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Toric Degenerations, Tropical Curve, and Gromov–Witten Invariants of Fano Manifolds

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Abstract. In this paper, we give a tropical method for computing Gromov–Witten type invariants of Fano manifolds of special type. This method applies to those Fano manifolds that admit toric degenerations to toric Fano varieties with singularities allowing small resolutions. Examples include (generalized) flag manifolds of type A and some moduli space of rank two bundles on a genus two curve.

1 Introduction

Since the appearance of tropical geometry, there are various kinds of applications of its ideas to problems in classical geometry. But in many cases the ambient spaces are toric varieties, and the number of applications of tropical geometry to problems in non-toric varieties is not large. In this paper, we try to extend the applicability of tropical geometry in the direction of the enumerative problems. More precisely, we give a method for computing Gromov–Witten type invariants of Fano manifolds of special type in terms of counting of tropical curves.

Our calculation is based on toric degenerations of Fano manifolds. Given such degenerations, we can compare holomorphic curves between Fano manifolds and (singular) toric varieties. On the other hand, the correspondence between curves in toric varieties and tropical curves is shown in [8, 12, 14]. Thus, we can count curves in Fano manifolds by counting appropriate tropical curves.

The content of this paper is as follows. After a short review of toric degenerations in Section 2, we try to define Gromov–Witten type invariants via tropical method in Section 3. In fact, we will define two types of invariants. The first one (Theorem 3.39) is directly related to Gromov–Witten invariants and defined when the incidence conditions (Subsection 3.2) satisfy suitable assumptions (Assumption 3.6 and the transversality assumption in Proposition 3.23). In particular, this invariant depends on the incidence conditions only through their homology classes. The second (Theorem 3.42) is the analogue of the one considered in [14] and is regarded as the "relative part" of the Gromov–Witten invariants. A priori, it might not be

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a homological invariant, but it gives a lower bound to the corresponding Gromov–Witten invariants (see Remark 3.43). This second invariant is defined only under Assumption 3.6 and contains the first invariant as a special case.

In the last section, we give two examples of the calculation of these invariants for varieties of particular interest (these are both the first type invariants, so we calculate actual Gromov–Witten invariants). In these cases, in principle, we can calculate all genus zero Gromov–Witten invariants by tropical method. In particular, we can calculate Gromov–Witten invariants including odd cohomology classes. Also, by the homological invariance of Gromov–Witten invariants, an interesting combinatorial invariance of the counting number of tropical curves is observed, which cannot be deduced purely by tropical method (Remark 4.8).

2 Toric Degeneration

Let us first recall the definition of toric degenerations of projective manifolds.

Definition 2.1 Let (X,ω) be a projective manifold X with a Kähler form ω. A *toric degeneration* $(\mathfrak{X},\widetilde{\omega})$ of (X,ω) is a flat family $f\colon \mathfrak{X} \to B$ of complex varieties over a connected complex variety B with two distinguished points p_0, p_1, and a Kähler form $\widetilde{\omega}$ on $\mathfrak{X}$ (defined on the smooth locus), such that (X_1,ω_1) is isomorphic to (X,ω) as a Kähler manifold, and (X_0,ω_0) is a toric variety with a torus invariant Kähler form. Here (X_0,ω_0), (X_1,ω_1) are the restrictions of $(\mathfrak{X},\widetilde{\omega})$ to the fibers over p_0, p_1, respectively. Note that in general, the toric variety X_0 has singular points. So, as in the case of $(\mathfrak{X},\widetilde{\omega})$, the form ω_0 is defined on the smooth locus of X_0.

If (X_0,ω_0) is a toric Fano variety, then we call $(\mathfrak{X},\widetilde{\omega})$ a toric Fano degeneration of (X,ω).

We note that, given such a family, if there is a holomorphic disk $\eta\colon D(2) = \{z \in \mathbb{C} \mid |z| < 2\} \to B$ in the base space B with $\eta(0) = p_0$ and $\eta(1) = p_1$, then there is a natural map, the *gradient Hamiltonian flow* ([15], see also [13])

$$\phi_{grH,t}\colon X_t \to X_0, \quad t \in [0,1],$$

which is a diffeomorphism away from the singular locus of X_0 and keeps the Kähler forms (also away from the singular points). Here X_t is the fiber over $\eta(t)$. We write $\phi_{grH,1}\colon X_1 \to X_0$ by ϕ_{grH}.

Definition 2.2 Let X be a (singular) toric variety defined by a fan Σ in $\mathbb{R}^n$. We say that X allows a small resolution when there is a refinement Σ' of Σ without adding a new ray such that the toric variety associated with Σ' is nonsingular.

An example of a class of Fano varieties having degenerations to toric Fano varieties allowing small resolutions is provided by flag manifolds of type A (including partial flag manifolds); see [13].

Remark 2.3 In [13], we developed the notion of *toric degeneration of integrable systems*. Using this notion and the methods in [11], the results in this paper can be extended to the counting of holomorphic disks with Lagrangian boundary conditions. Note that flag manifolds of type A allow degenerations of integrable systems.

Also, using the method of [12], we can partially extend the results of this paper to higher genus tropical curves. In fact, combining the methods of [11] and [12], we can even deal with curves of any genus and any number of boundary components.

3 Computing Gromov–Witten Type Invariants of Fano Manifolds by the Tropical Method

In this section, we explain that when we have a Fano manifold that has a degeneration to toric a Fano variety *allowing a small resolution* in the sense of Definition 2.1, we can count the number of appropriate holomorphic curves in the Fano manifold. This will be performed with the help of the tropical method developed in the case of toric varieties ([8, 12, 14]). Namely, in [8, 12, 14], equalities between suitable counts of tropical curves in affine spaces and those of holomorphic curves in toric varieties are shown. The latter are an analogue of Gromov–Witten invariants.

When there is a toric degeneration of a Fano manifold, we can compare the curves in the Fano manifold and those in its degeneration, that is, the toric variety. When this comparison is effective, we can compute the number of holomorphic curves in the Fano manifold by counting appropriate tropical curves. We will show that this can be performed when the toric variety has a small resolution.

3.1 Tropical Curves

First we recall some definitions about tropical curves; see [8, 12, 14] for more information. Let $\overline{\Gamma}$ be a weighted, connected, finite graph. Its sets of vertices and edges are denoted $\overline{\Gamma}^{[0]}, \overline{\Gamma}^{[1]}$. Then we denote the weight function by $w_{\overline{\Gamma}}\colon \overline{\Gamma}^{[1]} \to \mathbb{N} \setminus \{0\}$. An edge $E \in \overline{\Gamma}^{[1]}$ has adjacent vertices $\partial E = \{V_1, V_2\}$. Let $\overline{\Gamma}^{[0]}_{\infty} \subset \overline{\Gamma}^{[0]}$ be the set of one-valent vertices. We set $\Gamma = \overline{\Gamma} \setminus \overline{\Gamma}^{[0]}_{\infty}$. Non-compact edges of Γ are called *unbounded edges*. Let $\Gamma^{[1]}_{\infty}$ be the set of unbounded edges. Let $\Gamma^{[0]}, \Gamma^{[1]}, w_{\Gamma}$ be the sets of vertices and edges of Γ and the weight function of Γ (induced from $w_{\overline{\Gamma}}$ in an obvious way), respectively. Let N be a free abelian group of rank $n \geq 2$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 3.1 A *parameterized trivalent tropical curve* in $N_{\mathbb{R}}$ is a proper map $h\colon \Gamma \to N_{\mathbb{R}}$ satisfying the following conditions.

(i) Γ is a trivalent graph.

(ii) For every edge $E \subset \Gamma$ the restriction $h\big|_E$ is an embedding with the image $h(E)$ contained in an affine line with rational slope.

(iii) For every vertex $V \in \Gamma^{[0]}$, the following *balancing condition* holds. Let $E_1, \dots, E_m \in \Gamma^{[1]}$ be the edges adjacent to V and let $u_i \in N$ be the primitive integral vector emanating from $h(V)$ in the direction of $h(E_i)$. Then

$$\sum_{j=1}^{m} w(E_j)u_j = 0.$$

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- In this paper, we always assume that the graph Γ is trivalent, so hereafter a (parameterized) tropical curve means a (parameterized) trivalent tropical curve.

An isomorphism of parameterized tropical curves $h\colon \Gamma \to N_{\mathbb{R}}$ and $h'\colon \Gamma' \to N_{\mathbb{R}}$ is a homeomorphism $\Phi\colon \Gamma \to \Gamma'$ respecting the weights and $h = h' \circ \Phi$. A *tropical curve* is an isomorphism class of parameterized tropical curves. The *genus* of a tropical curve is the first Betti number of Γ. A *rational* tropical curve is a tropical curve of genus zero.

The set of *flags* of Γ is

$$\mathrm{F}\Gamma = \{(V, E) \mid V \in \partial E\}.$$

By (i) of the definition we have a map $u\colon \mathrm{F}\Gamma \to N$ sending a flag (V, E) to the primitive integral vector $u_{(V,E)} \in N$ emanating from V in the direction of $h(E)$.

An *l-marked tropical curve* is a tropical curve $h\colon \Gamma \to N_{\mathbb{R}}$ together with a choice of l not necessarily distinct edges

$$\mathbf{E} = (E_1, \dots, E_l) \in (\Gamma^{[1]})^l.$$

The *combinatorial type* of an l-marked tropical curve $(\Gamma, \mathbf{E}, h)$ is the marked graph $(\Gamma, \mathbf{E})$ together with the map $u\colon \mathrm{F}\Gamma \to N$.

The *degree* of a type $(\Gamma, \mathbf{E}, u)$ is a function $\Delta\colon N \setminus \{0\} \to \mathbb{N}$ with finite support defined by

$$\Delta(\Gamma, u)(v) := \sharp\big\{ (V, E) \in \mathrm{F}\Gamma \mid E \in \Gamma^{[1]}_{\infty}, w(E)u_{(V,E)} = v\big\}$$

Let

$$e = |\Delta| = \sum_{v \in N\setminus\{0\}} \Delta(v).$$

This is the same as the number of unbounded edges of the graph Γ. It is known that the space $\mathfrak{T}_{(\Gamma,\mathbf{E},u)}$ of marked tropical curves of given type $(\Gamma, \mathbf{E}, u)$, if nonempty, is a manifold with boundary (in fact, a convex polytope) of dimension larger than or equal to

$$e + (n-3)(1-g) - \mathrm{ov}(\Gamma).$$

Here $\mathrm{ov}(\Gamma)$ is the *overvalence* of Γ defined by

$$\mathrm{ov}(\Gamma) = \sum_{V \in \Gamma^{[0]}} \big(\sharp\{E \in \Gamma^{[1]} \mid (V, E) \in \mathrm{F}\Gamma\} - 3\big).$$

It is also known that if Γ is rational, the equality

$$\dim \mathfrak{T}_{(\Gamma,\mathbf{E},u)} = e + (n-3)(1-g) - \mathrm{ov}(\Gamma)$$

holds.

3.2 Incidence Conditions

To fix the counting problem, we have to define *incidence conditions* for tropical curves and holomorphic curves. We recall some terminologies concerning incidence conditions from [14]. See [14] for more details. We formulate them for any genus in view of Remark 2.3, although in this paper we almost always treat the genus zero case.

3.2.1 Incidence Conditions for Tropical Curves

We begin with the case of tropical curves.

