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Poincaré Lemma on Quaternion-like Heisenberg Groups

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Abstract. For smooth functions a_1, a_2, a_3, a_4 on a quaternion Heisenberg group, we characterize the existence of solutions of the partial differential operator system $X_1 f = a_1$, $X_2 f = a_2$, $X_3 f = a_3$, and $X_4 f = a_4$. In addition, a formula for the solution function f is deduced, assuming solvability of the system.

1 Introduction

Let $\mathbf{X} = \{X_1, X_2, \dots, X_m\}$ be m linearly independent vector fields defined on an n-dimensional manifold $\mathcal{M}_n$ with $m \le n$. The subspace $T_{\mathbf{X}}$ spanned by $X_1, \dots, X_m$ is called the *horizontal subspace*, and its complement is referred to as the *missing directions*. When $T_{\mathbf{X}} = T\mathcal{M}_n$, then one can conclude that $m = n$. In this case, $\mathcal{M}_n$ (associated with the Laplace–Beltrami operator $\Delta_{\mathbf{X}} = \sum_{j=1}^n X_j^2$) is basically a Riemannian manifold. Let $V = (a_1, a_2, \dots, a_n)$ be a vector-valued function defined on $\mathcal{M}_n$, where a_j, $j = 1, \dots, n$ are smooth functions. One interesting problem is to find necessary and sufficient conditions on a_j's so that V is conservative, *i.e.,* there exists a potential function f such that

$$X_1 f = a_1, \quad X_2 f = a_2, \quad \dots \quad X_n f = a_n.$$

For example, let $V = (a, b)$ be a vector-valued function defined on $\mathbb{R}^2$ where a and b are two smooth functions. Assume that $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y}$. Then V is conservative if and only if $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$. In fact, denote $\omega = a dx + b dy$ and

$$f(x, y) = \int_{r(t)} \omega = \int_0^1 \omega(r'(t)) dt = \int_0^1 a(tx, ty)x + b(tx, ty)y dt, \tag{1.1}$$

where $r(t) = t(x, y)$, $t \in [0, 1]$, is a straight line joining the origin and the point (x, y). Then by straightforward computations,

$$\frac{\partial f}{\partial x}(x, y) = a(x, y) + \int_0^1 ty\Big(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\Big) dt,$$
$$\frac{\partial f}{\partial y}(x, y) = b(x, y) + \int_0^1 tx\Big(\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\Big) dt.$$

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The result follows immediately. This is the famous Poincaré lemma. The potential function f in (1.1) can be interpreted as the work done by the force $\omega = adx + bdy$ from the origin to the point (x, y) connecting by the straight line $r(t)$.

Now let us turn to the case where $T_{\mathbf{X}} \neq T\mathcal{M}_n$. Since the complement of $T_{\mathbf{X}}$, by definition, is the missing directions, extra vector fields are needed to generate $T\mathcal{M}_n$. Assume that $\mathbf{X}$ satisfies the *bracket generating property*: "the horizontal vector fields $\mathbf{X}$ and their Lie brackets span $T\mathcal{M}_n$". Then by Chow's Theorem [4], we know that given any two points $A, B \in \mathcal{M}_n$, there is a piecewise C^1 horizontal curve $\gamma: [0,1] \to \mathcal{M}_n$, satisfying

$$\gamma(0) = A, \quad \gamma(1) = B, \quad \text{and} \quad \dot{\gamma}(s) = \sum_{k=1}^{m} a_k(s) X_k.$$

Then we can define the "length" of γ as usual:

$$\ell(\gamma) = \int_0^1 \sqrt{a_1^2(s) + a_2^2(s) + \cdots + a_m^2(s)}\, ds.$$

The shortest length $d_{cc}(A, B)$ is called the *Carnot–Carathéodory distance* between $A, B \in \mathcal{M}_n$, which is given by

$$d_{cc}(A, B) := \inf \ell(\gamma),$$

where the infimum is taken over all absolutely continuous horizontal curves joining A and B. Hence, we can define a geometry on $\mathcal{M}_n$ which is the so-called *subRiemannian geometry*. Under the bracket generating property, the sub-Laplace operator $\Delta_{\mathbf{X}} = \sum_{j=1}^{m} X_j^2$ is solvable and hypoelliptic, [5] and [6]. One notes that in place of $r(t)$ in $\mathbb{R}^2$, the horizontal curve γ and the Carnot-Carathéodory distance will play an essential role in proving the corresponding version of Poincaré's lemma in a subRiemannian setting on manifolds.

We are very interested in proving results similar to Poincaré's lemma in a sub-Riemannian setting. The first result was obtained in [1] and [2]. They obtained a so-called *integrability condition* for the 1-dimensional Heisenberg group $\mathcal{H}^1$. More precisely, given smooth functions a and b on $\mathcal{H}^1$, they found conditions on the functions a and b such that there exists a function f satisfying $X_1 f = a$ and $X_2 f = b$, where

$$X_1 = \partial_x - 2y\partial_z, \quad X_2 = \partial_y + 2x\partial_z, \tag{1.2}$$

are the Heisenberg vector fields. Note that $\{X_1, X_2\}$ satisfies the bracket generating property since $[X_1, X_2] = 4\frac{\partial}{\partial z}$. We recall related results for $\mathcal{H}^1$ in the following two theorems.

Theorem 1.1 ([1]) *Let X_1, X_2 be the Heisenberg vector fields. The system $X_1 f = a, X_2 f = b$ has a solution if and only if*

$$X_1^2 b = \big(X_1 X_2 + [X_1, X_2]\big) a, \quad X_2^2 a = \big(X_2 X_1 + [X_2, X_1]\big) b.$$

Theorem 1.2 ([2]) *Let X_1, X_2 be the Heisenberg vector fields and $\mathbf{p} = (x, y, z)$ in $\mathcal{H}^1$. Given any smooth functions a and b, set*

$$c = X_1 b - X_2 a, \quad a_1 = a + y\frac{c}{2}, \quad b_1 = b - x\frac{c}{2}, \quad c_1 = \frac{c}{4}.$$

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Consider

$$f(\mathbf{p}) = \int_0^1 \left[a_1(t\mathbf{p})x + b_1(t\mathbf{p})y + c_1(t\mathbf{p})z \right] dt.$$

Then

$$(X_1 f)(\mathbf{p}) = a(\mathbf{p}) + \int_0^1 \frac{tz}{4}\Big(X_1^2 b - (X_1X_2 + [X_1, X_2])a \Big)(t\mathbf{p})dt,$$

$$(X_2 f)(\mathbf{p}) = b(\mathbf{p}) - \int_0^1 \frac{tz}{4}\Big(X_2^2 a - (X_2X_1 + [X_2, X_1])b \Big)(t\mathbf{p})dt.$$

If the conditions

$$X_1^2 b = \big(X_1X_2 + [X_1, X_2] \big) a \quad \textit{and} \quad X_2^2 a = \big(X_2X_1 + [X_2, X_1] \big) b$$

hold, then $X_1 f = a, X_2 f = b$*, with*

$$f(\mathbf{p}) = \int_0^1 \left[a(t\mathbf{p})x + b(t\mathbf{p})y \right] dt.$$

Now a quaternion Heisenberg group is a subRiemannian manifold that we are going to work with. We wish to explore whether Poincaré's lemma remains true on such a setting. So let us recall some notation and definitions ([3,7]) as follows.

