nu)}(t^\nu \omega)$ to be a radical Banach algebra when $\lim_{t \to \infty} \omega(t)^{1/t} = 0$. Unfortunately, it happens that spaces $\mathcal{T}_+^{(\nu)}(t^\nu \omega)$ are not in general convolution algebras for decreasing weights ω. To see this, take any weight ω such that

$$0 < K_\omega := \int_0^\infty t \, \omega(t) \, dt < \infty. \tag{1}$$

For $\lambda > 0$, the function $e_\lambda(t) := e^{-\lambda t}$, $t > 0$, belongs to $\mathcal{T}_+^{(1)}(t\omega)$; indeed,

$$\|e_\lambda\|_{\mathcal{T}_+^{(1)}(t\omega)} = \lambda \int_0^\infty t e^{-\lambda t} \omega(t) \, dt \leq \lambda \int_0^\infty t \, \omega(t) \, dt = \lambda K_\omega.$$

Also, a simple calculation gives us that $(e_\lambda * e_\lambda)(t) = t e^{-\lambda t}$ for all $t > 0$. Hence,

$$\|e_\lambda * e_\lambda\|_{\mathcal{T}_+^{(1)}(t\omega)} = \int_0^\infty |1 - \lambda t| \, t e^{-\lambda t} \omega(t) \, dt.$$

If $\mathcal{T}_+^{(1)}(t\omega)$ were a Banach algebra for the convolution, we would have, for some constant $C > 0$,

$$\|e_\lambda * e_\lambda\|_{\mathcal{T}_+^{(1)}(t\omega)} \le C\|e_\lambda\|_{\mathcal{T}_+^{(1)}(t\omega)}^2;$$

that is to say,

$$\int_0^\infty |1 - \lambda t| t e^{-\lambda t} \omega(t)\, dt \le C\lambda^2 K_\omega^2, \quad \lambda > 0.$$

But this cannot be true, since as λ tends to 0 we get $K_\omega \le 0$, a contradiction. Then it follows that $\mathcal{T}_+^{(1)}(t\omega)$ is not a convolution algebra.

Note that the radical weight function $\omega(t) = e^{-t^2}$ satisfies the preceding condition (1). Therefore $\mathcal{T}_+^{(1)}(te^{-t^2})$ is not even an algebra for the convolution product. So we need to look for other candidates to get convolution radical algebras involving derivatives. Let us follow the model suggested by the Volterra algebra $L_*^1(0, 1)$.

Here we work with any $a > 0$ rather than merely with $a = 1$. Thus let define the subset

$$\mathcal{I}_a^{(\nu)} := \{f \in \mathcal{T}_+^{(\nu)}(t^\nu) : f \equiv 0 \text{ a.e. on } (0, a)\} = \{f \in \mathcal{T}_+^{(\nu)}(t^\nu) : \gamma(f) \ge a\},$$

where, within the second pair of braces, $\gamma(f) := \inf(\mathrm{supp} f)$.

LEMMA 1. *The subset $\mathcal{I}_a^{(\nu)}$ is a closed ideal of $\mathcal{T}_+^{(\nu)}(t^\nu)$.*

PROOF. Put $J_a := \{f \in L^1(\mathbb{R}^+) : f \equiv 0 \text{ a.e. on } (0, a)\}$. It is well known that J_a is a closed ideal of $L^1(\mathbb{R}^+)$. Thus the result is a consequence of the continuity of the inclusion mapping $\iota: \mathcal{T}_+^{(\nu)}(t^\nu) \hookrightarrow L^1(\mathbb{R}^+)$ (see [9, p. 16]) since $\mathcal{I}_a = \iota^{-1}(J_a)$. □

We call $\mathcal{I}_a^{(\nu)}$ a *standard ideal* of $\mathcal{T}_+^{(\nu)}(t^\nu)$ at a.

LEMMA 2. *The space $C_c^{(\infty)}(0, \infty)$ of $C^{(\infty)}$ functions with compact support in $(0, \infty)$ is dense in $\mathcal{T}_+^{(\nu)}(t^\nu)$ for all $\nu \ge 0$.*

PROOF. Let $\varphi \in C_c^{(\infty)}(0, \infty)$ be positive and such that $\int_0^\infty \varphi(t)\, dt = 1$. For $\varepsilon > 0$, put $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$, $x \in \mathbb{R}^+$. Then $(\varphi_\varepsilon)_{0<\varepsilon<1}$ is a bounded approximate identity in $\mathcal{T}_+^{(\nu)}(t^\nu)$ for every $\nu \ge 0$, that is, $\lim_{\varepsilon \to 0^+} f * \varphi_\varepsilon = f$ in $\mathcal{T}_+^{(\nu)}(t^\nu)$; see [11, Proposition 2.3]. Clearly, $h * \varphi_\varepsilon \in C_c^{(\infty)}(0, \infty)$ for every $h \in C_c^{(\infty)}([0, \infty))$, and then the lemma follows by density. □

Now let us consider the quotient Banach algebra $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$. Since density is preserved by passing to the quotient, we get the following theorem.

THEOREM 3. *The ideal of its nilpotent elements is dense in $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$. Hence, the Banach algebra $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ is radical.*

PROOF. For every $f \in C_c^{(\infty)}(0, \infty)$ there exists an integer N such that $\gamma(f^{*N}) > a$. Then the result follows from Lemma 2 and the commutativity of the algebras. □

REMARK 4. An alternative way to show that the quotient algebra $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ is radical is to check that the hull $h(\mathcal{I}_a^{(\nu)}) := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0,\ \mathcal{L}(g)(z) = 0\ (g \in \mathcal{I}_a^{(\nu)})\}$, where $\mathcal{L}$ is the Laplace transform on $\mathrm{Re}\, z \geq 0$, of the ideal $\mathcal{I}_a^{(\nu)}$ is empty. This is accomplished by a standard argument.

REMARK 5. Note that the continuous inclusions

$$\mathcal{T}_+^{(\mu)}(t^\mu) \hookrightarrow \mathcal{T}_+^{(\nu)}(t^\nu), \quad \mu \geq \nu \geq 0,$$

are inherited by the above quotient radical Banach algebras; that is, for $\mu \geq \nu \geq 0$,

$$\mathcal{T}_+^{(\mu)}(t^\mu)/\mathcal{I}_a^{(\mu)} \hookrightarrow \mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}.$$

In fact,

$$\begin{aligned}
\|f + \mathcal{I}_a^{(\nu)}\|_{\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}} &= \inf\{\|f + h\|_{\mathcal{T}_+^{(\nu)}(t^\nu)} : h \in \mathcal{I}_a \cap \mathcal{T}_+^{(\nu)}(t^\nu)\} \\
&\leq \inf\{\|f + h\|_{\mathcal{T}_+^{(\nu)}(t^\nu)} : h \in \mathcal{I}_a \cap \mathcal{T}_+^{(\mu)}(t^\mu)\} \\
&\leq C_{\nu,\mu} \inf\{\|f + h\|_{\mathcal{T}_+^{(\mu)}(t^\mu)} : h \in \mathcal{I}_a \cap \mathcal{T}_+^{(\mu)}(t^\mu)\} \\
&= C_{\nu,\mu} \|f + \mathcal{I}_a^{(\mu)}\|_{\mathcal{T}_+^{(\mu)}(t^\mu)/\mathcal{I}_a^{(\mu)}}.
\end{aligned}$$

The first part of the following result is not strictly necessary in the realm of this paper, but we include it here for the sake of completeness.