Definition 3.2 For $\mathbf{d} = (d_1, \ldots, d_l) \in \mathbb{N}^l$, an *affine constraint* of codimension $\mathbf{d}$ is an l-tuple $\mathbf{A} = (A_1, \ldots, A_l)$ of affine subspaces $A_i \subset N_{\mathbb{R}}$, defined over rational numbers, with

$$\dim A_i = n - d_i - 1.$$

An l-marked tropical curve $(\Gamma, \mathbf{E}, h)$ *matches* the affine constraint $\mathbf{A}$ if

$$h(E_i) \cap A_i \neq \varnothing, \quad i = 1, \ldots, l.$$

Let us fix a degree $\Delta \colon N \setminus \{0\} \to \mathbb{N}$. Now let $\mathbf{L} = (L_1, \ldots, L_l)$ be a set of linear subspaces of $N_{\mathbb{Q}}$, with $\operatorname{codim} L_i = d_i + 1$. Then the elements

$$\mathbf{A} = (A_1, \ldots, A_l), \quad A_i \in N_{\mathbb{Q}}/L_i$$

define affine constraints.

Definition 3.3 ([14, Definition 2.4]) Fix the genus g and a degree $\Delta \in \operatorname{Map}(N \setminus \{0\}, \mathbb{N})$ and write $|\Delta| = e$ as before. An affine constraint $\mathbf{A} = (A_1, \ldots, A_l)$ of codimension $\mathbf{d} = (d_1, \ldots, d_l)$ is *general* for Δ and g if

$$\sum_{i=1}^{l} d_i = e + (n-3)(1-g),$$

and if any l-marked tropical curve $(\Gamma, \mathbf{E}, h)$ of genus g and degree Δ matching $\mathbf{A}$ satisfies the following:

(i) $h(\Gamma^{[0]}) \cap \bigcup_i A_i = \varnothing$.

(ii) h is an embedding for $n > 2$. For $n = 2$, h is injective on the subset of vertices, and for any $x \in h(\Gamma)$, the inverse image is at most two points, and such $x \in h(\Gamma)$ with $\sharp\{h^{-1}(x)\} = 2$ is finite.

Proposition 3.4 ([14, Proposition 2.4]) *Fix the genus g and a degree Δ as above. Let $\mathfrak{A} := \prod_{i=1}^{l} N_{\mathbb{Q}}/L(A_i)$ be the space of affine constraints of codimension $\mathbf{d} = (d_1, \ldots, d_l)$ such that $\sum_{i=1}^{l} d_i = e + (n-3)(1-g)$. Then the subset*

$$\mathfrak{Z} := \{\mathbf{A}' \in \mathfrak{A} \mid \mathbf{A}' \textit{ is non-general for } \Delta \textit{ and } g\}$$

is nowhere dense in $\mathfrak{A}$.

This is proved for $g = 0$ in [14]. The argument extends to any genus with very little change, and we omit it.

For a marked tropical curve $(\Gamma, \mathbf{E}, h)$ matching the constraints $\mathbf{A}$, we have other important numbers: *weight* and *index*. The weight is defined by local data of the abstract graph Γ (weights and markings of the edges) as

$$w(\Gamma, \mathbf{E}) = \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_{\infty}} w_{\Gamma}(E) \cdot \prod_{i=1}^{l} w_{\Gamma}(E_i).$$

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There are two kinds of (lattice) indices, written as $\mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A})$ and $\delta_i(\Gamma, \mathbf{E}, h, \mathbf{A})$, respectively (see [14, Section 8]). The index $\mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A})$ is defined as the index of the inclusion of the lattices ([14], Proposition 5.7):

$$\operatorname{Map}(\Gamma^{[0]}, N) \to \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty} N/\mathbb{Z}u_{(\partial^- E, E)} \times \prod_{i=1}^{l} N/(\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N,$$
$$h \mapsto \big((h(\partial^+ E) - h(\partial^- E))_E, (h(\partial^- E_i))_i\big).$$

Here $\partial^\pm \colon \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty \to \Gamma^{[0]}$ is an arbitrary chosen orientation of the bounded edges, that is, $\partial E = \{\partial^- E, \partial^+ E\}$. For $E \in \Gamma^{[1]}_\infty$, $\partial^- E$ denotes the unique vertex adjacent to E.

The index $\delta_i(\Gamma, \mathbf{E}, h, \mathbf{A})$, for each marked edge E_i, is given by the product

$$\delta_i(\Gamma, \mathbf{E}, h, \mathbf{A}) = w_\Gamma(E_i) \cdot \big[\mathbb{Z}u_{(\partial^- E_i, E_i)} + L(A_i) \cap N : \big(\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)\big) \cap N\big].$$

3.2.2 Incidence Conditions for Holomorphic Curves in X_0 and X_1

Next, we define incidence conditions for holomorphic curves. Assume that X_0 is a toric variety defined by a complete fan in $N_\mathbb{Q}$. We use the same notation as above.

Definition 3.5 In the case of toric variety X_0, we take incidence conditions to be the subvarieties of X_0 given as the closures of the orbits of general points $\{q_1, \dots, q_l\}$ in X_0, by the subtori of the big torus acting on X_0 corresponding to the linear subspace $\{L_i\}$. We denote these subvarieties by $\mathbf{Z} = \{Z_i\}$.

For X_1, recall that there is a gradient Hamiltonian flow $\phi_{grH} \colon X_1 \to X_0$. Since $\phi_{grH} \colon X_1 \to X_0$ is diffeomorphic only away from the singular locus of X_0, if Z_i intersects the singular locus of X_0, the inverse image $\phi_{grH}^{-1}(Z_i)$ may not be of pure dimensional cycle, or may have boundary. So we assume the following.

Assumption 3.6 *Let us write* $\dim Z_i = m_i$. *Let* $\operatorname{int} X_0$ *be the complement of the union of toric divisors. Then the inverse image*

$$\phi_{grH}^{-1}(Z_i \cap \operatorname{int} X_0)$$

can be completed to an m_i*-dimensional cycle in* X_1 *in the following sense. Let* $W = X_0 \setminus \operatorname{int} X_0$ *be the union of toric divisors. Then there is a chain* C_i *of dimension at most* m_i *in* $\phi_{grH}^{-1}(W)$ *such that* $\phi_{grH}^{-1}(Z_i \cap \operatorname{int} X_0) \cup C_i$ *is an* m_i*-dimensional cycle in* X_1.

Definition 3.7 In the case of X_1, we assume that each Z_i satisfies Assumption 3.6, and take the cycle

$$\widetilde{Z}_i = \phi_{grH}^{-1}(Z_i \cap \operatorname{int} X_0) \cup C_i$$

as an incidence condition. There are choices of the chain C_i, and the homology class of $\widetilde{Z}_i$ may not be unique. We choose one from these choices and denote the cycle by $\widetilde{Z}_i$.

We note that the main results in this paper do not depend on the choice of $\widetilde{Z}_i$ (see Lemma 3.38).

When Assumption 3.6 holds for Z_i, we can assume that there is a continuous family of cycles $\mathcal{Z}_i \subset \mathfrak{X}$ over B such that:

- $\mathcal{Z}_i|_{X_1} = \widetilde{Z}_i$ and $\mathcal{Z}_i|_{X_0} = Z_i$;
- yhe complement in $\mathcal{Z}_i$ of the union

$$\bigcup_{t\in[0,1]} \phi^{-1}_{grH,t}(Z_i \cap \operatorname{int} X_0)$$

of the inverse images of $Z_i \cap \operatorname{int} X_0$ by the maps $\phi_{grH,t}$, $t \in [0,1]$, is contained in the union of the inverse images of the toric divisors $\bigcup_{t\in[0,1]} \phi^{-1}_{grH,t}(W)$.

3.3 Preliminary Arguments About Homology Classes

Let $(\mathfrak{X}, \widetilde{\omega})$ be a toric Fano degeneration of an n-dimensional Fano projective manifold (X,ω). Let X_0 be the toric Fano variety.

Assumption 3.8 *We assume that X_0 allows a small resolution. Also, we assume that the (pointed) base space $(B; p_0, p_1)$ of the degeneration $\mathfrak{X}$ is an open subset of $\mathbb{C}^m$ for some $m \in \mathbb{N}$ that is diffeomorphic to the pointed open ball, and the marked points are the origin ($= p_0$) and the point $(1,0,\dots,0) = p_1$. Let X_t be the fiber over $(t,0,\dots,0)$. We assume that X_t, $t \in (0,1]$ is nonsingular.*

The main point of Assumption 3.8 is the existence of a small resolution of X_0, which is the assumption of Theorem 3.28, the main ingredient of our study of holomorphic curves in this paper. It will also be used in Proposition 3.9(ii). Namely, we proved the following results in [13].

Proposition 3.9 ([13, Lemma 9.2])

(i) *When $(\mathfrak{X}, B)$ is a toric degeneration of X_1, there is a map $\phi\colon X_1 \to X_0$ which is natural up to homotopy (in particular, it is homotopic to ϕ_{grH}). The map ϕ is diffeomorphic away from the small neighbourhood of the singular locus of X_0.*

(ii) *When X_0 allows a small resolution, it induces an isomorphism*

$$\phi_*\colon \pi_2(X_1) \to \pi_2(X_0).$$

Since X_0 is a compact toric variety, it is simply connected. So the natural isomorphism $\pi_2(X_0) \simeq H_2(X_0,\mathbb{Z})$ holds. By Proposition 3.9(ii), the map ϕ induces an epimorphism of homology groups

$$\phi_*\colon H_2(X_1,\mathbb{Z}) \to H_2(X_0,\mathbb{Z}).$$

We note the following about this epimorphism.

Lemma 3.10 *The map ϕ_* has a natural splitting.*

Proof Let S be the set of singular points of X_0. Let $\pi\colon \widetilde{X}_0 \to X_0$ be a small resolution. Since X_0 allows a small resolution, any class in $\pi_2(X_0) \simeq H_2(X_0,\mathbb{Z})$ can be represented by a cycle in $X_0 \setminus S$. Namely, the representative $f\colon S^2 \to X_0$ can be taken so that $\pi^{-1}(f(S^2))$ is still the image of a map $\widetilde{f}\colon S^2 \to \widetilde{X}_0$. Then since $\pi^{-1}(S)$ has (real) codimension at least four, $\widetilde{f}$ can be deformed so that the the image of the composition with π is disjoint from S.

Moreover, if $f_1\colon S^2 \to X_0$ and $f_2\colon S^2 \to X_0$ are two maps representing the same class in $\pi_2(X_0) \simeq H_2(X_0,\mathbb{Z})$ whose images are disjoint from S, the homotopy between them can also be taken so that the image is disjoint from S, by the same dimensional reasoning as above.

The pullback of a map $f\colon S^2 \to X_0$ whose image is disjoint from S by $\phi_{grH,1}$ gives a well-defined natural splitting of ϕ_*. ■

Thus, the group $H_2(X_1,\mathbb{Z})$ can be written as

$$H_2(X_1,\mathbb{Z}) \simeq H_2' \oplus H_2'',$$

where $\phi_*|_{H_2'}$ is an isomorphism and $\phi_*|_{H_2''}$ is zero. In particular, the summand H_2' is torsion free.

