A note on small vibrations

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This note exhibits some old results in what is possibly a new form.

The simple harmonic oscillator, whose kinetic and potential energies are given in terms of a single coordinate x by

$$2T = \dot{x}^2,$$

$$2V = bx^2$$

where b is a positive constant, has the equation of motion

$$\ddot{x} + bx = 0.$$

The solution, appropriate to the initial conditions $x = x_0$, $\dot{x} = \dot{x}_0$ at $t = 0$ is conveniently written

$$x = x_0 \cos \sqrt{b}\, t + \frac{\dot{x}_0}{\sqrt{b}} \sin \sqrt{b}\, t. \tag{1}$$

The problem of the small oscillations of a conservative dynamical system of n degrees of freedom about a position of stable equilibrium is a generalisation of the problem of the oscillator; in seeking a generalisation of the solution (1) a matrix notation at once suggests itself. Let the dynamical system be defined by the two quadratic forms

$$2T = \dot{x}'\dot{x}$$

$$2V = x'Bx$$

where B is a symmetric square matrix, x a column matrix and x' its transpose. The form T is of course positive definite, and V is assumed to be non-negative. The equations of motion are

$$\ddot{x} + Bx = 0. \tag{2}$$

The standard method of solution is to find the latent roots $p_1^2, p_2^2 \ldots\ldots p_n^2$ of the matrix B, which will for the moment be assumed positive and distinct, and to construct the matrix L whose columns are the normalised latent vectors. Then $L'BL$ is a diagonal matrix, Λ, whose diagonal elements are the latent roots of B, and $L'L = I$, the unit matrix, *i.e.* L is orthogonal. Hence in terms of "normal coordinates" ξ given by

$$x = L\xi$$

the equations of motion become

$$\ddot{\xi} + \Lambda\,\xi = 0$$

i.e.

$$\ddot{\xi}_r + p_r^2\,\xi_r = 0, \qquad r = 1, 2 \,\ldots.\, n.$$

Each coordinate ξ_r is therefore given by a formula of type (1),

$$\xi_r = \xi_{r0} \cos p_r t + \frac{\dot{\xi}_{r0}}{p_r} \sin p_r t, \tag{3}$$

and the most general motion of the system is a superposition of n such normal modes with arbitrary amplitudes and phases. The set of equations (3) may be written as a single matrix equation

$$\xi = \begin{bmatrix} \cos p_1 t & . & . & \ldots \\ . & \cos p_2 t & . & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{bmatrix} \xi_0 + \begin{bmatrix} (\sin p_1 t)/p_1 & . & . & \ldots \\ . & (\sin p_2 t)/p_2 & . & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \\ . & . & . & \ldots \end{bmatrix} \dot{\xi}_0$$

$$= C\xi_0 + S\dot{\xi}_0,$$

where the diagonal matrices involving trigonometric functions have been denoted by C and S respectively for brevity.

The solution for x is obtained by writing $\xi = L^{-1}x$, which gives

$$x = LCL^{-1}x_0 + LSL^{-1}\dot{x}_0.$$

Now the transformation $L^{-1} \ldots . L$ which reduces B to diagonal form, with diagonal elements $p_1^2 \ldots . p_n^2$, will also reduce any function $f(B)$ of B to diagonal form, with diagonal elements $f(p_1^2) \ldots . f(p_n^2)$. Thus

$$L^{-1} \cos \sqrt{B}\,t\,L = C,$$

$$L^{-1} \frac{\sin \sqrt{B}\,t}{\sqrt{B}}\,L = S. \tag{4}$$

When these values of C and S are substituted in the solution it becomes

$$x = \cos\sqrt{B}\,t.\ x_0 + \frac{\sin\sqrt{B}\,t}{\sqrt{B}}\dot{x}_0 \quad (5)$$ [1]

in complete formal analogy with (1).

