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BOUNDED ENDOMORPHISMS OF LATTICES OF FINITE HEIGHT

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Every monoid M is isomorphic to the monoid $\mathrm{End}_{0,1}(L)$ of all (0,1)-preserving endomorphisms of a bounded lattice L, see [**3**]. The lattices L with $M \cong \mathrm{End}_{0,1}(L)$ that are constructed there are of an arbitrary infinite cardinality not smaller than that of M, and they all have infinite chains. The aim of the present article is to supplement these results. It will be shown that every finite monoid M is representable as $\mathrm{End}_{0,1}(L)$ of a finite lattice. In addition, an account of the difficulties involved in attempting to characterize endomorphism monoids of lattices of a fixed finite height will be given.

1. Definitions and notation. A *bounded lattice* L is, as usual, a lattice with a largest element 1 and a smallest element 0; only bounded lattices will be considered here. An *endomorphism* f of L is a *bounded endomorphism*, or a *(0,1)-preserving endomorphism* if $f(0) = 0$ and $f(1) = 1$; if $\{a, b\}$ is a complemented pair of elements of L, so is the pair $\{f(a), f(b)\}$. The set $C(L)$ of all complemented pairs of a bounded lattice L can be viewed as an undirected graph whose vertex set is the underlying set of L. If L is nontrivial, then $C(L)$ is a graph with no loops, and one of the *components* (i.e., maximal connected subsets) of $C(L)$ is the two-element set $\{0, 1\}$. It is clear that every bounded endomorphism of L induces a unique graph endomorphism of $C(L)$. Let $G = (X, R)$ be a graph without loops; i.e., let X be the vertex set of G and let the set R of all edges of G consist of two-element subsets of X. The *chromatic number* of G is the smallest number $n > 1$ for which there exists a graph homomorphism (compatible mapping) of G into the complete graph K_n without loops on n vertices. A finite set $Y \subseteq X$ is said to be an *independent set of* G if it contains no element of R. Thus $\emptyset$ and every singleton $\{x\}$ are independent sets of a graph G without loops and the system $I(G)$ of all independent sets of G is hereditary.

A chain C_{n+1} of $n + 1$ elements is said to have *length* n. A lattice L has *finite height* n if $C_{n+1} \subseteq L$ and no chain of L is of length exceeding n. Any finite lattice is an example of a bounded lattice of finite height.

A semigroup with an identity element is called a *monoid*; a *transformation monoid* is a pair (X, M) in which X is a set and $M \subseteq X^X$ is a set of transformations of X closed under composition and containing the identity transformation

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1_X of X. If L is a bounded lattice, then $(L, \mathrm{End}_{0,1}(L))$ is a transformation monoid.

2. Endomorphism monoids of lattices of finite height.

Definition. Let E_n be the class consisting of all (pairwise nonisomorphic) monoids M isomorphic to $\mathrm{End}_{0,1}(L)$ of a lattice L of height not larger than $n \geqq 1$. F_n will denote the set of all $M \in E_n$ representable by bounded endomorphisms of a finite lattice of height $\leqq n$.

It is clear that $E_1 = F_1$ (each contains only the trivial monoid), and that $E_n \subseteq E_{n+1}$ and $F_n \subseteq F_{n+1}$ for all $n \geqq 1$. The lattices of length two are exactly all lattices M_α with $\alpha > 0$ atoms. Since M_α is simple for every cardinal $\alpha > 2$, E_2 consists of the trivial monoid, $\mathrm{End}_{0,1}(C_3)$, $\mathrm{End}_{0,1}(C_2 \times C_2)$, and all monoids of all one-to-one mappings into itself of a set X with more than two elements. Thus, for instance, the cyclic group of order two is not in E_2, and F_2 contains every finite symmetric group with more than two elements. Every $E_n (n \geqq 2)$ has infinite members; hence F_n is a proper subset of E_n for all $n \geqq 2$.

The following lemma is needed to show that $E_n \neq E_{n+1}$.