Since X_1 is smooth, Poincaré duality holds, so there is a natural isomorphism

$$\big(H_{2n-2}(X_1,\mathbb{Z})\big)^* \simeq H_2' \oplus fH_2'',$$

where fH_2'' is the torsion free part of H_2''. Let

$$\big(H_{2n-2}(X_1,\mathbb{Z})\big)^* \simeq (H_{2n-2}')^* \oplus (H_{2n-2}'')^*$$

be the corresponding splitting of $(H_{2n-2}(X_1,\mathbb{Z}))^*$.

Let $\pi\colon \widetilde{X}_0 \to X_0$ be a small resolution.

Definition 3.11 Let $\mathcal{P}$ be the free abelian group generated by the toric prime divisors of X_0.

Since π is small, toric prime divisors of X_0 and $\widetilde{X}_0$ are in natural one-to-one correspondence. So we also denote by $\mathcal{P}$ the free abelian group generated by the toric prime divisors of $\widetilde{X}_0$.

Thus, there is a commutative diagram

$$\begin{tikzcd} & H_{2n-2}(\widetilde{X}_0,\mathbb{Z}) \arrow[d, "\pi_*"] \\ \mathcal{P} \arrow[ur, "p_1"] \arrow[r, "p_2"] & H_{2n-2}(X_0,\mathbb{Z}). \end{tikzcd} \tag{3.1}$$

Here p_1, p_2 are surjections, and π_* is an isomorphism, since π is small.

Since $\widetilde{X}_0$ is a toric variety, its second homotopy group is generated by holomorphic spheres, and a natural isomorphism

$$\pi_2(\widetilde{X}_0) \simeq H_2(\widetilde{X}_0,\mathbb{Z})$$

holds. In particular, $H_2(\widetilde{X}_0,\mathbb{Z})$ is free. Also, since $\widetilde{X}_0$ is smooth, we have a natural Poincaré duality isomorphism

$$(H_{2n-2}(\widetilde{X}_0,\mathbb{Z}))^* \simeq H_2(\widetilde{X}_0,\mathbb{Z}).$$

On the other hand, there is a natural surjection

$$H_2(\widetilde{X}_0,\mathbb{Z}) \to H_2(X_0,\mathbb{Z}).$$

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Combining with $H_2' \simeq H_2(X_0, \mathbb{Z})$, we see that there is a natural surjection

$$H_2(\widetilde{X}_0, \mathbb{Z}) \to H_2'.$$

Taking the Poincaré dual of the both sides, we have a surjection

$$P\colon \left(H_{2n-2}(\widetilde{X}_0, \mathbb{Z})\right)^* \to (H'_{2n-2})^*.$$

On the other hand, the dual of the map p_1 gives an inclusion

$$p_1^*\colon \left(H_{2n-2}(\widetilde{X}_0, \mathbb{Z})\right)^* \to \mathcal{P}^*.$$

3.4 Degrees and Homology Classes

First we recall the following definition from [14].

Definition 3.12 Let X_0 be an n-dimensional toric variety. A holomorphic curve $C \subset X_0$ is *torically transverse* if it is disjoint from all toric strata of codimension greater than one. A stable map $\varphi\colon C \to X_0$ is torically transverse if $\varphi^{-1}(\operatorname{int} X_0) \subset C$ is dense and $\varphi(C) \subset X_0$ is a torically transverse curve. Here $\operatorname{int} X_0$ is the complement of the union of toric divisors.

Let $\Sigma \subset N_{\mathbb{R}}$ be the fan defining X_0. For each ray of Σ, we have the generator $v \in N$ and its associated toric prime divisor D_v.

Definition 3.13 For a torically transverse curve $\varphi\colon C \to X_0$, the *degree* is given by the map

$$\Delta(\varphi)\colon N \setminus \{0\} \to \mathbb{N}$$

defined as follows. For a primitive $v \in N$ and $\lambda \in \mathbb{N}$, λv is mapped to 0 if $\mathbb{R}_{\geq 0} v$ is not a ray of Σ, and to the number of points of multiplicity λ in $\varphi^* D_v$ otherwise.

Recall that the degree $\Delta\colon N \setminus \{0\} \to \mathbb{N}$ of a tropical curve in $\mathbb{R}^n$ was given by the data of the direction vectors and multiplicity of the unbounded edges. We have to define the notion of degree for curves in a general fiber X_t.

Definition 3.14 We define the degree of a curve in general smooth fiber X_t, $t \neq 0$ to be its integral homology class.

Let $\varphi\colon C \to X_0$ be a torically transverse stable map of degree Δ. We define a map Δ_D from the set of primitive vectors in $N \setminus \{0\}$ to $\mathbb{N}$ as follows. Namely, with a primitive vector $v \in N$, we associate

$$\Delta_D(v) = \sum_{a>0} a\Delta(av) \in \mathbb{N}.$$

This can be regarded as an element of the dual space $\mathcal{P}^*$ of the space $\mathcal{P}$ introduced in Definition 3.11.

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Definition 3.15 A *coarse-degree* is a map $\widetilde{\Delta}$ from the set of nonzero primitive vectors in N to the set of nonnegative integers satisfying the condition

$$\sum_{v:\text{primitive}} \widetilde{\Delta}(v)v = 0.$$

This can be extended to an element of $\mathcal{P}^*$, and we denote it by the same letter $\widetilde{\Delta}$.

A degree Δ of a tropical curve or a torically transverse stable map in a toric variety naturally gives a coarse-degree.

Lemma 3.16 *For a degree Δ, the map Δ_D is a coarse-degree.*

Proof This follows from the balancing condition for the tropical curve. ■

Definition 3.17 A coarse-degree $\widetilde{\Delta}$ is called *rational* if it is induced from a degree of a rational tropical curve (or of a torically transverse rational stable map in a given toric variety). Let $\mathfrak{D} \subset \mathcal{P}^*$ be the set of rational coarse-degrees of X_0.

Recall that there is an inclusion

$$p_1^* \colon (H_{2n-2}(\widetilde{X}_0, \mathbb{Z}))^* \to \mathcal{P}^*.$$

Lemma 3.18 *The set $\mathfrak{D} \subset \mathcal{P}^*$ is a subset of (the image of) the space $(H_{2n-2}(\widetilde{X}_0, \mathbb{Z}))^*$.*

Proof Let $p_1 \colon \mathcal{P} \to H_{2n-2}(X_0; \mathbb{Z})$ be the quotient map defined in (3.1). Given an element $\widetilde{\Delta}$ of $\mathfrak{D}$, considered as a linear function on $\mathcal{P}$, define $\widetilde{\Delta}'(p_1(D)) = \widetilde{\Delta}(D)$. This gives a well-defined element of $H_{2n-2}(X_0; \mathbb{Z})^*$. Namely, we have to show that if $p_1(D)$ is zero, then $\widetilde{\Delta}(D)$ is also zero. Recall that $\widetilde{\Delta}(D)$ is the sum of the transversal intersection numbers between a rational curve and a linear sum D of toric divisors. So it must be zero when D is homologous to zero. The latter condition is the same as requiring $p_1(D) = 0$. ■

Lemma 3.19 *$\mathfrak{D}$ is a submonoid of $(H_{2n-2}(\widetilde{X}_0; \mathbb{Z}))^*$.*

Proof It suffices that the set of degrees of rational tropical curves forms a monoid. This follows because given two rational tropical curves, we can parallel transport one of them so that two tropical curves intersect. By taking a suitable union of graphs as a domain, we have a new rational tropical curve whose degree is the sum of the given two. ■

3.5 Degeneration of Rational Curves in X_1

Let X be an n-dimensional Fano manifold and $\mathfrak{X} \to \mathbb{C}$ be a toric degeneration with $X_1 = X$. Take incidence conditions $\widetilde{Z}_1, \dots, \widetilde{Z}_l$ satisfying Assumption 3.6, and let $m_i = \dim \widetilde{Z}_i$. Let $\beta \in H_2(X, \mathbb{Z})$. Then the moduli space of stable maps of class β from pre-stable rational curves without marked point has expected dimension

$n + c_1(X)(\beta) - 3$. We assume the equality

$$\sum_{i=1}^{l}(n - m_i - 1) = n + c_1(X)(\beta) - 3$$

holds. The expected dimension of the rational curve of class β satisfying the incidence conditions is 0.

Our goal is to understand the rational curves in X by the study of rational curves in the toric variety X_0. Moreover, it is desirable that we also understand the rational curves in X from the view point of tropical geometry. Since tropical curves capture only those curves in X_0 that intersects the dense torus orbit, it is favorable that we can understand the rational curves in X by studying only torically transverse curves in X_0. The purpose of this section is to prove that this is indeed the case.

To do it, we need to slightly perturb the complex structure on X to a nonintegrable one (that is, an almost complex structure) so that the moduli spaces of curves in X become smooth (in particular, of expected dimension).

Let $\mathcal{J}$ be the set of almost complex structures that tame the Kähler form of X.

Definition 3.20 ([7, Definition 3.1.5]) Let $\beta \in H_2(X, \mathbb{Z})$ be a homology class. An almost complex structure J on X is called *regular for* β if every J-holomorphic curve $\varphi\colon S^2 \to X$ of class β is Fredholm regular.

Theorem 3.21 ([7, Theorem 3.1.6]) *For a fixed class $\beta \in H_2(X, \mathbb{Z})$, the space of almost complex structures regular for β contains an intersection of countably many open and dense subsets in the space of all almost complex structures.*

Let J be a regular almost complex structure on X. Let $\varphi\colon C \to X$ be a stable map from a prestable rational curve with $[\varphi(C)] = \beta$ that satisfies generic incidence conditions $\{\widetilde{Z}_i\}$.

Lemma 3.22 *The domain curve C is a nonsingular rational curve.*

Proof Assume that C has two components, $C = C_1 \cup C_2$, where C_1 and C_2 are nonsingular rational curves. Then φ can be thought of as the pair of maps from pointed nonsingular rational curves

$$\varphi_1\colon (C_1, x_1) \to X, \quad \varphi_1\colon (C_2, x_2) \to X,$$

with the condition $\varphi(x_1) = \varphi_2(x_2)$. Let $[\varphi_1(C_1)] = \beta_1$ and $[\varphi_2(C_2)] = \beta_2$ with $\beta = \beta_1 + \beta_2$. Then the moduli spaces of the deformations of the maps φ_1 and φ_2 have dimension $n+c_1(X)(\beta_1)-2$ and $n+c_1(X)(\beta_2)-2$, respectively (note that there is a one dimensional degree of freedom in varying the point x_1 or x_2). When J is generic, the condition $\varphi(x_1) = \varphi_2(x_2)$ gives an n-dimensional constraint ([7, Proposition 6.2.8]). Thus, the moduli space of pair of maps (φ_1, φ_2) satisfying the above conditions has dimension

$$\bigl(n + c_1(X)(\beta_1) - 2\bigr) + \bigl(n + c_1(X)(\beta_2) - 2\bigr) - n = n + c_1(X)(\beta) - 4.$$

Since $n+c_1(X)(\beta)-3 = 0$ by assumption, $n+c_1(X)(\beta)-4 = -1$. Thus, this moduli space is empty.

Inductively, one sees that the moduli space of maps $\varphi\colon C \to X$ satisfying the incidence conditions $\{\widetilde{Z}_i\}$ is empty when C has more than one component.

