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INTERSECTIONS OF m-CONVEX SETS

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1. Introduction. Let S be a subset of some linear topological space. The set S is said to be *m-convex*, $m \geqq 2$, if and only if for every m-member subset of S, at least one of the $\binom{m}{2}$ line segments determined by these points lies in S. A point x in S is called a *point of local convexity* of S if and only if there is some neighborhood N of x such that if $y, z \in N \cap S$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) of S.

Several interesting decomposition theorems have been obtained for closed m-convex sets in the plane (Valentine [**9**], Stamey and Marr [**6**], Breen and Kay [**2**]). However, little work has been done on the problem of characterizing intersections of m-convex subsets of a set. Similar characterizations have been accomplished for intersections of maximal starshaped subsets of set S, where S is compact, simply connected and planar (Hare and Kenelly [**3**]), and for maximal L_n subsets of S (Sparks [**5**]). Also, for S a subset of an arbitrary linear topological space, Tattersall [**7**] has obtained conditions under which the intersection of all maximal m-convex subsets of S will be exactly the kernel of S. Unfortunately, in general such an intersection will not even be an m-convex set. Thus the purpose of this paper is to obtain conditions under which an intersection of m-convex subsets will be again m-convex. There are two main results: the first concerns 3-convex sets in R^d; the second, m-convex sets in the plane.

The following familiar terminology will be used: For points x, y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. Points $x_1, \ldots, x_n$ in S are *visually independent via S* if and only if for $1 \leqq i < j \leqq n$, x_i does not see x_j via S. Throughout the paper, conv S, aff S, cl S, bdry S, int S, rel int S, and ker S will be used to denote the convex hull, affine hull, closure, boundary, interior, relative interior, and kernel, respectively, of the set S. Also, if S is convex, dim S will denote the dimension of S.

2. Intersections of 3-convex sets in R^d. We begin with a series of preliminary lemmas.

LEMMA 1. *Let M be a closed m-convex subset of some linear topological space, and let Q denote the set of* lnc *points of M. Then $M = \text{cl}(M \sim Q)$.*

Proof. Let $x \in M$ and let N be an arbitrary neighborhood of x to show that N contains points in $M \sim Q$. Assume on the contrary that $N \cap M \subseteq Q$ to

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obtain a contradiction. Select points y_1, z_1 in $N \cap M$ such that $[y_1, z_1] \not\subseteq M$. Furthermore, since M is closed, we may select some neighborhood N_1 of y_1, $N_1 \subseteq N$, such that no point of $N_1 \cap M$ sees z_1 via M. Now $y_1 \in N \cap M \subseteq Q$, so we may select y_2, z_2 in $N_1 \cap M$ such that $[y_2, z_2] \not\subseteq M$. Continuing, by an obvious induction we may select a visually independent set $\{z_n\}$, contradicting the m-convexity of M. Our assumption is false, N contains points in $M \sim Q$, and $M \subseteq \text{cl}(M \sim Q)$. The reverse inclusion is obvious and the lemma is proved.

LEMMA 2. *Let M be a closed m-convex set in R^d, where $d = \dim \text{aff } M$, and let Q denote the set of* lnc *points of M. If $M \sim Q$ is connected, then $M = \text{cl}(\text{int } M)$.*

Proof. Let $x \in M$ and let N be any neighborhood of x to show that N contains points interior to M. By Lemma 1, $x \in \text{cl}(M \sim Q)$, so we may select y in $N \cap (M \sim Q)$. Choose a neighborhood N_1 of y such that $N_1 \subseteq N$ and $C \equiv N_1 \cap M$ is convex.

We assert that $\dim C = d$. Otherwise, there would be points of M not in aff C, and since $M = \text{cl}(M \sim Q)$, we could select z in $M \sim Q$ such that $z \notin \text{aff } C$. Since $M \sim Q$ is connected and locally convex, it is polygonally connected, and there would be a path λ in $M \sim Q$ from y to z. However, $(\text{aff } C) \cap \text{cl}(M \sim \text{aff } C) \subseteq Q$, so λ would contain a point of Q, impossible. Thus $\dim C = d$, and any point in $N \cap \text{int } C \neq \emptyset$ will be interior to M, finishing the argument.

LEMMA 3. *If $M = \text{cl}(\text{int } M)$, $\dim \text{aff } M = d$, and the set Q of* lnc *points of M lies in* ker M, *then either* conv Q *contains an interior point of M or Q is convex.*

Proof. Since $Q \subseteq \ker M$, clearly $\text{conv } Q \subseteq M$. If $\text{conv } Q \cap \text{int } M \neq \emptyset$, there is nothing to prove, so assume that $\text{conv } Q \subseteq \text{bdry } M$. Then $\dim \text{conv } Q \leqq d - 1$.