A quaternion number can be written as

$$x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4,$$

where $x_l's$ are real and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are imaginary units satisfying

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j};$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

Let $\mathbb{H}$ be the collection of all quaternion numbers. Denote

$$\operatorname{Im}\mathbb{H} = \{x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \in \mathbb{H} : x_1 = 0\} \cong \mathbb{R}^3.$$

The quaternion Heisenberg group $qH^1 \cong \mathbb{R}^7$ is a real 7-dimensional nilpotent Lie group isomorphic to $\mathbb{H} \times \operatorname{Im}\mathbb{H}$, equipped with the group law

$$\begin{aligned} p \cdot q &= (p', w) \cdot (q', v) \\ &= \Bigg(p' + q', w + v + \Big(\sum_{j,k=1}^{4} a_{jk}^1 x_j' x_k \Big)\mathbf{i} + \Big(\sum_{j,k=1}^{4} a_{jk}^2 x_j' x_k \Big)\mathbf{j} + \Big(\sum_{j,k=1}^{4} a_{jk}^3 x_j' x_k \Big)\mathbf{k} \Bigg), \end{aligned}$$

where $p = (p', w)$ and $q = (q', v)$ are in $\mathbb{H} \times \mathbb{R}^3$, $p' = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$, $q' = x_1' + \mathbf{i}x_2' + \mathbf{j}x_3' + \mathbf{k}x_4'$, and all a_{jk}^l are real with $a_{jk}^l = -a_{kj}^l$, for $l = 1, 2, 3$. This group can be considered as a translation group on the Szegö upper half space $\mathcal{U} \subset \mathbb{H}^2$,

$$\mathcal{U} = \left\{ (p', q') \in \mathbb{H}^2 : \operatorname{Re}(q') > |p'|^2 \right\},$$

with the boundary $\partial\mathcal{U} = \left\{ (p', q') \in \mathbb{H}^2 : \operatorname{Re}(q') = |p'|^2 \right\}$. We define the *height function* ρ on $\mathcal{U}$ as $\rho = x_1' - (x_1^2 + x_2^2 + x_3^2 + x_4^2)$. Then the quaternion Heisenberg group qH^1 acts transitively on each level set ρ = constant. In particular, $\partial\mathcal{U}$ can be viewed as an orbit of the origin under the action of qH^1.

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Consider the left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + \sum_{k=1}^{4}\sum_{l=1}^{3} a_{jk}^{l} x_k \frac{\partial}{\partial y_l}, \quad j = 1,2,3,4, \tag{1.3}$$

on qH^1. Given smooth functions a_1, a_2, a_3, and a_4, a necessary and sufficient condition for the solvability of the system $X_1 f = a_1$, $X_2 f = a_2$, $X_3 f = a_3$, and $X_4 f = a_4$, called the *integrability condition*, is going to be obtained in Section 2. The formula for the solution f will be derived using similar concepts as in Heisenberg groups [2]. The details of the proof will be presented in Section 3. We want to point out here that the situation in the quaternion Heisenberg group is more complicated than the Heisenberg group. We have three missing directions in this case, making the calculation much harder. The quaternion Heisenberg group that we are going to work with in the sequel is, in fact, in a large class of 7-dimensional nilpotent groups of codimension 3. Hence, all results in this paper are also true even in the large class.

2 Integrability Condition

We first use the bracket generating property by adding extra vector fields on $\mathbf{X} = \{X_1, X_2, X_3, X_4\}$ to form an orthonormal basis on qH^1. By (1.3),

$$\begin{aligned} X_n X_m = {} & \frac{\partial^2}{\partial x_n \partial x_m} + \sum_{k=1}^{4}\sum_{l=1}^{3} a_{mk}^{l} x_k \frac{\partial^2}{\partial x_n \partial y_l} + \sum_{l=1}^{3} a_{mn}^{l} \frac{\partial}{\partial y_l} \\ & + \sum_{k=1}^{4}\sum_{l=1}^{3} a_{nk}^{l} x_k \frac{\partial^2}{\partial y_l \partial x_m} + \Big(\sum_{k=1}^{4}\sum_{l=1}^{3} a_{nk}^{l} x_k \frac{\partial}{\partial y_l}\Big)\Big(\sum_{k=1}^{4}\sum_{l=1}^{3} a_{mk}^{l} x_k \frac{\partial}{\partial y_l}\Big), \end{aligned}$$

so that their Lie brackets are given by

$$[X_n, X_m] = X_n X_m - X_m X_n = -2\sum_{l=1}^{3} a_{nm}^{l} \frac{\partial}{\partial y_l}. \tag{2.1}$$

Further,

$$\big[X_j, [X_n, X_m]\big] = 0, \quad j = 1,2,3,4. \tag{2.2}$$

So $\{X_1, X_2, X_3, X_4\}$ satisfies the bracket generating property of step 2. For any smooth functions a_1, a_2, a_3, a_4, we have

$$\begin{cases} X_1 f = a_1, \\ X_2 f = a_2, \\ X_3 f = a_3, \\ X_4 f = a_4, \end{cases} \Longleftrightarrow \begin{cases} X_1 f = a_1, & X_2 f = a_2, \\ X_3 f = a_3, & X_4 f = a_4, \\ [X_1, X_2] f = c_{12}, & [X_1, X_3] f = c_{13}, \quad [X_1, X_4] f = c_{14}, \\ [X_2, X_3] f = c_{23}, & [X_2, X_4] f = c_{24}, \quad [X_3, X_4] f = c_{34}, \end{cases}$$

where $c_{ij} = -2\sum_{l=1}^{3} a_{ij}^{l} \frac{\partial f}{\partial y_l} = X_i a_j - X_j a_i, 1 \le i < j \le 4$. Each Lie bracket $[X_i, X_j], 1 \le i < j \le 4$ on the right of the last statement, as shown in (2.1), is spanned by $\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\}$, and thus lies in a 3-dimensional subbundle. From this, we have that the collection of $[X_i, X_j], 1 \le i < j \le 4$ are linearly dependent in the subbundle. For simplicity, suppose

$$a_{12}^2 = a_{12}^3 = a_{23}^1 = a_{23}^3 = a_{34}^1 = a_{34}^2 = 0 \tag{2.3}$$

