APPLICATION OF A WEIERSTRASS THEOREM
TO THE CONVERGENCE OF MOMENTS

L.K. Chan and E.R. Mead

In this note we apply a well-known theorem due to Weierstrass to show that under certain conditions convergence in distribution of a sequence of distribution functions implies the convergence of moments.

This note may be understood by an undergraduate student who has an introductory course of complex variables and a second course of statistics.

WEIERSTRASS THEOREM. (cf. [1, p. 174]). *Suppose for every positive integer* n, *the complex valued function* $\phi_n(z)$ *of a complex variable* z *is analytic in a region* Ω *and the sequence* $\{\phi_n(z)\}$ *converges to a limit function* $\phi(z)$ *in* Ω, *uniformly on every compact subset of* Ω. *Then* $\phi(z)$ *is analytic in* Ω. *Moreover, for every positive integer* k,

$$(1) \qquad \lim_{n\to\infty} \phi_n^{(k)}(z) = \phi^{(k)}(z)$$

uniformly on every compact subset of Ω, *where* $\phi_n^{(k)}(z)$ *and* $\phi^{(k)}(z)$ *denote, respectively, the kth derivatives of* $\phi_n(z)$ *and* $\phi(z)$.

Let $\phi_n(t) = \int_{-\infty}^{\infty} e^{ixt} dF_n(x)$, $\phi(t) = \int_{-\infty}^{\infty} e^{ixt} dF(x)$ be the characteristic functions of the distribution functions $F_n(x)$ and $F(x)$, respectively. Also let $\phi_n(z)$ and $\phi(z)$ be obtained by replacing the real variable t in $\phi_n(t)$ and $\phi(t)$ by the complex variable z, respectively. Based on the above theorem we prove the following proposition.

PROPOSITION. *Let* $\{F_n(x)\}$, $F(x)$ *and* $\{\phi_n(z)\}$, $\phi(z)$ *be as defined in the preceding paragraph. If the functions in* $\{\phi_n(z)\}$ *are analytic in a region* Ω *containing the origin and this sequence converges to* $\phi(z)$ *uniformly on every compact subset of* Ω, *then* $\phi(z)$ *is analytic and for every integer* $\alpha' > 0$ *and real* $\alpha > 0$

$$(2) \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} x^{\alpha'} dF_n(x) = \int_{-\infty}^{\infty} x^{\alpha'} dF(x),$$

https://doi.org/10.4153/CMB-1969-010-2 Published online by Cambridge University Press

$$(3) \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} |x|^{\alpha} dF_n(x) = \int_{-\infty}^{\infty} |x|^{\alpha} dF(x) .$$

<u>Proof of the Proposition.</u> That $\phi(z)$ is analytic follows immediately from the Weierstrass theorem. It is well known that if α' is a positive integer,

$$\int_{-\infty}^{\infty} x^{\alpha'} dF_n(x) = \frac{1}{i^{\alpha'}} \phi_n^{(\alpha')}(0), \quad i = \sqrt{-1} ;$$

it follows from (1) that (2) holds and when α' is an even integer (3) holds.

Now let us proceed with the proof of (3). Let $A > 0$ be a number such that F is continuous at A and $-A$. Then we write

$$(4) \quad \left| \int_{-\infty}^{\infty} |x|^{\alpha} dF_n - \int_{-\infty}^{\infty} |x|^{\alpha} dF \right| \leq \left| \int_{|x| \leq A} |x|^{\alpha} dF_n - \int_{|x| \leq A} |x|^{\alpha} dF \right|$$

$$+ \int_{|x| > A} |x|^{\alpha} dF_n + \int_{|x| > A} |x|^{\alpha} dF .$$

Let α' be the next even integer $\geq \alpha$. Then given any $\epsilon > 0$,

$$\int_{|x| > A} |x|^{\alpha'} dF_n \leq \left| \int_{-\infty}^{\infty} |x|^{\alpha'} dF_n - \int_{-\infty}^{\infty} |x|^{\alpha'} dF \right| + \int_{|x| > A} |x|^{\alpha'} dF$$

$$+ \left| \int_{|x| \leq A} |x|^{\alpha'} dF - \int_{|x| \leq A} |x|^{\alpha'} dF_n \right| < 3\epsilon$$

for sufficiently large n and A. In fact, for sufficiently large n, the first right-hand term is $< \epsilon$ since α' is an even integer, the third right-hand term is $< \epsilon$ by the Helley-Bray Lemma (cf. [3, p. 180]), while the second right-hand term is $< \epsilon$ since the α' moment of $F(x)$ exists and A is sufficiently large. We can let $A > 1$; then $|x|^{\alpha} < |x|^{\alpha'}$ if $|x| > A$. So in (4) the sum of the last two right hand terms is $<(3\epsilon + \epsilon) = 4\epsilon$ when n and A are sufficiently large. By the Helley-Bray Lemma the first right-hand term is $<\epsilon$ when n is sufficiently large. Therefore the left-hand term of (4) is $<5\epsilon$ when n is sufficiently large and hence (3) follows.