PROPOSITION 6. *For z such that $\mathrm{Re}\, z > 0$, let σ^z be the element of $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ defined by the function $x \mapsto \Gamma(z)^{-1} x^{z-1}$, $x > 0$. Then $(\sigma^z)_{\mathrm{Re}\, z > 0}$ is an analytic semigroup in $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ for every $\nu > 0$, such that:*

(i)
$$\sup_{t \in (0,1)} \|\sigma^t\|_{\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}} < \infty;$$

(ii) $\mathrm{span}\{\sigma^k : k \in \mathbb{N}\}$ *is dense in* $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$.

PROOF. The assertions are readily seen from the fact that for $\mathrm{Re}\, z > 0$ the function $x \mapsto \Gamma(z)^{-1} x^{z-1} e^{-x}$, $x \in (0, \infty)$, satisfies in the Banach algebra $\mathcal{T}_+^{(\nu)}(t^\nu)$ analogous properties to those of the statement; see [10, Proposition 1.1]. □

Thus the proposition tells us in particular that the subspace of polynomials is dense in $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$, and that $(\sigma^t)_{0 < t < 1}$ is a bounded approximate identity for $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ (there are many more bounded approximate identities in $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ according to the proof of Lemma 2).

The mapping $f + \mathcal{I}_a^{(\nu)} \mapsto f|_{(0,a)}$, $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)} \hookrightarrow L_*^1(0, a)$ is obviously well defined. The image of that mapping is studied in the next section.

3. Representation of $\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}$ on $(0, a]$

The task of representing the elements $f + \mathcal{I}_a^{(v)}$, for $f \in \mathcal{T}_+^{(v)}(t^v)$, as functions almost everywhere defined on the interval $(0, a)$ looks complicated for general fractional v (see the short discussion in Remark 9 at the end of this section). Here we settle the question for integer order of derivation.

So take $v = n \in \mathbb{N}$. Let us collect some properties of the elements of $\mathcal{T}_+^{(n)}(t^n)$. For $f \in \mathcal{T}_+^{(n)}(t^n)$, the function $x \mapsto f^{(k)}(x)x^k$ is integrable on $(0, \infty)$ for all $k = 0, 1, \ldots, n$ so that

$$f^{(k)}(x) = -\int_x^\infty f^{(k+1)}(y) \, dy, \quad x > 0, \quad \text{for } k = 0, \ldots, n - 1. \tag{2}$$

In particular, $f \in C^{(n-1)}(0, \infty)$ and $f^{(n-1)}$ is absolutely continuous with a.e. derivative $f^{(n)}$ on $(0, \infty)$. Integrating by parts $n - j - 1$ times in (2) with $k = n - 1$ we get, for $j = 0, \ldots, n - 1$,

$$f^{(j)}(x) = \frac{(-1)^{n+j}}{(n - j - 1)!} \int_x^\infty (y - x)^{n-j-1} f^{(n)}(y) \, dy,$$

whence, for $x > 0$ and $0 \le k \le n - 1$:

$$
\begin{aligned}
|x^{k+1} f^{(k)}(x)| &\le \frac{1}{(n - k - 1)!} \int_x^\infty y^{k+1}(y - x)^{n-(k+1)}|f^{(n)}(y)| \, dy \\
&= \frac{1}{(n - k - 1)!} \int_x^\infty y^{k+1} y^{n-(k+1)} \left(1 - \frac{x}{y}\right)^{n-(k+1)} |f^{(n)}(y)| \, dy \\
&\le \frac{1}{(n - k - 1)!} \int_x^\infty y^n |f^{(n)}(y)| \, dy
\end{aligned}
\tag{3}
$$

and therefore $\|x^{k+1} f^{(k)}\|_\infty \le C_n \|f\|_{\mathcal{T}_+^{(n)}(t^n)}$.

For $f \in \mathcal{T}_+^{(n)}(t^n)$, let $\|f + \mathcal{I}_a^{(n)}\|_{\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}}$ be the quotient norm of $f + \mathcal{I}_a^{(n)}$ in $\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}$, and put

$$\|\|f + \mathcal{I}_a^{(n)}\|\| := \int_0^a |f^{(n)}(x)|x^n \, dx + \max_{0 \le i \le n-1} \{|f^{(i)}(a)|\},$$

$$[[f + \mathcal{I}_a^{(n)}]] := \int_0^a |f^{(n)}(x)|x^n \, dx + \sum_{i=0}^{n-1} \|x^{i+1} f^{(i)}(x)\|_{(0,a]},$$

where $\|x^{i+1} f^{(i)}(x)\|_{(0,a]} := \sup_{0 < x \le a} |x^{i+1} f^{(i)}(x)|$.

THEOREM 7. *The (nonlinear) functionals $\|\|\cdot\|\|$ and $[[\cdot]]$ are both well defined on the quotient algebra $\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}$. Moreover, $\|\cdot\|_{\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}}$, $\|\|\cdot\|\|$ and $[[\cdot]]$ are equivalent norms on $\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}$.*

PROOF. It is clearly sufficient to show that the functionals of the statement are equivalent. We first start with $\| \cdot \|_{\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}}$ and $\||\cdot\||$. For $f \in \mathcal{T}_+^{(n)}(t^n)$ and $h \in \mathcal{I}_a^{(n)}$,

$$
\begin{aligned}
\|f - h\|_{\mathcal{T}_+^{(n)}(t^n)} &= \int_0^\infty |(f-h)^{(n)}(x)| x^n \, dx \\
&= \int_0^a |f^{(n)}(x)| x^n \, dx + \int_a^\infty |(f-h)^{(n)}(x)| x^n \, dx \\
&\geq \int_0^a |f^{(n)}(x)| x^n \, dx + C_{n,a} \max_{0 \leq k \leq n-1} |f^{(k)}(a)|
\end{aligned}
$$

by (3). Hence,

$$
\|f + \mathcal{I}_a^{(n)}\|_{\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}} := \inf_{h \in \mathcal{I}_a^{(n)}} \{\|f - h\|_{\mathcal{T}_+^{(n)}(t^n)}\} \geq C_{n,a} \||f + \mathcal{I}_a^{(n)}\||.
$$

This shows in particular that $\||f + \mathcal{I}_a^{(n)}\||$ is well defined on $\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}$.