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for the remainder of this paper. Then

$$[X_1, X_2] = -2a_{12}^1 \frac{\partial}{\partial y_1}, \quad [X_2, X_3] = -2a_{23}^2 \frac{\partial}{\partial y_2}, \quad [X_3, X_4] = -2a_{34}^3 \frac{\partial}{\partial y_3}$$

are linearly independent. Let $T_{ij} = [X_i, X_j]$. We can drop T_{13}, T_{14}, and T_{24} from $T_{ij}, 1 \le i < j \le 4$. Therefore, with a Riemannian metric g defined on qH^1, $\{X_1, X_2, X_3, X_4, T_{12}, T_{23}, T_{34}\}$ forms an orthonormal basis for qH^1. Let

$$U = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + c_{12}T_{12} + c_{23}T_{23} + c_{34}T_{34}.$$

The equivalence of $X_1 f = a_1, X_2 f = a_2, X_3 f = a_3, X_4 f = a_4$ becomes

$$\begin{aligned}
&\begin{cases} X_1 f = a_1, \\ X_2 f = a_2, \\ X_3 f = a_3, \\ X_4 f = a_4, \end{cases} \iff \begin{cases} X_1 f = a_1, & X_2 f = a_2, \\ X_3 f = a_3, & X_4 f = a_4, \\ T_{12} f = c_{12}, & T_{23} f = c_{23}, \quad T_{34} f = c_{34}. \end{cases} \\
&\iff \operatorname{grad} f = U \iff \operatorname{curl} U = 0 \\
&\iff A(X_i, X_j) = A(X_i, T_{k(k+1)}) = A(T_{k(k+1)}, T_{l(l+1)}) = 0
\end{aligned}$$

for $1 \le i < j \le 4$ and $1 \le k < l \le 3$, where $\operatorname{grad} f$ is defined by

$$\begin{aligned}
\operatorname{grad} f = (X_1 f)X_1 + (X_2 f)X_2 + (X_3 f)X_3 + (X_4 f)X_4 \\
+ (T_{12} f)T_{12} + (T_{23} f)T_{23} + (T_{34} f)T_{34},
\end{aligned}$$

and $\operatorname{curl} U$ is a 2-covariant antisymmetric tensor A on a pair of vector fields (X, Y) defined by

$$A(X, Y) = Yg(U, X) - Xg(U, Y) + g\big(U, [X, Y]\big).$$

Now we calculate the contents of $A(X_i, X_j)$, $A(X_i, T_{k(k+1)})$, and $A(T_{k(k+1)}, T_{l(l+1)})$ as follows. First,

$$A(X_i, X_j) = X_j a_i - X_i a_j + g(U, T_{ij}) = X_j a_i - X_i a_j + c_{ij}, \quad 1 \le i < j \le 4.$$

So $A(X_i, X_j) = 0$ is equivalent to

$$\begin{aligned}
c_{12} = X_1 a_2 - X_2 a_1, \quad c_{13} = X_1 a_3 - X_3 a_1, \quad c_{14} = X_1 a_4 - X_4 a_1, \\
c_{23} = X_2 a_3 - X_3 a_2, \quad c_{24} = X_2 a_4 - X_4 a_2, \quad c_{34} = X_3 a_4 - X_4 a_3.
\end{aligned}$$

Secondly, due to (2.2) we have

$$\begin{aligned}
A(X_1, T_{12}) &= T_{12} a_1 - X_1 c_{12} = [X_1, X_2] a_1 - X_1(X_1 a_2 - X_2 a_1) \\
&= \big(X_1 X_2 + [X_1, X_2]\big) a_1 - X_1^2 a_2.
\end{aligned}$$

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Similarly,

$$\begin{aligned}
A(X_1, T_{23}) &= [X_2, X_3]a_1 + X_1X_3a_2 - X_1X_2a_3,\\
A(X_1, T_{34}) &= [X_3, X_4]a_1 + X_1X_4a_3 - X_1X_3a_4,\\
A(X_2, T_{12}) &= X_2^2a_1 - \big(X_2X_1 + [X_2, X_1]\big)a_2,\\
A(X_2, T_{23}) &= \big(X_2X_3 + [X_2, X_3]\big)a_2 - X_2^2a_3,\\
A(X_2, T_{34}) &= [X_3, X_4]a_2 + X_2X_4a_3 - X_2X_3a_4,\\
A(X_3, T_{12}) &= X_3X_2a_1 - X_3X_1a_2 + [X_1, X_2]a_3,\\
A(X_3, T_{23}) &= X_3^2a_2 - \big(X_3X_2 + [X_3, X_2]\big)a_3,\\
A(X_3, T_{34}) &= (X_3X_4 + [X_3, X_4])a_3 - X_3^2a_4,\\
A(X_4, T_{12}) &= X_4X_2a_1 - X_4X_1a_2 + [X_1, X_2]a_4,\\
A(X_4, T_{23}) &= X_4X_3a_2 - X_4X_2a_3 + [X_2, X_3]a_4,\\
A(X_4, T_{34}) &= X_4^2a_3 - \big(X_4X_3 + [X_4, X_3]\big)a_4.
\end{aligned}$$

So $A(X_i, T_{k(k+1)}) = 0$ is equivalent to

$$\tag{2.4} \begin{cases}
X_1^2a_2 = \big(X_1X_2 + [X_1, X_2]\big)a_1, & X_2^2a_1 = \big(X_2X_1 + [X_2, X_1]\big)a_2,\\
X_2^2a_3 = \big(X_2X_3 + [X_2, X_3]\big)a_2, & X_3^2a_2 = \big(X_3X_2 + [X_3, X_2]\big)a_3,\\
X_3^2a_4 = \big(X_3X_4 + [X_3, X_4]\big)a_3, & X_4^2a_3 = \big(X_4X_3 + [X_4, X_3]\big)a_4,\\
[X_2, X_3]a_1 = X_1X_2a_3 - X_1X_3a_2, & [X_3, X_4]a_1 = X_1X_3a_4 - X_1X_4a_3,\\
[X_3, X_4]a_2 = X_2X_3a_4 - X_2X_4a_3, & [X_1, X_2]a_3 = X_3X_1a_2 - X_3X_2a_1,\\
[X_1, X_2]a_4 = X_4X_1a_2 - X_4X_2a_1, & [X_2, X_3]a_4 = X_4X_2a_3 - X_4X_3a_2.
\end{cases}$$

To calculate $A(T_{k(k+1)}, T_{l(l+1)})$, one notes that $[T_{k(k+1)}, T_{l(l+1)}] = 0$, and so

$$\begin{aligned}
A(T_{12}, T_{23}) &= T_{23}c_{12} - T_{12}c_{23}\\
&= [X_2, X_3](X_1a_2 - X_2a_1) - [X_1, X_2](X_2a_3 - X_3a_2)\\
&= -[X_2, X_3]X_2a_1 + \big([X_1, X_2]X_3 + [X_2, X_3]X_1\big)a_2 - [X_1, X_2]X_2a_3.
\end{aligned}$$

