Proceedings of the Edinburgh Mathematical Society (2014) **57**, 231–252
DOI:10.1017/S0013091513000850

3-FOLD EXTREMAL CONTRACTIONS OF TYPES (IC) AND (IIB)

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Dedicated to Vyacheslav Shokurov on the occasion of his 60th birthday

Abstract Let (X, C) be a germ of a 3-fold X with terminal singularities along an irreducible reduced complete curve C with a contraction $f\colon (X, C) \to (Z, o)$ such that $C = f^{-1}(o)_{\text{red}}$ and $-K_X$ is ample. Assume that (X, C) contains a point of type (IC) or (IIB). We complete the classification of such germs in terms of a general member $H \in |\mathscr{O}_X|$ containing C.

Keywords: terminal singularity; extremal contraction; $\mathbb{Q}$-conic bundle; divisorial contraction; flip

2010 *Mathematics subject classification:* Primary 14J30; 14E05; 14E30

1. Introduction

1.1. Let (X, C) be a germ of a 3-fold with terminal singularities along a reduced complete curve. We say that (X, C) is an *extremal curve germ* if there exists a contraction $f\colon (X, C) \to (Z, o)$ such that $C = f^{-1}(o)_{\text{red}}$ and $-K_X$ is f-ample.

If, furthermore, f is birational, then (X, C) is said to be an *extremal neighbourhood* [**5**]. In this case f is called *flipping* if its exceptional locus coincides with C (and then (X, C) is called *isolated*). Otherwise, the exceptional locus of f is two dimensional and f is called *divisorial*. If f is not birational, then $\dim Z = 2$ and (X, C) is said to be a $\mathbb{Q}$-*conic bundle germ* [**6**].

1.2. In this paper we consider only extremal curve germs with an irreducible central fibre C. For each singular point P of X, with $P \in C$, consider the germ $(P \in C \subset X)$. All such germs are classified into types (IA), (IC), (IIA), (IIB), (IA$^\vee$), (II$^\vee$), (ID$^\vee$), (IE$^\vee$) and (III), for the definitions of which we refer the reader to [**4**, **6**].

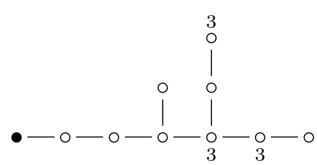
In this paper we complete the classification of extremal curve germs with irreducible central fibres containing points of type (IC) or (IIB). As in [**4**, **8**], the classification is done in terms of a general hyperplane section, that is, a general divisor H of $|\mathscr{O}_X|_C$, the linear subsystem of $|\mathscr{O}_X|$ consisting of sections containing C.

For a normal surface S and a curve $V \subset S$, we use the usual notation of graphs $\Delta(S, V)$ of the minimal resolution of S near V: each $\diamond$ corresponds to an irreducible component of V and each $\circ$ corresponds to an exceptional divisor on the minimal resolution of S, and we may use $\bullet$ instead of $\diamond$ if we want to emphasize that it is a complete (-1)-curve. A number attached to a vertex denotes the minus self-intersection number. For short, we may omit 2 if the self-intersection is -2.

Recall that if an extremal curve germ $(X, C \simeq \mathbb{P}^1)$ contains a point of type (IC), then (X, C) is not divisorial [**4**, Corollary 8.3.3]. For the remaining $\mathbb{Q}$-conic bundle case we prove the following.

Theorem 1.1. *Let (X, C) be a $\mathbb{Q}$-conic bundle germ of type* (IC) *with irreducible C and let $f\colon (X, C) \to (Z, o)$ be the corresponding contraction. Let $P \in X$ be a (unique) singular point. We then have the following.*

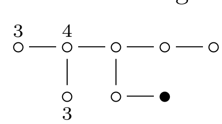
1.2.1. *The point $P \in X$ is of index* 5*. Moreover, the general member $H \in |\mathscr{O}_X|_C$ is normal, smooth outside of P, has only rational singularities, and the following is the only possibility for the dual graph of (H, C):*



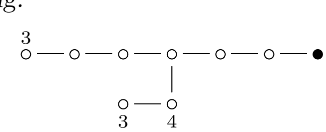
If an extremal curve germ $(X, C \simeq \mathbb{P}^1)$ contains a point of type (IIB), then it cannot be flipping [**4**, Theorem 4.5]. Remaining cases of divisorial contractions and $\mathbb{Q}$-conic bundles are covered by the following theorem.

Theorem 1.2. *Let (X, C) be an extremal curve germ of type* (IIB) *with irreducible C and let $f\colon (X, C) \to (Z, o)$ be the corresponding contraction. Let $P \in X$ be a (unique) singular point. The general member $H \in |\mathscr{O}_X|_C$ is then normal, smooth outside of P, and has only rational singularities. Moreover, the following are the only possibilities for the dual graph of (H, C).*

1.2.2. *If* (X, P) is a simple cAx/4 point *(see § 3.1), f is a divisorial contraction, $T := f(H)$ is Du Val of type* A_2*, we have the following:*



1.2.3. *If* (X, P) is a simple cAx/4 point*, f is a divisorial contraction, $T := f(H)$ is smooth, we have the following:*



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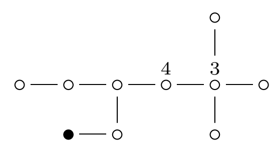
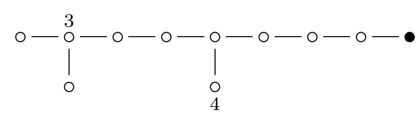
1.2.4. *If* (X, P) is a double cAx/4 point, f *is a divisorial contraction,* $T := f(H)$ *is Du Val of type* D_4*, we have the following:*



1.2.5. *If* (X, P) is a double cAx/4 point, f *is a* $\mathbb{Q}$*-conic bundle, we have the following:*



2. The case (IC)

In this section we prove Theorem 1.1. The techniques of [**4**, Chapter 8] will be used freely, sometimes without additional explanations.

2.1. Set-up

Let (X, P) be the germ of a three-dimensional terminal singularity and let $C \subset (X, P)$ be a smooth curve. Recall that the triple (X, C, P) is said to be of type (IC) if there exist analytic isomorphisms

$$(X, P) \simeq \mathbb{C}^3_{y_1,y_2,y_4}/\boldsymbol{\mu}_m(2, m-2, 1), \qquad C^\sharp \simeq \{y_1^{m-2} - y_2^2 = y_4 = 0\},$$

where m is odd and $m \geqslant 5$.

2.1.1. Let (X, C) be a $\mathbb{Q}$-conic bundle germ and let $f\colon (X, C) \to (Z, o)$ be the corresponding contraction. In this section we assume that C is irreducible and has a point P of type (IC). Recall that (X, C) is locally primitive at P [**5**, § 4.2]. Moreover, P is the only singular point on C [**6**, Theorem 8.6, Lemma 7.1.2]. Thus, the group $\mathrm{Cl}(Z, o)$ has no torsion. Moreover, the base point (Z, o) is smooth [**6**, Lemma 8.1.2].

2.2. We have an ℓ-splitting

$$\mathrm{gr}^1_C \mathscr{O} = (4P^\sharp) \,\tilde{\oplus}(-1 + (m-1)P^\sharp) \tag{2.1}$$

by [**7**, § 3], [**4**, § 2.10.2], and, hence, the unique $(4P^\sharp)$ in $\mathrm{gr}^1_C \mathscr{O}$. Since y_4 and $y_1^{m-2} - y_2^2$ form an ℓ-free ℓ-basis of $\mathrm{gr}^1_C \mathscr{O}$ at P, $(4P^\sharp)$ has an ℓ-free ℓ-basis of the form

$$u = \lambda_1 y_1^{(m-5)/2} y_4 + \mu_1 (y_1^{m-2} - y_2^2) \tag{2.2}$$

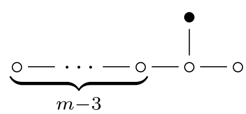
for some λ_1 and $\mu_1 \in \mathscr{O}_{C,P}$. It is easy to see that whether or not $\lambda_1(P) \neq 0$ does not depend on the choice of coordinates.

Remark 2.1. We have that

$$\mathscr{O}_C = \mathscr{O}_C(-H) \hookrightarrow \operatorname{gr}^1_C \mathscr{O} = \mathscr{O} \oplus \mathscr{O}(-1).$$

If $m \geqslant 7$, this implies that the term $y_1^2(y_1^{m-2} - y_2^2)$ appears in the equation of H. If $m = 5$, then either $y_1^2(y_1^3 - y_2^2)$ or $y_1^2 y_4$ appears in the equation of H.