Similarly, the moduli space is empty when C is irreducible but has nodes. ∎

Let J_t, $t \in (0,1]$ be regular family of almost complex structures on X_t (see [7, Definition 3.1.7]) that converges to the original toric complex structure of X_0 on the smooth part of X_0. Precisely, take J_t so that the family of almost complex structures

$$(\phi_{grH,t})_* J_t, \quad t \in (0,1]$$

on the smooth part of X_0 converges to the toric complex structure when $t \to 0$. Let

$$\varphi_t\colon C_t \to X_t, \quad t \in (0,1]$$

be a family of rational curves of class β that satisfies the incidence conditions $\{\mathcal{Z}_i\}$ (see the last paragraph of Subsection 3.2). By Lemma 3.22, each C_t is the nonsingular rational curve.

By Gromov's compactness theorem [4], there is a sequence $\{t_i\} \subset (0,1]$, converging to zero, such that the sequence of maps $\varphi_{t_i}\colon C_{t_i} \to X_{t_i}$ converges to a limit stable map $\varphi_0\colon C_0 \to X_0$ from a prestable rational curve. The following proposition is the main result of this subsection.

Proposition 3.23 *Assume that each $Z_i = \mathcal{Z}_i \cap X_0$ is transversal to each toric stratum. Then the prestable curve C_0 is the nonsingular rational curve. Moreover, φ_0 is torically transverse.*

Proof Since J_{t_i} is regular, each φ_{t_i} is a point on the moduli space of rational pseudo holomorphic curve of class β that is smooth and has dimension $n + c_1(X)(\beta) - 3$. Then by Gromov's compactness theorem, there is a family of holomorphic curves in X_0 that deforms φ_0 and has dimension at least $n + c_1(X)(\beta) - 3$.

On the other hand, by Theorem 3.28 below, any member of this family can be deformed into a torically transverse curve. Since torically transverse curves are Fredholm regular by Lemma 3.25 below, such curves can be deformed into X_t for t with the norm $|t|$ sufficiently small. Thus, if the above family in X_0 has dimension greater than $n + c_1(X)(\beta) - 3$, then we can construct the same dimensional family of curves in X_t, $t \neq 0$ that deforms φ_t, contracting the regularity of J_t. So this family in X_0 has dimension $n + c_1(X)(\beta) - 3$.

Those curves in this family whose domain is not nonsingular are contained in a strictly lower dimensional subfamily. However, since the incidence conditions $\{Z_i\}$ on X_0 are transverse to each toric stratum, these incidence conditions imply $n + c_1(X)(\beta) - 3$ dimensional conditions even for curves some of whose components are contained in toric divisors. Thus, the curves in the above subfamily cannot satisfy general incidence conditions. This proves the proposition. ∎

Corollary 3.24 *When each variety Z_i is a point, then any rational stable map $\varphi\colon C \to X_0$ satisfies the properties that φ is torically transverse and the domain curve C is the nonsingular rational curve.*

Proof When each Z_i is a point, the transversality assumption is equivalent to the statement that Z_i is not contained in the union of toric divisors. However, this is automatically satisfied when $\{Z_i\}$ is generic. ∎

We used the following simple result in the above proof.

Lemma 3.25 *Let C be a nonsingular rational curve, let X be a smooth toric variety, and let $\varphi\colon C \to X$ be a holomorphic map whose image intersects the dense torus orbit of X. Then φ is Fredholm regular.*

Proof It suffices to prove that the cohomology group $H^1(C, \varphi^* TX)$ vanishes. Since X is smooth, $\varphi^* TX$ is a sum of line bundles

$$\varphi^* TX \cong L_1 \oplus \cdots \oplus L_n.$$

Let $x \in C$ be a point that is mapped into the dense torus orbit of X by φ. Let $v \in (\varphi^* TX)_x$ be any vector in the stalk of $\varphi^* TX$ at x. Recall that an infinitesimal deformation of φ gives a section of $\varphi^* TX$. By the torus action on X, there is an infinitesimal deformation of φ such that the value of the corresponding section of $\varphi^* TX$ at x is v. This implies that each line bundle L_i, $i = 1, \dots, n$, above has a nontrivial section. This in turn implies that $H^1(C, L_i) = 0$, $i = 1, \dots, n$. Thus, $H^1(C, \varphi^* TX)$ vanishes. ∎

Remark 3.26 The moduli space of maps to X_0 of class $[\varphi_0(c_0)]$ from prestable rational curves can have larger dimensional components composed of curves with several components some of which are contained in toric divisors. However, they are not relevant to the counting of curves in X under the assumption of Proposition 3.23.

3.6 Rational Curves in a Toric Variety Admitting Small Resolutions

In this subsection, we prove the result (Theorem 3.28) that we used in the proof of Proposition 3.23. First we recall the result of Cho and Oh [1] regarding the explicit presentation of holomorphic discs in smooth projective toric varieties with Lagrangian torus boundary condition.

Theorem 3.27 (Cho and Oh [1, Theorem 5.3]) *Let L be a Lagrangian torus fiber of the moment map in a smooth projective toric variety*

$$X_\Sigma = (\mathbb{C}^r \setminus Z(\Sigma))/K.$$

Here r is the number of one dimensional cones of a fan Σ, the subset $Z(\Sigma) \subset \mathbb{C}^r$ is defined by the Stanley–Reisner ideal, and K is the kernel of the map $(\mathbb{C}^\times)^r \to (\mathbb{C}^\times)^N$ defined by one dimensional cones in Σ. Then any holomorphic map

$$\varphi\colon (D^2, \partial D^2) \to (X_\Sigma, L)$$

can be lifted to a holomorphic map

$$\widetilde{\varphi}\colon D^2 \to \mathbb{C}^r \setminus Z(\Sigma)$$

so that the homogeneous coordinate functions $(z_1(\widetilde{\varphi}),\dots,z_r(\widetilde{\varphi}))$ *are given by the Blaschke products with constant factors*

$$z_j(\widetilde{\varphi}) = c_j \cdot \prod_{k=1}^{\mu_j} \frac{z-\alpha_{j,k}}{1-\overline{\alpha}_{j,k}z},$$

where $c_j \in \mathbb{C}^\times$, $\alpha_{j,k} \in \operatorname{Int} D^2$ *and* μ_j *is a non-negative integer for* $j = 1,\dots,r$. *Moreover, the Maslov index of* φ *is given by*

$$\nu(\varphi) = 2\sum_{j=1}^{r} \mu_j.$$

We prove the following result about curves in a toric variety whose singularities admit small resolutions, which is the main result in this subsection.

Theorem 3.28 *Let* C *be a nonsingular rational curve and* X *be a toric variety whose singularities admit small resolutions. Let* $\varphi\colon C \to X$ *be a holomorphic map whose image intersects the dense torus orbit. Then* φ *can be deformed (through holomorphic maps) into a torically transverse map.*

We prove this theorem in several steps. First we note that this result was proved for discs with Lagrangian torus boundary condition in [13, Proposition 9.5]. We recall the proof for the reader's convenience.

Lemma 3.29 *Let* X *be a toric variety whose singularities admit small resolutions. Let* $\phi\colon D \to X$ *be a holomorphic disc with a boundary condition on a Lagrangian torus fiber of the moment map. Then* ϕ *can be deformed into a torically transverse disk with the same boundary condition.*

Proof Let $\widetilde{X} \to X$ be a small resolution of X. Let $\psi\colon D \to \widetilde{X}$ be the proper transform of ϕ. Since $\widetilde{X}$ is smooth, the map ψ has an explicit description

$$z_j(\widetilde{\psi}) = c_j \cdot \prod_{k=1}^{\mu_j} \frac{z-\alpha_{j,k}}{1-\overline{\alpha}_{j,k}z}$$

by Theorem 3.27. The map ψ intersects a toric stratum of codimension larger than one exactly when there are $j_1 \neq j_2$ such that $\alpha_{j_1,k_1} = \alpha_{j_2,k_2}$ for some k_1 and k_2. From this remark, and since $\widetilde{X}$ is a small resolution of X so that the exceptional locus has codimension larger than one, we can make ψ torically transverse by perturbing $\alpha_{j,k}$, Since the resolution is small, torically transverse disks in $\widetilde{X}$ project to torically transverse disks in X. ∎

On the other hand, the proof of [14, Theorem 8.3] shows that there is a degeneration of X together with a degeneration of the map φ that decomposes φ into a union of simple pieces. To state the precise statement, we recall some results from [14].

Let E be any toric stratum of X. By the construction of [14, Section 3], there is a variety $\mathfrak{X}$ over $\mathbb{C}$ with the following properties:

- $\mathfrak{X}$ is a toric variety and the map $\pi\colon \mathfrak{X} \to \mathbb{C}$ is toric.

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- Each fiber $X_t = \pi^{-1}(t)$, $t \neq 0$ is isomorphic to the toric variety X. The central fiber $X_0 = \pi^{-1}(0)$ is a union of toric varieties intersecting along torus orbits (see [14, Proposition 3.5] for precise description).
- There is a natural embedding $i\colon E \times \mathbb{C} \to \mathfrak{X}$ over $\mathbb{C}$.

We called such a family a *toric degeneration* of X in [14]. Such a family $\mathfrak{X}$ can be taken so that it satisfies the following properties. Let $\mathcal{E}$ be the union of all the toric strata of X that have codimension at least two.

Lemma 3.30 *There is a degeneration $\pi\colon \mathfrak{X} \to \mathbb{C}$ of X into a union of toric varieties $X_0 = \pi^{-1}(0)$ as above and a family of stable maps $\varphi_t\colon \mathcal{C} \to \mathfrak{X}$ over $\mathbb{C}$ $(t \in \mathbb{C})$ that satisfies the following properties:*

(i) $\varphi_1 = \varphi$.

(ii) *Let $\varphi_0\colon C_0 \to X_0$ be the stable map over $0 \in \mathbb{C}$. Let $C_{0,i}$ be a component of C_0. Then the restriction of φ_0 to $C_{0,i}$ is torically transverse except at $\varphi_0^{-1}(\mathcal{E})$, and $\varphi_0^{-1}(\mathcal{E})$ is a finite subset. In particular, the image $\varphi_0(C_{0,i})$ is not contained in a torus orbit of positive codimension.*

(iii) *Moreover, if $C_{0,i}$ is a component that contains a point mapped to $\mathcal{E}$, then the image of $C_{0,i}$ by ψ_0 is the closure of an orbit of a one parameter subgroup of the torus acting on the corresponding component of X_0.*

Here $\mathcal{C}$ is a suitable family of prestable rational curves over $\mathbb{C}$. Also, we regard $\mathcal{E}$ as a subset of X_0 using the natural embedding i above.