By virtue of (2.2),

$$\begin{aligned}
A(T_{12}, T_{23}) &= -X_2\big([X_2, X_3]a_1 + X_1X_3a_2 - X_1X_2a_3\big) + X_2X_1X_3a_2 - X_2X_1X_2a_3\\
&\quad + X_1X_2X_3a_2 - X_2X_1X_3a_2\\
&\quad + X_1\big([X_2, X_3]a_2 + X_2X_3a_2 - X_2^2a_3\big) - X_1X_2X_3a_2 + X_1X_2^2a_3\\
&\quad - X_1X_2^2a_3 + X_2X_1X_2a_3\\
&= -X_2A(X_1, T_{23}) + X_1A(X_2, T_{23}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
A(T_{12}, T_{34}) &= -X_2A(X_1, T_{34}) + X_1A(X_2, T_{34}),\\
A(T_{23}, T_{34}) &= -X_3A(X_2, T_{34}) + X_2A(X_3, T_{34}).
\end{aligned}$$

Applying

$$\begin{matrix}
A(X_1, T_{23}) = 0 & A(X_1, T_{34}) = 0 & A(X_2, T_{34}) = 0\\
A(X_2, T_{23}) = 0 & A(X_2, T_{34}) = 0 & A(X_3, T_{34}) = 0
\end{matrix}$$

to obtain $A(T_{k(k+1)}, T_{l(l+1)}) = 0$. In summary, $X_1 f = a_1, X_2 f = a_2, X_3 f = a_3, X_4 f = a_4$ is solvable if and only if (2.4) holds. We have proved the following theorem.

Theorem 2.1 *Let X_1, X_2, X_3, X_4 be the vector fields on qH^1 that are defined in* (1.3)*, with the properties* (2.1)*,* (2.2)*, and* (2.3)*. Then for any smooth functions a_1, a_2, a_3, a_4 we have*

$$\begin{aligned}
X_1^2 a_2 &= \big(X_1X_2 + [X_1, X_2]\big) a_1, & X_2^2 a_1 &= \big(X_2X_1 + [X_2, X_1]\big) a_2,\\
X_2^2 a_3 &= \big(X_2X_3 + [X_2, X_3]\big) a_2, & X_3^2 a_2 &= \big(X_3X_2 + [X_3, X_2]\big) a_3,\\
X_3^2 a_4 &= \big(X_3X_4 + [X_3, X_4]\big) a_3, & X_4^2 a_3 &= \big(X_4X_3 + [X_4, X_3]\big) a_4,\\
[X_2, X_3]a_1 &= X_1X_2a_3 - X_1X_3a_2, & [X_3, X_4]a_1 &= X_1X_3a_4 - X_1X_4a_3,\\
[X_3, X_4]a_2 &= X_2X_3a_4 - X_2X_4a_3, & [X_1, X_2]a_3 &= X_3X_1a_2 - X_3X_2a_1,\\
[X_1, X_2]a_4 &= X_4X_1a_2 - X_4X_2a_1, & [X_2, X_3]a_4 &= X_4X_2a_3 - X_4X_3a_2,
\end{aligned}$$

if and only if there exists a function f such that $X_1 f = a_1, X_2 f = a_2, X_3 f = a_3$, and $X_4 f = a_4$.

3 The Poincaré Lemma

The solvability of $X_1 f = a_1$, $X_2 f = a_2, X_3 f = a_3, X_4 f = a_4$, by Theorem 2.1, is characterized by (2.4). Let $\mathbf{p} = (x_1, x_2, x_3, x_4, y_1, y_2, y_3)$ be a point in qH^1. Denote the straight line connecting the origin and $\mathbf{p}$ by

$$r(t) = t\mathbf{p} = (tx_1, tx_2, tx_3, tx_4, ty_1, ty_2, ty_3), \quad 0 \le t \le 1.$$

By (1.3) with $c_l = \frac{\partial f}{\partial y_l}, l = 1, 2, 3,$

$$\begin{aligned}
\begin{cases} X_1 f = a_1,\\ X_2 f = a_2,\\ X_3 f = a_3,\\ X_4 f = a_4, \end{cases}
&\iff \begin{cases}
\frac{\partial}{\partial x_1} f + a_{12}^1 c_1 x_2 + \sum_{k=3}^4 \sum_{l=1}^3 a_{1k}^l c_l x_k = a_1,\\
\frac{\partial}{\partial x_2} f + a_{21}^1 c_1 x_1 + a_{23}^2 c_2 x_3 + \sum_{l=1}^3 a_{24}^l c_l x_4 = a_2,\\
\frac{\partial}{\partial x_3} f + \sum_{l=1}^3 a_{31}^l c_l x_1 + a_{32}^2 c_2 x_2 + a_{34}^3 c_3 x_4 = a_3,\\
\frac{\partial}{\partial x_4} f + \sum_{k=1}^2 \sum_{l=1}^3 a_{4k}^l c_l x_k + a_{43}^3 c_3 x_3 = a_4,
\end{cases}\\
&\iff \begin{cases}
\frac{\partial}{\partial x_1} f - \frac{1}{2}(c_{12}x_2 + c_{13}x_3 + c_{14}x_4) = a_1,\\
\frac{\partial}{\partial x_2} f - \frac{1}{2}(-c_{12}x_1 + c_{23}x_3 + c_{24}x_4) = a_2,\\
\frac{\partial}{\partial x_3} f - \frac{1}{2}(-c_{13}x_1 - c_{23}x_2 + c_{34}x_4) = a_3,\\
\frac{\partial}{\partial x_4} f - \frac{1}{2}(-c_{14}x_1 - c_{24}x_2 - c_{34}x_3) = a_4,
\end{cases}\\
&\iff \begin{cases}
\frac{\partial}{\partial x_1} f = a_1^*,\\
\frac{\partial}{\partial x_2} f = a_2^*,\\
\frac{\partial}{\partial x_3} f = a_3^*,\\
\frac{\partial}{\partial x_4} f = a_4^*,
\end{cases}
\end{aligned}$$

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where

$$a_1^* = a_1 + \tfrac{1}{2}(c_{12}x_2 + c_{13}x_3 + c_{14}x_4), \qquad a_2^* = a_2 + \tfrac{1}{2}(-c_{12}x_1 + c_{23}x_3 + c_{24}x_4),$$
$$a_3^* = a_3 + \tfrac{1}{2}(-c_{13}x_1 - c_{23}x_2 + c_{34}x_4), \qquad a_4^* = a_4 + \tfrac{1}{2}(-c_{14}x_1 - c_{24}x_2 - c_{34}x_3).$$

