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ON TOTALLY REAL SUBMANIFOLDS IN A 6-SPHERE

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ABSTRACT. The 6-dimensional sphere S^6 has an almost complex structure induced by properties of Cayley algebra. With respect to this structure S^6 is a nearly Kaehlerian manifold. We investigate 2-dimensional totally real submanifolds in S^6. We prove that a 2-dimensional totally real submanifold in S^6 is flat.

1. **Introduction.** A Riemannian submanifold (M, Ψ) of an almost Hermitian manifold $(\tilde{M}, J, \langle,\rangle)$ is called totally real if $J_{\Psi(P)}(d\Psi_P(X))$ belongs to the normal bundle ν for any $X \epsilon T_P M$, $P \epsilon M$. The almost Hermitian manifold $(\tilde{M}, J, \langle,\rangle)$ is called a nearly Kaehlerian manifold provided that $(\tilde{\nabla}_U J)U = 0$ for any $U \epsilon X(\tilde{M})$.

The six-dimensional sphere S^6 is the most typical example of nearly Kaehlerian manifolds. The existence of such a nearly Kaehlerian structure for the 6-sphere was proved by Fukami and Ishihara [2] by making use of the properties of the Cayley division algebra. The almost complex submanifolds of the 6-dimensional sphere were studied by Gray and Sekigawa. A. Gray [3] proved that with respect to the Canonical nearly Kaehlerian structure, S^6 has no 4-dimensional almost complex submanifolds. On the other hand Sekigawa studied the 2-dimensional almost complex submanifolds of S^6; [4]. He proved, among other things, that a 2-dimensional almost complex submanifold of S^6 with Gaussian curvature $K < 1$ is either diffeomorphic to a 2-dimensional torus or a 2-dimensional sphere.

Concerning totally real submanifolds of S^6, on which this paper is about, N. Ejiri proved the following [1]:

THEOREM 1. *A 3-dimensional totally real submanifold of S^6 is orientable and minimal.*

THEOREM 2. *Let M be a 3-dimensional totally real submanifold of constant curvature C in S^6. Then either $C = 1$ (i.e., M is totally geodesic) or $C = 1/16$.*

In this paper we consider the 2-dimensional totally real submanifolds of S^6. For these submanifolds we obtain the following:

THEOREM. *Let M be a complete, 2-dimensional totally real submanifold of the 6-dimensional sphere S^6. Then M is flat.*

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2. **The canonical nearly Kaehlerian structure on S^6.** The 6-dimensional unit sphere S^6 does not admit any Kaehlerian structure. However, it admits a nearly Kaehlerian structure [2]. The Riemannian metric $\bar{g}$ on S^6 induced from $\mathcal{R}^7$, is a Hermitian metric with respect to the nearly Kaehlerian structure J.

Let $\bar{\nabla}$ be the covariant derivative with respect to the Riemannian connection on S^6. Then we have the following

LEMMA 1. *For all vector fields X on S^6 $(\bar{\nabla}_X J)X = 0$.*

Define a skew-symmetric tensor field G of type (1, 2) by

$$G(X,Y) = (\bar{\nabla}_X J)Y. \tag{2.1}$$

Then one can see that

$$G(X,JY) = -JG(X,Y). \tag{2.2}$$

3. **2-dimensional totally real submanifolds of S^6.** Let M be a 2-dimensional totally real submanifold of S^6. Let ∇ be the Riemannian connection on M and R be the Riemannian curvature tensor of M in S^6. Then the Gauss formula, Weingarten formula are given respectively by

$$\sigma(X,Y) = \bar{\nabla}_X Y - \nabla_X Y \tag{3.1}$$

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \qquad X, Y \epsilon \mathcal{X}(M) \tag{3.2}$$

where ξ is a local field of normal vector to M, and $-A_\xi X$ (resp. $\nabla^\perp_X \xi$) denotes the tangential part (resp. normal part) of $\bar{\nabla}_X \xi$.

The tangential part $A_\xi X$ is related to the second fundamental form σ as follows:

$$\langle \sigma(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle \qquad X, Y \epsilon \mathcal{X}(M) \tag{3.3}$$

We denote by $R^\perp$ the curvature tensor of the normal connection i.e. $R^\perp(X,Y) = [\nabla^\perp_X, \nabla^\perp_Y] - \nabla^\perp_{[X,Y]}$. Then the Gauss equation is given by

$$\begin{aligned} \langle R(X,Y)Z, W\rangle = {} & \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle + \langle \sigma(X,Z), \sigma(Y,W)\rangle \\ & - \langle \sigma(X,W), \sigma(Y,Z)\rangle \end{aligned} \tag{3.4}$$

Write the normal bundle ν as $\nu = \mu \oplus J(TM)$ where $J\mu = \mu$ (μ is an invariant subbundle of ν). Then we have the following