Proof In fact, φ_0 is a generalization of what in [14] we called a *maximally degenerate curve* where restriction of φ_0 to each component of C_0 (and also all φ_t, $t \neq 0$) was required to be torically transverse. The extension to the case where we allow intersection with the locus $\mathcal{E}$ is straightforward. Namely, blow up X so that the proper transform of φ becomes torically transverse (in particular, this will be different from small resolutions of X in general). Then apply the argument of the proof of [14, Theorem 8.3] so that we obtain a maximally degenerate curve. Now blow down the divisors over $E \times \mathbb{C}$ to obtain a degeneration of X (not of its blow up). The resulting curve satisfies our requirements. ■

Let $C_{0,i}$ be a component of C_0 that contains a point mapped to the locus $\mathcal{E}$. Let $X_{0,i}$ be the component of X_0 to which $C_{0,i}$ is mapped. Then by Lemma 3.30(iii), the intersection of the image of $C_{0,i}$ by φ_0 with a Lagrangian torus fiber of the moment map is, if nonempty, a circle (possibly multiply covered). In particular, such a circle divides the restriction $\varphi_{0,i}$ of φ_0 to $C_{0,i}$ into two holomorphic disks with Lagrangian torus boundary condition. Let

$$\varphi_{0,i}|_{D_i}\colon D \to X_{0,i}, \quad i = 1, 2$$

be these holomorphic disks and let $\varphi_{0,i}|_{D_1}\colon D \to X_{0,i}$ be the one whose image contains the point mapped to the locus $\mathcal{E}$ (we can assume such a point is unique on $C_{0,i}$ by suitably modifying the degeneration $\mathfrak{X}$ if necessary).

By Lemma 3.29, $\varphi_{0,i}|_{D_1}$ can be deformed into a torically transverse disk $\varphi'_{0,i}|_{D_1}\colon D \to X_{0,i}$. However, although $\varphi'_{0,i}|_{D_1}$ satisfies the same Lagrangian torus boundary condition as $\varphi_{0,i}|_{D_1}$, $\varphi_{0,i}|_{\partial D_1}$ and $\varphi'_{0,i}|_{\partial D_1}$ are different maps. So we cannot glue $\varphi'_{0,i}|_{D_1}$ and $\varphi_{0,i}|_{D_2}$ to obtain a map from a rational curve into $X_{0,i}$. But this point can be fixed by the argument in [11, Section 9.2.3]. Let us recall it briefly.

Namely, in [11, Section 8], we constructed a degeneration of holomorphic disks in a toric variety with Lagrangian torus boundary condition.

Lemma 3.31 *There is a degeneration $\pi_i\colon \mathfrak{X}_i \to \mathbb{C}$ of $X_{0,i}$ into a union of toric varieties and a family of stable maps $\widetilde{\varphi}_{t,i}\colon \mathcal{D} \to \mathfrak{X}_i$ over $\mathbb{C}$ that satisfies the following properties:*

(i) $\widetilde{\varphi}_{1,i} = \varphi'_{0,i}|_{D_1}$.
(ii) *For all $t \in \mathbb{C}$, $\widetilde{\varphi}_{t,i}$ is a torically transverse map.*
(iii) *The pre-stable disk D_0 has only one component $D_{0,b}$ that has boundary. The restriction of $\widetilde{\varphi}_{0,i}$ to $\partial D_{0,b}$ is the same as $\varphi_{0,i}|_{\partial D_1}$ under the natural identification of the Lagrangian tori to which $\widetilde{\varphi}_{0,i}|_{\partial D_{0,b}}$ and $\varphi_{0,i}|_{\partial D_1}$ are mapped.*

Here $\mathcal{D}$ is a suitable family of prestable disks (see [11, Definition 4.5] *for definition of prestable disks) and D_t, $t \in \mathbb{C}$, is the fiber of $\mathcal{D}$ over t.* ■

The degeneration can be torically embedded in $\mathbb{P}^d \times \mathbb{C} \to \mathbb{C}$ for some integer d (see [11, Section 8.1]). Here $\mathbb{P}^d$ is a projective toric manifold. So a metric is induced on the total space $\mathfrak{X}_i$ of the degeneration. The following is a consequence of results of [11, Section 8.2].

Lemma 3.32 *Let ϵ be an arbitrary small positive constant. After a suitable base change of the family π_i, the degeneration of Lemma 3.31 can be taken so that the Gromov–Hausdorff distance between the images of $\widetilde{\varphi}_{1,i}$ and $\widetilde{\varphi}_{0,i}$ is smaller than ϵ, considered as maps to the same target space $\mathbb{P}^d$.* ■

The degenerations $\pi\colon \mathfrak{X} \to \mathbb{C}$ and $\pi_i\colon \mathfrak{X}_i \to \mathbb{C}$ can be combined to a single degeneration $\pi'\colon \mathfrak{X}' \to \mathbb{C}$ of the original toric variety X. On the central fiber of this degeneration π', there are two stable disks $\varphi_0|_{C_0 \setminus \operatorname{int} D_1}$ and $\widetilde{\varphi}_{0,i}$. Here $\operatorname{int} D_1$ is the set $D_1 \setminus \partial D_1$. The Lagrangian tori to which the boundary of these maps are mapped are in general different. But the image of $\varphi'_{0,i}|_{D_1}$ can be taken as close to the image of $\varphi_{0,i}|_{D_1}$ as we like (with respect to the Gromov–Hausdorff distance), so by Lemmas 3.31 and 3.32, slightly deforming $\widetilde{\varphi}_{0,i}$ by the torus action, these two maps glue to give a stable map $\widetilde{\varphi}'_{0,i}$ from a nodal rational curve to X'_0, the central fiber of the degeneration $\pi'\colon \mathfrak{X}' \to \mathbb{C}$.

Performing this construction at all the points of φ where it intersects the singular locus E of X, we obtain a torically transverse rational map $\widetilde{\varphi}'_0$ to X'_0.

Then by [14, Theorem 8.3], such a map can be (not uniquely) smoothed to a map in X. Moreover, as in Lemma 3.32, the Gromov–Hausdorff distance between the images of the original map φ in Theorem 3.28 and such a smoothed map $\widetilde{\varphi}'$ can be taken as small as we like (after a suitable base change if necessary).

Proof of Theorem 3.28 Let φ'' and $\widetilde{\varphi}''$ be the proper transform of these maps to a small resolution $\widetilde{X}$ of X. By the above argument, the distance of the images of φ'' and $\widetilde{\varphi}''$ can be taken as small as we like. Since $\widetilde{X}$ is a smooth toric manifold, by the regularity (Lemma 3.25) there is a family of stable maps that contains both φ'' and $\widetilde{\varphi}''$. Projecting this family to X, we obtain a desired deformation of φ. ■

3.7 Coarse Degree for X_0 Associated with a Class in H_2'

By Lemma 3.22 and Proposition 3.23, when the incidence conditions Z_i in X_0 are transversal to each toric stratum, then we only need to consider stable maps with the properties to study the relationship between the counting problems in X_1 and X_0:

- The domain is the nonsingular rational curve.
- For the case of curves in X_0, the map is torically transverse.

So in the following the domain of stable maps will always be the nonsingular rational curve, and stable maps in X_0 are torically transverse unless otherwise specified.

A torically transverse rational stable map $\varphi\colon \mathbb{P}^1 \to X_0$ lifts to a torically transverse rational stable map $\widetilde{\varphi}\colon \mathbb{P}^1 \to \widetilde{X}_0$, where $\widetilde{X}_0$ is a small resolution of X_0. Since $\widetilde{X}_0$ is smooth, the image of $\widetilde{\varphi}$ gives an element of $(H_{2n-2}(\widetilde{X}_0, \mathbb{Z}))^*$. On the other hand, Since φ is Fredholm regular by Lemma 3.25, φ lifts to a torically transverse stable map $\varphi_1\colon C \to X_1$. Since X_1 is smooth, too, φ_1 gives an element of $(H_{2n-2}(X_1, \mathbb{Z}))^*$, in fact, an element of $(H'_{2n-2})^*$ using the notation of the previous subsection.

Recall that there is a natural surjection

$$P\colon \big(H_{2n-2}(\widetilde{X}_0, \mathbb{Z})\big)^* \to (H'_{2n-2})^*$$

and a submonoid $\mathfrak{D} \subset (H_{2n-2}(\widetilde{X}_0, \mathbb{Z}))^*$. Using the construction so far, we have the following proposition.

Proposition 3.33 *Let $[\varphi'] \in H_2'$ be an element represented by a rational stable map in X_1 that is a lift of some torically transverse rational stable map φ in X_0. Then the intersection of $P^{-1}([\varphi'])$ and $\mathfrak{D}$ in $(H_{2n-2}(\widetilde{X}_0, \mathbb{Z}))^*$, or its image in $\mathcal{P}^*$, is given as follows:*

(i) *Consider all the deformation classes of torically transverse rational stable maps*

$$\varphi_i\colon \mathbb{P}^1 \to X_0, \quad i = 1, \dots, m$$

whose lifts to X_1 give the class $[\varphi']$.

(ii) *Lift φ_i to $\widetilde{X}_0$. This gives an element a_i of $\mathcal{P}^*$.*

(iii) *Then we have $P^{-1}([\varphi']) \cap \mathfrak{D} = \{a_1, \dots, a_m\}$.*

Definition 3.34 For $[\varphi'] \in H_2'$ as in the proposition above, we call the subset

$$P^{-1}([\varphi']) \cap \mathfrak{D}$$

the set of *coarse-degrees corresponding to* $[\varphi']$. We may also consider it as a subset of $\mathcal{P}^*$ by the natural inclusion p_1^*.

3.8 Stable Maps in X_0 and their Deformations to X_1

In Subsection 3.5, we studied the degeneration of holomorphic curves in a Fano manifold $X = X_1$ to curves in X_0. In this subsection, we study the converse and establish that there is a one-to-one correspondence between curves in X_1 and in X_0 in a suitable sense.

Let us assume that we have fixed a degree Δ of a torically transverse rational stable map in X_0. Let $e = |\Delta|$ be the number of the unbounded edges of the

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associated tropical curve as before. Take generic incidence conditions $(Z_1, \dots, Z_l)$ in X_0 corresponding to a sequence of linear subspaces $(L_1, \dots, L_l)$ of $N_{\mathbb{R}}$. Write $\operatorname{codim} L_i = \operatorname{codim} Z_i = d_i + 1$ and assume that $\sum_{i=1}^{l} d_i = e + n - 3$. We also assume that the domain of a stable map is the nonsingular rational curve and such a map is torically transverse when it is a map to X_0, as noted in the previous subsection.

Recall that we are considering regular family of almost complex structures on X_t, $(t \geq 0)$, converging to the toric complex structure on the smooth part of X_0. We assume that X_1 is sufficiently close to X_0, by suitable base change as in Lemma 3.32. Since a torically transverse rational stable map φ in X_0 is Fredholm regular by Lemma 3.25, φ lifts to a stable map φ_1 in X_1.

Lemma 3.35 *The virtual dimension of the moduli spaces at φ is the same as that of φ_1.*

Proof By assumption the map φ is torically transverse. In particular, there is a neighborhood U of the image of φ in X_0 on which the restriction

$$\phi_{grH}|_{\phi_{grH}^{-1}(U)} \colon \phi_{grH}^{-1}(U) \to U$$

is a diffeomorphism. Since the virtual dimension of a stable map is calculated by a characteristic number, which is a topological invariant, the virtual dimensions of the moduli spaces at φ and φ_1 are the same. ∎

Thus, if $(Z_1, \dots, Z_l)$ are incidence conditions on X_0 for curves of class $\beta = [\varphi]$ satisfying Assumption 3.6, then $(\widetilde{Z}_1, \dots, \widetilde{Z}_l)$ can be used as incidence conditions on X_1 for curves of class $[\varphi_1]$. By the Fredholm regularity, the maps around φ in the moduli space of stable maps deforming φ locally (and noncanonically) correspond to the maps around φ_1 in a one-to-one manner. In particular, we have the following proposition.