Let $\omega = \sum_{j=1}^4 a_j^* dx_j + \sum_{l=1}^3 c_l dy_l$. Then

$$f(\mathbf{p}) = \int_{r(t)} \omega = \int_0^1 \omega(r'(t))dt = \int_0^1 \Big[\sum_{j=1}^4 a_j^*(r(t))x_j + \sum_{l=1}^3 c_l(r(t))y_l\Big]dt. \tag{3.1}$$

Taking $\frac{\partial}{\partial x_i}, 1 \le i \le 4$ and $\frac{\partial}{\partial y_\alpha}, 1 \le \alpha \le 3$ to (3.1) yields

$$\frac{\partial f}{\partial x_i}(\mathbf{p}) = \int_0^1 \Big\{ t x_i \frac{\partial}{\partial x_i} a_i^*(r(t)) + a_i^*(r(t)) + \sum_{\substack{j=1\\ j\neq i}}^4 t x_j \frac{\partial}{\partial x_i} a_j^*(r(t)) + \sum_{l=1}^3 t y_l \frac{\partial}{\partial x_i} c_l(r(t))\Big\} dt,$$
$$\frac{\partial f}{\partial y_\alpha}(\mathbf{p}) = \int_0^1 \Big\{ t y_\alpha \frac{\partial}{\partial y_\alpha} c_\alpha(r(t)) + c_\alpha(r(t)) + \sum_{j=1}^4 t x_j \frac{\partial}{\partial y_\alpha} a_j^*(r(t)) + \sum_{\substack{l=1\\ l\neq \alpha}}^3 t y_l \frac{\partial}{\partial y_\alpha} c_l(r(t))\Big\} dt.$$

Since

$$x_i \frac{\partial}{\partial x_i} a_i^*(r(t)) = \frac{d}{dt} a_i^*(r(t)) - \sum_{\substack{j=1\\ j\neq i}}^4 x_j \frac{\partial}{\partial x_j} a_i^*(r(t)) - \sum_{l=1}^3 y_l \frac{\partial}{\partial y_l} a_i^*(r(t)),$$
$$y_\alpha \frac{\partial}{\partial y_\alpha} c_\alpha(r(t)) = \frac{d}{dt} c_\alpha(r(t)) - \sum_{j=1}^4 x_j \frac{\partial}{\partial x_j} c_\alpha(r(t)) - \sum_{\substack{l=1\\ l\neq \alpha}}^3 y_l \frac{\partial}{\partial y_l} c_\alpha(r(t)),$$

it follows that

$$\begin{aligned}\frac{\partial f}{\partial x_i}(\mathbf{p}) &= \int_0^1 \Big\{ t\Big[\frac{d}{dt} a_i^*(r(t)) - \sum_{\substack{j=1\\ j\neq i}}^4 x_j \frac{\partial}{\partial x_j} a_i^*(r(t)) - \sum_{l=1}^3 y_l \frac{\partial}{\partial y_l} a_i^*(r(t))\Big] \\ &\qquad + a_i^*(r(t)) + \sum_{\substack{j=1\\ j\neq i}}^4 t x_j \frac{\partial}{\partial x_i} a_j^*(r(t)) + \sum_{l=1}^3 t y_l \frac{\partial}{\partial x_i} c_l(r(t))\Big\} dt \\ &= a_i^*(\mathbf{p}) + \int_0^1 \Big\{ \sum_{\substack{j=1\\ j\neq i}}^4 \Big[\frac{\partial}{\partial x_i} a_j^*(r(t)) - \frac{\partial}{\partial x_j} a_i^*(r(t))\Big] t x_j \\ &\qquad + \sum_{l=1}^3 \Big[\frac{\partial}{\partial x_i} c_l(r(t)) - \frac{\partial}{\partial y_l} a_i^*(r(t))\Big] t y_l\Big\} dt,\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(3.3)\quad \frac{\partial f}{\partial y_\alpha}(\mathbf{p}) &= \int_0^1 \Big\{ t\Big[\frac{d}{dt}c_\alpha(r(t)) - \sum_{j=1}^{4} x_j \frac{\partial}{\partial x_j}c_\alpha(r(t)) - \sum_{\substack{l=1\\ l\neq\alpha}}^{3} y_l \frac{\partial}{\partial y_l}c_\alpha(r(t))\Big] \\ &\quad + c_\alpha(r(t)) + \sum_{j=1}^{4} t x_j \frac{\partial}{\partial y_\alpha}a_j^*(r(t)) + \sum_{\substack{l=1\\ l\neq\alpha}}^{3} t y_l \frac{\partial}{\partial y_\alpha}c_l(r(t))\Big\}\, dt \\ &= c_\alpha(\mathbf{p}) + \int_0^1 \Big\{ \sum_{j=1}^{4}\Big[\frac{\partial}{\partial y_\alpha}a_j^*(r(t)) - \frac{\partial}{\partial x_j}c_\alpha(r(t))\Big] t x_j \\ &\quad + \sum_{\substack{l=1\\ l\neq\alpha}}^{3}\Big[\frac{\partial}{\partial y_\alpha}c_l(r(t)) - \frac{\partial}{\partial y_l}c_\alpha(r(t))\Big] t y_l \Big\}\, dt.\end{aligned}$$

Let I_4 denote the 4×4 identity matrix and let $B = (I_4 \mid B_1)$ be a 4×7 matrix with

$$B_1 = \begin{pmatrix} a_{12}^1 x_2 + a_{13}^1 x_3 + a_{14}^1 x_4 & a_{13}^2 x_3 + a_{14}^2 x_4 & a_{13}^3 x_3 + a_{14}^3 x_4 \\ -a_{12}^1 x_1 + a_{24}^1 x_4 & a_{23}^2 x_3 + a_{24}^2 x_4 & a_{24}^3 x_4 \\ -a_{13}^1 x_1 & -a_{13}^2 x_1 - a_{23}^2 x_2 & -a_{13}^3 x_1 + a_{34}^3 x_4 \\ -a_{14}^1 x_1 - a_{24}^1 x_2 & -a_{14}^2 x_1 - a_{24}^2 x_2 & -a_{14}^3 x_1 - a_{24}^3 x_2 - a_{34}^3 x_3 \end{pmatrix}.$$

Then

$$\begin{pmatrix} (X_1 f)(\mathbf{p}) \\ (X_2 f)(\mathbf{p}) \\ (X_3 f)(\mathbf{p}) \\ (X_4 f)(\mathbf{p}) \end{pmatrix} = B \begin{pmatrix} \partial_{x_1} f(\mathbf{p}) \\ \partial_{x_2} f(\mathbf{p}) \\ \partial_{x_3} f(\mathbf{p}) \\ \partial_{x_4} f(\mathbf{p}) \\ \partial_{y_1} f(\mathbf{p}) \\ \partial_{y_2} f(\mathbf{p}) \\ \partial_{y_3} f(\mathbf{p}) \end{pmatrix}.$$