For the converse inequality, take

$$
g(x) := \begin{cases} f(x), & x \in (0, a], \\ p(x), & x \in [a, a+1], \\ 0, & x \in [a+1, \infty), \end{cases}
$$

where

$$
p(x) = c_{2n-1}(a+1-x)^{2n-1} + c_{2n-2}(a+1-x)^{2n-2} + \cdots + c_n(a+1-x)^n
$$

and $p^{(i)}(a) = f^{(i)}(a)$ for $i = 0, \ldots, n-1$. The polynomial p exists and is unique since its coefficients are the solutions of the Hermite problem of $n \times n$ linear equations

$$
\left.\begin{aligned}
c_{2n-1} + c_{2n-2} + \cdots + c_n &= f(a) \\
-c_{2n-1}(2n-1) - c_{2n-2}(2n-2) - \cdots - c_n n &= f'(a) \\
\cdots &= \cdots \\
(-1)^{n+1} c_{2n-1}(2n-1)(2n-2)\cdots(n+1) + \cdots + (-1)^{n+1} c_n n! &= f^{(n-1)}(a)
\end{aligned}\right\}
$$

for which the matrix

$$
A_n := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
-(2n-1) & -(2n-2) & \cdots & -n \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{n+1}(2n-1)\cdots(n+1) & (-1)^{n+1}(2n-2)\cdots n & \cdots & (-1)^{n+1} n!
\end{pmatrix}
$$

is invertible. In fact, it is readily seen by induction that

$$
|A_n| = \prod_{k=1}^n (k-1)! \neq 0.
$$

It is straightforward to check that $g \in \mathcal{T}_+^{(n)}(t^n)$. Now, since $f|_{(0,a)} \equiv g|_{(0,a)}$,

$$\|f + \mathcal{I}_a^{(n)}\|_{\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}} = \|g + \mathcal{I}_a\|_{\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}} \leq \|g\|_{\mathcal{T}_+^{(n)}(t^n)} = \int_0^\infty |g^{(n)}(x)|x^n \, dx$$

$$= \int_0^a |f^{(n)}(x)|x^n \, dx + \int_a^{a+1} |p^{(n)}(x)|x^n \, dx.$$

To estimate the second integral we use the fact that the expression for the nth derivative of p is

$$p^{(n)}(x) = (-1)^n c_{2n-1}(2n-1)(2n-2)\cdots(n+1)n(a+1-x)^{n-1}$$
$$+ (-1)^n c_{2n-2}(2n-2)(2n-3)\cdots n(n-1)(a+1-x)^{n-2}$$
$$+ \cdots + (-1)^n c_n n!,$$

so if $x \in (a, a+1)$ then

$$|p^{(n)}(x)| \leq \max_{n \leq i \leq 2n-1} \{|c_i|\}(2n-1)(2n-2)\cdots(n+1)n$$
$$\cdot [(a+1-x)^{n-1} + (a+1-x)^{n-2} + \cdots + 1]$$
$$\leq 2^n n^n n \max_{n \leq i \leq 2n-1} \{|c_i|\} = C_n \max_{n \leq i \leq 2n-1} \{|c_i|\}.$$

On the other hand, the coefficients c_i are linear combinations of the images $f^{(j)}(a)$, because of Cramer's rule,

$$c_{2n-i} = |A_n|^{-1} \begin{vmatrix} 1 & \cdots & f(a) & \cdots & 1 \\ -(2n-1) & \cdots & f'(a) & \cdots & -n \\ \vdots & & \vdots & & \vdots \\ (-1)^{n+1}(2n-1)\cdots(n+1) & \cdots & f^{(n-1)}(a) & \cdots & (-1)^{n+1}n! \end{vmatrix}$$
$$= |A_n|^{-1}(\mathrm{Cof}_{1,i} \cdot f(a) + \mathrm{Cof}_{2,i} \cdot f'(a) + \cdots + \mathrm{Cof}_{n,i} \cdot f^{(n-1)}(a))$$
$$= b_{i,0}f(a) + b_{i,1}f'(a) + \cdots + b_{i,n-1}f^{(n-1)}(a),$$

where the column of the $f^{(j)}(a)$ is the ith, $\mathrm{Cof}_{k,i}$ is the (k,i) cofactor and $b_{i,j} := |A_n|^{-1}\mathrm{Cof}_{j+1,i}$. Note that the $b_{i,j}$ only depend on n. Hence,

$$\max_{n \leq i \leq 2n-1} \{|c_i|\} \leq C_n \max_{0 \leq i \leq n-1} \{|f^{(i)}(a)|\},$$

and then

$$\int_a^{a+1} |p^{(n)}(x)|x^n \, dx \leq C_n \max_{0 \leq i \leq n-1} \{|f^{(i)}(a)|\} \int_a^{a+1} x^n \, dx$$
$$= C_{n,a} \max_{0 \leq i \leq n-1} \{|f^{(i)}(a)|\}.$$

In this way, we have obtained that

$$\|f + \mathcal{I}_a\|_{\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}} \leq \int_0^a |f^{(n)}(x)|x^n \, dx + C_{n,a} \max_{0\leq i\leq n-1}\{|f^{(i)}(a)|\}$$
$$\leq C\||f + \mathcal{I}_a^{(n)}\||,$$

as required.

Finally, notice that the inequality $\|x^{k+1}f^{(k)}\|_\infty \leq C_n\|f\|_{\mathcal{T}_+^{(n)}(t^n)}$, $k = 0, 1, \ldots, n-1$, and Remark 5 imply that

$$C_{n,a}\||f + \mathcal{I}_a^{(n)}\|| \leq [[f + \mathcal{I}_a^{(n)}]] \leq C_{n,a}\||f + \mathcal{I}_a^{(n)}\||.$$

This concludes the proof. □

Let $\mathcal{V}^{(n)}(0, a)$ denote the space of functions $f: (0, a] \to \mathbb{C}$ such that there exist $f', \ldots, f^{(n-1)}$ on $(0, a]$, the function $f^{(n-1)}$ is absolutely continuous on $(0, a]$, and

$$\int_0^a |f^{(n)}(x)|x^n \, dx < \infty.$$

COROLLARY 8. *The space $\mathcal{V}^{(n)}(0, a)$, endowed with the convolution product*

$$(f * g)(x) = \int_0^x f(x - y)g(y) \, dy, \quad x \in (0, a], f, g \in \mathcal{V}^{(n)}(0, a),$$

and the norm

$$\||f\|| = \int_0^a |f^{(n)}(x)|x^n \, dx + \max_{0\leq i\leq n-1} |f^{(i)}(a)|, \quad f \in \mathcal{V}^{(n)}(0, a),$$

is a radical Banach algebra isomorphic to $\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)}$.

PROOF. By Theorem 7 we have the isomorphism

$$\mathcal{T}_+^{(n)}(t^n)/\mathcal{I}_a^{(n)} \cong \mathcal{V}^{(n)}(0, a)$$

as Banach spaces and algebras. Then the result follows by Theorem 3. □

We call the radical Banach algebra $\mathcal{V}^{(n)}(0, a)$ the *Volterra algebra of absolutely continuous functions of order n* on $(0, a]$, or the *Sobolev–Volterra algebra* for short. From now on, we denote by $\|\cdot\|_{\mathcal{V}^{(n)}(0,a)}$ the previous norm $\|\|\cdot\|\|$ on $\mathcal{V}^{(n)}(0, a)$.