Proposition 3.36 *Let $\varphi \colon \mathbb{P}^1 \to X_0$ be a torically transverse stable map of given degree Δ as above, satisfying the incidence conditions $(Z_1, \dots, Z_l)$, and assume that each Z_i is transversal to each toric stratum, as in Proposition 3.23. Assume also that each Z_i satisfies Assumption 3.6 and choose $\widetilde{Z}_i \subset X_1$ for each i. Then φ can be uniquely deformed into a stable map $\varphi_1 \colon \mathbb{P}^1 \to X_1$ satisfying the incidence conditions $(\widetilde{Z}_1, \dots, \widetilde{Z}_l)$. Its degree β satisfies the property that the coarse-degree $\Delta_D \in \mathcal{P}^*$ associated with Δ is contained in the set of coarse-degrees corresponding to β (Definition 3.34).*

Proof This follows from the transversality of φ with respect to the incidence conditions. See the proof of [14, Proposition 7.3]. ∎

For each i, we take a family of cycles $\mathcal{Z}_i$ over the base B as in Subsection 3.2.2. Then the above proposition can be formulated for this family, giving lifts $\varphi_t \colon C \to X_t$ for every $t \in [0, 1]$. There may be several choices for $\mathcal{Z}_i$ as in Definition 3.7. However, this does not matter (see Lemma 3.38).

We can prove the converse of Proposition 3.36.

Proposition 3.37 *Consider the same situation as Proposition 3.36. Then there is a positive number T with the following property: For any t with $0 < t < T$, assume that there is a rational stable map $\varphi_t\colon \mathbb{P}^1 \to X_t$ of degree β satisfying the incidence conditions given by the restrictions $\mathcal{Z}_i|_{X_t}$. Then φ_t is contained in the ones constructed in Proposition 3.36 (with X_1 replaced by X_t).*

Proof Suppose the statement is false. Then there is a sequence of rational stable maps $\psi_{t_i}\colon \mathbb{P}^1 \to X_{t_i}$ of degree β, with $t_i \to 0$ as $i \to \infty$ satisfying the incidence conditions, but not contained in the ones constructed in Proposition 3.36. By Gromov's compactness theorem, we may assume that ψ_{t_i} converges to a limit rational stable map $\psi\colon C' \to X_0$ satisfying the incidence conditions $(Z_1, \dots, Z_l)$, whose homology class is $\phi_*(\beta)$.

However, by Proposition 3.23 the domain of the map ψ must be the nonsingular rational curve and ψ is torically transverse. Then, by Fredholm regularity (Lemma 3.25), the sequence ψ_{t_i} must be contained in one of the families in Proposition 3.36, a contradiction. ∎

Lemma 3.38 *The family φ_t does not depend on the choice of $\mathcal{Z}_i$*

Proof A torically transverse map $\varphi\colon \mathbb{P}^1 \to X_0$ of the given degree satisfying constraints $\{Z_i\}$ does not intersect any Z_i on the toric boundary if the constraints $\{Z_i\}$ are generic, by dimensional reasoning (the transversality with respect to the incidence conditions, see the proof of [14, Proposition 7.3]). Since the choice of $\mathcal{Z}_i$ only determines how to lift Z_i around its intersection with the singular locus of X_0, the choice does not affect the configuration of rational curves satisfying the constraints. ∎

3.9 Counting Invariants via Toric Fano Degeneration

We define two counting invariants. One is directly related to the genuine Gromov–Witten invariants, and the other is the transversal Gromov–Witten type invariants considered in [14].

3.9.1 Gromov–Witten Invariants

As before, reparameterizing the base B, we assume X_1 satisfies the condition of Proposition 3.37.

We briefly recall the definition of Gromov–Witten invariants. See [2] for more information. Let X be a projective algebraic manifold and let $\beta \in H_2(X, \mathbb{Z})$. Let $\overline{\mathcal{M}}_{g,n+1}(X, \beta)$ be the moduli stack of stable maps of genus g with target X, where the homology class of the image is β, and let $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{virt}$ be its virtual fundamental class. $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{virt}$ has dimension

$$(1 - g)(\dim X - 3) - \int_\beta \omega_X + n.$$

Let $\pi_1\colon \overline{\mathcal{M}}_{g,n}(\beta, X) \to X^n$ be the natural map given by evaluations. Let $\alpha_1, \dots, \alpha_n$ be elements of $H^*(X, \mathbb{Q})$. Then the Gromov–Witten invariant $I_{g,n,\beta}(\alpha_1, \dots, \alpha_n)$ is defined by

$$I_{g,n,\beta}(\alpha_1,\dots,\alpha_n)=\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{virt}}\pi_1^*(\alpha_1\otimes\cdots\otimes\alpha_n).$$

We are concerned with the case when $g=0$.

To state the main theorem, we recall what we count and at the same time we introduce some notations. Let us fix a class $\beta\in H_2'$ (see Subsection 3.3) represented by a rational stable map. Let $\Delta_\beta=\{a_1,\dots,a_m\}$ be the set of coarse-degrees corresponding to β in the sense of Definition 3.34.

On X_0, we count (a priori not necessarily torically transverse) stable maps $\varphi\colon C\to X_0$ satisfying the following:

(a) The domain C is a connected, l-pointed rational prestable curve.
(b) φ satisfies general incidence conditions $\mathbf{Z}=(Z_1,\dots,Z_l)$ of codimension $\mathbf{d}=(d_1,\dots,d_l)$, satisfying Assumption 3.6 and the transversality assumption of Proposition 3.23.
(c) Then by Proposition 3.23, φ has to be torically transverse. In particular, the degree Δ of φ in the sense of Definition 3.13 is defined. The degree Δ should satisfy the property that the associated coarse-degree Δ_D is contained in the set Δ_β.

By Proposition 3.23, C has to be the nonsingular rational curve.

For the case of X_1, we count stable maps $\psi\colon C\to X_1$ such that the following hold:

(a) C is a connected, l-pointed rational prestable curve.
(b) ψ satisfies the incidence conditions determined by Definition 3.7 (which we denote by $\widetilde{\mathbf{Z}}$).
(c) ψ has degree $\beta\in H_2'\subset H_2(X_1,\mathbb{Z})$.

In this case too, C has to be the nonsingular rational curve (Lemma 3.22).

For the case of tropical curves, we count tropical curves $h\colon\Gamma\to N_{\mathbb{R}}$ such that the following hold:

(a) The graph Γ is connected, rational, and l-marked: $\mathbf{E}=(E_1,\dots,E_l)$.
(b) $h\colon\Gamma\to\mathbb{R}^n$ satisfies generic incidence conditions $\mathbf{A}=(A_1,\dots,A_l)$ of codimension $\mathbf{d}$ such that Z_i is the closure of an orbit of the subtorus corresponding to L_i, the linear subspace of $N_{\mathbb{R}}$ parallel to A_i.
(c) (Γ,h) has degree Δ and this Δ satisfies the same property as in the case of X_0.

Tropical curves have to be counted with multiplicity

$$\text{(W)}\qquad w(\Gamma,\mathbf{E})\cdot\mathfrak{D}(\Gamma,\mathbf{E},h,\mathbf{A})\cdot\prod_{i=1}^{l}\delta_i(\Gamma,\mathbf{E},h,\mathbf{A})$$

(see Subsection 3.2.1).

We denote these counting numbers by $N_{X_0}(\Delta_\beta,\mathbf{Z})$, $N_{X_1}(\beta,\widetilde{\mathbf{Z}})$, and $N_{\mathrm{trop}}(\Delta_\beta,\mathbf{L})$ respectively. If the virtual dimension of the moduli spaces of the curves is not equal to the codimension of the incidence conditions, we define these numbers to be 0. Our main theorem follows.

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Theorem 3.39 *The following equalities hold:*

$$N_{X_1}(\beta, \mathbf{Z}) = N_{X_0}(\Delta_\beta, \widetilde{\mathbf{Z}}) = N_{\mathrm{trop}}(\Delta_\beta, \mathbf{A}).$$

These numbers do not depend on the choice of $\mathbf{Z}$*,* $\widetilde{\mathbf{Z}}$ *or* $\mathbf{A}$*, if they are general.*

Proof In [14, Theorem 8.3 and Corollary 8.4], the equality between $N_{X_0}(\Delta_\beta, \mathbf{Z})$ and $N_{\mathrm{trop}}(\Delta_\beta, \mathbf{A})$, as well as the independence to the choice of incidence conditions are shown. So it suffices to show the equality between $N_{X_1}(\beta, \widetilde{\mathbf{Z}})$ and $N_{X_0}(\Delta_\beta, \mathbf{Z})$.

By Proposition 3.36, we see that any rational stable map of degree Δ such that Δ_D is contained in Δ_β and satisfying the incidence conditions $\mathbf{Z}$ in X_0 uniquely lifts to a rational stable map in X_1 satisfying the incidence conditions $\widetilde{\mathbf{Z}}$. By Lemma 3.35, the virtual dimensions of the rational stable map in X_0 and its lift in X_1 are the same. This shows that the inequality $N_{X_1}(\beta, \widetilde{\mathbf{Z}}) \geq N_{X_0}(\Delta_\beta, \mathbf{Z})$ holds. So the problem is to show that this lift is surjective, which is done in Proposition 3.37. ■

Corollary 3.40 $N_{X_1}(\beta, \widetilde{\mathbf{Z}})$ *is the Gromov–Witten invariant* $I_{0,l,\beta}(\alpha_1, \dots, \alpha_l)$*, where* α_i *is the Poincaré dual of the homology class of* $\widetilde{Z}_i$*.*

Proof By the theorem, the stable maps contributing to $N_{X_0}(\Delta_\beta, \mathbf{Z})$ and $N_{X_1}(\beta, \widetilde{\mathbf{Z}})$ are in one-to-one correspondence. Recall that the stable maps contributing to $N_{X_0}(\Delta_\beta, \mathbf{Z})$ are Fredholm regular. So the stable maps contributing to $N_{X_1}(\beta, \widetilde{\mathbf{Z}})$, which are small perturbations of those contributing to $N_{X_0}(\Delta_\beta, \mathbf{Z})$, are Fredholm regular, too (recall that we assume X_1 is sufficiently close to X_0). So the moduli space of rational stable maps of degree β intersecting with general incidence conditions $\widetilde{\mathbf{Z}}$ in X_1 is $N_{X_1}(\beta, \widetilde{\mathbf{Z}})$ points, and its virtual fundamental cycle is equal to itself. ■

Theorem 3.41 *Using the same notation as in Corollary 3.40, The Gromov–Witten invariant* $I_{0,l,\beta}(\alpha_1, \dots, \alpha_l)$ *for* $X = X_1$ *is equal to* $N_{\mathrm{trop}}(\Delta_\beta, \mathbf{A})$*.*

Proof This follows from Theorem 3.39 and Corollary 3.40. ■

3.9.2 Transversal Gromov–Witten Type Invariants

The transversality assumption of Proposition 3.23 for an incidence condition Z_i in X_0 does not hold in general. Nevertheless, even without this assumption we can define counting invariants of curves by restricting attention only to torically transversal curves, as we did in [14]. When the transversality assumption holds, all the curves satisfying the incidence conditions have to be torically transverse, so under this assumption, these two invariants are equal.