https://doi.org/10.4153/CMB-1969-010-2 Published online by Cambridge University Press

Although the conditions on $\{\phi_n(z)\}$ in the Proposition imply that $\{\phi_n(t)\}$ converges to $\phi(t)$ on an interval $|t| <$ a constant, $\{F_n(x)\}$ may not converge in distribution to $F(x)$ except when either the sets $\{x: 0 < F_n(x) \leq 1\}$ or the sets $\{x: 0 \leq F_n(x) < 1\}$ are uniformly bounded from below or from above, respectively (c.f. [4, p. 141]).

APPLICATION OF THE PROPOSITION (Convergence of moments of the Central Limit Theorem). Let $\{Y_k\}$ be a sequence of independent, identically distributed random variables with mean zero and variance 1. Let $\phi_Y(t)$ be the characteristic function of Y_k. If there exists an open disk $|z| < \rho' < \infty$ such that $\phi_Y(z)$ is analytic in the disk, then (2) and (3) hold, where F_n and F are, respectively, the distribution functions of $(\Sigma_{k=1}^{n} Y_k)/\sqrt{n}$ and the normal random variable with mean zero and variance 1.

Proof of the Application. Let $\phi_n(t)$ be the characteristic function of $(\Sigma_{k=1}^{n} Y_k)/\sqrt{n}$. By the Proposition it is sufficient to prove that

$$\lim_{n\to\infty} \phi_n(z) = \exp(-z^2/2) \text{ uniformly on every closed disk } |z| \leq \rho < \rho' . \tag{5}$$

Since $\phi_Y(z)$ is analytic in $|z| < \rho'$, it has the Taylor expansion (about $z = 0$) when $|z| \leq \rho$

$$\phi_Y(z) = 1 - \frac{z^2}{2} + R_3(z) , \tag{6}$$

where the remainder term satisfies

$$|R_3(z)| \leq \frac{r_1 M}{r_1 - |z|} \left(\frac{|z|}{r_1}\right)^3 , \quad \rho < r_1 < \rho' , \tag{7}$$

and

$$M = \max_{|z| = r_1} |\phi_Y(z)| < \infty .$$

Since the Y_k's are independent, we have by (6)

https://doi.org/10.4153/CMB-1969-010-2 Published online by Cambridge University Press

$$\phi_n(z) = [1 - \frac{z^2}{2n} + R_3(\frac{z}{\sqrt{n}})]^n, \ |z| \leq \rho$$

and by (7)

$$|R_3(\frac{z}{\sqrt{n}})| \leq \frac{r_1 M}{r_1 - |z/\sqrt{n}|} \left(\frac{z/\sqrt{n}}{r_1}\right)^3 \leq \frac{K}{n^{3/2}}, \text{ where } K \text{ is}$$

independent of z and n. Then it can be proved that

$$\lim_{n\to\infty} \log \phi_n(z) = \lim_{n\to\infty} n \log[1 - z^2/(2n) + R_3(z/\sqrt{n})]$$

$$= - z^2/2 \text{ uniformly on } |z| \leq \rho .$$

The above limit implies (5).

Although the condition of analyticity seems to be quite strong, for most distributions we have in standard text books and applications their characteristic functions are analytic. The characteristic functions of the standardized random variables of the following distributions satisfy the condition that $\phi_Y(z)$ is analytic in $|z| < \rho' < \infty$ for some ρ' : Beta, Binomial, Gamma, Laplace, Negative Binomial, Normal, Poisson and Rectangular (for the expressions of the characteristic functions of the above distributions; see e.g., [4, p. 26]).

Students interested in further studies about the relation between convergence in distribution and convergence of moments can consult [2] and [3, pp. 180-185]). Rate of convergence of moments in the central limit theorem under weaker conditions and proved by using more advanced and complicated techniques can be found in [5].

REFERENCES

1. L.V. Ahlfors, Complex analysis. (McGraw-Hill, Second Edition, 1966).

2. M. Fréchet and J. Shohat, A proof of the generalized second-limit theorem in the theory of probability. Trans. Amer. Math. Soc. 33 (1931) 533-543.

3. M. Loève, Probability theory (Van Nostrand, Second Edition, 1960).

4. E. Lukacs, Characteristic functions. (Charles Griffin and Co. Ltd. 1960).

5. B. von Bahr, On the convergence of moments in the central limit theorem. Ann. Math. Statist. 36 (1965) 808-818.

University of Western Ontario