We will show that Q is a convex subset of M. Suppose, on the contrary, that there is some z in $\text{conv } Q \sim Q$. It is easy to see that Q is closed, so $\text{conv } Q \sim Q$ is open in conv Q, and z may be selected in rel int conv Q. Using the fact that $z \notin Q$, select a neighborhood N of z for which $N \cap M$ is convex. Then since $z \in \text{bdry } M$, there is a hyperplane H supporting $N \cap M$ at z, with $N \cap M$ in $\text{cl}(H_1)$ (where H_1, H_2 denote distinct open halfspaces determined by H). Since $z \in \ker M$, clearly no point of M lies in H_2. Also, since $z \in \text{rel int conv } Q$, Q must lie in H. (Otherwise, z would lie in $(\text{conv } Q) \cap H \subseteq \text{rel bdry conv } Q$.) Therefore, for p, q in $M \sim H$, $[z, p] \cup [z, q] \subseteq M$, no lnc point of M lies in conv $\{z, p, q\}$, so by a lemma of Valentine [**8**, Corollary 1], $\text{conv } \{z, p, q\} \subseteq M$ and $[p, q] \subseteq M \sim H$. Hence $M \sim H$ is convex, and since $M \subseteq \text{cl}(H_1)$, the set $\text{cl}(M \sim H) = \text{cl}(\text{int } M) = M$ is convex. But this implies that $Q = \emptyset$, a contradiction. Thus Q must be convex, completing the proof.

LEMMA 4. *Let M be a closed 3-convex set in R^d, where $d = \dim \text{aff } M$, and let Q*

denote the set of lnc *points of* M. *If* $M \sim Q$ *is connected and* Q *lies in a hyperplane, then* M *is a union of two convex sets.*

Proof. By Lemma 2, $M = \text{cl}(\text{int } M)$. Also, since M is 3-convex, it is easy to show that $Q \subseteq \ker M$, so by Lemma 3, either conv Q contains an interior point of M or Q is convex.

Suppose, for the moment, that $w \in \text{conv } Q \cap \text{int } M \neq \emptyset$. For H a hyperplane containing Q, with H_1 and H_2 the corresponding open halfspaces, we assert that $\text{cl}(M \cap H_1)$, $\text{cl}(M \cap H_2)$ are convex sets whose union is M: If x, y are in $M \cap H_1$, then $[x, w] \cup [w, y] \subseteq M$, no lnc point of M can be in conv $\{x, y, w\}$, so by Valentine's lemma, conv $\{x, y, w\} \subseteq M$ and $[x, y] \subseteq H_1 \cap M$. Hence $H_1 \cap M$ is convex, as is $\text{cl}(H_1 \cap M)$. Similarly $\text{cl}(H_2 \cap M)$ is convex, and since $M = \text{cl}(\text{int } M)$, clearly

$$M = \text{cl}(H_1 \cap M) \cup \text{cl}(H_2 \cap M),$$

the desired result.

In case conv $Q \cap \text{int } M = \emptyset$, then Q must be convex by Lemma 3. We will show that Q satisfies the definition of essential given in [**1**, Definition 1]. Precisely, if $q \in Q$ and N is any convex neighborhood of q, we assert that $(N \cap M) \sim Q$ is connected: Let r, s belong to $(\text{int } M) \cap N$. Since $M \sim Q$ is connected and $M = \text{cl}(\text{int } M)$, by standard arguments, int M is connected. Also, int M is locally convex and hence polygonally connected, so there is a polygonal path λ in int M from r to s. Let T denote a neighborhood of λ, $T \subseteq \text{int } M$. Since $q \in Q \subseteq \ker M$, $\text{conv}(T \cup \{q\}) \subseteq M$, and $\text{conv}(T \cup \{q\})$ contains a path λ' in $(\text{int } M) \cap N$ from r to s. Thus $(\text{int } M) \cap N$ is polygonally connected and hence connected. Since

$$(\text{int } M) \cap N \subseteq (M \cap N) \sim Q \subseteq \text{cl}[(\text{int } M) \cap N],$$

it follows that $(M \cap N) \sim Q$ is also connected, and the assertion is proved. Therefore, we may apply arguments given in [**1**, Theorem 3] to conclude that M is a union of two convex sets, finishing the proof of the lemma.