The more general problem in which

$$2T = \dot{x}'A\dot{x}$$
$$2V = x'Bx,$$

where A and B are symmetric matrices, and T is positive-definite and V non-negative as before, can be treated in the same way. The equations of motion are

$$A\ddot{x} + Bx = 0$$

or

$$\ddot{x} + A^{-1}Bx = 0. \quad (6)$$

Here $A^{-1}B$ is no longer in general a symmetric matrix, but there still exists a transformation $H^{-1}(A^{-1}B)H$ which diagonalises it, although the transformation is no longer orthogonal. For if A is definite, it is a well-known result that H can be found such that simultaneously

$$H'AH = \mathrm{I}$$
$$H'BH = \Pi,$$

where Π is a diagonal matrix, whose elements are the squares of the normal frequencies. From these equations it follows that

$$H^{-1}(A^{-1}B)H = \Pi,$$

and the columns of H are the latent vectors of $A^{-1}B$. Thus the general solution of the equations (6) is

$$x = \cos\sqrt{(A^{-1}B)}\,t.\quad x_0 + \frac{\sin\sqrt{(A^{-1}B)}\,t}{\sqrt{(A^{-1}B)}}\dot{x}_0. \quad (7)$$

In the foregoing the matrix functions $\cos\sqrt{B}\,t$, $\frac{\sin\sqrt{B}\,t}{\sqrt{B}}$, etc., have in effect been defined by the relations

$$\cos\sqrt{B}\,t = LCL^{-1}$$
$$\frac{\sin\sqrt{B}\,t}{\sqrt{B}} = LSL^{-1} \quad (8)$$

which follow from (4). These express *e.g.* $\cos\sqrt{B}\,t$ as a linear

[1] Note added in proof: This solution is given in Chapter XII of the third volume of W. D. MacMillan's *Theoretical Mechanics* (McGraw-Hill, 1936), with acknowledgement to unpublished work by W. Bartky.

combination of $\cos p_1 t$, $\cos p_2 t \ldots \cos p_n t$, showing that it is allowable to write the time-derivates of $\cos \sqrt{B}\,t$ and $\sin \sqrt{B}\,t$ as $-\sqrt{B} \sin \sqrt{B}\,t$ and $\sqrt{B} \cos \sqrt{B}\,t$ respectively, and thus to verify the solutions (5) or (7) by direct substitution in the differential equations (2) or (6).

The expansions (8) of the matrix functions are of course equivalent to Sylvester's interpolation formula. This formula is most conveniently arrived at by noting that if B is a matrix with latent roots $\lambda_1, \lambda_2 \ldots \lambda_n$, assumed distinct, the determinant

$$\begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ \lambda_1 & \lambda_2 & & B \\ \lambda_1^2 & \lambda_2^2 & & B^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f(\lambda_1) & f(\lambda_2) & & f(B) \end{vmatrix} \tag{9}$$

vanishes. For $(B - \lambda_1 I)(B - \lambda_2 I) \ldots\ldots (B - \lambda_n I)$ is obviously a factor, and by the Cayley-Hamilton identity this is the null matrix. The expression for $f(B)$ follows by expanding from the last row. Thus for a matrix B of the third order, with latent roots p_1^2, p_2^2, p_3^2

$$\cos \sqrt{B}\,t \equiv \frac{(B - p_2^2 I)(B - p_3^2 I)}{(p_1^2 - p_2^2)(p_1^2 - p_3^2)} \cos p_1 t + \frac{(B - p_3^2 I)(B - p_1^2 I)}{(p_2^2 - p_3^2)(p_2^2 - p_1^2)} \cos p_2 t$$
$$+ \frac{(B - p_1^2 I)(B - p_2^2 I)}{(p_3^2 - p_1^2)(p_3^2 - p_2^2)} \cos p_3 t, \tag{10}$$

with a similar formula for $\dfrac{\sin \sqrt{B}\,t}{\sqrt{B}}$, the matrix coefficients being the same in both.

The process may be illustrated by the example of three equal masses m spaced at equal intervals a along a light string at tension T_1 and vibrating transversely. In this case

$$A = I, \qquad B = \sigma^2 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \qquad \sigma^2 \equiv T_1/am.$$

The latent roots of B are

$$p_1^2 = (2 - \sqrt{2})\,\sigma^2$$
$$p_2^2 = 2\,\sigma^2$$
$$p_3^2 = (2 + \sqrt{2})\,\sigma^2$$

and the expression for $\cos\sqrt{B}\,t$, by (10), is

$$\cos\sqrt{B}\,t = \tfrac{1}{4}\begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix}\cos p_1 t + \tfrac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\cos p_2 t$$

$$+ \tfrac{1}{4}\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}\cos p_3 t.$$

A similar expression, with the same matrix coefficients, holds for $(\sin\sqrt{B}\,t)/\sqrt{B}$, and the general solution (5) can therefore be written down at once *in extenso*. The matrix L, if required, can be constructed by taking a non-vanishing column from each of the matrix coefficients, and normalising each column to unity:

$$L = \tfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{bmatrix}$$

The same rule gives the diagonalising matrix H when (7) is the form of solution, but the appropriate normalisation is then with respect to A instead of I.