2.3. According to [**7**, § 3] (cf. [**4**, § 2.10]) a general member $F \in |-K_X|$ contains C, has only Du Val singularities, and $\Delta(F, C)$ is the following graph of (-2)-curves:



(2.3)

where $\bullet$ corresponds to C. We can choose coordinates y_1, y_2, y_4 in a neighbourhood of P such that $F = \{y_4 = 0\}/\boldsymbol{\mu}_m$. In particular, the ℓ-splitting (2.1) has the form

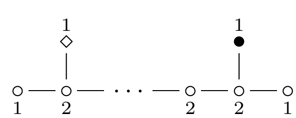
$$\operatorname{gr}^1_C \mathscr{O} = (4P^\sharp) \,\tilde{\oplus}\, \mathscr{O}_C(-F). \tag{2.4}$$

Lemma 2.2. *A general member $H \in |\mathscr{O}_X|_C$ is normal, has only rational singularities, and is smooth outside of P.*

Proof. This is similar to § 3.3.4. Let $T := f(H)$ and let $\varGamma := H \cap F$. As in § 3.3.2, consider the Stein factorization

$$f_F \colon (F, C) \xrightarrow{f_1} (F_Z, o_Z) \xrightarrow{f_2} (Z, o). \tag{2.5}$$

Set $\varGamma_Z := f_1(\varGamma)$. We may assume that, in some coordinate system, the germ (F_Z, o_Z) is given by $z^2 + xy^2 + x^{m-1} = 0$. Then, by [**2**], up to coordinate change the double cover $(F_Z, o_Z) \to (Z, o)$ is just the projection to the (x, y)-plane. Hence, we may assume that $\varGamma_Z$ is given by $x = y$. By § 2.3 we see that the fundamental cycle of the graph $\Delta(F, \varGamma)$ is given by



where the number attached to each vertex denotes its coefficient in the fundamental cycle. Therefore, $\varGamma$ is reduced, so H is smooth outside of P. The restriction $f_H \colon H \to T$ is a rational curve fibration. Hence, H has only rational singularities. □

2.4. Let J be the C-laminal ideal such that $I_C \supset J \supset F^2_C \mathscr{O}$ and $J/F^2_C \mathscr{O} = (4P^\sharp)$ in (2.4). Since J is locally a nested complete intersection (c.i.) on $C \setminus \{P\}$, and (y_4, u) is a (1,2)-monomializing ℓ-basis of $I_C \supset J$ at P with u, as in (2.2), we have an ℓ-exact sequence

$$0 \to \mathscr{O}_C(-2F) \to \operatorname{gr}^0_C J \to (4P^\sharp) \to 0 \tag{2.6}$$

and an ℓ-isomorphism $\mathscr{O}_C(-2F) \simeq (-1+(m-2)P^\sharp)$. Thus, we have $\mathrm{gr}_C^0 J \simeq \mathscr{O} \oplus \mathscr{O}(-1)$ as $\mathscr{O}_C$-modules. The unique $\mathscr{O}$ in $\mathrm{gr}_C^0 J$ is generated near P by

$$y_1^2 u + \alpha y_2 y_4^2 \mod F^3(\mathscr{O}, J) \tag{2.7}$$

for some $\alpha \in \mathscr{O}_{C,P}$.

Proofs of the following two lemmas given in [**4**] apply to our situation without any changes.

Lemma 2.3 (Kollár and Mori [4, Lemma 8.5.3]).

$$F^3(\mathscr{O}, J)^\sharp \subset ((y_1^{m-2} - y_2^2)^2, (y_1^{m-2} - y_2^2)y_4, \lambda_1 y_1^{(m-5)/2} y_4^2, y_4^3).$$

Lemma 2.4 (Kollár and Mori [4, Lemma 8.6]). *The ℓ-exact sequence (2.6) is ℓ-split if and only if $\alpha(P) = 0$.*

Proposition 2.5. *If $m \geqslant 7$, then $\alpha(P) \neq 0$.*

Proof. Assume that $\alpha(P) = 0$, that is, (2.6) is ℓ-split. Then, $\mathrm{gr}_C^0 J$ contains a unique $(4P^\sharp)$. Let $\mathscr{K}$ be the C-laminal ideal such that $J \supset \mathscr{K} \supset F_C^1(J)$ and $\mathscr{K}/F_C^1(J) = (4P^\sharp)$. By [**5**, § 8.14], $\mathscr{K}$ is locally a nested c.i. on $C \setminus \{P\}$ and $(1,3)$-monomializable at P, and we have the ℓ-isomorphisms

$$\mathrm{gr}_C^i(\mathscr{O}, \mathscr{K}) \simeq (-1 + (m-i)P^\sharp), \quad i = 1, 2, \tag{2.8}$$

and an ℓ-exact sequence

$$0 \to (-1 + (m-3)P^\sharp) \to \mathrm{gr}_C^3(\mathscr{O}, \mathscr{K}) \to (4P^\sharp) \to 0. \tag{2.9}$$

By (2.8) $\tilde{\otimes}\, \omega_X$, we see that $\mathrm{gr}_C^i(\omega_X, \mathscr{K}) \simeq (-1+(m-i-1)P^\sharp)$, so $H^j(\mathrm{gr}_C^i(\omega_X, \mathscr{K})) = 0$ for $i = 1, 2$, $j = 0, 1$ because

$$m-2, m-3 \in 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+.$$

Now, using (2.9) $\tilde{\otimes}\, \omega_X$, we obtain that

$$0 \to (-2 + (2m-4)P^\sharp) \to \mathrm{gr}_C^3(\omega_X, \mathscr{K}) \to (-1 + (m+3)P^\sharp) \to 0.$$

We note that $(-1 + (m+3)P^\sharp) \simeq \mathscr{O}(-1)$ as $\mathscr{O}_C$-modules because $3 \notin 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+$ for $m \geqslant 7$. We similarly note that $(-2 + (2m-4)P^\sharp) \simeq \mathscr{O}(-2)$ because $m - 4 \notin 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+$. Hence, $H^1(\mathrm{gr}_C^3(\omega_X, \mathscr{K})) \neq 0$. Note that $\omega_X/F^1(\omega_X, \mathscr{K}) = \mathrm{gr}_C^0\, \omega \simeq \mathscr{O}(-1)$. Using the standard exact sequences

$$0 \to \mathrm{gr}_C^i(\omega_X, \mathscr{K}) \to \omega_X/F^{i+1}(\omega_X, \mathscr{K}) \to \omega_X/F^i(\omega_X, \mathscr{K}) \to 0$$

we obtain that $H^1(\omega_X/F^4(\omega_X, \mathscr{K})) \neq 0$. By [**6**, § 4.4] we have that

$$-K_X \cdot V = 5/m \geqslant -K_X \cdot f^{-1}(o) = 2,$$

where $V = \mathrm{Spec}_X\, \mathscr{O}_X/F^4(\mathscr{O}_X, \mathscr{K})$, which is a contradiction. $\square$

Proposition 2.6.

(i) *$\mathscr{O}_F(-C)$ is an ℓ-invertible $\mathscr{O}_F$-module with an ℓ-free ℓ-basis $y_1^{m-2} - y_2^2$ at P and an ℓ-isomorphism*

$$\mathscr{O}_C \tilde{\otimes} \mathscr{O}_F(-C) \simeq (4P^\sharp).$$

(ii) $H^0(\mathscr{O}_F(-\nu C)) \twoheadrightarrow H^0(\mathscr{O}_C \tilde{\otimes} \mathscr{O}_F(-\nu C))$ *for all* $\nu \geqslant 0$.

(iii) *There exist sections $s_1, s_2 \in H^0(I_C)$ such that*

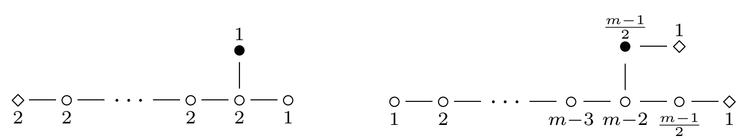
$$s_1 \equiv (\text{unit}) \cdot (y_1 + \xi_1 y_2^{m-1})^2 (y_1^{m-2} - y_2^2) \mod y_4 \qquad \textit{near } P,$$
$$s_2 \equiv (\text{unit}) \cdot (y_2 + \xi_2 y_1^{m-1}) (y_1^{m-2} - y_2^2)^{(m-1)/2} \mod y_4 \quad \textit{near } P,$$

where $\xi_1, \xi_2 \in \mathscr{O}_{X^\sharp}$ are invariants.

(iv) $H^0(I_C) \twoheadrightarrow H^0(\mathrm{gr}_C^0 J) = H^0(I_C / F^3(\mathscr{O}, J)) \simeq \mathbb{C}$.

Proof. Part (i) follows from the construction of F. Hence, $H^1(\mathscr{O}_C \tilde{\otimes} \mathscr{O}_F(-\nu C)) = 0$ for all $\nu \geqslant 0$, and $H^1(\mathscr{O}_F(-\nu C)) = 0$ since C is a fibre of proper f. Thus we have (ii).

To prove (iii) consider the Stein factorization (2.5) and, as in the proof of Lemma 2.2, we take an embedding $(F_Z, o_Z) \subset \mathbb{C}^3_{x,y,z}$ such that (F_Z, o_Z) is given by the equation $z^2 + xy^2 + x^{m-1}$, and the map $f_2\colon (F_Z, o_Z) \to (Z, o)$ is just the projection to the (x, y)-plane. Take $s_1 = f^*x$ and $s_2 = f^*y$. The weighted blow-up of (F_Z, o_Z), with weights $(2, m-2, m-1)$, extracts the central vertex of the $\mathrm{D_m}$-diagram (2.3). The multiplicity of the corresponding exceptional curve in f_2^*x and f_2^*y is equal to 2 and $m-2$, respectively. Using this, one can easily show that the multiplicities of all exceptional curves in f_2^*x and f_2^*y, respectively, are given by the following diagrams:



where the vertex $\bullet$, as usual, corresponds to C and the vertices $\diamond$ correspond to components of the proper transforms of $\{f_2^*x = 0\}$ and $\{f_2^*y = 0\}$. The multiplicity of C is exactly the exponent of $y_1^{m-2} - y_2^2$ in $s_i \mod y_4$. Therefore,

$$s_1 \equiv \gamma_1 (y_1^{m-2} - y_2^2), \qquad s_2 \equiv \gamma_2 (y_1^{m-2} - y_2^2)^{(m-1)/2} \mod y_4,$$

where $\gamma_i \in \mathscr{O}_{X^\sharp}$ are semi-invariants. Using the above diagrams, we see that

$$(\{\gamma_1 = 0\} \cdot C)_F = -4/m \quad \text{and} \quad (\{\gamma_2 = 0\} \cdot C)_F = (m-2)/m$$

because $(C^2)_F = 4/m$ by (i). Since $y_1 y_2$ is of weight 0, we have that

$$\gamma_1 = (\text{unit}) \cdot (y_1 + y_2^{m-1} \xi_1)^2 \mod y_4$$

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for some $\xi_1 \in \mathscr{O}_X$. Indeed, since $\gamma_1 = 0$ defines a double curve on F, one has that $\gamma_1 = (\text{unit}) \cdot \delta^2 \mod y_4$ for some $\delta \in \mathscr{O}_{X^\sharp}$ with weight $\equiv 2$ such that $\delta|_C = y_1|_C$.