THEOREM 1. *Let* S *be a closed subset of* R^k, *and assume that* S *contains all triangles whose boundaries lie in* S. *Let* $\mathscr{M}$ *denote any collection of closed* 3-*convex subsets of* S *such that for* M *in* $\mathscr{M}$ *and* Q_M *the corresponding set of* lnc *points of* M, *each member of* Q_M *is an* lnc *point for* $S \cap \text{aff } Q_M$ *and* $M \sim Q_M$ *is connected. Then*

$$\cap \{M : M \in \mathscr{M}\} \equiv \cap \mathscr{M}$$

is 3-*convex.*

Proof. Let M belong to $\mathscr{M}$, let dim aff $M = d$, and let $Q_M = Q$ denote the set of lnc points of M. Since M is 3-convex, $Q \subseteq \ker M$. We will show that if $x, y \in M$ and $[x, y] \subseteq S$, then $[x, y] \subseteq M$. There are three cases to consider.

Case 1. In case int conv $Q \neq \emptyset$ (as a subset of the d-dimensional space aff M), then select $w \in$ int conv Q and let N be a d-dimensional neighborhood of z for which $N \subseteq$ conv Q. Since conv $Q \subseteq$ ker M, conv$(N \cup \{x\}) \subseteq M$ and conv$(N \cup \{y\}) \subseteq M$. Therefore, since S contains all triangles whose boundaries lie in S, conv$(N \cup [x, y]) \subseteq S$, and (conv$\{x, y, w\}) \sim [x, y]$ can contain no lnc point of $S \cap$ aff Q. Hence (conv$\{x, y, w\}) \sim [x, y]$ can contain no lnc point of M, $[w, x] \cup [w, y] \subseteq M$, and by a generalization of Valentine's lemma, conv$\{x, y, w\} \subseteq M$ and $[x, y] \subseteq M$.

Case 2. Assume that int conv $Q = \emptyset$ and that conv Q contains an interior point of M. Then clearly we may select a point w in (rel int conv Q) $\cap$ int M. Unfortunately, there are three subcases to consider, depending upon whether x, y belong to aff Q:

Case 2a. If $x, y \notin$ aff Q, then no point of $(w, x]$ is in aff Q, and to each point of $(w, x]$ we may associate a convex neighborhood disjoint from aff Q. Also, since $w \in$ int M, there is some neighborhood of w disjoint from Q. Hence by using a compactness argument, we may select a convex cylinder about $[w, x]$ disjoint from Q. Finally, let N_x be a convex neighborhood of w contained in the cylinder, $N_x \subseteq M$. For z in N_x, $[z, w] \cup [w, x] \subseteq M$, clearly no lnc point of M lies in conv$\{z, w, x\}$, so again by Valentine's lemma, $[z, x] \subseteq M$. Thus conv$(N_x \cup \{x\}) \subseteq M$. Repeating the argument for y, we obtain a neighborhood N_y of w with conv$(N_y \cup \{y\}) \subseteq M$. Then $N = N_x \cap N_y$ is a neighborhood of w with conv$(N \cup \{x\}) \subseteq M$ and conv$(N \cup \{y\}) \subseteq M$. By repeating an argument used in Case 1, conv$\{x, y, w\}$ contains no lnc point of M and $[x, y] \subseteq M$, the desired result.

Case 2b. If both x and y are in aff Q, then consider the set $M_0 \equiv M \cap$ aff Q as a subset of the flat aff Q. Since $w \in$ rel int conv Q, w is interior to ker M_0, and we may select a neighborhood N of w in aff Q for which $N \subseteq$ ker M_0. Repeating the argument in Case 1, (conv$\{x, y, w\}) \sim [x, y]$ can contain no lnc point of $S \cap$ aff Q and hence no lnc point of M, so $[x, y] \subseteq M$.

Case 2c. In case exactly one of x and y, say y, is in aff Q, then use the argument in Lemma 4 to write M as a union of the convex sets $M_1 \equiv \text{cl}(M \cap H_1)$ and $M_2 \equiv \text{cl}(M \cap H_2)$, where H_1 and H_2 are open halfspaces determined by a hyperplane H, with $Q \subseteq H$. Since $w \in$ (rel int conv Q) $\cap$ int M, w is in $M_1 \cap M_2$, and if N is a convex neighborhood of w in M, then $N \cap H_1 \neq \emptyset$, $N \cap H_2 \neq \emptyset$, and $N \cap H \subseteq M_1 \cap M_2$.