LEMMA 1. *For every $n > 1$ there is a finite monoid A_n such that $A_n \subseteq M \in E_k$ only if $k > n$.*

Proof. Let A_n be generated by $\{a\}$ and let $a^n = a^{n+1}$, that is, $A_n = \{1, a, a^2, \ldots, a^n\}$. Let A_n be isomorphic to a submonoid of $\mathrm{End}_{0,1}(L)$ and let $f \in \mathrm{End}_{0,1}(L)$ represent a; as $a^{n-1} \neq a^n$, there is an $x \in L$ such that $f^{n-1}(x) \neq f^n(x)$. Denote $x = f^0(x)$ and for every $k \in \{0, 1, \ldots, n\}$ set

$$b_k = f^k(x) \vee f^{k+1}(x) \vee \ldots \vee f^n(x),$$
$$c_k = f^k(x) \wedge f^{k+1}(x) \wedge \ldots \wedge f^n(x).$$

It is easy to see that $f(b_k) = b_{k+1}, f(c_k) = c_{k+1}$ for every $k \in \{0, \ldots, n-1\}$, and that

$$(1) \qquad c_0 \leqq c_1 \leqq \ldots \leqq c_n = f^n(x) = b_n \leqq b_{n-1} \leqq \ldots \leqq b_0.$$

We will show that the set of elements listed in (1) contains a chain of length n; this is certainly true if all elements b_j, c_k are distinct. If, on the other hand, $b_j = b_{j-1}$ for some $j \in \{1, \ldots, n\}$, then $f^{n-1}(x) \vee f^n(x) = b_{n-1} = f^{n-j}(b_{j-1}) = f^{n-j}(b_j) = f^n(x)$ and hence $f^{n-1}(x) \leqq f^n(x)$. In a similar way, $c_{k-1} = c_k$ for some $k \in \{1, \ldots, n\}$ implies $f^{n-1}(x) \geqq f^n(x)$. Thus either $\{b_n, \ldots, b_0\}$ or $\{c_0, \ldots, c_n\}$ is a chain of length n contained in L. If the height of L is n, then either $f^n(0) = 1$ or $f^n(1) = 0$; therefore the height of L is strictly larger than n.

PROPOSITION 2. *$E_n \neq E_{n+1}$ and $F_n \neq F_{n+1}$ for all $n \geqq 1$.*

Proof. The claim is clearly true for $n = 1$. If $n > 1$, let $X = \{x_0, x_1, \ldots, x_n\}$ be an $(n+1)$-element set. It is well known that the distributive lattice 2^X has height $n + 1$ and that $\mathrm{End}_{0,1}(2^X) = B_{n+1}$ is dually isomorphic to the monoid

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X^X of all transformations of X into itself. If $f \in X^X$ is the mapping defined by $f(x_i) = x_{i+1}$ for every $i \in \{0, \ldots, n-1\}$, $f(x_n) = x_n$, then f generates a submonoid of X^X isomorphic to the monoid A_n from Lemma 1. A_n is a commutative monoid and therefore $A_n \subseteq B_{n+1}$ as well. Applying Lemma 1 now shows that $B_{n+1} \in F_{n+1} \backslash E_n$, which proves both claims.

Little else is known about the classes E_n and F_n. It will be seen, however, that E_3 is considerably larger than E_2, and that lattices of height three are no longer in a one-to-one correspondence with their monoids of bounded endomorphisms.

Definition. A one-to-one graph homomorphism $f : (X_1, R_1) \to (X_2, R_2)$ is called *algebraic* if $\{x, y\} \in R_1$ is equivalent to $\{f(x), f(y)\} \in R_2$. Let $\mathbf{A}$ denote the category whose objects are all undirected graphs (X, R) without loops such that for every $x \in X$ there are distinct y_1, y_2, y_3 with $\{x, y_1\} \in R$, $\{x, y_2\} \in R$, $\{x, y_3\} \notin R$; the morphisms of $\mathbf{A}$ will be all one-to-one algebraic homomorphisms.

Note that every isomorphism is algebraic and that the (algebraic) one-to-one endomorphisms of a finite graph are exactly its automorphisms.

LEMMA 3. *There is a one-to-one functor H from $\mathbf{A}$ to the category of all lattices of height three and all their bounded homomorphisms such that every $g : H(X_1, R_1) \to H(X_2, R_2)$ is of the form $g = H(f)$ for some $f : (X_1, R_1) \to (X_2, R_2)$ in $\mathbf{A}$; in other words, H is a full embedding.*

Proof. Let $I_2(X, R)$ denote the set of all those independent sets of an $(X, R) \in \mathbf{A}$ which have less than three elements. Set $H(X, R) = I_2(X, R) \cup \{1\}$, where $1 \notin I_2(X, R)$. Let $H(X, R)$ be ordered by the inclusion order of elements of $I_2(X, R)$ together with the requirement that 1 be the largest element of $H(X, R)$. Thus $H(X, R)$ is a lattice of height three, with $\emptyset$ as its least element. For every morphism $f : (X_1, R_1) \to (X_2, R_2)$ of $\mathbf{A}$ define $H(f)$ by $H(f)(1) = 1$, $H(f)(\emptyset) = \emptyset$, $H(f)(\{x\}) = \{f(x)\}$ for all $x \in X$, and $H(f)(\{x, y\}) = \{f(x), f(y)\}$ for every two-element independent subset of (X_1, R_1). As f is a one-to-one algebraic homomorphism, $H(f)$ is well-defined; it is easy to see that $H(f)$ is a bounded lattice homomorphism and that H is, indeed, a one-to-one functor.