Similarly, we have that $\gamma_2|_C = y_2|_C$. Hence,

$$\gamma_2 = (\text{unit}) \cdot (y_2 + y_1^{m-1}\xi_2) \mod y_4.$$

Finally, (iv) follows from (iii) because $H^0(\text{gr}_C^0 J) \simeq \mathbb{C}$. □

2.5. By Proposition 2.5 there are four cases to treat.

2.5.1. The case $m \geqslant 7$, $\alpha(P) \neq 0$.

2.5.2. The case $m = 5$, $\lambda_1(P) \neq 0$.

2.5.3. The case $m = 5$, $\lambda_1(P) = 0$, $\alpha(P) \neq 0$.

2.5.4. The case $m = 5$, $\lambda_1(P) = 0$, $\alpha(P) = 0$.

We show that cases 2.5.1–2.5.3 do not occur and that case 2.5.4 implies case 1.2.1.

2.6. Proof of Theorem 1.1 for cases 2.5.1 and 2.5.3. By (2.7) and Proposition 2.6, a general section $s \in H^0(I_C)$ satisfies

$$s \equiv (\text{unit}) \cdot (y_1^2 u + \alpha y_2 y_4^2) \mod F^3(\mathscr{O}, J) \quad \text{at } P,$$

where $\alpha(P) \neq 0$ by assumption. We take s_2 as given in Proposition 2.6 (iii). We claim that s_2 belongs to $H^0(F^3(\mathscr{O}, J))$. Indeed, it is obvious that $s \notin \mathbb{C} \cdot s_2 + F^3(\mathscr{O}, J)$ near P. Hence, by $H^0(I_C/F^3(\mathscr{O}, J)) = \mathbb{C} \cdot s$ we have $s_2 \in H^0(F^3(\mathscr{O}, J))$, as claimed. By Lemma 2.3, we see that the coefficient of $y_2 y_4^2$ (respectively, y_2^m) in the Taylor expansion of s_2 at $P^\sharp$ is 0 (respectively, non-zero) because $m \geqslant 7$ or $\lambda_1(P) = 0$. We now analyse the set $H = \{s = 0\}$. By Bertini's theorem, H is smooth outside of C. Since $\mathscr{O} \cdot s$ is the unique $\mathscr{O}$ in $\text{gr}_C^1 \mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}(-1)$, H is smooth on $C \setminus \{P\}$. To study (H, P), we can apply [**4**, § 10.7]. Indeed, if $\lambda_1(P) = 0$, then $\mu_1(P) \neq 0$ by the construction in § 2.2. Thus, [**4**, § 10.7.1] holds by Lemma 2.3. Replacing s with a general linear combination of s and s_2 we see that [**4**, § 10.7.2] is satisfied. Since $m \geqslant 7$ or $\lambda_1(P) = 0$, we can now apply [**4**, § 10.7]. One can see that the contraction $f_H \colon H \to T$ must be birational in this case, which is a contradiction.

2.7. Proof of Theorem 1.1 for case 2.5.2. The argument is the same as that in § 2.6 except that we need to check the conditions of [**4**, § 10.7]. Note that (2.2) has the form $u = \lambda_1 y_4 + \mu_1(y_1^3 - y_2^2)$. Since $\lambda_1(P) \neq 0$, by a coordinate change we can assume that $\mu_1(P) \neq 0$. Let $D := \{y_1 = 0\}/\boldsymbol{\mu}_m \in |-2K_X|$ and let

$$\phi_D := \frac{u - \lambda_1(P) y_4}{\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_4} = \frac{(\lambda_1 - \lambda_1(P)) y_4 + \mu_1(y_1^3 - y_2^2)}{\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_4} \in \mathscr{O}_D(-K_X).$$

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Arguments in [**7**, § 3.1] show that there exists a section $\phi \in H^0(\mathscr{O}(-K_X))$ sent to ϕ_D modulo ω_Z. Thus the image of ϕ under the homomorphism

$$I_C \mathbin{\tilde{\otimes}} \mathscr{O}_X(-K_X) \twoheadrightarrow \operatorname{gr}^1_C \mathscr{O}_X(-K_X) = (1) \mathbin{\tilde{\oplus}} (0) \twoheadrightarrow (0)$$

is non-zero because $\lambda_1(P) \neq 0$. Hence, $F' = \{\phi = 0\} \in |-K_X|$ is smooth outside of P and we may choose ϕ such that F' is, furthermore, normal by Bertini's theorem. We have an ℓ-splitting

$$\operatorname{gr}^1_C \mathscr{O} = (4P^\sharp) \mathbin{\tilde{\oplus}} \mathscr{O}_C(-F').$$

By the construction of F', we see that $(F', P) = \{v = 0\}/\boldsymbol{\mu}_m$, where $v = y_1^3 - y_2^2 + \lambda_1' y_4$ for some $\lambda_1' \in \mathscr{O}_{C,P}$ such that $\lambda_1'(P) = 0$. As in Proposition 2.6, we see that $\mathscr{O}_{F'}(-C)$ is an ℓ-invertible $\mathscr{O}_{F'}$-module with an ℓ-free ℓ-basis u at P, and there exists an ℓ-isomorphism

$$\mathscr{O}_C \mathbin{\tilde{\otimes}} \mathscr{O}_{F'}(-C) \simeq (4P^\sharp).$$

We similarly see that

$$H^0(\mathscr{O}_{F'}(-\nu C)) \twoheadrightarrow H^0(\mathscr{O}_C \mathbin{\tilde{\otimes}} \mathscr{O}_{F'}(-\nu C)) \quad \text{for all } \nu \geqslant 0.$$

We note that $y_1^2 u$ and $y_2 u^2$ are bases of $\mathscr{O}_C \mathbin{\tilde{\otimes}} \mathscr{O}_{F'}(-\nu C)$ at P for $\nu = 1$ and 2, respectively. Thus, for arbitrary $a_1, a_2 \in \mathbb{C}$, there exists a section $s_0' \in H^0(\mathscr{O}_{F'}(-C))$ such that

$$s_0' \equiv a_1 y_1^2 u + a_2 y_2 u^2 \mod (v, u^3).$$

Recall that the map $H^0(\mathscr{O}_X) \to H^0(\mathscr{O}_{F'})$ is surjective modulo $f^*\omega_Z$ [**7**, Proposition 2.1]. In our situation, sections of $f^*\omega_Z$ lifted to $\mathbb{C}^3_{y_1,y_2,y_4}$ are contained in $\bigwedge^2 \Omega^1_X$. We claim that

$$\bigwedge^2 \Omega^1_X \subset (y_1, y_2, y_4)^3 \cdot \Omega^2_{X^\sharp} \subset (y_1, y_2, y_4)^4 \cdot \omega_{F'^\sharp} \tag{2.10}$$

on the index-1 cover $F'^\sharp \subset X^\sharp$ of $F' \subset X$.

Note first that the local coordinates of X at P are

$$y_1 y_2, \quad y_1^5, \quad y_2^5, \quad y_1^2 y_4, \quad y_2^3 y_4, \quad y_2 y_4^2.$$

Since $y_1 y_2$ is the only term of degree 2, and the rest are of degree greater than or equal to 3, we see that $\bigwedge^2 \Omega^1_X \subset (y_1, y_2, y_4)^3 \cdot \Omega^2_{X^\sharp}$, the first inclusion.

Since $\phi = \beta_1(y_1^3 - y_2^2) + \beta_2 y_4$ with $\beta_1, \beta_2 \in \mathscr{O}_X$ such that $\beta_2(P) = 0$, we have that $\Omega^2_{X^\sharp}|_{F'^\sharp} \subset (y_1, y_2, y_4) \cdot \omega_{F'^\sharp}$ because

$$\Omega := \left.\frac{\mathrm{d}y_2 \wedge \mathrm{d}y_4}{\partial\phi/\partial y_1}\right|_{F'^\sharp} = \pm \left.\frac{\mathrm{d}y_1 \wedge \mathrm{d}y_4}{\partial\phi/\partial y_2}\right|_{F'^\sharp} = \pm \left.\frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{\partial\phi/\partial y_4}\right|_{F'^\sharp} \in \omega_{F'^\sharp},$$

which settles the second inclusion.