The case of coincident roots of the secular determinant is of historic interest. If the equations of motion are of the form (2) as before, each of the coordinates x_r will satisfy an equation, obtained by elimination of the remaining variables, which is simply

$$(D^2 + p_1^2)\,(D^2 + p_2^2)\,\ldots\,(D^2 + p_n^2)\,x_r = 0, \qquad D^2 \equiv d^2/dt^2,$$

where some of the p^2's are now assumed to coincide. The general solution might therefore be expected to contain "secular" terms $h(t)\cos pt$, $k(t)\sin pt$ corresponding to a repeated root p, $h(t)$ and $k(t)$ being polynomials of degree one less than the multiplicity of the root, in addition to the usual terms in $\cos pt$, $\sin pt$, for the single roots. This is indeed what Lagrange and others who followed him supposed. It turns out, however, that on substituting the assumed solutions into the original differential equations (2) in accordance with the usual procedure for determining the relations between the constants, all the coefficients in $h(t)$ and $k(t)$ vanish identically, with the exception of their constant terms. The reason for this is not directly obvious, although of course the matter has been well understood since Weierstrass discussed it in 1858. The matrix solution given above concisely demonstrates the disappearance of the "secular" terms in the dynamical problem, as will now be shown.

When repeated roots occur, (5) is still formally the solution of (2), but as there is now a certain arbitrariness in the matrix L in the definitions (8), the trigonometric functions will be assumed to be defined by Sylvester's theorem, in its confluent form. This may be obtained from (9) by a simple limiting process; if a matrix B of the fourth order with latent roots $\alpha, \beta, \beta, \beta$ is taken as an illustration, the resulting confluent formula is

$$(\beta-\alpha)^3 f(B) = -\begin{vmatrix} 1 & 1 & \cdot & \cdot & 1 \\ \alpha & \beta & 1 & \cdot & B \\ \alpha^2 & \beta^2 & 2\beta & 1 & B^2 \\ \alpha^3 & \beta^3 & 3\beta^2 & 3\beta & B^3 \\ f(\alpha) & f(\beta) & f'(\beta) & \frac{1}{2}f''(\beta) & \cdot \end{vmatrix}. \tag{11}$$

The expansion of *e.g.* $\cos\sqrt{B}\,t$ in the solution (5) would thus contain terms in $\cos\sqrt{\beta}\,t$, $t\sin\sqrt{\beta}\,t$, $t^2\cos\sqrt{\beta}\,t$, arising from the terms $f(\beta), f'(\beta)$, $f''(\beta)$ in (11) respectively. It will now be shown that the coefficients of $f'(\beta), f''(\beta)$ in (11) vanish, so that the secular terms do not appear.

The peculiarity of the dynamical case is that if the matrix B has repeated roots it satisfies an identical equation which is of lower degree than the order of B. This reduced Cayley-Hamilton equation is obtained by counting each distinct linear factor in the ordinary Cayley-Hamilton equation once only. In the illustration just chosen the reduced equation is $(B-\alpha I)(B-\beta I)\equiv 0$ instead of $(B-\alpha I)(B-\beta I)^3\equiv 0$. The truth of the reduced equation follows at once from consideration of its diagonalised form; its existence depends essentially on the fact that B (or $A^{-1}B$) can be reduced to pure diagonal form, whether there are repeated roots or not (linearity of the elementary divisors). The determinant on the right of (11) may now be simplified by pre-multiplying it matrix-wise by the following determinant whose value is unity:

$$\begin{vmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \alpha\beta & -(\alpha+\beta) & 1 & \cdot & \cdot \\ \cdot & \alpha\beta & -(\alpha+\beta) & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix}.$$

The result is, in consequence of the relation $B^2-(\alpha+\beta)B+\alpha\beta I\equiv 0$,

$$(\beta-\alpha)^3 f(B) = -\begin{vmatrix} 1 & 1 & \cdot & \cdot & 1 \\ \alpha & \beta & 1 & \cdot & B \\ \cdot & \cdot & -(\alpha-\beta) & 1 & \cdot \\ \cdot & \cdot & -\beta(\alpha-\beta) & -\alpha+2\beta & \cdot \\ f(\alpha) & f(\beta) & \underline{f'(\beta)} & \underline{\frac{1}{2}f''(\beta)} & \cdot \end{vmatrix}.$$