To show that H is full, choose an arbitrary bounded lattice homomorphism $g : H(X_1, R_1) \to H(X_2, R_2)$ and assume that $g(a) = g(b)$ for some elements $a < b$ of $H(X_1, R_1)$. As every nonzero element of $H(X_1, R_1)$ is a join of its atoms, there is an $x_0 \in X_1$ such that $\{x_0\} \leqq b$ and $a \wedge \{x_0\} = \emptyset$. If $y_1 \neq y_2$ are such that $\{x_0, y_1\}, \{x_0, y_2\} \in R$, then $\emptyset = g(\emptyset) = g(\{x_0\}) \wedge g(a) = g(\{x_0\} \wedge b) = g(\{x_0\})$, and $1 = g(1) = g(\{x_0\} \vee \{y_i\}) = g(\{y_i\})$ for $i = 1, 2$. However, $\{y_1\} \wedge \{y_2\} = \emptyset$ and, consequently, $\emptyset = g(\{y_1\} \wedge \{y_2\}) = g(\{y_1\}) \wedge g(\{y_2\}) = 1$, which is a contradiction. Thus g is a one-to-one homomorphism. Since every atom $\{x\}$ of $H(X_1, R_1)$ is covered by some $\{x, y_3\}$, $g(\{x\})$ must be an atom again; denote $g(\{x\}) = \{f(x)\}$. Accordingly, f is a one-to-one mapping of X_1

into X_2 and if $\{x, y\} \in R_1$, then $\{x\} \vee \{y\} = 1$ in $H(X_1, R_1)$. Hence $\{f(x)\} \vee \{f(y)\} = g(\{x\}) \vee g(\{y\}) = g(1) = 1$, which means that $\{f(x), f(y)\} \in R_2$. If, on the other hand, $\{x, y\} \notin R_1$, then $\{x, y\} < 1$ in $H(X_1, R_1)$ and $\{f(x)\} \vee \{f(y)\} = g(\{x\}) \vee g(\{y\}) = g(\{x, y\}) < g(1) = 1$. Hence $g(\{x, y\}) \notin R_2$ and $g(\{x, y\}) = \{f(x), f(y)\}$. Altogether, f is an algebraic one-to-one graph homomorphism and $H(f) = g$.

A monoid M is *left cancellative* if $xy = xz$ implies $y = z$ for all $x, y, z \in M$, or, equivalently, if all left translations of M are one-to-one mappings of M into itself. Since the left translations of M form the endomorphism monoid of the multi-unary algebra on M whose operations are all right translations of M, Lemma 7 and Lemma 8 of [**5**] imply that M is isomorphic to the monoid of all one-to-one endomorphisms of an algebra B with two unary operations; B is finite for a finite (group) M, and the cardinalities of M and B are equal if M is an infinite left-cancellative monoid. The main result of [**4**] implies that for every infinite cardinal $c \geqq \operatorname{card}(M)$ there are exactly 2^c pairwise nonisomorphic algebras B_i with two unary operations and $\operatorname{card}(B_i) = c$ that represent M in this way. If M is a finite group, then an application of [**4**] will give $\aleph_0$ pairwise nonisomorphic finite such algebras B_n. The full embedding $\Psi \circ \Phi$ defined in [**5**] translates one-to-one homomorphisms of algebras with two unary operations into one-to-one algebraic homomorphisms of graphs from **A**. Since the composite full embedding $H \circ \Psi \circ \Phi$ does not increase infinite cardinalities and since it assigns finite lattices to finite algebras, these remarks can be combined to obtain the following statement.

PROPOSITION 4. *E_3 contains all left-cancellative monoids and F_3 contains all finite groups. Moreover, for any left-cancellative monoid M and any infinite cardinal $c \geqq \operatorname{card}(M)$ there are exactly 2^c pairwise nonisomorphic lattices L_i of height three and $\operatorname{card}(L_i) = c$ such that $M \cong \operatorname{End}_{0,1}(L_i)$ for every $i \in 2^c$. If M is a finite group, then there are infinitely many pairwise nonisomorphic finite lattices L_n of height three with $M \cong \operatorname{End}_{0,1}(L_n)$ for every $n \in \omega$.*

Hence, in particular, E_3 includes all groups and every group appears as the endomorphism monoid of a proper class of pairwise nonisomorphic lattices of height three.