From (2.10) and $(v, u^3) \subset (y_1^3, y_2^2, y_4^3)$ we see that there exists $s' \in H^0(I_C)$ such that

$$s' \equiv a_1 y_2 y_4 + a_2 y_2 y_4^2 \mod (y_1, y_2, y_4)^4 + (y_1^3, y_2^2, y_4^3).$$

By this, we obtain non-vanishing of the coefficient of $x_2 x_3^2$ in [**4**, § 10.7]. Note that [**4**, § 10.7.1] is satisfied because $\lambda_1(P) \neq 0$, and [**4**, § 10.7.3] is satisfied because the term y_2^5 appears and $y_1^2 y_2^2$ does not appear in s_2. The rest of the proof is the same as in § 2.6.

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Remark 2.7. In [**4**], the explanation at the beginning of [**4**, § 8.11] was not appropriate: the non-vanishing of the coefficient of $x_2x_3^2$ of [**4**, § 10.7] as well as [**4**, § 10.7.3] should have been verified. The last three lines of our § 2.7 supplement the insufficient treatment in [**4**, § 8.11].

2.8. The case 2.5.4

In this case $m = 5$ and $\lambda_1(P) = \alpha(P) = 0$. Since $\lambda_1(P) = 0$, we have that $\mu_1(P) \neq 0$ because u is an ℓ-basis (see (2.2)). Since $\alpha(P) = 0$, we have that $\alpha y_2 = \lambda_2 y_1^4$ for some $\lambda_2 \in \mathscr{O}_{C,P}$, as in Lemma 2.4. Thus, a general section $s \in H^0(I_C)$ satisfies the following relation near P:

$$s \equiv (\text{unit}) \cdot y_1^2(u + \lambda_2 y_1^2 y_4^2) \mod F^3(\mathscr{O}, J). \tag{2.11}$$

Hence, s does not contain any of the terms y_1y_2, $y_1^2y_4$, $y_2y_4^2$ and contains terms y_1^5, $y_1^2y_2^2$. By the lemma below, s also contains $y_2^3y_4$.

Lemma 2.8. *Let τ be the weight $\tau = \frac{1}{5}(4,1,2)$ and let $(H,P) \subset \mathbb{C}^3/\boldsymbol{\mu}_5(2,3,1)$ be a normal surface singularity given by $\phi(x_1,x_2,x_3) = 0$, where ϕ is a $\boldsymbol{\mu}_5$-invariant that does not contain any terms of τ-weight less than 2. Then, (H,P) is not a rational singularity.*

Proof. According to [**3**] we may assume that the coefficients of ϕ are general under the assumption that $\phi_{\tau=1} = 0$. Consider the weighted blow-up with weight τ. The exceptional divisor $\varUpsilon$ is given in $\mathbb{P}(4,1,2)$ by the equation $\phi_{\tau=2}(x_1,x_2,x_3) = 0$ or, equivalently, in $\mathbb{P}(2,1,1)$ by $\phi_{\tau=2}(x_1,x_2^{1/2},x_3) = 0$. Thus, $\varUpsilon \in |\mathscr{O}_{\mathbb{P}(2,1,1)}(5)|$ is a general member. By Bertini's theorem $\varUpsilon$ is smooth and the pair $(\mathbb{P}(2,1,1),\varUpsilon)$ is purely log terminal (PLT). By the subadjunction formula,

$$2p_a(\varUpsilon) - 2 = (K_{\mathbb{P}(2,1,1)} + \varUpsilon) \cdot \varUpsilon - \tfrac{1}{2} = 2.$$

Hence, $\varUpsilon$ is not rational. □

Lemma 2.9. *The equation s contains the term $y_1y_4^3$.*

Proof. Since $\alpha(P) = 0$, we can write that $\alpha = y_1y_2\beta$ for some $\beta \in \mathscr{O}_{C,P}$. The unique $\mathscr{O} \subset \mathrm{gr}_C^0 J$ is generated near P by

$$y_1^2u + (y_1y_2\beta)y_2y_4^2 = y_1^2u + y_1^4\beta y_4^2 = y_1^2(u + y_1\beta y_4^2) \in F^3(\mathscr{O}, J).$$

By Lemma 2.4, the sequence (2.6) splits and we have

$$\begin{array}{c} \mathrm{gr}_C^0 J \simeq (4P^\sharp)\,\tilde{\oplus}\,\mathscr{O}_C(-2F) \\ \qquad\qquad\qquad\quad \big\| \\ \qquad\qquad\qquad\quad (-1 + (3P^\sharp)) \end{array}$$

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Let $\mathscr{K}$ be the C-laminal ideal such that $J \supset \mathscr{K} \supset F^3(\mathscr{O}_C, J)$ and $\mathscr{K}/F^3(\mathscr{O}, J) = (4P^\sharp)$. Then, $\mathscr{K}$ is locally a nested c.i. on $C \setminus \{P\}$ and (y_4, u) is a $(1,3)$-monomializable ℓ-basis of $I_C \supset \mathscr{K}$ at P (where u is given by (2.2)). We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (-1+2P^\sharp) & \longrightarrow & \operatorname{gr}^0_C \mathscr{K} & \longrightarrow & (4P^\sharp) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & \mathscr{O}_C(-3F) & & & & & & \end{array}$$

Since $H^1(\mathscr{O}_C(-3F) \tilde{\otimes} \omega) \neq 0$, as in the proof of Proposition 2.5, the sequence does not split. So, locally near P, the sheaf $\operatorname{gr}^0_C \mathscr{K}$ has a section $y_1^2 u + \gamma y_1 y_4^3$ with $\gamma(P) \neq 0$. $\square$

Thus, by Lemmas 2.8 and 2.9, s does not contain any of the terms $y_1 y_2$, $y_1^2 y_4$, $y_2 y_4^2$ and contains terms y_1^5, $y_1^2 y_2^2$, $y_2^3 y_4$, $y_1 y_4^3$. Therefore, [**4**, § 10.8] can be applied to (H, P). It is easy to see that the whole configuration contracts to a curve. We get the case 1.2.1. This completes the proof of Theorem 1.1.

3. The case (IIB)

3.1. Set-up

Let (X, P) be the germ of a three-dimensional terminal singularity and let $C \subset (X, P)$ be a smooth curve. Recall that the triple (X, C, P) is said to be of type (IIB) if (X, P) is a terminal singularity of type cAx/4 and there exist analytic isomorphisms

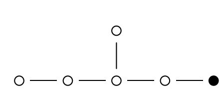
$$\begin{gathered} (X, P) \simeq \{y_1^2 - y_2^3 + \alpha = 0\}/\boldsymbol{\mu}_4(3,2,1,1) \subset \mathbb{C}^4_{y_1,\dots,y_4}/\boldsymbol{\mu}_4(3,2,1,1), \\ C \simeq \{y_1^2 - y_2^3 = y_3 = y_4 = 0\}/\boldsymbol{\mu}_4(3,2,1,1), \end{gathered}$$

where $\alpha = \alpha(y_1, \dots, y_4) \in (y_3, y_4)$ is a semi-invariant with $\operatorname{wt} \alpha \equiv 2 \mod 4$ and $\alpha_2(0, 0, y_3, y_4) \neq 0$ (see [**5**, A.3]).

Definition 3.1. We say that (X, P) is a *simple* (respectively, *double*) cAx/4-point if $\operatorname{rk} \alpha_2(0, 0, y_3, y_4) = 2$ (respectively, $\operatorname{rk} \alpha_2(0, 0, y_3, y_4) = 1$).

3.1.1. Let (X, C) be an extremal curve germ and let $f\colon (X, C) \to (Z, o)$ be the corresponding contraction. In this section we assume that C is irreducible and has a point P of type (IIB). According to [**4**, Theorem 4.5] the germ (X, C) is not flipping. Recall that (X, C) is locally primitive at P [**5**, § 4.2]. Moreover, P is the only singular point [**5**, Theorem 6.7], [**6**, Theorem 8.6, Lemma 7.1.2]. Thus, the group $\operatorname{Cl}(Z, o)$ has no torsion. Therefore, f is either a divisorial contraction to a cDV (compound Du Val) point or a conic bundle over a smooth base [**6**, Proposition 8.4].

3.2. According to [**4**, Theorem 2.2] and [**7**], a general member $F \in |-K_X|$ contains C, has only Du Val singularities, and the graph $\Delta(F, C)$ has the form



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where all the vertices correspond to (-2)-curves and $\bullet$ corresponds to C. Under the identifications of § 3.1, a general member $F \in |-K_X|$ near P is given by $\lambda y_3 + \mu y_4 = 0$ for some $\lambda, \mu \in \mathscr{O}_X$ such that $\lambda(0)$, $\mu(0)$ are general in $\mathbb{C}^*$ [**4**, § 2.11], [**7**, § 4].

3.3. Let H be a general member of $|\mathscr{O}_X|_C$, let $T := f(H)$, and let $\Gamma := H \cap F$.

3.3.1. If f is divisorial, we set $F_Z := f(F)$ and $\Gamma_Z := f(\Gamma)$. Then, $F_Z \in |-K_Z|$, T is a general hyperplane section of (Z, o) and Γ_Z is a general hyperplane section of F_Z.

3.3.2. If f is a $\mathbb{Q}$-conic bundle, we consider the Stein factorization

$$f_F : (F, C) \xrightarrow{f_1} (F_Z, o_Z) \xrightarrow{f_2} (Z, o).$$

Here we set $\Gamma_Z := f_1(\Gamma)$.

In both cases F_Z is a Du Val singularity of type E_6 by § 3.2.

Lemma 3.2.