Using (3.2) and (3.3), $((X_1 f)(\mathbf{p}), (X_2 f)(\mathbf{p}), (X_3 f)(\mathbf{p}), (X_4 f)(\mathbf{p}))^T$ becomes

$$(3.4)\qquad B\begin{pmatrix} a_1^*(\mathbf{p}) \\ a_2^*(\mathbf{p}) \\ a_3^*(\mathbf{p}) \\ a_4^*(\mathbf{p}) \\ c_1(\mathbf{p}) \\ c_2(\mathbf{p}) \\ c_3(\mathbf{p}) \end{pmatrix} + \int_0^1 BMr(t)^T dt = \begin{pmatrix} a_1(\mathbf{p}) \\ a_2(\mathbf{p}) \\ a_3(\mathbf{p}) \\ a_4(\mathbf{p}) \end{pmatrix} + \int_0^1 tBM\mathbf{p}^T dt,$$

where $M = (m_{ij})$ is a 7×7 skew-symmetric matrix with entries

$$m_{ij} = \begin{cases} \partial_{x_i} a_j^* - \partial_{x_j} a_i^*, & 1 \le i < j \le 4, \\ \partial_{x_i} c_{j-4} - \partial_{y_{j-4}} a_i^*, & 1 \le i \le 4, 5 \le j \le 7, \\ \partial_{y_{i-4}} c_{j-4} - \partial_{y_{j-4}} c_{i-4}, & 5 \le i < j \le 7. \end{cases} \tag{3.5}$$

The integrand $tBM\mathbf{p}^T$ in (3.4) is a 4×1 matrix

$$tBM\mathbf{p}^T = t\Big((BM\mathbf{p}^T)_1, (BM\mathbf{p}^T)_2, (BM\mathbf{p}^T)_3, (BM\mathbf{p}^T)_4 \Big)^T.$$

Using m_{ij} as of (3.5), each $(BM\mathbf{p}^T)_j, 1 \le j \le 4$ is calculated as follows.

$$\begin{aligned}
&(BM\mathbf{p}^T)_1 \\
&= m_{12}x_2 + m_{13}x_3 + m_{14}x_4 + m_{15}y_1 + m_{16}y_2 + m_{17}y_3 \\
&\quad + (-m_{15}x_1 - m_{25}x_2 - m_{35}x_3 - m_{45}x_4 + m_{56}y_2 + m_{57}y_3)(a_{12}^1 x_2 + a_{13}^1 x_3 + a_{14}^1 x_4) \\
&\quad + (-m_{16}x_1 - m_{26}x_2 - m_{36}x_3 - m_{46}x_4 - m_{56}y_1 + m_{67}y_3)(a_{13}^2 x_3 + a_{14}^2 x_4) \\
&\quad + (-m_{17}x_1 - m_{27}x_2 - m_{37}x_3 - m_{47}x_4 - m_{57}y_1 - m_{67}y_2)(a_{13}^3 x_3 + a_{14}^3 x_4) \\
&= \{-x_1\partial_{x_1} - x_2\partial_{x_2} - x_3\partial_{x_3} - x_4\partial_{x_4} - y_1\partial_{y_1} - y_2\partial_{y_2} - y_3\partial_{y_3} + x_1X_1\}a_1 \\
&\quad + x_2X_1a_2 + x_3X_1a_3 + x_4X_1a_4 \\
&\quad + \{y_1\partial_{x_1} - (a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4)(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4}) \\
&\quad - [y_2(a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4) - y_1(a_{13}^2x_3 + a_{14}^2x_4)]\partial_{y_2} \\
&\quad - [y_3(a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4) - y_1(a_{13}^3x_3 + a_{14}^3x_4)]\partial_{y_3} \\
&\quad + (a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4)(2 + x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4} + y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3})\}c_1 \\
&\quad + \{y_2\partial_{x_1} - (a_{13}^2x_3 + a_{14}^2x_4)(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4}) \\
&\quad + [y_2(a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4) - y_1(a_{13}^2x_3 + a_{14}^2x_4)]\partial_{y_1} \\
&\quad - [y_3(a_{13}^2x_3 + a_{14}^2x_4) - y_2(a_{13}^3x_3 + a_{14}^3x_4)]\partial_{y_3} \\
&\quad + (a_{13}^2x_3 + a_{14}^2x_4)(2 + x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4} + y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3})\}c_2 \\
&\quad + \{y_3\partial_{x_1} - (a_{13}^3x_3 + a_{14}^3x_4)(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4}) \\
&\quad + [y_3(a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4) - y_1(a_{13}^3x_3 + a_{14}^3x_4)]\partial_{y_1} \\
&\quad + [y_3(a_{13}^2x_3 + a_{14}^2x_4) - y_2(a_{13}^3x_3 + a_{14}^3x_4)]\partial_{y_2} \\
&\quad + (a_{13}^3x_3 + a_{14}^3x_4)(2 + x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4} + y_1\partial_{y_1} + y_2\partial_{y_2} + y_3\partial_{y_3})\}c_3 \\
&= (-x_1\partial_{x_1} - x_2\partial_{x_2} - x_3\partial_{x_3} - x_4\partial_{x_4} - y_1\partial_{y_1} - y_2\partial_{y_2} - y_3\partial_{y_3})a_1 \\
&\quad + x_1X_1a_1 + x_2X_1a_2 + x_3X_1a_3 + x_4X_1a_4 \\
&\quad + [y_1X_1 + 2(a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4)]c_1 \\
&\quad + [y_2X_1 + 2(a_{13}^2x_3 + a_{14}^2x_4)]c_2 \\
&\quad + [y_3X_1 + 2(a_{13}^3x_3 + a_{14}^3x_4)]c_3 \\
&= \{x_1[-X_1 + (a_{12}^1x_2 + a_{13}^1x_3 + a_{14}^1x_4)\partial_{y_1} + (a_{13}^2x_3 + a_{14}^2x_4)\partial_{y_2} + (a_{13}^3x_3 + a_{14}^3x_4)\partial_{y_3}] \\
&\quad + x_2[-X_2 + (-a_{12}^1x_1 + a_{24}^1x_4)\partial_{y_1} + (a_{23}^2x_3 + a_{24}^2x_4)\partial_{y_2} + a_{24}^3x_4\partial_{y_3}] \\
&\quad + x_3[-X_3 - a_{13}^1x_1\partial_{y_1} + (-a_{13}^2x_1 - a_{23}^2x_2)\partial_{y_2} + (-a_{13}^3x_1 + a_{34}^3x_4)\partial_{y_3}] \\
&\quad + x_4[-X_4 + (-a_{14}^1x_1 - a_{24}^1x_2)\partial_{y_1} + (-a_{14}^2x_1 - a_{24}^2x_2)\partial_{y_2} + (-a_{14}^3x_1 - a_{24}^3x_2 - a_{34}^3x_3)\partial_{y_3}]
\end{aligned}$$