As no E_n contains all monoids, it is, perhaps, natural to ask whether bounded lattices whose chains are all finite (with no common bound for their lengths) can be utilized as representing lattices for the class of all monoids. The last theorem of this section gives a negative answer to this question.

Notation. Define a binary operation $*$ on the interval $[0, 1]$ of real numbers by $a * b = \min(a + b, 1)$. The operation $*$ is associative and commutative; the set D of all rational numbers $d = m \cdot 2^{-n} \in [0, 1]$ contains 0 and is closed under $*$. Therefore, $M = (D, *)$ is a countable commutative monoid.

PROPOSITION 5. *If $M \cong M' \subseteq \operatorname{End}_{0,1}(L)$, then L has an infinite chain.*

Proof. For every $n \geqq 0$, let f_n denote the endomorphism of L corresponding to $2^{-n} \in D$. Note that $f_n{}^2 = f_{n-1}$ for all $n \geqq 1$, so that $0 \leqq m < n$ implies that $f_m = f_n{}^k$ for some $k \geqq 2$. Further, observe that if $y \in L$ is such that $f_{n+1}{}^k(y) \leqq f_0(y)$, then $f_{n+1}{}^j(y) = f_{n+1}{}^{j-k}f_{n+1}{}^k(y) \leqq f_{n+1}{}^{j-k}f_0(y) = f_0(y)$ for every $j \geqq k$.

There must be an $x \in L$ for which $f_0(x) \neq f_1(x)$. For every $n \geqq 1$ set

$$b_n = f_1(x) \vee f_2(x) \vee \ldots \vee f_n(x),$$
$$c_n = f_1(x) \wedge f_2(x) \wedge \ldots \wedge f_n(x)$$

and observe that

$$(2) \qquad \ldots \leqq c_{n+1} \leqq c_n \leqq \ldots \leqq c_1 = f_1(x) = b_1 \leqq b_2 \leqq \ldots \leqq b_n \leqq \ldots$$

We will show that the set in (2) contains an infinite chain. If all elements listed under (2) are distinct, there is nothing to prove. If $b_{n+1} = b_n$ for some $n \geqq 1$, then

$$(3) \qquad f_{n+1}(x) \leqq f_1(x) \vee \ldots \vee f_n(x).$$

By a previous observation, there is a k_0 such that $f_{n+1}{}^{k_0+1}(x) \leqq f_0(x)$, namely $k_0 = 2^{n+1} - 1$. Assuming that $k \geqq 2$ and $f_{n+1}{}^{k+1}(x) \leqq f_0(x)$, we will first show that $f_{n+1}{}^k(x) \leqq f_0(x)$. To this end, apply the mapping $f_{n+1}{}^{k-1}$ to both sides of (3) to obtain

$$(4) \qquad f_{n+1}{}^k(x) \leqq f_{n+1}{}^{k-1}f_1(x) \vee f_{n+1}{}^{k-1}f_2(x) \vee \ldots \vee f_{n+1}{}^{k-1}f_n(x).$$

If $m \leqq n$, then $f_{n+1}{}^{k-1}f_m = f_{n+1}{}^j$ for some $j \geqq k + 1$ and, consequently, using a previous observation and the inductive hypothesis, $f_{n+1}{}^{k-1}f_m(x) \leqq f_0(x)$ for all $m \in \{1, \ldots, n\}$. This, together with (4), gives $f_{n+1}{}^k(x) \leqq f_0(x)$. A simple induction based on this argument leads to the inequality $f_n(x) = f_{n+1}{}^2(x) \leqq f_0(x)$. As $n \geqq 1$, $f_1 = f_n{}^j$ for some $j \geqq 1$; thus $f_1(x) = f_n{}^{j-1}f_n(x) \leqq f_n{}^{j-1}f_0(x) = f_0(x)$. Now $\{c_n : n \geqq 1\}$ is the desired infinite chain, for if $c_m = c_{m+1}$ for some $m \geqq 1$, then an argument dual to that given above yields $f_1(x) \geqq f_0(x)$, contradicting the choice of x.

3. Endomorphisms of finite lattices. This section is concerned with the construction of a finite lattice L with $\mathrm{End}_{0,1}(L)$ isomorphic to a given finite monoid M.