(i) *H is normal, has only rational singularities, and is smooth outside of P.*

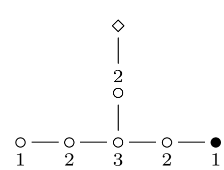
(ii) *$\Gamma = C + \Gamma_1$ (as a scheme), where Γ_1 is a reduced irreducible curve.*

(iii) *If f is birational, then $T = f(H)$ is a Du Val singularity of type* E_6, D_5, D_4, $\mathrm{A}_4, \dots, \mathrm{A}_1$ *(or smooth).*

Proof. Consider the following two cases.

3.3.3. *The case when f is divisorial*

Since the point (Z, o) is terminal of index 1, the germ (T, o) is a Du Val singularity. Since Γ_Z is a general hyperplane section of F_Z, we see that the graph $\Delta(F, \Gamma)$ has the following form:



(3.1)

where, as usual, $\diamond$ corresponds to the proper transform of Γ_Z and the numbers attached to vertices are the coefficients of the corresponding exceptional curves in the pull-back of Γ_Z. By Bertini's theorem, H is smooth outside of C. Since the coefficient of C is equal to 1, $F \cap H = C + \Gamma$ (as a scheme), H is smooth outside of P. In particular, H is normal. Since $f_H : H \to T$ is a birational contraction and (T, o) is a Du Val singularity, the singularities of H are rational.

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3.3.4. *The case when f is a $\mathbb{Q}$-conic bundle*

We may assume that, in some coordinate system, the germ (F_Z, o_Z) is given by $x^2 + y^3 + z^4 = 0$. Then, by [**2**], up to coordinate change the double cover $(F_Z, o_Z) \to (Z, o)$ is just the projection to the (y, z)-plane. Hence, we may assume that Γ_Z is given by $z = 0$. As in the case 3.3.3 we see that the graph $\Delta(F, \Gamma)$ has the form (3.1). Therefore, H is smooth outside of P. The restriction $f_H\colon H \to T$ is a rational curve fibration. Hence, H has only rational singularities.

Lemma 3.2 (iii) follows by the fact that there exists a hyperplane section F_Z of (Z, o) that is Du Val of type E_6 (see, for example, [**1**]). □

We need a more detailed description of (H, C) near P.

Lemma 3.3. *In the notation of § 3.1 the surface $H \subset X$ is given locally near P by the equation $y_3v_3 + y_4v_4 = 0$, where $v_3, v_4 \in \mathscr{O}_{P^\sharp, X^\sharp}$ are semi-invariants with $\operatorname{wt} v_i \equiv 3$ and at least one of v_3 or v_4 contains a linear term in y_1.*

Proof. Since H is normal and $\operatorname{gr}^1_C \mathscr{O} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$, we have that $\mathscr{O}_C(-H) = \mathscr{O} \subset \operatorname{gr}^1_C \mathscr{O}$, i.e. the local equation of H must be a generator of $\mathscr{O} \subset \operatorname{gr}^1_C \mathscr{O}$. □

3.4. Let σ be the weight $\frac{1}{4}(3, 2, 1, 1)$. By Lemma 3.3 the surface germ (H, P) can be given in $\mathbb{C}^4/\boldsymbol{\mu}_4(3, 2, 1, 1)$ by the two equations

$$\left.\begin{aligned} y_1^2 - y_2^3 + \eta(y_3, y_4) + \phi(y_1, y_2, y_3, y_4) = 0,\\ y_1 l(y_3, y_4) + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi(y_1, y_2, y_3, y_4) = 0, \end{aligned}\right\} \tag{3.2}$$

where η, l, q and ξ are homogeneous polynomials of degree 2, 1, 2 and 4, respectively, $\eta \neq 0$, $l \neq 0$, $\phi, \psi \in (y_3, y_4)$, $\sigma\text{-}\operatorname{ord} \phi \geqslant \frac{3}{2}$, $\sigma\text{-}\operatorname{ord} \psi \geqslant 2$. Moreover, $\operatorname{rk} \eta = 2$ (respectively, $\operatorname{rk} \eta = 1$) if (X, P) is a simple (respectively, double) cAx/4-point.

3.4.1. Consider the weighted blow-up

$$g\colon (W \supset \tilde{X} \supset \tilde{H}) \to (\mathbb{C}^4/\boldsymbol{\mu}_4(3, 2, 1, 1) \supset X \supset H)$$

with weight σ. Let E be the g-exceptional divisor, let $\varXi := E \cap \tilde{H}$ be the exceptional divisor of $g_H := g|_{\tilde{H}}$, and let $\tilde{C}$ be the proper transform of C. Define

$$\varXi_0 := \{y_3 = y_4 = 0\} \subset E.$$

If $\tilde{H}$ is normal, let $g_1\colon \hat{H} \to \tilde{H}$ be the minimal resolution. Thus, in this case, we have the morphisms

$$h\colon \hat{H} \xrightarrow{g_1} \tilde{H} \xrightarrow{g_H} H \xrightarrow{f_H} T.$$

Lemma 3.4.

(i) *$E \simeq \mathbb{P}(3, 2, 1, 1)$ and $\varXi$ is given in this $\mathbb{P}(3, 2, 1, 1)$ by*

$$\eta(y_3, y_4) = y_1 l(y_3, y_4) + y_2 q(y_3, y_4) + \xi(y_3, y_4) = 0.$$

(ii) *$\tilde{C}$ of C meets E at $Q := (1:1:0:0) \in \Xi_0$.*

(iii) *Ξ_0 is a component of Ξ and $(\Xi_0 \cdot \Xi)_{\tilde{H}} = -\frac{2}{3}$.*

(iv) *If $\tilde{H}$ is normal, then $K_{\tilde{H}} = g^* K_H - \frac{3}{4}\Xi$.*

Proof. Statements (i) and (ii) are obvious; (iii) follows from

$$(\Xi_0 \cdot \Xi)_{\tilde{H}} = (\Xi_0 \cdot E)_W = (\Xi_0 \cdot \mathscr{O}_E(E))_E = (\Xi_0 \cdot \mathscr{O}_E(-4))_E = -\tfrac{2}{3},$$

and (iv) follows from $K_W = g^* K_{\mathbb{C}^4/\boldsymbol{\mu}_4} + \frac{3}{4}E$. □

3.5. The case of a simple cAx/4-point

After a coordinate change, we may assume that $\eta = y_3 y_4$. We may also assume that the term y_3 appears in $l(y_3, y_4)$ with coefficient 1, that is, $l(y_3, y_4) = y_3 + c y_4$, $c \in \mathbb{C}$. Thus, (3.2) for (H, P) have the form

$$\left.\begin{aligned} y_1^2 - y_2^3 + y_3 y_4 + \phi = 0, \\ y_1(y_3 + c y_4) + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi = 0. \end{aligned}\right\} \tag{3.3}$$

It is easy to see that in this case $\tilde{X}$ has only isolated (terminal) singularities. Indeed, $\tilde{X} \cap E$ is given by $y_3 y_4 = 0$ in $E \simeq \mathbb{P}(3,2,1,1)$. Hence, $\operatorname{Sing}(\tilde{X}) \subset \Xi_0 \cup \operatorname{Sing}(E)$. There exist the following subcases.

3.5.1. *The subcase when (X, P) is a simple* cAx/4*-point and $c \neq 0$*

We show that only the case 1.2.2 occurs. We may assume that in (3.3) $l(y_3, y_4) = y_3 + y_4$. In this case, $\Xi = 2\Xi_0 + \Xi' + \Xi''$, where Ξ' and Ξ'' are given in $E \simeq \mathbb{P}(3,2,1,1)$ as

$$\begin{aligned} \Xi' &:= \{y_3 = y_1 + y_2 q(0, y_4)/y_4 + \xi(0, y_4)/y_4 = 0\}, \\ \Xi'' &:= \{y_4 = y_1 + y_2 q(y_3, 0)/y_3 + \xi(y_3, 0)/y_3 = 0\}. \end{aligned}$$

All the components of Ξ pass through $(0:1:0:0)$ and do not meet each other elsewhere.

Claim 3.5. *The surface $\tilde{H}$ is normal and has the following singularities (in natural weighted coordinates on $E \simeq \mathbb{P}(3,2,1,1)$):*

- *$O_1 := (1:0:0:0)$, which is of type* A_2,
- *$Q := \Xi_0 \cap \tilde{C} = (1:1:0:0)$, which is of type* A_1,
- *$O_2 := \Xi_0 \cap \Xi' \cap \Xi'' = (0:1:0:0)$, which is a log terminal point of index 2 (a cyclic quotient singularity of type $(1, 2k-1)/4k$).*

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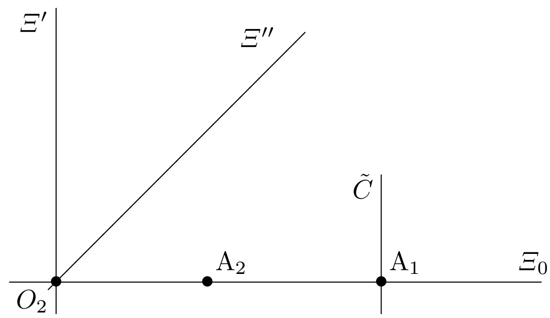
The pairs $(\tilde{H}, \Xi_0 + \Xi' + \tilde{C})$ *and* $(\tilde{H}, \Xi_0 + \Xi'' + \tilde{C})$ *are log canonical (LC). Moreover, they are PLT at all points of* $\Xi_0 \setminus \{O_2, Q\}$*. Thus, the surface* $\tilde{H}$ *looks as follows:*



Proof. Since $\Xi = \tilde{H} \cap E$ is reduced along Ξ' and Ξ'', the singular locus of $\tilde{H}$ is contained in $\Xi_0 = \{y_3 = y_4 = 0\}$.