If both x and y lie in M_1 (or M_2), the argument is complete. Otherwise, without loss of generality, assume that $x \in M_1$, $y \in M_2$. The convex cone C at x emanating through $N \cap H$ necessarily contains some point z in $N \cap H_2$, and $[x, z] \subseteq M$. We may select a neighborhood N' of z with $N' \subseteq C \cap N \cap H_2$. Then for z' in N', $[x, z] \cup [z, z'] \subseteq M$, there are no lnc points of M in $C \cap H$ and hence no lnc points of M in conv$\{x, z, z'\}$, so again by Valentine's lemma, $[x, z'] \subseteq M$. Thus conv$(N' \cup \{x\}) \subseteq M$. Since $N' \subseteq M_2$ and $y \in M_2$, conv$(N' \cup \{y\}) \subseteq M$. Repeating an argument from Case 1, (conv$\{x, y, z\}) \sim [x, y]$ contains no lnc point of M and $[x, y] \subseteq M$, finishing the proof of Case 2.

Case 3. Finally, consider the case in which conv $Q \cap$ int $M = \emptyset$. By Lemma 2, $M =$ cl(int M), and by an earlier remark, $Q \subseteq$ ker M. Hence we may use Lemma 3 to conclude that Q is convex. By remarks in the proof of Lemma 4, we may apply arguments given in [**1**, Theorem 3] to conclude that M is a union of two convex sets cl$(M \cap H_1)$ and cl$(M \cap H_2)$, where H_1 and H_2 are distinct open halfspaces determined by an appropriate hyperplane H, and $Q \subseteq H$. By [**1**, Lemma 4], int $M \sim$ aff Q is connected, so clearly $(H \cap$ int $M)$ $\sim$ aff $Q \neq \emptyset$. Then by adapting an argument in [**1**, Theorem 3], for w any point in $(H \cap$ int $M) \sim$ aff Q, w is in ker M.

We assert that there is some neighborhood N of w for which conv$(N \cup \{x\})$ $\subseteq M$: If $x \in M \sim H$ or if $x \in (M \cap$ aff $Q) \sim Q$, then $[w, x)$ contains no member of aff Q, $x \notin Q$, and we may employ an argument used in Case 2a to select an appropriate neighborhood N of w. If $x \in (M \cap H) \sim$ aff Q, then by an argument in [**1**, Theorem 3], x is in ker M; thus any neighborhood N of w in M has the required property. A similar result holds if $x \in Q \subseteq$ ker M, and the assertion is proved. A parallel statement holds for y, and an argument from Case 1 may be used to show that $[x, y] \subseteq M$, finishing Case 3 and completing this portion of the proof.

The remaining steps are easy. For points x, y, z in $\bigcap \mathscr{M}$, since every member of $\mathscr{M}$ is 3-convex, at least one of the corresponding segments, say $[x, y]$, lies in S. But then by our previous argument, $[x, y]$ lies in every M in $\mathscr{M}$, $\bigcap \mathscr{M}$ is again 3-convex, and Theorem 1 is proved.

It is interesting to notice that if $M \sim Q$ is not connected or if members of Q are not lnc points of S, then the result in Theorem 1 fails, as later examples will reveal.

3. Intersections of m-convex sets. The following result is an analogue of Theorem 1 for m-convex sets in the plane.

THEOREM 2. *Let S be a closed, simply connected subset of the plane. Let $\mathscr{M}$ be any collection of closed m-convex subsets of S such that for M in $\mathscr{M}$ and Q_M the corresponding set of* lnc *points of M, each member of Q_M is an* lnc *point of S and $M \sim Q_M$ is connected. Then $\bigcap \mathscr{M}$ is again an m-convex set.*

Proof. Let M belong to $\mathscr{M}$ with $Q_M \equiv Q$ the corresponding set of lnc points of M. As in the proof of Theorem 1, we will show that if x and y are points of M with $[x, y] \subseteq S$, then $[x, y] \subseteq M$.

By [**4**, Lemma 2], M is locally starshaped, so there is a neighborhood N of x such that x sees each point of $N \cap M$ via M. Also, by Lemma 2, $M =$ cl(int M), so we may choose a point x_0 in $N \cap$ int M and a corresponding neighborhood N' of x_0, with $N' \subseteq N \cap$ int M. Then conv$(N' \cup \{x\}) \subseteq M$ and $[x_0, x) \subseteq$ int M. Using a parallel argument select y_0 with $[y_0, y) \subseteq$ int M. Clearly $x_0, y_0 \in M \sim Q$. Since $M \sim Q$ is connected and locally convex, it is polygonally connected, and there is a polygonal path in $M \sim Q$ from x_0 to y_0.