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$$
\begin{aligned}
&- y_1\partial_{y_1} - y_2\partial_{y_2} - y_3\partial_{y_3}\}a_1\\
&+ x_1X_1a_1 + x_2X_1a_2 + x_3X_1a_3 + x_4X_1a_4\\
&+ y_1X_1c_1 + y_2X_1c_2 + y_3X_1c_3 - x_2(X_1a_2 - X_2a_1) - x_3(X_1a_3 - X_3a_1) - x_4(X_1a_4 - X_4a_1)\\
=\ & y_1(X_1c_1 - \partial_{y_1}a_1) + y_2(X_1c_2 - \partial_{y_2}a_1) + y_3(X_1c_3 - \partial_{y_3}a_1)\\
=\ & \frac{y_1}{2a^1_{12}}((X_1X_2 + [X_1, X_2])a_1 - X_1^2a_2) + \frac{y_2}{2a^2_{23}}([X_2, X_3]a_1 - X_1X_2a_3 + X_1X_3a_2)\\
&+ \frac{y_3}{2a^3_{34}}([X_3, X_4]a_1 - X_1X_3a_4 + X_1X_4a_3).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(BM\mathbf{p}^T)_2 &= y_1(X_2c_1 - \partial_{y_1}a_2) + y_2(X_2c_2 - \partial_{y_2}a_2) + y_3(X_2c_3 - \partial_{y_3}a_2)\\
&= \frac{y_1}{2a^1_{12}}\big(X_2^2a_1 - (X_2X_1 + [X_2, X_1])a_2\big)\\
&\quad + \frac{y_2}{2a^2_{23}}\big((X_2X_3 + [X_2, X_3])a_2 - X_2^2a_3\big)\\
&\quad + \frac{y_3}{2a^3_{34}}\big([X_3, X_4]a_2 - X_2X_3a_4 + X_2X_4a_3\big),
\end{aligned}
$$

$$
\begin{aligned}
(BM\mathbf{p}^T)_3 &= y_1(X_3c_1 - \partial_{y_1}a_3) + y_2(X_3c_2 - \partial_{y_2}a_3) + y_3(X_3c_3 - \partial_{y_3}a_3)\\
&= \frac{y_1}{2a^1_{12}}\big([X_1, X_2]a_3 - X_3X_1a_2 + X_3X_2a_1\big)\\
&\quad + \frac{y_2}{2a^2_{23}}\big(X_3^2a_2 - (X_3X_2a + [X_3, X_2])a_3\big)\\
&\quad + \frac{y_3}{2a^3_{34}}\big((X_3X_4 + [X_3, X_4])a_3 - X_3^2a_4\big),
\end{aligned}
$$

$$
\begin{aligned}
(BM\mathbf{p}^T)_4 &= y_1(X_4c_1 - \partial_{y_1}a_4) + y_2(X_4c_2 - \partial_{y_2}a_4) + y_3(X_4c_3 - \partial_{y_3}a_4)\\
&= \frac{y_1}{2a^1_{12}}\big([X_1, X_2]a_4 - X_4X_1a_2 + X_4X_2a_1\big)\\
&\quad + \frac{y_2}{2a^2_{23}}\big([X_2, X_3]a_4 - X_4X_2a_3 + X_4X_3a_2\big)\\
&\quad + \frac{y_3}{2a^3_{34}}\big(X_4^2a_3 - (X_4X_3 + [X_4, X_3])a_4\big).
\end{aligned}
$$

We have completed the calculations of $tBM\mathbf{p}^T$. To calculate (3.1) further, note that from the first equality of (3.1), $\omega = df$. By the fundamental theorem of calculus,

$$\int_\gamma \omega = f(\mathbf{p}) - f(\mathbf{0})$$

for any horizontal curve γ joining the origin and $\mathbf{p}$. Let

$$\gamma(t) = \big(x_1(t), x_2(t), x_3(t), x_4(t), y_1(t), y_2(t), y_3(t)\big), \quad 0 \le t \le 1$$

be any curve on qH^1 with $\gamma(0) = \mathbf{0}$ and $\gamma(1) = \mathbf{p}$. Then

$$\dot\gamma = \sum_{j=1}^4 \dot x_j \frac{\partial}{\partial x_j} + \sum_{l=1}^3 \dot y_l \frac{\partial}{\partial y_l} = \sum_{j=1}^4 \dot x_j X_j + \sum_{l=1}^3 \Big[\dot y_l - \sum_{k=1}^4 \Big(\sum_{j=1}^4 \dot x_j a^l_{jk}\Big) x_k\Big] \frac{\partial}{\partial y_l}.$$

The curve γ being horizontal means that $\dot{\gamma}$ can be represented by the vector fields X_1, X_2, X_3, and X_4 only, in which case,

$$\dot{y}_l = \sum_{k=1}^{4}\Big(\sum_{j=1}^{4}\dot{x}_j a_{jk}^{l}\Big)x_k, \quad l = 1,2,3. \tag{3.6}$$

Combining (3.1) and (3.6), we have

$$\begin{aligned}
&\int_0^1 \omega(\gamma'(t))dt \\
&\quad= \int_0^1 \Big\{\sum_{j=1}^{4} a_j^*(\gamma(t))\dot{x}_j + \sum_{l=1}^{3} c_l(\gamma(t))\dot{y}_l\Big\}dt \\
&\quad= \int_0^1 \Big\{\Big[a_1(\gamma(t)) + \frac{1}{2}(c_{12}(\gamma(t))x_2 + c_{13}(\gamma(t))x_3 + c_{14}(\gamma(t))x_4)\Big]\dot{x}_1 \\
&\qquad+ \Big[a_2(\gamma(t)) + \frac{1}{2}(-c_{12}(\gamma(t))x_1 + c_{23}(\gamma(t))x_3 + c_{24}(\gamma(t))x_4)\Big]\dot{x}_2 \\
&\qquad+ \Big[a_3(\gamma(t)) + \frac{1}{2}(-c_{13}(\gamma(t))x_1 - c_{23}(\gamma(t))x_2 + c_{34}(\gamma(t))x_4)\Big]\dot{x}_3 \\
&\qquad+ \Big[a_4(\gamma(t)) + \frac{1}{2}(-c_{14}(\gamma(t))x_1 - c_{24}(\gamma(t))x_2 - c_{34}(\gamma(t))x_3)\Big]\dot{x}_4 \\
&\qquad+ \sum_{l=1}^{3} c_l(\gamma(t))\sum_{k=1}^{4}\Big(\sum_{j=1}^{4}\dot{x}_j a_{jk}^{l}\Big)x_k\Big\}dt \\
&\quad= \int_0^1 \sum_{j=1}^{4} a_j(\gamma(t))\dot{x}_j dt = \int_0^1 g(U(\gamma(t)),\gamma'(t))dt,
\end{aligned}$$

where $U = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$ and $g(\cdot,\cdot)$ is the sub-Riemannian metric. Therefore, we have the following theorem.