LEMMA 2. *Let X, Y be tangent to M. Then the vector $G(X,Y) \epsilon \mu$.*

PROOF. It suffices to prove the lemma for tangent basis vectors. So for the time being assume that X and Y are tangent basis vectors for M. In order to prove the above lemma one needs to show that

$$\langle(\bar{\nabla}_X J)Y, X\rangle = \langle(\bar{\nabla}_X J)Y, Y\rangle = 0$$

and

$$\langle(\bar{\nabla}_X J)Y, JX\rangle = \langle(\bar{\nabla}_X J)Y, JY\rangle = 0.$$

This is because M is 2-dimensional totally real in S^6 and the normal bundle ν is spanned by orthonormal frame field of the form $\{JX, JY, N_1, N_2\}$, for some unit vectors N_1, $N_2 \epsilon \mu$. Note that $\langle(\bar{\nabla}_X J)Y, X\rangle = -\langle Y, (\bar{\nabla}_X J)X\rangle = 0$, using the fact that $\bar{\nabla}J$ is skew-symmetric with respect to $\langle,\rangle$ and the skew symmetry of G. For the same reason we also have

$$\langle(\bar{\nabla}_X J)Y, Y\rangle = -\langle(\bar{\nabla}_Y J)X, Y\rangle = \langle X, (\bar{\nabla}_Y J)Y\rangle = 0$$

Now using (2.1) and the skew symmetry of G we get

$$\langle(\bar{\nabla}_X J)Y, JX\rangle = -\langle Y, (\bar{\nabla}_X J)JX\rangle = \langle Y, J(\bar{\nabla}_X J)X\rangle = 0$$

and

$$\begin{aligned}\langle(\bar{\nabla}_X J)Y, JY\rangle &= -\langle(\bar{\nabla}_Y J)X, JY\rangle = \langle X, (\bar{\nabla}_Y J)JY\rangle \\ &= -\langle X, J(\bar{\nabla}_Y J)Y\rangle = 0\end{aligned}$$

which completes the proof of the lemma. □

4. **Proof of the theorem.** Using equation (3.2) with $\xi = JY$ we have

$$J\bar{\nabla}_X Y + (\bar{\nabla}_X J)Y = -A_{JY}X + \nabla_X^{\perp} JY \tag{4.1}$$

and using equations (3.1) and (2.1) in equation (4.1) we get

$$J\sigma(X, Y) = -A_{JY}X + \nabla_X^{\perp} JY - G(X, Y) - J\nabla_X Y \tag{4.2}$$

Assume now, in particular, that $\{X, Y\}$ is an orthonormal frame field for M, chosen in such a way that $\nabla_X X = 0$. The existence of such a frame for our complete submanifold M is possible. This follows from Gauss Lemma [5] which is in fact valid for any complete Riemannian manifold. To construct our orthonormal frame field $\{X, Y\}$ in this case, we may just choose X to be any unit vector field on M satisfying $\nabla_X X = 0$. and then we apply Gram-Schmidt to any frame field orthogonal to X to obtain Y. For the frame field $\{X, Y\}$ we first prove that

(i) $$\langle\nabla_X^{\perp} JY, JY\rangle = 0$$

and

(ii) $$\langle \nabla_X^{\perp} JY, JX\rangle = 0$$

(i) is trivial since the frame field is orthonormal. For (ii) note that $\langle , \rangle$ is Hermitian. Then using the fact that $\langle X, Y\rangle = 0$, $\nabla_X X = 0$ and $(\bar{\nabla}_X J)(X) = 0$, equation (ii) follows.

Since the normal bundle $\nu = \mu \oplus J(TM)$, the vector $J\sigma(X, Y)\epsilon\mu \oplus (TM)$. Thus the vector in the right hand side of equation (4.2) namely $-A_{JY}X + \nabla_X^{\perp} JY - G(X, Y) - J\nabla_X Y$ belongs to $\mu \oplus (TM)$. From lemma (2) $G(X, Y)\epsilon\mu$, and we have just proved that $\nabla_X^{\perp} JY\epsilon\mu$. Since $-A_{JY}X\epsilon(TM)$ we have to have

(4.4) $$\nabla_X Y = 0$$

Switching X and Y in (4.2) we also get

(4.5) $$\nabla_Y X = 0$$

By virtue of the frame being orthonormal and equation (4.5) we get

(4.6) $$\langle \nabla_Y Y, Y\rangle = 0$$

and

(4.7) $$\langle \nabla_Y Y, X\rangle = 0$$

Then it follows from (4.6) and (4.7) that

(4.8) $$\nabla_Y Y = 0$$

The sectional curvature K of M is given by

(4.9) $$K(X, Y) = R(X, Y, Y, X) = \langle \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y, X\rangle$$

Hence it follows from (4.3), (4.4), (4.5), (4.8) and (4.9) that M is flat. □

As an immediate consequence of the above theorem, we have the following:

COROLLARY. *A 2-dimensional sphere S^2 is not totally real in S^6.*

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