Moreover, since $[x_0, x) \cup [y_0, y) \subseteq M \sim Q$, there is a polygonal path λ in M from x to y, with $\lambda \sim \{x, y\} \subseteq M \sim Q$. Let

$$x = t_0, t_1, \ldots, t_k = y$$

denote the consecutive vertices of λ, and assume that λ has been selected so that k is minimal for all such paths in M.

For the moment, assume that λ contains no point of (x, y). Now if $k \geqq 3$, then using the fact that S is simply connected, for some pair of adjacent segments $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$,

$$(\text{int conv}\{t_{i-1}, t_i, t_{i+1}\}) \cup (t_{i-1}, t_{i+1})$$

contains no lnc point of S (and hence no lnc point of M). Furthermore, since x and y are the only points of λ which might lie in Q, $(t_{i-1}, t_i] \cup [t_i, t_{i+1})$ contains no lnc point of M, so by a generalization of Valentine's lemma, $\text{conv}\{t_{i-1}, t_i, t_{i+1}\} \subseteq M$. However, then $[t_{i-1}, t_{i+1}] \subseteq M$, and x and y are the only points of $[t_{i-1}, t_{i+1}]$ which might lie in Q. (Clearly $[t_{i-1}, t_{i+1}] \cap Q \neq \emptyset$ only if $i = 1$ and $x \in Q$ or if $i = k - 1$ and $y \in Q$.) Letting λ' denote the path having vertices $t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k$, $\lambda' \sim \{x, y\} \subseteq M \sim Q$ and λ' has length $k - 1$, contradicting the minimality of k. Hence $k \leqq 2$. Similarly, if $k = 2$, then $[t_0, t_1] \cup [t_1, t_2] \subseteq M$, there is no lnc point of M in $(\text{conv}\{t_0, t_1, t_2\}) \sim [t_0, t_2]$, so again by Valentine's lemma, $\text{conv}\{t_0, t_1, t_2\} \subseteq M$ and $[t_0, t_2] = [x, y] \subseteq M$, the desired result. Of course if $k = 1$, then $\lambda = [x, y] \subseteq M$.

In case λ contains points of (x, y), the argument above may be adapted suitably for subsets of λ having only their endpoints x', y' on $[x, y]$ to show that $[x', y'] \subseteq M$. Then again $[x, y] \subseteq M$, and this portion of the argument is complete.

Finally, for any m points in $\bigcap \mathscr{M}$, at least one of the corresponding segments must lie in S. Then by the argument above, this segment lies in every member of $\mathscr{M}$, and $\bigcap \mathscr{M}$ is an m-convex set, finishing the proof of the theorem.

The following example shows that the results in Theorems 1 and 2 fail without the requirement that $M \sim Q$ be connected for $M \in \mathscr{M}$.

Example 1. Let S denote the simply connected set in Figure 1, A and B the indicated vertical strips, C and D the horizontal ones. Then $A \cup B$, $C \cup D$ are 3-convex subsets of S having no lnc points, yet their intersection is not 3-convex.

Furthermore, the results of Theorems 1 and 2 require that members of Q be lnc points of S, as Example 2 reveals.

Example 2. Let S denote the simply connected set in Figure 2, $P = \text{conv}\{p_i : 1 \leqq i \leqq 4\}$, $R = \text{conv}\{r_i : 1 \leqq i \leqq 4\}$, $M_R = \text{cl}(R \sim \text{conv}\{a, b, c\})$, $M_P = \text{cl}(P \sim \text{conv}\{x, y, z\})$. Then M_R and M_P are 3-convex, but the lnc points b and y are not lnc points of S, and $M_1 \cap M_2$ is not 3-convex.

The final result concerns maximal m-convex subsets of a set.