Note that, for $n = 0$, we have that $\mathcal{V}^{(n)}(0, a) := L^1(0, a)$, and the norm is, in this case, just the integral part. For $n > 0$ and $f \in \mathcal{V}^{(n)}(0, a)$, the norm

$$\|f\|_{\mathcal{V}^{(n)}(0,a)} = \int_0^a |f^{(n)}(x)|x^n \, dx + \sup_{0\leq k\leq n-1} |f^{(k)}(a)|$$
$$\sim \int_0^a |f^{(n)}(x)|x^n \, dx + \sum_{k=0}^{n-1} \|x^{k+1}f^{(k)}(x)\|_{(0,a]},$$

is a mixture of L^1 norm and sup-norm. Indeed, the projection

$$p\colon f \mapsto (f(a), \ldots, f^{(n-1)}(a))$$

yields a direct sum decomposition $\mathcal{V}^{(n)}(0, a) = \ker p \oplus \mathbb{C}^n$, through which the norm $\|\cdot\|_{\mathcal{V}^{(n)}(0,a)}$ becomes the standard coordinatewise topology on $\mathbb{C}^n$ and just the L^1 norm type $\int_0^a |f^{(n)}(x)| x^n \, dx$ on $\ker p$.

Next, we state some automatic properties of Sobolev–Volterra algebras as regarding discontinuous homomorphisms and Esterle's classification of radical Banach algebras.

(i)   Since $\mathcal{V}^{(n)}(0, a)$ is radical with bounded approximate identities it belongs to class 8 defined in [8], and not in class 9; that is, there is no analytic semigroup in $\mathcal{V}^{(n)}(0, a)$ that is bounded on $\{|z| < 1, \operatorname{Re} z > 0\}$. (Otherwise $L_*^1(0, a)$ would also be in class 9 because $\mathcal{V}^{(n)}(0, a) \hookrightarrow L_*^1(0, a)$, but $L_*^1(0, a)$ cannot belong to class 9 by [3, Corollary 1].)

In the following three points we assume the continuum hypothesis.

(ii)   Once again notice that $\mathcal{V}^{(n)}(0, a)$ is radical with bounded approximate identities. Then it contains a copy of $L^1(e^{-t^2})$; see [6, Theorem 5.1].

(iii)   Since $\mathcal{V}^{(n)}(0, a)$ is separable as well, there exists a discontinuous homomorphism $\mathcal{V}^{(n)}(0, a) \to L_*^1(0, a)$; see [6, Corollary 6.6].

(iv)   Let $\mathbb{C}[[X]]$ denote the algebra of complex formal series in one variable X. Since $\mathcal{V}^{(n)}(0, a)$ is in class 8 it is also in class 5 (see [8]), so that there exists a one-to-one homomorphism $\mathbb{C}[[X]] \to \mathcal{V}_1^{(n)}(0, a) := \mathcal{V}^{(n)}(0, a) \oplus \mathbb{C}$. Equivalently, there exists a discontinuous homomorphism $C(K) \to \mathcal{V}_1^{(n)}(0, a)$. Furthermore, there is a discontinuous homomorphism $A \to \mathcal{V}_1^{(n)}(0, a)$ for every unital commutative separable Banach algebra A. In particular, we can take $A = C^{(m)}[0, a]$ for all $m \in \mathbb{N}$. See [7, Theorems 6.4 and 6.5] and [8, pp. 59, 60] as a basis for the above results.

REMARK 9. Similarly to the integer case, we would like to have a representation of the quotient radical Banach algebra $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ which only took into account the behavior of the functions on the interval $(0, a]$; that is, to have a Volterra-type algebra on $(0, a]$ formed by absolutely continuous functions of *fractional* order ν on $(0, a]$. To obtain such an algebra it seems sensible to search for an equivalent norm in $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ given in terms of the restriction of $W^\nu f$ on $(0, a]$, for example,

$$\|f + \mathcal{I}_a^{(\nu)}\|_{(\nu),a} := \int_0^a |W^\nu f(x)| x^\nu \, dx + \sup_{0 \le \beta \le \nu - 1} \{|W^\beta f(a)|\}.$$

However, this does not work. In fact this question must face a serious obstacle; namely, whereas $\mathcal{I}_a^{(\nu)}$ is invariant under the usual derivation, that is, $W^n(\mathcal{I}_a^{(n)}) \subseteq \mathcal{I}_a^{(n)}$,
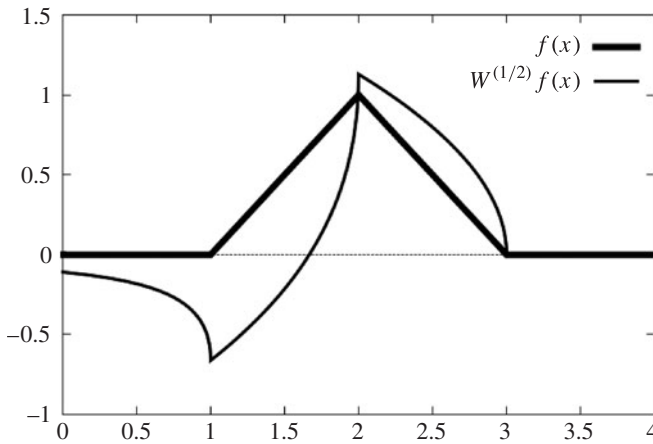
FIGURE 1. The case $\nu = 1/2$ and $a = 1$.

this does not hold for fractional ν. For example, let

$$
f(x) := \begin{cases} 0, & x \in (0, a], \\ x - a, & x \in [a, a + 1], \\ -x + a + 2, & x \in [a + 1, a + 2], \\ 0, & x \in [a + 2, \infty). \end{cases}
$$

We have that $f \in \mathcal{T}_+^{(1)}(t)$, and then $f \in \mathcal{T}_+^{(\nu)}(t^\nu)$ for all $0 < \nu < 1$. Moreover, for $0 < \nu < 1$ and $0 < x < a$,

$$
\begin{aligned}
W^\nu f(x) &= \frac{-1}{\Gamma(1 - \nu)} \int_x^\infty (y - x)^{-\nu} f'(y) \, dy \\
&= \frac{-1}{\Gamma(1 - \nu)} \int_a^{a+1} (y - x)^{-\nu} \, dy + \frac{1}{\Gamma(1 - \nu)} \int_{a+1}^{a+2} (y - x)^{-\nu} \, dy \\
&= \frac{1}{\Gamma(2 - \nu)} ((a - x)^{1-\nu} - 2(a + 1 - x)^{1-\nu} + (a + 2 - x)^{1-\nu}),
\end{aligned}
$$

which means that, while $f|_{(0,a]} \equiv 0$, the derivative $W^\nu f$ is such that $W^\nu f|_{(0,a]} \not\equiv 0$ a.e. In other words, $W^\nu(\mathcal{I}_a^{(\nu)}) \not\subseteq \mathcal{I}_a^{(\nu)}$. See the Figure 1 for the case $\nu = 1/2$ and $a = 1$.

QUESTION. Is it possible to characterize the elements of the algebra $\mathcal{T}_+^{(\nu)}(t^\nu)/\mathcal{I}_a^{(\nu)}$ intrinsically as functions on $(0, a]$?

4. Closed ideals and derivations of the Sobolev algebra

We will show that the standard ideals $\mathcal{I}_x^{(n)}$, $0 \leq x \leq a$, are the only closed ideals of $\mathcal{V}^{(n)}(0, a)$. Then, because of this and a result of [15], it follows that all

derivations $D \colon \mathcal{V}^{(n)}(0, a) \to \mathcal{V}^{(n)}(0, a)$ are automatically continuous. Recall that such a derivation is by definition a linear map such that $D(f * g) = f * D(g) + D(f) * g$, for $f, g \in \mathcal{V}^{(n)}(0, a)$.