Consider the chart $U_1 = \{y_1 \neq 0\} \subset W$, $U_1 \simeq \mathbb{C}^4/\boldsymbol{\mu}_3(1,1,2,2)$. The equations of $\tilde{H}$ have the form

$$\begin{aligned} y_1 - y_1 y_2^3 + y_3 y_4 + y_1 \phi_{3/2}(1, y_2, y_3, y_4) + y_1^2 (+\cdots) &= 0, \\ y_3 + y_4 + y_2 q(y_3, y_4) + \xi(y_3, y_4) + y_1 \psi_2(1, y_2, y_3, y_4) + y_1^2 (+\cdots) &= 0, \end{aligned}$$

and $\tilde{C}$ is cut out on $\tilde{H}$ by $y_3 = y_4 = 0$. Using the condition that $y_1 = y_3 = y_4 = 0$, one can obtain that the surface $\tilde{H} \cap U_1$ has two singular points on the exceptional divisor $\{y_1 = 0\}$: $Q = \{y_1 = y_3 = y_4 = 1 - y_2^3 = 0\}$ and the origin O_1. It is easy to see that $(\tilde{H}, Q)$ is a Du Val singularity of type A_1 and $(\tilde{H}, O_1)$ is a Du Val singularity of type A_2. Since Ξ_0 and $\tilde{C}$ are smooth curves meeting each other transversely, the pair $K_{\tilde{H}} + \Xi_0 + \tilde{C}$ is LC at Q.

Consider the chart $U_2 = \{y_2 \neq 0\} \subset W$, $U_2 \simeq \mathbb{C}^4/\boldsymbol{\mu}_2(1,0,1,1)$. The equations of $\tilde{H}$ have the form

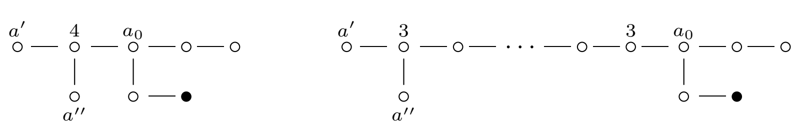
$$\begin{aligned} y_1^2 y_2 - y_2 + y_3 y_4 + y_2 \phi_{3/2}(y_1, 1, y_3, y_4) + y_2^2 (+\cdots) &= 0, \\ y_1(y_3 + y_4) + q(y_3, y_4) + \xi(y_3, y_4) + y_2 \psi_2(y_1, 1, y_3, y_4) + y_2^2 (+\cdots) &= 0. \end{aligned}$$

We then get only one new singular point: the origin O_2 where the singularity of $\tilde{H}$ is analytically isomorphic to a singularity in $\mathbb{C}^3_{y_1,y_3,y_4}/\boldsymbol{\mu}_2(1,1,1)$ given by

$$\{y_1(y_3 + y_4) + q(y_3, y_4) + (\text{terms of degree} \geqslant 3) = 0\}. \tag{3.4}$$

Hence, $(\tilde{H}, O_2)$ is a log terminal singularity of index 2. □

Therefore, for the graph $\Delta(H, C)$ we have only the following two possibilities:



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where the vertex marked by a_0 (respectively, a', a'') corresponds to Ξ_0 (respectively, Ξ', Ξ'') and • corresponds to $\hat{C}$.

Using Lemma 3.4 (iii) one can easily obtain that $a_0 = 2$. Similarly,

$$(\Xi' \cdot \Xi)_{\tilde{H}} = (\Xi'' \cdot \Xi)_{\tilde{H}} = -2.$$

This gives us that $a' = a'' = 3$. However, the right-hand configuration above is not contractible. We get the case 1.2.2.

Corollary 3.6. *We have that $q(0, y_4) \neq 0$.*

Proof. Assume that $q(0, y_4) = 0$. Take H such that in (3.2) the functions η, ϕ, l, q, ξ and ψ are sufficiently general under this assumption. Let X' be a general one-parameter deformation family of H. According to [**4**, Proposition 11.4] there exists a contraction $f' \colon X' \to Z'$, so (X', C') is an extremal curve germ. Moreover, (X', C') is of type (IIB). By 3.5.1 we get a contradiction (otherwise (3.4) is not a point of type $\frac{1}{4}(1, 1)$). □

3.5.2. *The subcase when (X, P) is a simple* cAx/4*-point and $c = 0$*

We show that only the case 1.2.3 occurs. Equations (3.3) have the form

$$y_1^2 - y_2^3 + y_3 y_4 + \phi = 0,$$
$$y_1 y_3 + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi = 0.$$

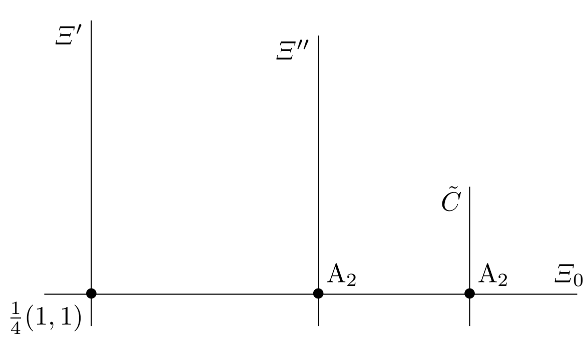
In this case, $\Xi = 3\Xi_0 + \Xi' + \Xi''$, where Ξ' and Ξ'' are given in $E \simeq \mathbb{P}(3, 2, 1, 1)$ as

$$\Xi' = \{y_4 = y_1 + y_2 q(y_3, 0)/y_3 + \xi(y_3, 0)/y_3 = 0\},$$
$$\Xi'' = \{y_3 = y_2 q(0, y_4)/y_4^2 + \xi(0, y_4)/y_4^2 = 0\}.$$

Claim 3.7. *The surface $\tilde{H}$ is normal and has the following singularities (in natural weighted coordinates on $E \simeq \mathbb{P}(3, 2, 1, 1)$):*

- $O_1 := \Xi_0 \cap \Xi'' = (1 : 0 : 0 : 0)$, *which is of type* A_2,
- $Q := \Xi_0 \cap \tilde{C} = (1 : 1 : 0 : 0)$, *which is of type* A_2,
- $O_2 := \Xi_0 \cap \Xi' = (0 : 1 : 0 : 0)$, *which is of type* $\frac{1}{4}(1, 1)$.

The pair $(\tilde{H}, \Xi_0 + \Xi' + \Xi'' + \tilde{C})$ is LC. Thus, $\tilde{H}$ looks as follows:



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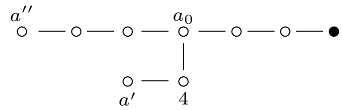
The proof is similar to the proof of Claim 3.5, so we omit it.

By the above claim, $\Delta(H, C)$ has the form



Since

$$(\Xi' \cdot \Xi)_{\tilde{H}} = -2, \qquad (\Xi'' \cdot \Xi)_{\tilde{H}} = -\tfrac{4}{3}$$

(cf. Lemma 3.4 (iii)), we have that $a_0 = 2$ and $a' = a'' = 3$. Thus, we get the case 1.2.3.

3.6. The case of a double cAx/4-point

We may assume that $\eta = y_3^2$. By Corollary 3.6, $q(0, y_4) \neq 0$, so we also may assume that $q(0, y_4) = y_4^2$. Thus, Equations (3.2) for (H, P) have the form

$$\begin{aligned} y_1^2 - y_2^3 + y_3^2 + \phi &= 0, \\ y_1 l(y_3, y_4) + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi &= 0, \end{aligned}$$

where ϕ does not contain any terms of degree less than or equal to 2. This case is more complicated because $\tilde{X}$ has non-isolated singularities.

Remark 3.8. $\operatorname{Sing}(\tilde{X})$ has exactly one one-dimensional irreducible component

$$\Lambda := \{y_3 = y_1^2 - y_2^3 + \phi_{\sigma=3/2}(y_1, y_2, 0, y_4) = 0\} \subset E \simeq \mathbb{P}(3, 2, 1, 1).$$

There exist the following subcases.

3.6.1. *The subcase when (X, P) is a double* cAx/4*-point and $l(0, y_4) \neq 0$*

We show that only the case 1.2.4 occurs. After a coordinate change, we may assume that $l(y_3, y_4) = y_4$, so Equations (3.2) for (H, P) have the form

$$\left.\begin{aligned} y_1^2 - y_2^3 + y_3^2 + \phi &= 0, \\ y_1 y_4 + y_2 q(y_3, y_4) + \xi(y_3, y_4) + \psi &= 0. \end{aligned}\right\} \tag{3.5}$$

In this case, $\Xi = 2\Xi_0 + 2\Xi'$, where

$$\Xi' = \{y_3 = y_1 + y_2 q(0, y_4)/y_4 + \xi(0, y_4)/y_4 = 0\} \subset E \simeq \mathbb{P}(3, 2, 1, 1).$$

Claim 3.9. *The surface $\tilde{H}$ is normal and has the following singularities on Ξ_0 (in natural weighted coordinates on $E \simeq \mathbb{P}(3, 2, 1, 1)$):*

- $O_1 := (1 : 0 : 0 : 0)$, *which is of type* A_2,
- $Q := \Xi_0 \cap \tilde{C} = (1 : 1 : 0 : 0)$, *which is of type* A_1,
- $O_2 := \Xi_0 \cap \Xi' = (0 : 1 : 0 : 0)$, *which is a log terminal point of index* 2.