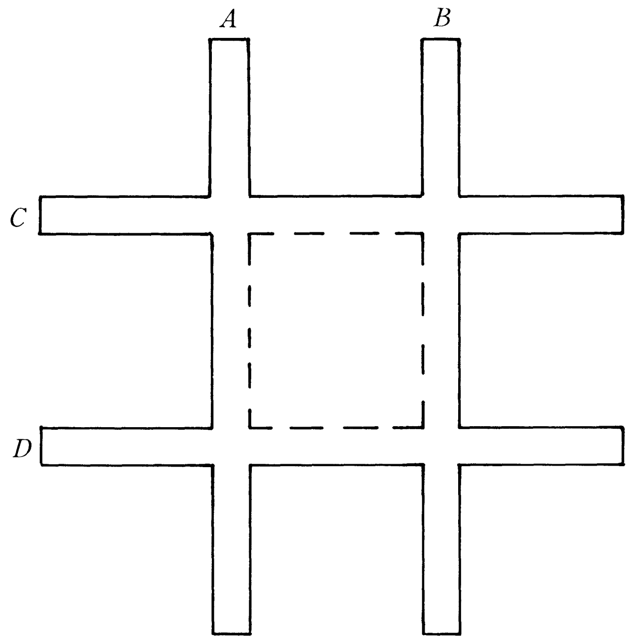

FIGURE 1

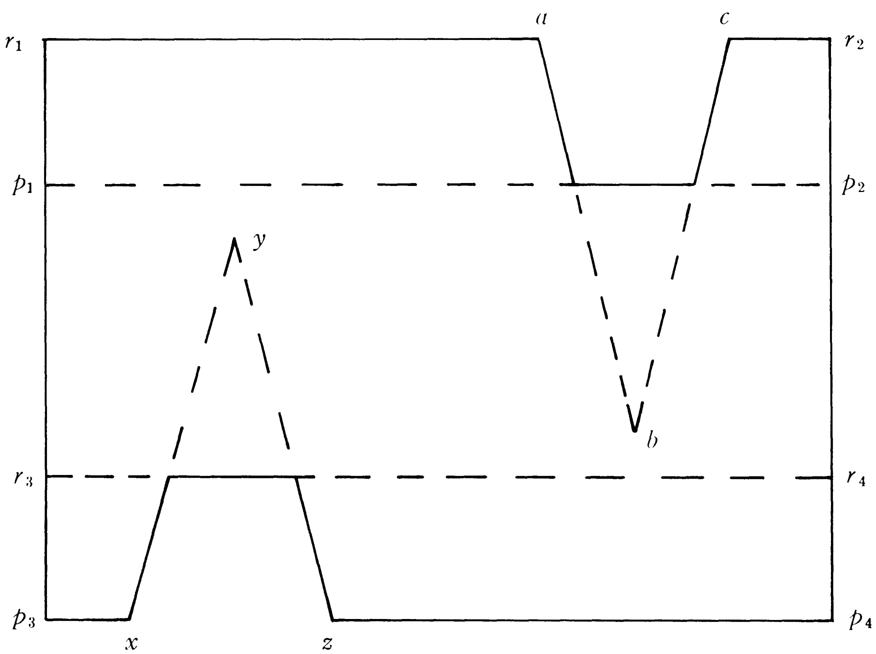

FIGURE 2

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THEOREM 3. *Let S be a closed subset of R^d,* int ker $S \neq \emptyset$, *with Q the set of* lnc *points of S. Let $\mathcal{N}$ denote the collection of all maximal m-convex subsets of S, and let $\mathcal{M}$ denote any subcollection of $\mathcal{N}$ such that for M in $\mathcal{M}$, the* lnc *points of M are in Q. Then $\bigcap \mathcal{M}$ is m-convex.*

Proof. By an obvious use of Zorn's lemma, it is easy to show that every m-convex subset of S lies in a maximal m-convex subset of S, so the collection $\mathcal{N}$ is not empty. Also, since S is closed, each member of $\mathcal{N}$ is closed. Further, it is not hard to prove that if $M \in \mathcal{N}$ and $s \in$ ker S, then $sM \equiv \bigcup \{[s, t] : t \text{ in } M\}$ is m-convex. Hence $M = sM$, $s \in$ ker M, and ker $S \subseteq \bigcap \mathcal{N} \subseteq \bigcap \mathcal{M}$.

If $\mathcal{M} = \emptyset$, there is nothing to prove. Otherwise, let M belong to $\mathcal{M}$, and let $x, y \in \bigcap \mathcal{M}$ with $[x, y] \subseteq S$. Then for any $z \in$ int ker $S \subseteq$ ker M and any neighborhood N of z with $N \subseteq$ ker S, conv$(N \cup [x, y]) \subseteq S$. Hence using techniques employed in the proof of Theorem 1, $[x, y] \subseteq M$, and $\bigcap \mathcal{M}$ is m-convex.

In conclusion, we note that the maximality of members of $\mathcal{M}$ in Theorem 3 may be replaced by the following requirement: For each M in $\mathcal{M}$, ker M contains a point in int ker S.

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