PROPOSITION 10. *Each closed ideal of* $\mathcal{V}^{(n)}(0, a)$ *is standard.*

PROOF. Let I be a closed ideal of $\mathcal{V}^{(n)}(0, a)$. Then $J := q^{-1}(I)$ is a closed ideal of $\mathcal{T}_+^{(1)}(t)$, where $q \colon \mathcal{T}_+^{(1)}(t) \to \mathcal{V}^{(n)}(0, a)$ is the canonical quotient mapping. Let $h(J)$ be the hull, or zero set, of the ideal J in the Gelfand spectrum of $\mathcal{T}_+^{(1)}(t)$. Any $\xi \in h(J)$ is a character of $\mathcal{T}_+^{(1)}(t)$ such that $\xi(J) = 0$. Since $q(\mathcal{I}_a^{(n)}) = (0) \subseteq I$, we have that $\mathcal{I}_a^{(n)} \subseteq J$. Hence there is a character $\tilde{\xi} \colon \mathcal{V}^{(n)}(0, a) \equiv \mathcal{T}_+^{(1)}(t)/\mathcal{I}_a^{(n)} \to \mathbb{C}$ with $\xi = \tilde{\xi} \circ q$. As $\mathcal{V}^{(n)}(0, a)$ is radical it must be the case that $\tilde{\xi} = 0$, and thus $\xi = 0$. In conclusion, $h(J) = \emptyset$. This implies by [13, Theorem 3.2] that J is standard in $\mathcal{T}_+^{(1)}(t)/\mathcal{I}_a^{(n)}$; that is, $J = I_x$ for some $x \in [0, \infty)$. If $x \geq a$ then $I = q(J) = (0)$; if $0 \leq x < a$ then $I = q(J) = \mathcal{I}_x^{(n)}$ as required.   □

COROLLARY 11. *Epimorphisms from Banach algebras onto* $\mathcal{V}^{(n)}(0, a)$ *and derivations* $\mathcal{V}^{(n)}(0, a) \to \mathcal{V}^{(n)}(0, a)$ *are continuous.*

PROOF. Since every closed ideal of $\mathcal{V}^{(n)}(0, a)$ is standard, it is enough to apply [15, Theorem 2] with an argument similar to that of [15, Corollary 4], just using test functions $\varphi \in C_c^{(n)}(0, a)$ instead characteristic (indicator) functions.   □

We would like to find a characterization of all derivations from $\mathcal{V}^{(n)}(0, a)$ into itself, as has been done for the Volterra algebra $L_*^1(0, a)$ in [16]. Unfortunately, it is not clear to us how to deal with that question completely. We give some partial results.

Let $D \colon \mathcal{V}^{(n)}(0, a) \to \mathcal{V}^{(n)}(0, a)$ be a (bounded) derivation. Let $\mathbf{1}$ denote the constant function $\mathbf{1}(x) = 1$, $x \in (0, a]$. Then, as $\mathbf{1}^{*m} = x^{m-1}/(m-1)!$ we have $Dx^m = m!D(\mathbf{1}^{*(m+1)}) = (m+1)!\mathbf{1}^{*m} * D\mathbf{1} = (m+1)mx^{m-1} * D\mathbf{1}$. Hence, $Dp = (xp)'' * g$ for every polynomial p, with the convention $x'' = \delta_0$ (the Dirac delta at 0), where $g := D\mathbf{1} \in \mathcal{V}^{(n)}(0, a)$. At this point, one can look at getting an expression for the derivation D acting on the (holomorphic) semigroup σ^z defined in Proposition 6. As before,

$$D(\sigma^z) = (x^z/\Gamma(z))'' * g = z(\sigma^{z-1} * g),$$

so that $\sigma^2 * D(\sigma^z) = z\sigma^{z+1} * g = (x\sigma^z) * g$ whenever $\operatorname{Re} z > 0$. Now the question is to identify the quotient g/σ^2.

By reasoning along the same lines as in [16] one can try an approximation argument. It is not difficult to see that the above equality for polynomials also holds for test functions $f \colon Df = (xf)'' * g$, $f \in C_c^{(n+2)}(0, a]$.

Take $(\varphi_m)_{m=1}^\infty \subseteq C_c^{(\infty)}((0, a))$ as a bounded approximate identity for $\mathcal{V}^{(n)}(0, a)$ and put $g_m := g * \varphi_m$. Then $g_m \in C^\infty((0, a))$ and $g_m'' = g * \varphi_m''$. Moreover, integration by parts gives us that $(xf)'' * g_m = xf * g_m''$ for all m. Since one may assume that $g_m \to g$ a.e., it is to be expected that g_m'' should converge in some suitable way to a certain measure or distribution μ on $(0, a)$ analogously to the case $n = 0$; see [16]. (Then we would have $g = \mu * \sigma^2$.) However, by following an argument similar to that of [16],

one gets a gap caused by the fact that the algebra $\mathcal{V}^{(n)}(0, a)$ *is not* invariant under right translations.

In the opposite direction, we have the following results.

LEMMA 12. *The application*

$$d : \mathcal{V}^{(n)}(0, a) \to \mathcal{V}^{(n)}(0, a)$$
$$f(x) \mapsto xf(x)$$

is a (bounded) derivation on $\mathcal{V}^{(n)}(0, a)$.

PROOF. Obviously, if $f \in \mathcal{V}^{(n)}(0, a)$ then $d(f)$ is absolutely continuous of order n, and $d(f) \in \mathcal{V}^{(n)}(0, a)$ as well. Thus the mapping d is well defined.

Finally, d satisfies the derivation rule. Given $x \in (0, a)$,

$$d(f * g)(x) = x(f * g)(x) = \int_0^x (x - t + t)f(x - t)g(t)\,dt$$
$$= \int_0^x (x - t)f(x - t)g(t)\,dt + \int_0^x f(x - t)\,tg(t)\,dt$$
$$= (df * g + f * dg)(x).$$

This concludes the proof. □

From now on, $(tf)^{(k)}(u)$ is used to denote the value at u of the function

$$t \mapsto \frac{d^k(tf(t))}{dt^k}(t), \quad t > 0.$$

The next lemma is needed in order to prove the main result of this section.

LEMMA 13.

(i) *Let* $k \geq 1$ *and* $0 < u < a$. *Then*

$$(tf)^{(k)}(u) = kf^{(k-1)}(u) + uf^{(k)}(u), \quad \forall f \in \mathcal{V}^{(k)}(0, a).$$

(ii) *Let* $k \geq 1$, $l \in \{0, \dots, k - 1\}$ *and* $0 < b < a$. *Then*

$$\int_0^b y^l(tf)^{(k)}(y)\,dy = \sum_{j=0}^{l} (-1)^j \frac{l!}{(l - j)!} b^{l-j}(tf)^{(k-1-j)}(b)$$

for all $f \in \mathcal{V}^{(k)}(0, a)$ *null near* 0.