The starting point is a result, used already in [**3**], stating that every finite monoid M is isomorphic to the full endomorphism monoid of a finite connected graph G without loops and of a chromatic number at least three. The full embedding $\Psi \circ \Phi$ of [**5**] can again be used (in conjunction with arguments analogous to those of the preceding section) to show that every finite monoid possesses infinitely many pairwise nonisomorphic finite graphs G_n of this type with $\mathrm{End}(G_n) \cong M$ for each G_n.

To explain the method used here, we return to the basic construction of [**3**]: given a graph $G = (X, R)$ satisfying the requirements listed above, define a congruence θ_G on the bounded lattice $F(X)$ freely generated by X as the smallest congruence that identifies $x \vee y$ with 1 and $x \wedge y$ with 0 for any pair $\{x, y\} \subseteq X$ which belongs to R; let $M(G) = F(X)/\theta_G$, and let $\pi : F(X) \to M(G)$ be the canonical homomorphism whose kernel is θ_G. If $F(f)$ is the free extension of a graph endomorphism f of G, then θ_G is contained in the kernel of $\pi \circ F(f)$; hence there is a unique endomorphism $M(f)$ of $M(G)$ defined by $\pi \circ F(f) = M(f) \circ \pi$. It is easy to see that $f \mapsto M(f)$ is a homomorphism of End(G) into $\mathrm{End}_{0,1}(M(G))$. It is proved in [**3**] that the restriction of π to the set X of generators of $F(X)$ is a one-to-one mapping, so that M is a one-to-one monoid homomorphism. If $\pi(x)$ is denoted as x for every $x \in X$, then $C(M(G)) \supseteq R \cup \{\{0, 1\}\}$. Once it is known [**3**] that $C(M(G)) = R \cup \{\{0, 1\}\}$, then every bounded endomorphism g of $M(G)$ must map the set $X \cup \{0, 1\}$ into itself. There are just two components of $C(M(G))$: the component (X, R) whose chromatic number is larger than two, and the 2-colourable component $\{0, 1\}$. Therefore $g(X) \subseteq X$ and the restriction f of g to X is an endomorphism of G. As X generates $M(G)$, $g = M(f)$; hence M is an isomorphism of End(G) onto $\mathrm{End}_{0,1}(M(G))$.

This construction can obviously be performed in any nontrivial variety $\mathbf{V}$ of lattices once $F(X)$ is replaced by the bounded lattice $V(X)$ generated freely by X in $\mathbf{V}$. The relation θ_G is now interpreted as the smallest congruence on $V(X)$ which turns edges of G into complemented pairs. If, moreover, $\mathbf{V}$ is a locally finite variety, then $V(G) = V(X)/\theta_G$ will be finite for every finite graph G. If θ_G separates elements of X and if $C(V(G)) = R \cup \{\{0, 1\}\}$, then an argument identical to that concerning $M(G)$ will show that $\mathrm{End}(G) \cong \mathrm{End}_{0,1}(V(G))$. The task on hand is, therefore, to find a locally finite variety $\mathbf{V}$ of lattices in which $V(G)$ will satisfy these requirements for a given finite graph G.

Definition and notation. Let $I(G)$ denote the set of independent sets of a nontrivial connected finite graph $G = (X, R)$ without loops. Set $I^*(G) = I(G) \cup \{1\}$, where $1 \notin I(G)$. If the elements of $I(G)$ are ordered by inclusion and if 1 is the largest element of $I^*(G)$, then $I^*(G)$ is a lattice in which the join of independent sets a, b is their union $a \cup b$ if the latter set is independent, and $a \vee b = 1$ if $a \cup b$ is dependent. Let $I_*(G)$ denote the dual of $I^*(G)$ and let $A(G)$ be the sublattice of $I^*(G) \times I_*(G)$ generated by the set

$$S = \{\langle \emptyset, \emptyset \rangle\} \cup \{\langle \{x\}, \{x\} \rangle : x \in X\}.$$

Figure 1 gives an example of the lattice $A(G)$ for the four-element graph G shown there. If $\delta^* : F(X) \to I^*(G)$ and $\delta_* : F(X) \to I_*(G)$ are homomorphisms for which $\delta^*(x) = \delta_*(x) = \{x\}$ for all $x \in X \subset F(X)$, then $\delta = \delta^* \times \delta_*$ maps $F(X)$ into $A(G)$; let Δ be the kernel of δ.