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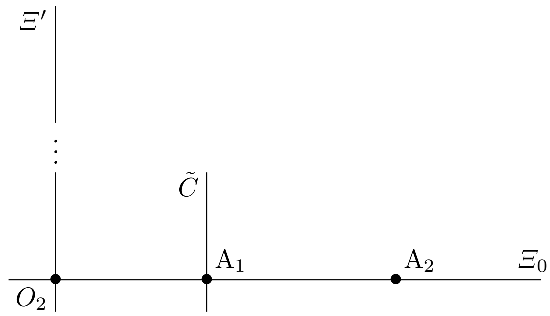
The pair $(\tilde{H}, \Xi_0 + \Xi' + \tilde{C})$ *is LC along* Ξ_0. *Moreover, it is PLT at all points of* $\Xi_0 \setminus \{O_2, Q\}$. *Thus,* $\tilde{H}$ *looks as follows:*

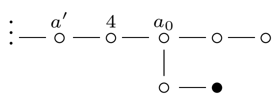


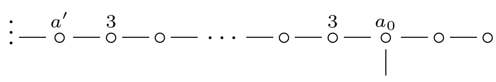
where there are more singular points sitting on $\Xi' \setminus \{O_2\}$ *that must be Du Val.*

The proof is similar to the proof of Claim 3.5.

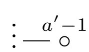
Remark 3.10. For a general choice of ξ and ϕ, the surface $\tilde{H}$ has exactly three singular points on $\Xi' \setminus \{O_2\}$ and these points are of type A_1.

Hence, the dual graph $\Delta(H, C)$ has one of the following forms:

(a)


(b)


where ⋮ corresponds to some Du Val singularities sitting on Ξ'. Since the whole configuration is contractible to either a Du Val point or a curve, we have that $a_0 = 2$ and case (b) does not occur. In case (a), contracting black vertices successively, we get the following:



Hence, $a' = 2$ or 3.

3.6.1.1. Let (S, o) be a normal surface singularity and let $\mu\colon \hat{S} \to S$ be its resolution. Recall that the *codiscrepancy divisor* is a unique $\mathbb{Q}$-divisor $\Theta = \sum \theta_i \Theta_i$ on $\hat{S}$ with support in the exceptional locus such that $\mu^* K_S = K_{\hat{S}} + \Theta$. If μ is the minimal resolution, then Θ must be effective. The coefficient θ_i is called the *codiscrepancy* of Θ_i. We denote it by $\operatorname{cdisc}(\Theta_i)$. If (S, o) is a rational singularity, then $\theta_i = \operatorname{cdisc}(\Theta_i)$ can be found from the system of linear equations

$$\sum_i \theta_i \Theta_i \cdot \Theta_j = -K_{\hat{S}} \cdot \Theta_j = 2 + \Theta_j^2.$$

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Let $a_i := -\Theta_i^2$. The system can then be rewritten as

$$a_j\theta_j = a_j - 2 + \sum\nolimits' \theta_i,$$

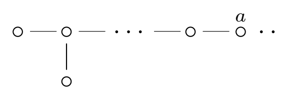
where $\sum'$ runs through all exceptional curves Θ_i meeting Θ_j.

Lemma 3.11. *Let Δ be the dual graph of a resolution of a rational singularity and let Δ' be its subgraph consisting of one vertex of weight $a \geqslant 2$ and $n-1$ vertices of weight 2. Assume that the remaining part $\Delta \setminus \Delta'$ is attached to $\overset{a}{\circ}$.*

(i) *If Δ' has the form*

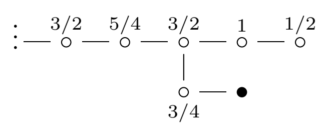
then the codiscrepancies of the components in Δ', indexed from left to right, are computed by $\alpha_k = k\alpha_1$, $k \leqslant n$.

(ii) *If Δ' has the form*



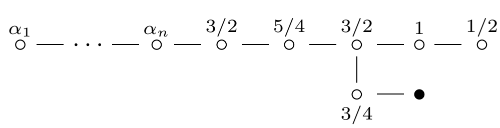
then the codiscrepancies of the components in Δ' are computed by $2\alpha_1 = 2\alpha_2 = \alpha_3$ and $\alpha_k = \alpha_3$ for $3 \leqslant k \leqslant n$, when the bottom component is indexed first and the rest are indexed from left to right.

3.6.1.2. By Lemma 3.4 (iv) we have that $\mathrm{cdisc}(\Xi_0) = \mathrm{cdisc}(\Xi') = \frac{3}{2}$. Using 3.6.1.1 we compute the codiscrepancies of exceptional divisors over $\tilde{H}$:

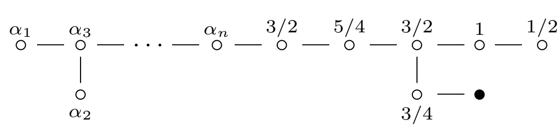


3.6.1.3. If $a' = 2$, then the configuration $\vdots\!\!-\!\!\overset{a'-1}{\circ}$ is contracted either to a smooth point or to a curve. Therefore, we have one of the following possibilities:

(a1)



(a2) for $n \geqslant 2$,



We then get a contradiction by Lemma 3.11.

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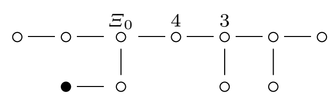
3.6.1.4. Thus, $a' = 3$. Then, f is divisorial and the configuration $\vdots\!-\!\overset{a'-1}{\circ}$ is exactly the dual graph of the minimal resolution of (T, o), which is a Du Val graph of type E_6, D_5, D_4, A_4, A_3, A_2 or A_1. If the graph $\Delta(H, C)$ has the form (a1), then, as above, $\frac{3}{2} = \alpha_{n+1} = (n+1)\alpha_1$, $3 \cdot \frac{3}{2} = 1 + \alpha_n + \frac{5}{4}$. This gives us that $n\alpha_1 = \frac{9}{4}$, $\alpha_1 = \frac{3}{2} - \frac{9}{4} < 0$, which is a contradiction. Similarly, in case (a2) with $n \geqslant 3$ we obtain that $\alpha_n = \frac{3}{2}$, $3 \cdot \frac{3}{2} = 1 + \alpha_n + \frac{5}{4}$, which is a contradiction.

If there exist three connected components of the exceptional divisor attached to Ξ', then for corresponding codiscrepancies $\alpha_n, \beta_m, \gamma_l$ we have that $3 \cdot \frac{3}{2} = 1 + \alpha_n + \beta_m + \gamma_l + \frac{5}{4}$, $\alpha_n + \beta_m + \gamma_l = \frac{9}{4}$. On the other hand, $2\alpha_n \geqslant \frac{3}{2}$, $2\beta_m \geqslant \frac{3}{2}$, $2\gamma_l \geqslant \frac{3}{2}$. Hence, the equalities $\alpha_n = \beta_m = \gamma_l = \frac{3}{4}$ hold and we get the case 1.2.4.

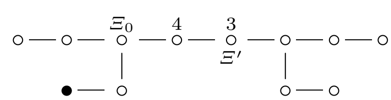
In the remaining cases, by direct computations we obtain that the exceptional divisors have codiscrepancies whose denominators divide 4 only in cases 3.6.1.5 or 3.6.1.6.

3.6.1.5. (T, o) is Du Val of type D_5, and $\Delta(H, C)$ has the form:



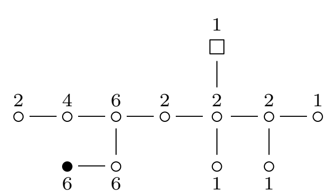
Here, $\tilde{H}$ has two singular points on $\Xi' \setminus \Xi_0$ and these points are of types A_1 and A_3.

3.6.1.6. (T, o) is Du Val of type E_6, and $\Delta(H, C)$ has the form:



Here, $\tilde{H}$ has exactly one singular point on $\Xi' \setminus \Xi_0$ and this point is of type A_5.

3.6.2. We now show that in cases 3.6.1.5 and 3.6.1.6 the chosen element $H \in |\mathscr{O}_X|_C$ is not general. Consider the case 3.6.1.5 (case 3.6.1.6 can be treated similarly). Take a divisor D on $\hat{H}$, whose coefficients are as follows:



where $\square$ corresponds to an arbitrary smooth analytic curve $\hat{G}$ meeting Ξ' transversely, so $\operatorname{Supp} D$ is a simple normal crossing divisor. It is easy to verify that D is numerically trivial, so $D = h^* G_Z$, where G_Z is a Cartier divisor on T. There exists an exact sequence

$$0 \to \mathscr{O}_X(-H) \to \mathscr{O}_X \to \mathscr{O}_H \to 0.$$

Since D corresponds to a section in $H^0(\mathscr{O}_H)$ and $R^1 f_* \mathscr{O}_X(-H) \simeq R^1 f_* \mathscr{O}_X = 0$, there exists a member $H' \in |\mathscr{O}_X|_C$ such that $H' \cap H = D$ and, in particular, H' contains C.