(iii) *Let* $m \geq 0$, $n \in \{0, \dots, m\}$ *and* $0 < u < a$. *Then*

$$u^n(tf)^{(m)}(u) = \left(t \sum_{j=0}^{n} c_{j,n,m} t^j f^{(j)} \right)^{(m-n)}(u), \quad \forall f \in \mathcal{V}^{(k)}(0, a),$$

for certain coefficients $c_{j,n,m} \in \mathbb{R}$.

Proof. We proceed by induction.

(i) The case $k = 1$ and the inductive step $(k) \Rightarrow (k + 1)$ are straightforward.

(ii) The case $l = 0$ is trivial for all $k \geq 1$. The inductive step $(k - 1, l) \Rightarrow (k, l + 1)$, for $k \geq 2$, is as follows:

$$
\begin{aligned}
\int_0^b y^{l+1}(tf)^{(k)}(y)\, dy &= b^{l+1}(tf)^{(k-1)}(b) - (l+1)\int_0^b y^l (tf)^{(k-1)}(y)\, dy \\
&= b^{l+1}(tf)^{(k-1)}(b) - (l+1)\sum_{j=0}^{l}(-1)^j \frac{l!}{(l-j)!} b^{l-j}(tf)^{(k-2-j)}(b) \\
&= \sum_{m=0}^{l+1}(-1)^m \frac{(l+1)!}{(l+1-m)!} b^{l+1-m}(tf)^{(k-1-m)}(b),
\end{aligned}
$$

where we have integrated by parts and applied the induction hypothesis at level $(k - 1, l)$.

(iii) The case $n = 0$ is trivial for all $m \geq 0$, with $c_{0,0,m} = 1$. Also the case $n = m$, with $m \geq 1$, is straightforward, by using part (i), with $c_{j,m,m} = 0$ for $j \in \{0, \ldots, n-2\}$ (if $m \geq 2$), $c_{m-1,m,m} = m$ and $c_{m,m,m} = 1$. Now the inductive step is

$$
\left.\begin{array}{c}(m, n-1) \\ (m, n)\end{array}\right\} \Rightarrow (m+1, n)
$$

for $m \geq n \geq 1$. Then

$$
\begin{aligned}
u^n(tf)^{(m+1)}(u) &= (t^n(tf)^{(m)})'(u) - nu^{n-1}(tf)^{(m)}(u) \\
&= \left(\left(t\sum_{j=0}^{n} c_{j,n,m} t^j f^{(j)}\right)^{(m-n)}\right)'(u) - n\left(t\sum_{j=0}^{n-1} c_{j,n-1,m} t^j f^{(j)}\right)^{(m-(n-1))}(u) \\
&= \left(t\sum_{j=0}^{n} c_{j,n,m+1} t^j f^{(j)}\right)^{(m+1-n)}(u)
\end{aligned}
$$

with $c_{n,n,m+1} = c_{n,n,m}$ (and therefore $c_{n,n,m+1} = c_{n,n,m} = \cdots = c_{n,n,n} = 1$), and

$$
c_{j,n,m+1} = c_{j,n,m} - nc_{j,n-1,m} \quad \text{for } j = 0, \ldots, n-1.
$$

This concludes the proof. $\square$

Remark 14. As a matter of fact, the coefficients $c_{j,n,m}$ are as follows. If $m = n = 0$,

$$
c_{0,0,0} = 1.
$$

If $m = n \geq 1$,

$$
c_{n,n,n} = 1, \quad c_{n-1,n,n} = n \quad \text{and} \quad c_{j,n,n} = 0 \quad \text{for } j = 0, \ldots, n-2, \text{ if } n \geq 2.
$$

If $m = n + 1 \geq 2$,

$$c_{n,n,n+1} = 1 \quad \text{and} \quad c_{j,n,n+1} = 0 \quad \text{for } j = 0, \ldots, n - 1.$$

Finally, if $m - n \geq 2$,

$$c_{j,n,m} = (-1)^{n-j} \binom{n}{j} \frac{(m - 2 - j)!}{(m - 2 - n)!} \quad \text{for } j = 0, \ldots, n.$$

We have not included the value of the coefficients in the formulation of the lemma in order not to make the statement (and its proof) too long. Note that the exact expression for the coefficients is not important to establish the estimates.

Through the proof of the following theorem we will assume that $f \in C^{(n)}(0, a)$, vanishing near the origin. Then for a continuous function μ on $[0, a)$, there exists the function defined on $[0, a)$ given by the convolution $xf * \mu$ and it is derivable up to order n on $[0, a)$, with

$$(xf * \mu)^{(j)} = (xf)^{(j)} * \mu \quad \text{for each } j = 0, \ldots, n.$$

As usual we will identify $d\mu_j(t)$ and $\mu_j(t)\, dt$ when necessary.

THEOREM 15. *Fix $n \geq 1$. Let $\mu_0, \ldots, \mu_{n-1}$ be n derivable functions on $[0, a)$, and let μ_n be a Borel measure on $[0, a)$ satisfying:*

(i)

$$\sup_{0 < s < a} s \int_0^{a-s} |d\mu_j|(t) < \infty, \quad j = 0, \ldots, n;$$

(ii)

$$\int_0^s d\mu_{j+1}(t) = s\mu_j(s) - (j + 1) \int_0^s \mu_j(t)\, dt, \quad s \in [0, a); \ 0 \leq j \leq n - 1.$$

Then:

(1)   *for each $k = 0, \ldots, n$ and $j \in \{0, \ldots, n - k\}$,*

$$\int_0^a |(xf * \mu_j)^{(k)}(x)| x^k\, dx \leq C\|f\|_{\mathcal{V}^{(k)}(0,a)}, \quad \forall f \in \mathcal{V}^{(k)}(0, a);$$

(2)   *for each $k = 1, \ldots, n$ and $j \in \{0, \ldots, n - k\}$,*

$$\|x^k (xf * \mu_j)^{(k-1)}(x)\|_{(0,a]} \leq C\|f\|_{\mathcal{V}^{(k)}(0,a)}, \quad \forall f \in \mathcal{V}^{(k)}(0, a).$$

In consequence, the operators $f \mapsto xf * \mu_j$, $j = 0, \ldots, n - k$, are bounded from $\mathcal{V}^{(k)}(0, a)$ to $\mathcal{V}^{(k)}(0, a)$ for each $k = 0, \ldots, n$.

Proof. We proceed by induction on k. By density, it will be enough to prove the inequalities for functions $f \in C^{(k)}(0, a)$ null near the origin.