If $\theta \supseteq \theta_G$ is a congruence on $F(X)$ and $\tau : F(X) \to F(X)/\theta = T(G)$ is the canonical homomorphism, then $T(G)$ is a bounded lattice generated by $\tau(X)$ such that $0 \leqq \tau(x) \leqq 1$ for all $x \in X$; note also that $\bigvee(\tau(x) : x \in Y) = 1$, $\bigwedge(\tau(x) : x \in Y) = 0$ if Y is a dependent set of G.

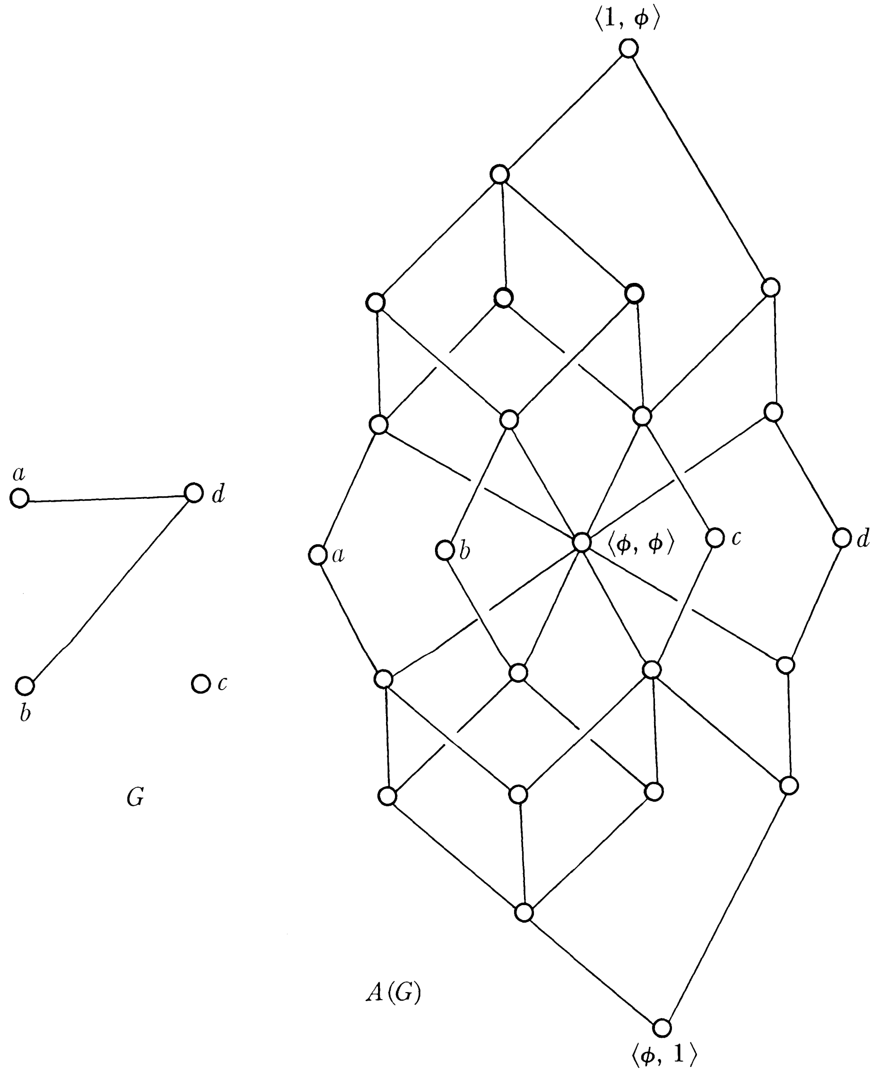

FIG. 1. A generator $\langle \{x\}, \{x\} \rangle$ of $A(G)$ is denoted by x.

LEMMA 6. *For every $x \in X$ and every $P \in F(X)$*

(i) $\{x\} \leqq \delta^*(P)$ *implies* $\tau(x) \leqq \tau(P)$,

(ii) $\{x\} \geqq \delta_*(P)$ *implies* $\tau(x) \geqq \tau(P)$.

Proof. We will prove (i) by an induction on the rank of P—see, for instance, [**2**].

If rank$(P) = 1$, then $P = y \in X$ and $\{x\} \leqq \delta^*(P) = \{y\}$ implies $x = y$ as the elements of $I^*(G)$ are ordered by inclusion. Thus, in this case, $\tau(x) = \tau(y)$.

Assume (i) to be true for all elements of $F(X)$ of rank smaller than n and let $\operatorname{rank}(P) = n > 1$.