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The proper transform $\tilde{H}'$ of H' by g satisfies $\tilde{H}' = g^*H' - E|_{\tilde{X}}$. Since $\Xi = E \cap \tilde{H}$ and $\Xi = 2\Xi_0 + 2\Xi'$, we have that $\tilde{H}'|_{\tilde{H}} = 4\Xi_0 + g_1(\hat{G})$. In particular, Ξ' is not a component of $\tilde{H}'|_{\tilde{H}}$. Note that $|g_1(\hat{G})|$ is a base-point-free linear system on $\tilde{H}$ (because $H^1(\mathscr{O}_{\tilde{H}}) = 0$). Thus, we can take H' such that $\tilde{H}'$ does not pass through points in $\tilde{H} \cap \Lambda \setminus \Xi_0$. Now let H_ε be a general member of the pencil generated by H and H'. Note that $\Lambda \cap \Xi_0 = \{Q\}$ and that Λ meets $\tilde{H}$ and $\tilde{H}_\varepsilon$ transversely at Q. By Bertini's theorem the proper transform $\tilde{H}_\varepsilon$ of H_ε on $\tilde{X}$ also meets Λ transversely along Ξ'. Since $(\tilde{H}_\varepsilon \cdot \Lambda)_{\tilde{X}} = (\mathscr{O}(4) \cdot \Lambda)_{\mathbb{P}(3,2,1,1)} = 4$, the intersection $\tilde{H}_\varepsilon \cap \Lambda$ consists of four distinct points. Therefore, $\tilde{H}_\varepsilon$ has three Du Val points on $\tilde{H}_\varepsilon \cap \Lambda \setminus \Xi_0$. This shows that for H_ε the situation of § 1.2.4 holds, so the chosen H is not general in the case 3.6.1.5.

3.6.3. *The subcase when* (X, P) *is a double* cAx/4*-point and* $l(0, y_4) = 0$

We show that only the case 1.2.5 occurs. We may assume that $l(y_3, y_4) = y_3$, so Equations (3.2) for (H, P) have the form

$$\left.\begin{aligned} y_1^2 - y_2^3 + y_3^2 + \phi = 0, \\ y_1y_3 + y_2q(y_3, y_4) + \xi(y_3, y_4) + \psi = 0. \end{aligned}\right\} \tag{3.6}$$

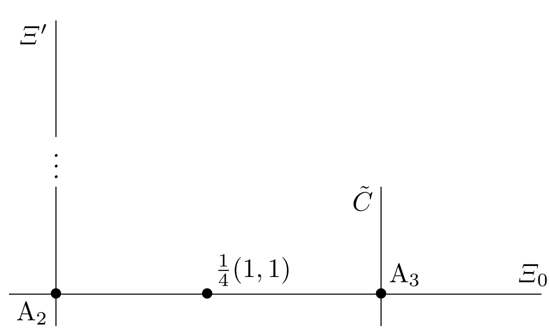
In this case, $\Xi = 4\Xi_0 + 2\Xi'$, where

$$\Xi' = \{y_3 = y_2q(0, y_4)/y_4^2 + \xi(0, y_4)/y_4^2 = 0\} \subset E \simeq \mathbb{P}(3, 2, 1, 1).$$

Claim 3.12. *The surface* $\tilde{H}$ *is normal and has the following singularities on* Ξ_0 *(in natural weighted coordinates on* $E \simeq \mathbb{P}(3, 2, 1, 1)$*):*

- $O_1 := \Xi_0 \cap \Xi' = (1 : 0 : 0 : 0)$*, which is of type* A_2*,*
- $Q := \Xi_0 \cap \tilde{C} = (1 : 1 : 0 : 0)$*, which is of type* A_3*,*
- $O_2 := (0 : 1 : 0 : 0)$*, which is a cyclic quotient singularity of type* $\frac{1}{4}(1, 1)$*.*

The pair $(\tilde{H}, \Xi_0 + \Xi' + \tilde{C})$ *is LC along* Ξ_0*. Moreover, it is PLT at all points of* $\Xi_0 \setminus \{O_1, Q\}$*. Thus,* $\tilde{H}$ *looks as follows:*



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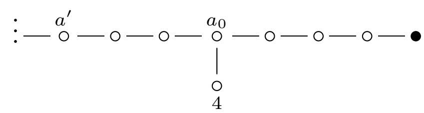
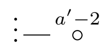
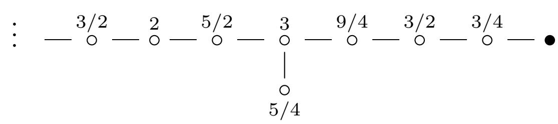
Hence, the dual graph $\Delta(H, C)$ has the following form:



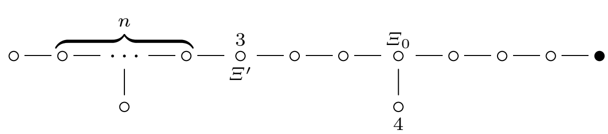
where $\vdots$ corresponds to some Du Val singularities sitting on Ξ'. Since the whole configuration is contractible to either a Du Val point or a curve, we have that $a_0 = 2$. Contracting black vertices successively on some step we get the following:



Recall that $\vdots$ is not empty. Hence, $a' = 3$ or 4. By Lemma 3.4 (iv) we have that $\mathrm{cdisc}(\Xi_0) = 3$, $\mathrm{cdisc}(\Xi') = \frac{3}{2}$. Using 3.6.1.1 we compute the codiscrepancies of exceptional divisors over $\tilde{H}$ to give the following:

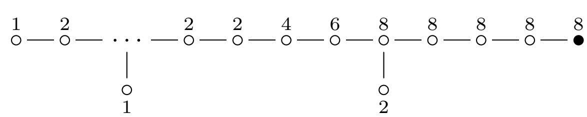


If $a' = 4$, we get a contradiction as in § 3.6.1.4. If $a' = 3$, then the whole configuration contracts to a curve, i.e. f is a $\mathbb{Q}$-conic bundle. As in § 3.6.1.3, we infer that the graph $\Delta(H, C)$ has the following form:



where $n \geqslant 0$.

We show that $n = 0$, that is, the case 1.2.5 holds. As in § 3.6.2, take a divisor D on $\hat{H}$ whose coefficients are as follows:



Then, $D = h^*o$ is a scheme fibre of $h\colon \hat{H} \to T$. There exists a member $H' \in |\mathscr{O}_X|_C$ such that $H'|_H = g_{H*}g_{1*}D = f_H^* o$. Since $\Xi = 4\Xi_0 + 2\Xi'$, we have that $\tilde{H}'|_{\tilde{H}} = g_{1*}D - \Xi = 4\Xi_0$. In particular, the curve Ξ' is not a component of $\tilde{H}'|_{\tilde{H}}$. Hence, the base locus of the pencil generated by $\tilde{H}$ and $\tilde{H}'$ coincides with Ξ_0. As in § 3.6.2 a general member $\tilde{H}_\varepsilon$ of this pencil meets the curve Λ transversely outside of Ξ_0. Note that $\Lambda \cap \Xi_0 = \{Q\}$ and the local intersection number of Λ and $\tilde{H}_\varepsilon$ at Q is equal to 2. By Bertini's theorem, the proper transform $\tilde{H}_\varepsilon$ of H_ε on $\tilde{X}$ meets Λ transversely along Ξ'. Since $(\tilde{H}_\varepsilon \cdot \Lambda)_{\tilde{X}} = (\mathscr{O}(4) \cdot \Lambda)_{\mathbb{P}(3,2,1,1)} = 4$, the intersection $\tilde{H}_\varepsilon \cap \Lambda$ consists of three distinct points. Therefore, $\tilde{H}_\varepsilon$ has two Du Val points on $\tilde{H}_\varepsilon \cap \Lambda \setminus \Xi_0$. This shows that for H_ε the situation of § 1.2.5 holds, so the chosen H is not general if $n > 0$.

Example 3.13. Let H be given by the equations

$$y_1^2 - y_2^3 + y_3^2 = 0,$$
$$y_1 y_3 + y_2 y_4^2 + y_4^4 = 0.$$

Then, a one-parameter deformation of H is a $\mathbb{Q}$-conic bundle as in § 1.2.5.

Acknowledgements. The paper was written during Y.P.'s stay at RIMS, Kyoto University, in February and March 2011. Y.P. thanks the institute for the invitation, their hospitality and the nice working environment. S.M. was partly supported by the Japan Society for the Promotion of Science Grant-in-Aid for Scientific Research (B)(2), (Grant 20340005). Y.P. was partly supported by the RFBR (Grants 11-01-00336-a, 11-01-92613-KO_a), Leading Scientific Schools (Grant 4713.2010.1) and AG Laboratory SU-HSE, RF Government (Grant Ag. 11.G34.31.0023).

References

1. V. I. Arnold, Normal forms for functions near degenerate critical points, the Weyl groups of A_k, D_k, E_k and Lagrangian singularities, *Funct. Analysis Applic.* **6** (1972), 254–272.
2. F. Catanese, Automorphisms of rational double points and moduli spaces of surfaces of general type, *Compositio Math.* **61** (1987), 81–102.
3. R. Elkik, Singularités rationnelles et déformations, *Invent. Math.* **47** (1978), 139–147.
4. J. Kollár and S. Mori, Classification of three-dimensional flips, *J. Am. Math. Soc.* **5**(3) (1992), 533–703.
5. S. Mori, Flip theorem and the existence of minimal models for 3-folds, *J. Am. Math. Soc.* **1**(1) (1988), 117–253.
6. S. Mori and Y. Prokhorov, On $\mathbb{Q}$-conic bundles, *Publ. Res. Inst. Math. Sci.* **44** (2008), 315–369.
7. S. Mori and Y. Prokhorov, On $\mathbb{Q}$-conic bundles, III, *Publ. Res. Inst. Math. Sci.* **45** (2009), 787–810.
8. S. Mori and Y. Prokhorov, Threefold extremal contractions of type IA, *Kyoto J. Math.* **51**(2) (2011), 393–438.