(1) The case $k = 0$ is given in [16], but we include it here for the convenience of the reader. Let $j \in \{0, \ldots, n\}$. Then

$$\|xf * \mu_j\|_1 \leq \int_0^a \int_0^x (x - t)|f(x - t)||d\mu_j|(t) \, dx = \int_0^a \int_0^{a-t} s|f(s)|ds|d\mu_j|(t)$$
$$= \int_0^a |f(s)|\left(s \int_0^{a-s} |d\mu_j|(t)\right) ds \ \leq \ C\|f\|_1$$

where we have applied condition (i). Now let $k \in \{1, \ldots, n\}$ and suppose that the statement is true for $0, 1, \ldots, k - 1$. Let $f \in \mathcal{V}^{(k)}(0, a)$ be null near 0, and $j \in \{0, \ldots, n - k\}$. Then

$$\int_0^a |(xf * \mu_j)^{(k)}(x)|x^k \, dx \leq I_{k,1} + I_{k,2},$$

where

$$I_{k,1} := \int_0^a \left|\int_0^x (tf)^{(k)}(y)(x^k - y^k)\mu_j(x - y) \, dy\right| dx,$$
$$I_{k,2} := \int_0^a \left|\int_0^x (tf)^{(k)}(y)\mu_j(x - y)y^k \, dy\right| dx.$$

Now, to estimate the first integral $I_{k,1}$, notice that applying the cyclotomic identity

$$x^k - y^k = (x - y) \sum_{l=0}^{k-1} x^{k-1-l}y^l,$$

condition (ii), and Fubini's theorem, give us

$$I_{k,1} = \int_0^a \left|\int_0^x \left(\sum_{l=0}^{k-1} x^{k-1-l}y^l\right)(tf)^{(k)}(y)\left(\int_0^{x-y} d\mu_{j+1}(s) + (j + 1) \int_0^{x-y} d\mu_j(s)\right) dy\right| dx$$
$$= \int_0^a \left|\int_0^x \left(\int_0^{x-s} \left(\sum_{l=0}^{k-1} x^{k-1-l}y^l\right)(tf)^{(k)}(y) \, dy\right)(d\mu_{j+1}(s) + (j + 1) \, d\mu_j(s))\right| dx.$$

From now on, to simplify our notation, denote

$$\mu_\bullet := \mu_{j+1} + (j + 1)\mu_j, \quad \text{so that } d\mu_\bullet(s) := d\mu_{j+1}(s) + (j + 1) \, d\mu_j(s).$$

We can use Lemma 13(ii) and (iii) to get

$$
\begin{aligned}
I_{k,1} &= \int_0^a \left| \sum_{l=0}^{k-1} x^{k-1-l} \int_0^x \sum_{j=0}^{l} (-1)^j \frac{l!}{(l-j)!} (x-s)^{l-j} (tf)^{(k-1-j)} (x-s)\, d\mu_\bullet(s) \right| dx \\
&= \int_0^a \left| \sum_{l=0}^{k-1} \sum_{j=0}^{l} (-1)^j \frac{l!}{(l-j)!} x^{k-1-l} (t^{l-j} (tf)^{(k-1-j)} * \mu_\bullet)(x) \right| dx \\
&= \int_0^a \left| \sum_{l=0}^{k-1} \sum_{j=0}^{l} (-1)^j \frac{l!}{(l-j)!} x^{k-1-l} \left(\left(t \sum_{m=0}^{l-j} C_{m,l,j,k} t^m f^{(m)} \right)^{(k-1-l)} * \mu_\bullet \right)(x) \right| dx.
\end{aligned}
$$

By the comment prior to the statement of the theorem, the $(k-1-l)$th derivative affects the whole convolution product, therefore

$$
\begin{aligned}
I_{k,1} &\leq C_k \sum_{l=0}^{k-1} \sum_{j=0}^{l} \left\| \left(t \sum_{m=0}^{l-j} C_{m,l,j,k} t^m f^{(m)} \right) * \mu_\bullet \right\|_{\mathcal{V}^{(k-1-l)}(0,a)} \\
&\leq C_k \sum_{l=0}^{k-1} \sum_{j=0}^{l} \left\| \sum_{m=0}^{l-j} C_{m,l,j,k} t^m f^{(m)} \right\|_{\mathcal{V}^{(k-1-l)}(0,a)} \leq C_k \sum_{l=0}^{k-1} \sum_{j=0}^{l} \sum_{m=0}^{l-j} \| t^m f^{(m)} \|_{\mathcal{V}^{(k-1-l)}(0,a)} \\
&\leq C_k \sum_{l=0}^{k-1} \sum_{m=0}^{l} \| t^m f^{(m)} \|_{\mathcal{V}^{(k-1-l)}(0,a)}.
\end{aligned}
$$

Here we have applied the induction hypothesis over μ_j and μ_{j+1} at levels $0, 1, \ldots, k-1$. By Remark 5, we have the continuous inclusions

$$
\mathcal{V}^{(k)}(0,a) \hookrightarrow \mathcal{V}^{(k-1)}(0,a) \hookrightarrow \cdots \hookrightarrow \mathcal{V}^{(0)}(0,a) = L^1(0,a),
$$

so to get the bound for $I_{k,1}$ it suffices to prove that

$$
\| t^j f^{(j)} \|_{\mathcal{V}^{(k-1-j)}} \leq C_k \| f \|_{\mathcal{V}^{(k-1)}} \quad \text{for all } j = 0, \ldots, k-1.
$$

This is just a direct calculation:

$$
\begin{aligned}
\| t^j f^{(j)} \|_{\mathcal{V}^{(k-1-j)}} &= \int_0^a \left| \left(t^j f^{(j)} \right)^{(k-1-j)} (u) \right| u^{k-1-j} du \\
&= \int_0^a \left| \sum_{m=0}^{k-1-j} \binom{k-1-j}{m} (t^j)^{(m)}(u) (f^{(j)})^{(k-1-j-m)}(u) \right| u^{k-1-j} du \\
&= \int_0^a \left| \sum_{m=0}^{\min\{j,k-1-j\}} \binom{k-1-j}{m} \frac{j!}{(j-m)!} u^{j-m} f^{(k-1-m)}(u) \right| u^{k-1-j} du \\
&\leq C_k \sum_{m=0}^{\min\{j,k-1-j\}} \| f \|_{\mathcal{V}^{(k-1-m)}(0,a)} \leq C_k \| f \|_{\mathcal{V}^{(k-1)}(0,a)}.
\end{aligned}
$$

As regards the second integral, $I_{k,2}$, one has

$$I_{k,2} \le \int_0^a \int_0^x (y^k |f^{(k)}(y)| + k y^{k-1} |f^{(k-1)}(y)|) y |\mu_j(x-y)| \, dy \, dx$$

$$= \int_0^a (y^k |f^{(k)}(y)| + k y^{k-1} |f^{(k-1)}(y)|) y \int_0^{a-y} |\mu_j(s)| \, ds \, dy$$

$$\le C(\|f\|_{\mathcal{V}^{(k)}(0,a)} + k \|f\|_{\mathcal{V}^{(k-1)}(0,a)}) \le C_{n,a} \|f\|_{\mathcal{V}^{(k)}(0,a)}.$$

Here we have applied Lemma 13(i), Fubini's theorem, condition (i) and Remark 5.

(2) In the base case $k = 1$, for $j \in \{0, \dots, n-1\}$,

$$\|x(xf * \mu_j)(x)\|_{(0,a]} \le J_{1,1} + J_{1,2},$$

with

$$J_{1,1} := \sup_{0 < x < a} \left| \int_0^x (x-y) \mu_j(x-y) y f(y) \, dy \right|,$$

and

$$J_{1,2} := \sup_{0 < x < a} \left| \int_0^x y^2 \mu_j(x-y) f(y) \, dy \right|.$$

For the first supremum,

$$J_{1,1} \le \sup_{0 < x < a} \left(\int_0^x |f(y)| \left(y \int_0^{x-y} |d\mu_{j+1}|(s) + (j+1) y \int_0^{x-y} |d\mu_j|(s) \right) dy \right)$$

$$\le C \|f\|_1 \le C \|f\|_{\mathcal{V}^{(1)}(0,a)},$$

where we have applied conditions (ii) and (i) and Remark 5.