The elements underlined have vanishing co-factors, and may be suppressed, so that the appearance of secular terms is only apparent. The formula (11) thus reduces to

$$(\beta - a) f(B) = - \begin{matrix} 1 & 1 & 1 \\ a & \beta & B \\ f(a) & f(\beta) & . \end{matrix} ,$$

which might have been written down at once in the form

$$\left| \begin{matrix} 1 & 1 & 1 \\ a & \beta & B \\ f(a) & f(\beta) & f(B) \end{matrix} \right| = 0 \tag{12}$$

from a knowledge of the reduced Cayley-Hamilton equation; but the derivation by "confluence" throws more light on the disappearance of the secular terms.

It can hardly have escaped notice that the confluent form of Sylvester's expansion of a function $f(B)$ of a matrix B can be expressed by a simple formula of type (12) whenever the elementary divisors are linear, as they always are in the dynamical case[1].

Scarcely any modification is needed if a latent root of B vanishes, say $a = 0$. It is only necessary to replace $\cos \sqrt{a}\, t$ by 1 and $(\sin \sqrt{a}\, t)/\sqrt{a}$ by t. The solution of (2) will then contain a term linear in t. The motion in the corresponding normal mode is one of constant velocity, which may be regarded a simple harmonic motion of infinite period.

The following example illustrates the occurrence of a vanishing root, a pair of repeated roots, and a single root of the secular equation. Four beads, each of mass m, slide on a smooth wire bent into a circle of radius r. They are connected by four light springs lying along the circle, each of natural length $\pi r/2$ and modulus λ, so as to form a closed chain.

If the angular displacements of the beads from their equilibrium positions are θ_1, θ_2, θ_3, θ_4, the kinetic and potential energies are given by

$$2T = mr^2 \dot{\theta}' \dot{\theta},$$
$$2V = (2\lambda r/\pi)\, \theta' B \theta,$$

[1] With regard to the general case in which the elementary divisors are not necessarily linear, the writer is indebted to the referee for the remark that the confluent determinantal form (*cf.* (11)) *essentially* involving f', f'' ...... arises when and only when the reduced characteristic equation has repeated roots. The order of the determinant exceeds by unity that of the reduced characteristic equation.

so that the equations of motion are

$$\ddot{\theta} + \sigma^2 B\theta = 0,$$

with

$$B = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}, \qquad \sigma^2 \equiv 2\lambda/\pi r m.$$

The latent roots of B are 0, 2, 2, 4, so that the reduced Cayley-Hamilton equation is $B(B-2I)(B-4I)=0$. The confluent form of Sylvester's formula is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & B \\ 0 & 2^2 & 4^2 & B^2 \\ f(0) & f(2) & f(4) & f(B) \end{vmatrix} = 0,$$

or

$$f(B) = \tfrac{1}{8}(B-2I)(B-4I)f(0) - \tfrac{1}{4}B(B-4I)f(2) + \tfrac{1}{8}B(B-2I)f(4),$$

so that the solution is

$$\theta = \tfrac{1}{4}\{C_1 + 2C_2 \cos\sqrt{2}\,\sigma t + C_3 \cos 2\sigma t\}\,\theta_0$$
$$+ \tfrac{1}{4}\{C_1 t + 2C_2(\sin\sqrt{2}\,\sigma t)/(\sqrt{2}\,\sigma) + C_3(\sin 2\sigma t)/(2\sigma)\}\,\dot{\theta}_0$$

where

$$C_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & . & -1 & . \\ . & 1 & . & -1 \\ -1 & . & 1 & . \\ . & -1 & . & 1 \end{bmatrix}, \; C_3 = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

The ratios of the coordinates in the various modes can be read off from the matrix coefficients as

1 : 1 : 1 : 1 for the mode of zero frequency, corresponding to a uniform rotation;

$\left.\begin{matrix} 1 : 0 : -1 : 0 \\ \text{or } 0 : 1 : 0 : -1 \end{matrix}\right\}$ for the degenerate modes of frequency $\sqrt{2}\,\sigma$;

and 1 : −1 : 1 : −1 for the non-degenerate mode of frequency 2σ.

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