If $P = Q \vee R$ in $F(X)$, then Q and R have rank smaller than n. Let $\{x\} \leqq \delta^*(P)$. If $\delta^*(Q) = 1$, then $\{x\} \leqq \delta^*(Q)$ and, by the induction hypothesis, $\tau(x) \leqq \tau(Q) \leqq \tau(P)$. We may therefore assume that $\delta^*(Q) < 1$ and $\delta^*(R) < 1$. If $\delta^*(P) = 1$, then $\delta^*(Q)$ and $\delta^*(R)$ are independent sets of G whose union is dependent; the induction hypothesis implies that $\tau(y) \leqq \tau(Q)$ for all $y \in \delta^*(Q)$, and that $\tau(y) \leqq \tau(R)$ for all $y \in \delta^*(R)$. Hence $\tau(y) \leqq \tau(Q) \vee \tau(R) = \tau(P)$ for all $y \in \delta^*(Q) \cup \delta^*(R)$ and, consequently,

$$\tau(P) \geqq \bigvee(\tau(y) : y \in \delta^*(Q) \cup \delta^*(R)) = 1,$$

since $\delta^*(Q) \cup \delta^*(R)$ is dependent. Thus $\tau(x) \leqq \tau(P)$ for all $x \in X$. If $\delta^*(P) < 1$, then $\delta^*(P) = \delta^*(Q) \cup \delta^*(R)$ is independent. Hence $\{x\} \leqq \delta^*(P)$ and the induction hypothesis imply that $\tau(x) \leqq \tau(Q)$ or $\tau(x) \leqq \tau(R)$; in either case, $\tau(x) \leqq \tau(Q) \vee \tau(R) = \tau(P)$ as required.

If $P = Q \wedge R$ and $\{x\} \leqq \delta^*(P)$, then $\{x\} \leqq \delta^*(Q)$ and $\{x\} \leqq \delta^*(R)$; the induction hypothesis gives $\tau(x) \leqq \tau(Q)$ and $\tau(x) \leqq \tau(R)$, so that $\tau(x) \leqq \tau(Q) \wedge \tau(R) = \tau(P)$ as was to be shown.

This finishes the proof, for (ii) is a statement which is dual to (i).

LEMMA 7. *If $\theta \subseteq \Delta$, then $\tau(X)$ is an antichain of $T(G)$ bijective to X; if $\tau(x)$ is denoted as x for every $x \in X$, then $C(T(G)) = R \cup \{\{0, 1\}\}$.*

Proof. $\operatorname{Ker} \tau \subseteq \Delta$ implies the existence of a unique homomorphism $\epsilon : T(G) \to A(G)$ such that $\delta = \epsilon \circ \tau$; as τ is onto, $\epsilon = \epsilon_1 \times \epsilon_2$, where $\epsilon_1 \circ \tau = \delta^*$ and $\epsilon_2 \circ \tau = \delta_*$.

If $\tau(x) \leqq \tau(y)$, then $\{x\} = \delta^*(x) = \epsilon_1(\tau(x)) \leqq \epsilon_1(\tau(y)) = \delta^*(y) = \{y\}$, that is, $x = y$. Thus $\tau(X)$ is an antichain bijective to X; set $x = \tau(x)$.

If $\{x, y\} \in R$, then $\epsilon_1(1) = \epsilon_1(x \vee y) = \epsilon_1(\tau(x \vee y)) = \delta^*(x \vee y) = 1$ and, similarly, $\epsilon_2(0) = 1$ in $I_*(G)$.

Let $a, b \in T(G)$ be such that $a \vee b = 1$ and $a \wedge b = 0$, let $a = \tau(A)$, $b = \tau(B)$, where $A, B \in F(X)$. Then $1 = \epsilon_1(a \vee b) = \delta^*(A) \vee \delta^*(B)$ in $I^*(G)$ and if $\delta^*(A) = 1$, then $\{x\} \leqq \delta^*(A)$ and $\{y\} \leqq \delta^*(A)$ for every $\{x, y\} \in R$. Lemma 6 implies that $\tau(x) \vee \tau(y) \leqq \tau(A) = a$; since $\{x, y\}$ is dependent, $\tau(x) \vee \tau(y) = 1$ in $T(G)$. Hence $a = 1$ and $b = 0$. Now if $\delta^*(A), \delta^*(B)$ are independent sets of G whose union is dependent, then there is an $\{x_1, y_1\} \in R$ such that $\{x_1\} \leqq \delta^*(A)$ and $\{y_1\} \leqq \delta^*(B)$. Lemma 6 now implies that $x_1 = \tau(x_1) \leqq \tau(A) = a$ and $y_1 = \tau(y_1) \leqq \tau(B) = b$.