Thus, here we only assume Assumption 3.6 for the incidence conditions Z_i. In this case, we count the following. As before, we fix a class $\beta \in H_2'$ represented by a rational stable map.

On X_0, we count stable maps $\varphi\colon C \to X_0$ such that the following hold:

(a) The domain C is a connected, l-pointed rational prestable curve.
(b) φ is torically transverse, satisfying general incidence conditions $\mathbf{Z} = (Z_1, \dots, Z_l)$ of codimension $\mathbf{d} = (d_1, \dots, d_l)$ that satisfy Assumption 3.6.

(c) φ has a fixed degree Δ such that the associated coarse-degree Δ_D is contained in the set Δ_β.

By Proposition 3.23, C has to be the nonsingular rational curve.

For the case of X_1, we count stable maps $\psi\colon C \to X_1$ such that the following hold:

(a) C is a connected, l-pointed rational prestable curve.
(b) ψ satisfies the incidence conditions determined by Definition 3.7 (which we denote by $\widetilde{\mathbf{Z}}$) and ψ is a lift of a torically transverse stable map φ in X_0 as in Proposition 3.36. Note that in Proposition 3.36, the transversality assumption for Z_i is assumed, but for torically transverse φ, the conclusion holds without this assumption.
(c) ψ has degree $\beta \in H_2' \subset H_2(X_1, \mathbb{Z})$.

For the case of tropical curves, we count the same objects as before, namely, tropical curves $h\colon \Gamma \to N_{\mathbb{R}}$ such that the following hold:

(a) The graph Γ is connected, rational, and l-marked: $\mathbf{E} = (E_1, \dots, E_l)$.
(b) $h\colon \Gamma \to \mathbb{R}^n$ satisfies generic incidence conditions $\mathbf{A} = (A_1, \dots, A_l)$ of codimension $\mathbf{d}$, such that Z_i is the closure of an orbit of the subtorus corresponding to L_i, the linear subspace of $N_{\mathbb{R}}$ parallel to A_i.
(c) (Γ, h) has degree Δ and this Δ satisfies the same property as in the case of X_0.

As we mentioned before, tropical curves have to be counted with multiplicity (W).

We denote these three counting numbers by $N_{X_0}^{\mathrm{trans}}(\Delta, \mathbf{Z})$, $N_{X_1}^{\mathrm{trans}}(\beta, \widetilde{\mathbf{Z}})$, and $N_{\mathrm{trop}}(\Delta, \mathbf{L})$ respectively. Then the conclusion as Theorem 3.39 is proved by the same proof.

Theorem 3.42 *The following equalities hold:*

$$N_{X_1}^{\mathrm{trans}}(\beta, \mathbf{Z}) = N_{X_0}^{\mathrm{trans}}(\Delta_\beta, \widetilde{\mathbf{Z}}) = N_{\mathrm{trop}}(\Delta_\beta, \mathbf{A}).$$

These numbers do not depend on the choice of $\mathbf{Z}$, $\widetilde{\mathbf{Z}}$ *or* $\mathbf{A}$*, if they are general.*

Remark 3.43 Note that $N_{X_1}^{\mathrm{trans}}(\beta, \mathbf{Z})$ may not be a homological invariant, that is, if we change the cycle $\widetilde{Z}_i$ within the same homology class, but not necessarily related to Z_i as in Assumption 3.6, the number of rational stable maps incident to them may change. However, by the Fredholm regularity of the maps contributing to $N_{X_1}^{\mathrm{trans}}(\beta, \mathbf{Z})$, it gives a lower bound for the Gromov–Witten invariants

$$N_{X_1}^{\mathrm{trans}}(\beta, \mathbf{Z}) \le I_{0,l,\beta}(\alpha_1, \dots, \alpha_l),$$

where α_i is the Poincaré dual of the homology class of $\widetilde{Z}_i$.

4 Examples

4.1 Flag Manifold F_3

In this section, we give an example of calculation of Gromov–Witten invariants by tropical method in the case of flag manifold F_3, which parameterizes the full flags in $\mathbb{C}^3$. It is embedded in $\mathbb{P}^2 \times \mathbb{P}^2$ by Plücker embedding, and its toric degeneration is given explicitly by

$$\mathfrak{X} = \Big\{ \big([Z_1 : Z_2 : Z_3], [Z_{12} : Z_{13} : Z_{23}], t \big) \;\Big|\; Z_1 Z_{23} - Z_2 Z_{13} + t Z_3 Z_{12} = 0 \Big\}.$$

with the central fiber

$$X_0 = \Big\{ \big([Z_1 : Z_2 : Z_3], [Z_{12} : Z_{13} : Z_{23}] \big) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\Big|\; Z_1 Z_{23} - Z_2 Z_{13} = 0 \Big\}.$$

The Gelfand–Cetlin polytope (see [13] for details) of F_3 is defined by the inequalities

$$\begin{array}{ccccccccc}
\lambda_1 & & > & & \lambda_2 & & > & & \lambda_3 \\
& \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & \\
& & \lambda_1^{(2)} & & & & \lambda_2^{(2)} & & \\
& & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}$$

and graphically given by Figure 1. This is the same as the polytope associated with

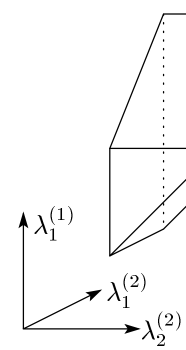

Figure 1: The Gelfand-Cetlin polytope for $n = 3$

the toric Fano variety X_0.

We use the coordinate $(x, y, z) = (\lambda_2^{(2)}, \lambda_1^{(2)}, \lambda_1^{(1)})$ for $M_{\mathbb{R}} \simeq \mathbb{R}^3$. Let π_1, π_2 be the natural projections of F_3 to the factors of $\mathbb{P}^2 \times \mathbb{P}^2$. The generic fiber of π_i, $i = 1, 2$, is isomorphic to $\mathbb{P}^1$, and the corresponding homology classes generate the second homology group $H_2(\mathrm{F}_3, \mathbb{Z})$, which is isomorphic to $\mathbb{Z}^2$. We denote these generators by $(1, 0)$ and $(0, 1)$. Then clearly

$$I_{0,1,(1,0)}(pt) = I_{0,1,(0,1)}(pt) = 1.$$

The two point function $I_{0,1,(1,1)}(pt, pt)$ is calculated by counting tropical curves with degree

$$(-1, 0, 0), (0, 1, 0), (0, -1, 1), (1, 0, -1).$$

These are the same as the tropical curves corresponding to the lines in $\mathbb{P}^3$, and so there is one such tropical curve through generic two points. Thus,

$$I_{0,1,(1,1)}(pt, pt) = 1.$$

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We calculate the three point functions $I_{0,3,\beta}(pt, pt, pt)$ for $\beta = (1,2), (2,1)$. It is easy to see that the tropical curves corresponding to $(1,2)$ have degree

$$(-1,0,0), (0,-1,1), (0,-1,0), (0,1,0), (0,1,0), (1,0,-1).$$

Projecting such a tropical curve $\bar{\Gamma}$ to the xz-plane, we still have a tropical curve Γ. It is a plane tropical curve with degree

$$(-1,0), (0,1), (1,-1).$$

That is, it is a tropical line with unbounded edges of these directions. Three unbounded edges (directions $(0,-1,0), (0,1,0), (0,1,0)$) are projected to points on the xz-plane.

Projecting three generic points $\bar{a}, \bar{b}, \bar{c}$ in $\mathbb{R}^3$ to this plane, we have three generic points a, b, c on $\mathbb{R}^2$. If the original tropical curve intersects generic three points, the projected curve also intersects three points. However, there is no tropical line on $\mathbb{R}^2$ that is incident to generic three points, so it follows that there is no rational tropical curve in $\mathbb{R}^3$of the degree given above which is incident to $\bar{a}, \bar{b}, \bar{c}$. This means

$$I_{0,3,(1,2)}(pt, pt, pt) = 0.$$

The identity $I_{0,3,(2,1)}(pt, pt, pt) = 0$ can be shown in the same way. By the same proof, it follows that

$$I_{0,k,(1,k-1)}(pt, \dots, pt) = I_{0,k,(k-1,1)}(pt, \dots, pt) = 0$$

for all $k > 2$.

In fact, we can calculate all genus zero Gromov–Witten invariants by tropical method. As noted above, the second homology group of the flag manifold F_3 is rank two, and it is generated by rational curves whose classes we denoted by $(1,0)$ and $(0,1)$. We rewrite these as l_1, l_2.

In X_0, these classes are tropically represented by lines with direction vectors $(1,0,0)$ and $(0,1,0)$. These curves lift to X_1, so, considered as incidence conditions, they satisfy Assumption 3.6 and the transversality condition of Proposition 3.23. Real four dimensional classes are evaluated using the divisor axiom. Generators of these classes s_1, s_2 are uniquely determined by the relations $s_i \cap l_j = \delta_{ij}$.

For a class $\beta = (s,t) \in H_2(\mathrm{F}_3, \mathbb{Z})$, the set of coarse-degrees Δ_β corresponding to β (Definition 3.34) is given by the following set of degrees for rational tropical curves.

Lemma 4.1 *A coarse-degree $\widetilde{\Delta}$ belongs to $\Delta_\beta = \Delta_{(s,t)}$ if and only if the following conditions are satisfied.*

Representing a coarse-degree as an integer valued function on the set of primitive vectors in $N = \mathbb{Z}^3$,

$$\begin{aligned} &\widetilde{\Delta}(-1,0,0) = s, &&\widetilde{\Delta}(0,1,0) = t,\\ &\widetilde{\Delta}(1,0,0) = s_1, &&\widetilde{\Delta}(1,0,-1) = s_2,\\ &\widetilde{\Delta}(0,-1,0) = t_1, &&\widetilde{\Delta}(0,-1,1) = t_2, \end{aligned}$$

where $s_1 + s_2 = s$, $t_1 + t_2 = t$ and $t_2 = s_2$. The others are zero.

Then we have the following proposition.