For the second supremum,

$$J_{1,2} = \sup_{0 < x < a} \left| - \int_0^x \left(\int_0^{x-y} d\mu_j(s) \right) [t^2 f(t)]'(y) \, dy \right|$$

$$\le \sup_{0 < x < a} \left(\int_0^x \left(y \int_0^{x-y} |d\mu_j|(s) \right) (2|f(y)| + y|f'(y)|) \, dy \right)$$

$$\le C(2\|f\|_1 + \|f\|_{\mathcal{V}^{(1)}(0,a)}) \le C \|f\|_{\mathcal{V}^{(1)}(0,a)}.$$

Now take $k \in \{2, \dots, n\}$ and suppose that the statement is true for $0, 1, \dots, k-1$. For $j \in \{0, \dots, n-k\}$,

$$\|x^k (xf * \mu_j)^{(k-1)}(x)\|_{(0,a]} \le J_{k,1} + J_{k,2},$$

where

$$J_{k,1} := \sup_{0 < x < a} \left| \int_0^x (x^k - y^k) \mu_j(x-y)(tf)^{(k-1)}(y) \, dy \right|,$$

and

$$J_{k,2} := \sup_{0 < x < a} \left| \int_0^x y^k \mu_j(x-y)(tf)^{(k-1)}(y) \, dy \right|.$$

With a similar argument to that used to estimate $I_{k,1}$, we get

$$J_{k,1} = \sup_{0<x<a} \left| \int_0^x \sum_{l=0}^{k-1} x^{k-1-l} \left(\int_0^{x-s} y^l (tf)^{(k-1)}(y)\, dy \right) d\mu_\bullet(s) \right|$$
$$\leq A_{k,1} + B_{k,1},$$

where

$$A_{k,1} := \sup_{0<x<a} \left| \int_0^x \sum_{l=0}^{k-2} x^{k-1-l} \left(\int_0^{x-s} y^l (tf)^{(k-1)}(y)\, dy \right) d\mu_\bullet(s) \right|$$

and

$$B_{k,1} := \sup_{0<x<a} \left| \int_0^x \left(\int_0^{x-s} y^{k-1}(tf)^{(k-1)}(y)\, dy \right) d\mu_\bullet(s) \right|.$$

For $A_{k,1}$, we proceed as we did for $I_{k,1}$:

$$A_{k,1} = \sup_{0<x<a} \left| \int_0^x \sum_{l=0}^{k-2} x^{k-1-l} \sum_{j=0}^{l} (-1)^j \frac{l!}{(l-j)!} (x-s)^{l-j} (tf)^{k-2-j}(x-s)\, d\mu_\bullet(s) \right|$$

$$= C_k \sum_{l=0}^{k-2} \sum_{j=0}^{l} \left\| x^{k-1-l} \left(\left(t \sum_{i=0}^{l-j} C_{i,j,l,k} t^i f^{(i)} \right) * \mu_\bullet \right)^{(k-2-l)}(x) \right\|_{(0,a]}$$

$$\leq C_k \sum_{l=0}^{k-2} \sum_{j=0}^{l} \left\| \sum_{i=0}^{l-j} C_{i,j,l,k} t^i f^{(i)} \right\|_{\mathcal{V}^{(k-1-l)}(0,a)} \leq C_k \|f\|_{\mathcal{V}^{(k-1)}(0,a)}.$$

For $B_{k,1}$, we apply Fubini's theorem and get

$$B_{k,1} = \sup_{0<x<a} \left| \int_0^x \left(\int_0^{x-y} d\mu_\bullet(s) \right) y^{k-1}(tf)^{(k-1)}(y)\, dy \right|$$

$$\leq \sup_{0<x<a} \int_0^x \left(\int_0^{x-y} |d\mu_\bullet|(s) \right) y^{k-1} |(tf)^{(k-1)}(y)|\, dy$$

$$\leq \sup_{0<x<a} \int_0^x \left(\sup_{0<y<x} y \int_0^{x-y} |d\mu_\bullet|(s) \right) y^{k-2} |(tf)^{(k-1)}(y)|\, dy$$

$$\leq C((k-1)\|f\|_{\mathcal{V}^{(k-2)}(0,a)} + \|f\|_{\mathcal{V}^{(k-1)}(0,a)}) \leq C_k \|f\|_{\mathcal{V}^{(k-1)}(0,a)}$$

where we have used condition (i), Lemma 13(i) and Remark 5.

Finally, for the second supremum,

$$
\begin{aligned}
J_{k,2} &= \sup_{0<x<a}\left|\int_0^x\int_0^y d\mu_j(s)(k(x-y)^{k-1}(tf)^{(k-1)}(x-y)\right.\\
&\qquad\left. + (x-y)^k(tf)^{(k)}(x-y))\,dy\right|\\
&\leq \sup_{0<x<a}\left(\int_0^x r\int_0^{x-r}|d\mu_j|(s)(k(k-1)r^{k-2}|f^{(k-2)}(r)|\right.\\
&\qquad\left. + 2kr^{k-1}|f^{(k-1)}(r)| + r^k|f^{(k)}(r)|)\,dr\right)\\
&\leq C(k(k-1)\|f\|_{\mathcal{V}^{(k-2)}(0,a)} + 2k\|f\|_{\mathcal{V}^{(k-1)}(0,a)} + \|f\|_{\mathcal{V}^{(k)}(0,a)})\\
&\leq C_{n,a}\|f\|_{\mathcal{V}^{(k)}(0,a)},
\end{aligned}
$$

where we have again applied condition (i) and Remark 5.

With all the above estimates, the proof is done.   □

From Theorem 15 one immediately gets the following result.

COROLLARY 16. *Take $n \geq 1$. Let $\mu_0, \ldots, \mu_{n-1}$ be n derivable functions on $[0, a)$, and let μ_n be a Borel measure on $[0, a)$ satisfying*

(i)

$$
\sup_{0<s<a} s\int_0^{a-s}|d\mu_j|(t) < \infty, \quad j = 0, \ldots, n,
$$

(ii)

$$
\int_0^s d\mu_{j+1}(t) = s\,\mu_j(s) - (j+1)\int_0^s \mu_j(t)\,dt, \quad s \in [0, a);\ 0 \leq j \leq n-1.
$$

Then the linear mapping $f \mapsto xf * \mu_0$ is a (bounded) derivation from $\mathcal{V}^{(n)}(0, a)$ to $\mathcal{V}^{(n)}(0, a)$.

QUESTION. Is every derivation $D\colon \mathcal{V}^{(n)}(0, a) \to \mathcal{V}^{(n)}(0, a)$ of the form given in Corollary 16?

If we knew how to describe all the derivations on $\mathcal{V}^{(n)}(0, a)$ then we could pose naturally in this setting the problem of finding the automorphisms of the algebra $\mathcal{V}^{(n)}(0, a)$, and whether or not the group of such automorphisms is connected in the operator norm topology on $\mathcal{V}^{(n)}(0, a)$ (see [14] for $n = 0$).

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