Theorem 3.1 *Let X_1, X_2, X_3, X_4 be the vector fields on qH^1 given in (1.3), equipped with the assumption (2.3). Consider any smooth functions a_1, a_2, a_3, a_4,*

$$c_1 = \frac{X_1a_2 - X_2a_1}{-2a_{12}^1}, \quad c_2 = \frac{X_2a_3 - X_3a_2}{-2a_{23}^2}, \quad c_3 = \frac{X_3a_4 - X_4a_3}{-2a_{34}^3},$$

$$a_j^* = a_j + \frac{1}{2}\sum_{k=1}^{4} x_k(X_ja_k - X_ka_j), \quad j = 1,2,3,4,$$

and let

$$f(\mathbf{p}) = \int_0^1 \Big[\sum_{j=1}^{4} a_j^*(t\mathbf{p})x_j + \sum_{l=1}^{3} c_l(t\mathbf{p})y_l\Big]dt,$$

where $\mathbf{p} = (x_1, x_2, x_3, x_3, y_1, y_2, y_3)$. Then

$$\begin{aligned}
(X_1f)(\mathbf{p}) = a_1(\mathbf{p}) + \int_0^1 \Big\{&\frac{ty_1}{2a_{12}^1}((X_1X_2 + [X_1,X_2])a_1 - X_1^2a_2)(t\mathbf{p}) \\
&+ \frac{ty_2}{2a_{23}^2}([X_2,X_3]a_1 - X_1X_2a_3 + X_1X_3a_2)(t\mathbf{p}) \\
&+ \frac{ty_3}{2a_{34}^3}([X_3,X_4]a_1 - X_1X_3a_4 + X_1X_4a_3)(t\mathbf{p})\Big\}dt,
\end{aligned}$$

$$(X_2 f)(\mathbf{p}) = a_2(\mathbf{p}) + \int_0^1 \Big\{ \frac{ty_1}{2a_{12}^1}(X_2^2 a_1 - (X_2X_1 + [X_2, X_1])a_2)(t\mathbf{p}) + \frac{ty_2}{2a_{23}^2}((X_2X_3 + [X_2, X_3])a_2 - X_2^2 a_3)(t\mathbf{p}) + \frac{ty_3}{2a_{34}^3}([X_3, X_4]a_2 - X_2X_3a_4 + X_2X_4a_3)(t\mathbf{p})\Big\}\, dt,$$

$$(X_3 f)(\mathbf{p}) = a_3(\mathbf{p}) + \int_0^1 \Big\{ \frac{ty_1}{2a_{12}^1}([X_1, X_2]a_3 - X_3X_1a_2 + X_3X_2a_1)(t\mathbf{p}) + \frac{ty_2}{2a_{23}^2}(X_3^2 a_2 - (X_3X_2a + [X_3, X_2])a_3)(t\mathbf{p}) + \frac{ty_3}{2a_{34}^3}((X_3X_4 + [X_3, X_4])a_3 - X_3^2 a_4)(t\mathbf{p})\Big\}\, dt,$$

$$(X_4 f)(\mathbf{p}) = a_4(\mathbf{p}) + \int_0^1 \Big\{ \frac{ty_1}{2a_{12}^1}([X_1, X_2]a_4 - X_4X_1a_2 + X_4X_2a_1)(t\mathbf{p}) + \frac{ty_2}{2a_{23}^2}([X_2, X_3]a_4 - X_4X_2a_3 + X_4X_3a_2)(t\mathbf{p}) + \frac{ty_3}{2a_{34}^3}(X_4^2 a_3 - (X_4X_3 + [X_4, X_3])a_4)(t\mathbf{p})\Big\}\, dt.$$

If the integrability conditions (2.4) hold, then the system of equations $X_1 f = a_1, X_2 f = a_2, X_3 f = a_3, X_4 f = a_4$ is solvable and

$$f(\mathbf{p}) = \int_0^1 g(U(\gamma(t)), \gamma'(t))\,dt,$$

where $U = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$, $\gamma(t)$ is a horizontal curve connecting the origin and $\mathbf{p}$, and $g(\cdot,\cdot)$ is the sub-Riemannian metric.

Assume that $a_{jk}^l = -2$ for $j = l = 1, k = 2$, and $a_{jk}^l = 0$ otherwise. Consider the hyperplane

$$\widetilde{\mathcal{H}^1} = \{\mathbf{p} \in qH^1 : x_3 = x_4 = y_2 = y_3 = 0\};$$

then, by (1.3), the vector fields $X_3 = X_4 = 0$ and X_1, X_2 reduce to the Heisenberg vector fields (1.2); $\widetilde{\mathcal{H}^1}$ turns into the Heisenberg group $\mathcal{H}^1$. In this case, we have the following corollary.

Corollary 3.2 *Under the hypotheses of the above paragraph, Theorems 2.1 and 3.1 recover Theorems 1.1 and 1.2, respectively.*

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References

[1] O. Calin, D. C. Chang, and M. Eastwood, *Integrability conditions for Heisenberg and Grushin-type distributions*. Anal. Math. Phys. **4**(2014), no. 1–2, 99–114. http://dx.doi.org/10.1007/s13324-014-0073-1

[2] O. Calin, D. C. Chang, and J. Hu, *Poincaré's lemma on the Heisenberg group*. Adv. in Appl. Math. **60**(2014), 90–102. http://dx.doi.org/10.1016/j.aam.2014.08.003

[3] D. C. Chang, I. Markina, and W. Wang, *On the Cauchy-Szegö kernel for quaternion Siegel upper half-space*. Complex Anal. Oper. Theory 7(2013), no. 5, 1623–1654. http://dx.doi.org/10.1007/s11785-012-0282-2

[4] W.-L. Chow, *Uber Systeme von linearen partiellen Differentialgleichungen erster Ordnung*. Math. Ann. **117**(1939), 98–105. http://dx.doi.org/10.1007/BF01450011

[5] C. Fefferman and D. H. Phong, *The uncertainty principle and sharp Garding inequalities*. Comm. Pure and Applied Math. **34**(1981), 285–331. http://dx.doi.org/10.1002/cpa.3160340302

[6] L. Hörmander, *Hypoelliptic second order differential equations*. Acta Math. **119**(1967), 147–171. http://dx.doi.org/10.1007/BF02392081

[7] W. Wang, *The tangential Cauchy-Fueter complex on the quaternionic Heisenberg group*. J. Geom. Phys. **61**(2011), 363–380. http://dx.doi.org/10.1016/j.geomphys.2010.10.006

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