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Proposition 4.2 *Let*

$$pt \in H_0(\mathrm{F}_3, \mathbb{Z}), l_1, l_2 \in H_2(\mathrm{F}_3, \mathbb{Z}), s_1, s_2 \in H_4(\mathrm{F}_3, \mathbb{Z})$$

be the generators as above and let $pt^\vee, l_1^\vee, l_2^\vee, s_1^\vee, s_2^\vee$ *be their dual cohomology classes.*
Let $\beta = (s, t)$ *be the degree of the curve as above. Then, when*

$$2(s+t) = 2a + b + c,$$

we have,

$$I_{0,k,\beta}((pt^\vee)^a, (l_1^\vee)^b, (l_2^\vee)^c, (s_1^\vee)^d, (s_2^\vee)^e) = s^d t^e I_{0,k',\beta}((pt^\vee)^a, (l_1^\vee)^b, (l_2^\vee)^c),$$

where

$$k = a + b + c + d + e, \quad k' = a + b + c.$$

The number

$$I_{0,k',\beta}\big((pt^\vee)^a, (l_1^\vee)^b, (l_2^\vee)^c\big)$$

is the same as the number of k'*-marked rational tropical curves of degree* Δ *incident to generic* a *points, generic* b *lines with direction* $(1, 0, 0)$ *and generic* c *lines with direction* $(0, 1, 0)$*. The degree* Δ *should satisfy the condition that the associated coarse-degree* Δ_D *belongs to* $\Delta_{(s,t)}$*. When* $2(s + t) \neq 2a + b + c$*, the invariant is zero.*

4.2 Moduli Space of Rank 2 Bundles on a Curve of Genus Two

In this subsection, we give another example of calculation of Gromov–Witten invariants of a Fano manifold of particular interest via tropical method. Newstead [10] and Narasimhan and Ramanan [9] showed that the moduli space of stable rank two vector bundles with a fixed determinant of odd degree on a genus two curve defined as the double cover of $\mathbb{P}^1$ branched over $\{\omega_0, \dots, \omega_5\} \subset \mathbb{C}$, is a Fano complete intersection

$$X = Q_1 \cap Q_2$$

of two quadrics

$$Q_1 : \sum_{i=0}^{5} x_i^2 = 0, \quad Q_2 : \sum_{i=0}^{5} \omega_i x_i^2 = 0$$

in $\mathbb{P}^5$. The Betti numbers of X are easy to calculate:

$$b_0 = b_2 = b_4 = b_6 = 1, \quad b_3 = 4,$$

and the others are 0. Then X has several different structures of integrable systems, known as Goldman systems [3], and X can be degenerated (without considering integrable system) to Fano toric varieties in several ways. Nevertheless, there seems to be no known toric degeneration of X as a Fano integrable system in the sense of [13]. In fact, we see that, from the view point of toric degenerations of integrable systems and enumerations of holomorphic curves associated with them, Goldman systems are not good ones (see below).

By deforming the defining equations Q_1 and Q_2 (and changes of variables), we have, for example, three toric degenerations, whose toric varieties are given by

(a) $xy = zw, zw = uv$,
(b) $x^2 = zw, zw = uv$,
(c) $x^2 = yz, w^2 = uv$,

where u, v, w, x, y, z are homogeneous coordinates of $\mathbb{P}^5$. The moment polytopes of them are octahedron, quadrangular pyramid and tetrahedron respectively. Jeffrey and Weitsman [5] explicitly described the Goldman system in this case and showed that the moment polytope will be either a quadrangular pyramid or a tetrahedron, depending on the pants decomposition of the genus two curve. However, the toric varieties associated with these polytopes do not satisfy Assumption 3.8 (namely, they do not admit small resolutions), so they are not very good from the enumerative point of view.

Here we investigate the case of the octahedron. The torus action is given by

$$[x{:}y{:}z{:}w{:}u{:}v] \mapsto [\alpha x{:}\beta y{:}\gamma z{:}\alpha\beta\gamma^{-1}w{:}\alpha\beta u : v], \alpha, \beta, \gamma \in \mathbb{C}^*.$$

The moment polytope is the convex hull of

$$\{(\lambda, 0, 0), (0, \lambda, 0), (0, 0, \lambda), (\lambda, \lambda, -\lambda), (\lambda, \lambda, 0), (0, 0, 0)\}, \lambda \geq 0,$$

and the defining inequalities are given by

$$\begin{array}{ll} \ell_1(u) = \langle (0, 1, 1), u\rangle \geq 0, & \ell_2(u) = \langle (-1, 0, 0), u\rangle + \lambda \geq 0, \\ \ell_3(u) = \langle (0, -1, 0), u\rangle + \lambda \geq 0, & \ell_4(u) = \langle (1, 0, 1), u\rangle \geq 0, \\ \ell_5(u) = \langle (0, 1, 0), u\rangle \geq 0, & \ell_6(u) = \langle (-1, 0, -1), u\rangle + \lambda \geq 0, \\ \ell_7(u) = \langle (0, -1, -1), u\rangle + \lambda \geq 0, & \ell_8(u) = \langle (1, 0, 0), u\rangle \geq 0. \end{array}$$

It is easy to see that X_0 has a small resolution. In fact, all the singularities of X_0 are locally isomorphic to the singularity of the degeneration of F_3. As before, by the Fredholm regularity of torically transversal rational curve in toric varieties ([1]), we have the following.

Proposition 4.3 *Any torically transverse rational stable map in X_0 can be deformed to a rational stable map in X.*

By Proposition 3.9, it is easy to see that in this case there is a natural isomorphism

$$H_2(X, \mathbb{Z}) \cong H_2(X_0, \mathbb{Z}).$$

One sees that there are four families of $\mathbb{P}^1$s on X_0 corresponding to the pairs of parallel facets of the octahedron. These curves have two dimensional freedom (given by the parallel transport) to move in X_0. These are all homologous, and generate $H_2(X_0, \mathbb{Z})$. By the isomorphism above, any of their lifts generate $H_2(X, \mathbb{Z})$, due to Proposition 4.3.

Now since the singular points of X_0 are locally isomorphic to that of the degeneration of F_3, the inverse images of them under the gradient flow $\phi_{grH,t}$ are three dimensional spheres. On the other hand, X is ruled by any one of the two dimensional families of $\mathbb{P}^1$ that are the lifts of the rulings on X_0. General members of these rulings on X_0 do not intersect with the singular points, so general members of the corresponding rulings on X do not intersect with the three dimensional spheres above.

Let p_i, $i = 1, \dots, a$ be general points on X_0 and l_j, $j = 1, \dots, b$ be general lines in one of the rulings of X_0, $a, b \in \mathbb{Z}_{\geq 0}$. Extend them to a holomorphic family $p_{i,t}, l_{j,t}$ on $\mathfrak{X} \to \mathbb{C}$ around the origin of $\mathbb{C}$. The following is a consequence of Propositions 3.36 and 3.37. Note that when $\psi\colon C \to X_0$ is a torically transverse holomorphic map, it

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gives a natural map $\widetilde{\psi}\colon X \to \widetilde{X}_0$, where $\widetilde{X}_0$ is a small resolution of X_0. The proof of Lemma 3.10 shows that this gives a map

$$\pi^*\colon H_2(X_0, \mathbb{Z}) \to H_2(\widetilde{X}_0, \mathbb{Z}),$$

that splits the pushforward map

$$\pi_*\colon H_2(\widetilde{X}_0, \mathbb{Z}) \to H_2(X_0, \mathbb{Z}).$$

Proposition 4.4 *There is a natural one-to-one correspondence between the families of rational stable maps $\varphi\colon C_t \to X_t$ with $c_1(X_t)(\varphi(C_t)) = 2a + b$ intersecting $p_{i,t}, l_{j,t}$ for $t \neq 0$, and the rational stable maps $\psi\colon C' \to X_0$ with*

$$c_1(\widetilde{X}_0)(\pi^*(\psi(C'))) = 2a + b$$

intersecting p_i, l_j. The domains C_t, C' are in fact $\mathbb{P}^1$, and ψ is torically transverse. ■

Remark 4.5 In the case of degeneration to the quadrangular pyramid or to the tetrahedron, the rulings of X_t are broken and it seems difficult to calculate the invariants of X_t directly from the tropical calculation on X_0. Also, as we mentioned above, X_0 does not satisfy Assumption 3.8, and the behavior of holomorphic curves may change when we move from X_t to X_0. This is the reason that Goldman systems may not be good from the view point of toric degenerations of integrable systems (or from the view point of enumerative geometry).

Concerning this point, it is an interesting problem to find a natural construction of a structure of an integrable system on X that torically degenerate to X_0 (note that we can pull-back the toric integrable system structure on X_0 to X by ϕ_{grH}^{-1}. However, its geometric meaning is not clear, compared to Goldman systems).

By Propositions 4.3 and 4.4, we can compute the Gromov–Witten invariants of X by counting curves in X_0, which in turn can be calculated by tropical method. In this case, $H_2(X, \mathbb{Z})$ is free of rank one and so we can parameterize it by the set of integers. Let l be the generator of $H_2(X, \mathbb{Z})$ represented by a rational stable map. Let h be the generator of $H_4(X, \mathbb{Z})$ such that the intersection pairing satisfies $l \cdot h = 1$.

In this case, the set $\Delta_{a\cdot l}$ of coarse-degrees, $a \in \mathbb{N}$ is described as follows.

Lemma 4.6 *A coarse-degree $\widetilde{\Delta}$ belongs to $\Delta_{a\cdot l}$ if and only if the following conditions are satisfied.*

Representing a coarse-degree as an integer valued function on the set of primitive vectors in $N = \mathbb{Z}^3$,

$$\begin{aligned}
\widetilde{\Delta}(0,1,1) &= \widetilde{\Delta}(0,-1,-1) = a_1,\\
\widetilde{\Delta}(1,0,0) &= \widetilde{\Delta}(-1,0,0) = a_2,\\
\widetilde{\Delta}(0,1,0) &= \widetilde{\Delta}(0,-1,0) = a_3,\\
\widetilde{\Delta}(1,0,1) &= \widetilde{\Delta}(-1,0,-1) = a_4,
\end{aligned}$$

where $a_1 + a_2 + a_3 + a_4 = a$. The others are zero.

Then we have the following proposition.

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Proposition 4.7 *When $2m = 2a + b$, we have*

$$I_{0,k,m\cdot l}\big((pt^\vee)^a,(l^\vee)^b,(h^\vee)^c\big) = m^c I_{0,k',m\cdot l}\big((pt^\vee)^a,(l^\vee)^b\big),$$

where $k = a + b + c$, $k' = a + b$. The number $I_{0,k',m\cdot l}((pt^\vee)^a,(l^\vee)^b)$ is the same as the number of k'-marked rational tropical curves of degree Δ incident to generic a points and generic b lines parallel to either $(1,0,0),(0,1,0),(0,1,1)$ or $(1,0,1)$. The degree Δ must satisfy the condition that the associated coarse-degree Δ is contained in $\Delta_{m\cdot l}$. When $2m \neq 2a + b$,

$$I_{0,k,m\cdot l}((pt^\vee)^a,(l^\vee)^b,(h^\vee)^c) = 0.$$

Remark 4.8 It might be interesting that the counting number of the tropical curves in the proposition is the same however we choose the number of lines parallel to $(1,0,0),(0,1,0),(0,1,1)$, or $(1,0,1)$, provided the total number is b. This is not a purely tropical consequence, but follows from the homological invariance of the Gromov–Witten invariants.

Using our method, we can also calculate the Gromov–Witten invariants with odd degree arguments. First, as we remarked above, the generators σ_i, $i = 1,2,3,4$ of $H_3(X)$ are collapsed to singular points of X_0 by the map ϕ_{grH}. If some $I_{0,l,\beta} \neq 0$ with σ_i in the argument, then there is a family of rational stable maps $\psi_t\colon C_t \to X_t$, $t \in (0,1]$ satisfying the incidence conditions. Assume that there are other arguments other than σ_is. By Gromov's compactness theorem, we can assume that ψ_t converges to a rational stable map in X_0.

By the above remark about the classes σ_i, the limit curve must intersect the singular points of X_0. On the other hand, it satisfies the incidence conditions induced from the classes other than σ_i. However, as in the proof of Proposition 3.23, such a curve belongs to a lower dimensional subvariety of the moduli space, and so it cannot satisfy these incidence conditions. The case where all the arguments are from σ_is can be dealt with by similar dimension counting argument. So we have the following proposition.

Proposition 4.9 *The number $I_{0,l,\beta}(\alpha_1,\dots,\alpha_k) = 0$ if one of α_i is a cohomology class of odd degree.*

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