Furthermore, $\delta_*(A) \wedge \delta_*(B) = \epsilon_2(a \wedge b) = 1$ in $I_*(G)$. If $\delta_*(A) = 1$, then $\{x\} \geqq \delta_*(A)$ and $\{y\} \geqq \delta_*(A)$ hold in $I_*(G)$ for every $\{x, y\} \in R$. An application of Lemma 6 yields $0 = \tau(x) \wedge \tau(y) \geqq \tau(A) = a$ and this contradicts the inequality $a \geqq x_1$ obtained earlier. Therefore $\delta_*(A), \delta_*(B)$ are independent sets; as $\delta_*(A) \cup \delta_*(B)$ is dependent, there must be $x_2 \in \delta_*(A)$, $y_2 \in \delta_*(B)$ such that $\{x_2, y_2\} \in R$. Now Lemma 6 implies $x_2 = \tau(x_2) \geqq \tau(A) = a$ and $y_2 = \tau(y_2) \geqq$

$\tau(B) = b$. Altogether, since X forms an antichain in $T(G)$, $x_1 \leqq a \leqq x_2$ and $y_1 \leqq b \leqq y_2$ imply $a = x_1 = x_2$ and $b = y_1 = y_2$. This finishes the proof.

Remark. It should be pointed out that since θ_G itself is contained in Δ, Lemma 7 applied to $T(G) = M(G)$ implies that $M(G)$ has only the complements induced by the graph G. If the definition of $A(G)$ is suitably extended in order to accommodate infinite connected graphs, the basic result of [**3**] would become a special case of Lemma 7.

THEOREM 8. *For every finite monoid M there are countably many pairwise nonisomorphic finite lattices L_n with* $\mathrm{End}_{0,1}(L_n) \cong M$.

Proof. Let $(G_n : n \in \omega)$ be a set of pairwise nonisomorphic finite connected graphs of chromatic number at least three, let $\mathrm{End}_{0,1}(G_n) \cong M$ for every $n \in \omega$. Let $\mathbf{V}_n$ be the variety generated by $A(G_n)$; as $A(G_n)$ is finite, $\mathbf{V}_n$ is a locally finite variety. The fully invariant congruence Θ_n, such that $F(X_n)/\Theta_n = F_n(X_n)$ is the lattice freely generated by X_n in $\mathbf{V}_n$, is clearly contained in Δ; the same is true for θ_G. Thus the finite lattice $V_n(G_n) \cong F(X_n)/\theta_{Gn} \vee \Theta_n$ satisfies the hypothesis of Lemma 7. To show that $\mathrm{End}_{0,1}(V_n(G_n)) \cong \mathrm{End}(G_n) \cong M$, it is enough to use the conclusion of Lemma 7 in an argument duplicating that which concerns $M(G)$. Finally, if $V_n(G_n) \cong V_m(G_m)$, then the components of $C(V_n(G_n))$, $C(V_m(G_m))$ of chromatic number >2 must be isomorphic, that is, $G_n \cong G_m$ or, equivalently, $n = m$.

4. Concluding remarks. Interestingly enough, lattices $A(G)$ themselves cannot be used to represent $\mathrm{End}(G)$: all but finitely many lattices $I^*(G)$ are simple and, consequently, $\mathrm{End}_{0,1}(A(G)) \cong \mathrm{Aut}(G)$ for almost every finite connected graph G. For a similar reason, no essential advantage is to be gained by an increase in height, obtained by including larger independent sets, of lattices used to prove Lemma 3.

A naturally arising question of whether all lattices L_n in Theorem 8 may be chosen in a single locally finite variety, thereby turning the present construction into a functorial one, is answered positively in [**1**], where lattices $A(G)$ and somewhat more general *cover set lattices* are investigated and employed to generalize some well-known theorems on $\mathscr{C}$-reduced free products.

Theorem 8 together with Lemma 1 enable us to define meaningfully a *lattice rank* of a finite monoid M as the smallest natural n for which $M \in F_n$ (e.g., the dual of the monoid X^X with $\mathrm{card}(X) = n$ is of lattice rank n). Thus it appears to be natural enough to ask for a characterization of finite monoids of rank n; a problem little understood by the authors.

A different approach to that given here is to consider nonconstant endomorphisms of lattices. However, for a given *finite* lattice L, it is not too difficult to construct a finite lattice, extending L, whose nonconstant endomorphisms are closed under composition and form a monoid isomorphic to $\mathrm{End}_{0,1}(L)$.

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