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REGULAR CYCLIC ACTIONS ON COMPLEX PROJECTIVE SPACE WITH CODIMENSION-TWO FIXED POINTS

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Abstract

If M^{2n} is a cohomology $\mathbb{C}P^n$ and p is an odd prime, let G_p be the cyclic group of order p. A Type II_0 G_p action on M^{2n} is an action with fixed point set a codimension-2 submanifold and an isolated point. A Type II_0 G_p action is standard if it is regular and the degree of the fixed codimension-2 submanifold is one. If n is odd and M^{2n} admits a standard G_p action of Type II_0, then every Type II_0 G_p action on M^{2n} is standard and so, if n is odd, $\mathbb{C}P^n$ admits a G_p action of Type II_0 if and only if the action is standard.

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1. Introduction

A cohomology complex projective n-space is a smooth, closed, orientable $2n$-manifold, M^{2n}, such that there is a class $x \in H^2(M;\mathbb{Z})$ with the property that $H^*(M;\mathbb{Z}) = \mathbb{Z}[x]/\left(x^{n+1}\right)$. Let p be an odd prime and let G_p denote the cyclic group of order p. If M^{2n} admits a smooth G_p action, then the number of components of the fixed point set is at most p ([2], p. 378). Actions of Type II are actions with a fixed point set, M^{G_p}, consisting of two components, $F_1^{2k_1}$ and $F_2^{2k_2}$. Note that $n-1 = k_1 + k_2$ ([2], p. 378) and so we will say that M^{2n} admits an action of Type II_k if $M^{G_p} = F_1^{2k_1} \cup F_2^{2k_2}$ and $k = \min(k_1, k_2)$. The focus of this paper will be G_p actions of Type II_0, that is, G_p actions fixing a codimension-2 submanifold F^{2n-2}, and an isolated point.

Two invariants are associated with a G_p action of Type II_0: the degree of F^{2n-2}, the codimension-2 component of the fixed point set, and the complex representation of G_p determined by the tangent space at the isolated fixed point, $\tau_*(M^{2n})$. If i : $K^{2n-2} \subset M^{2n}$ is the inclusion map of a closed, connected, orientable submanifold

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and d is an integer, we say that the degree of K^{2n-2} is d if $i_*[K]$ is dual to dx. The degree of K^{2n-2} depends on the choice of a generator $x \in H^2(M; \mathbb{Z})$ and orientations of K^{2n-2} and M^{2n}. The orientations of M^{2n} and K^{2n-2} are chosen as follows. A generator $x \in H^2(M; \mathbb{Z})$ is chosen and then M^{2n} is oriented by requiring that $x^n[M]$ is positive; K^{2n-2} is oriented by requiring that $(i^*x)^{n-1}[K]$ is positive if it is nonzero. With these conventions, the degree of K^{2n-2} is positive if it is nonzero. The degree of F^{2n-2}, d, is one invariant associated with a G_p action of Type II_0. The normal bundle of $F^{2n-2} \subset M^{2n}$ has a complex structure and the eigenvalue of the action of a generator g of G_p on the normal bundle of $F^{2n-2} \subset M^{2n}$ is $\lambda = \exp(2\pi i/p)$ if g is chosen properly. The second invariant associated with a G_p action of Type II_0, the tangent space at the isolated fixed point, $\tau_*(M^{2n})$, is a complex representation of G_p, and with the right choice of complex structure, the eigenvalues of the differential of g are contained in the set $\{\lambda^j : 1 \leq j \leq \mu\}$, where $\mu = (p-1)/2$. Let m_j be the multiplicity of the eigenvalue λ^j, $1 \leq j \leq \mu$. Note that $m_1 + m_2 + \cdots + m_\mu = n$. A $(\mu+1)$-tuple of integers, $(d; m_1, m_2, \ldots, m_\mu)$ is associated in this way with a G_p action of Type II_0 on M^{2n}.

A G_p action of Type II_0 on M^{2n} is said to be *regular* if the $(\mu+1)$-tuple associated with the action has the form $(d; n, 0, \ldots, 0)$, that is, $m_1 = n$ and $m_j = 0, 2 \leq j \leq \mu$. Note that a G_3 action of Type II_0 is automatically regular since $\mu = 1$. A G_p action of Type II_0 on M^{2n} is *standard* if the action is regular and $d = 1$, that is, the $(\mu+1)$-tuple associated with the action has the form $(1; n, 0, \ldots, 0)$. A G_3 action of Type II_0 is standard if $d = 1$. It is not known if every G_p action of Type II_0 on $\mathbb{C}P^n$, complex projective n-space, is standard. Our first result will allow us to settle this question about $\mathbb{C}P^n$ if n is odd.

THEOREM A. *Suppose that M^{2n} is a cohomology complex projective n-space and that p is an odd prime. If n is odd and M^{2n} admits a standard G_p action of* Type II_0, *then every* Type II_0 *G_p action on M^{2n} is a standard action.*

If n is arbitrary, then $\mathbb{C}P^n$ admits standard G_p actions of Type II_k for every prime p and every k such that $0 \leq k \leq [(n-1)/2]$. Our next theorem follows from this observation and Theorem A.

THEOREM B. *Suppose that p is an odd prime. If n is odd, then $\mathbb{C}P^n$ admits a G_p action of* Type II_0 *if and only if the action is standard.*

A cohomology projective n-space, M^{2n}, is said to have a *standard* Pontrjagin class $p_*(M^{2n}) \in H^*(M; \mathbb{Z})$ if $p_*(M^{2n}) = (1+x^2)^{n+1}$. Note that $\mathbb{C}P^n$ has a standard Pontrjagin class but not every smooth manifold in the homotopy type of $\mathbb{C}P^n$ has a standard Pontrjagin class ([11], [7], Theorem 3.1). If $n \leq 4$ and M^{2n} admits a G_p action of Type II_0, then it is a standard action and the Pontrjagin class of M^{2n} is standard ([3], Theorem A (i) (ii) ($n \leq 3$, $p \geq 3$, $n = 4$, $p > 3$), [5] Theorem E ($n = 4$, $p = 3$)). For some primes p, the existence of a G_p action of Type II_0 on a cohomology $\mathbb{C}P^5$, M^{10}, implies that the action is a standard action and the Pontrjagin class of M^{10} is standard ([3], Theorem A (iii)). In our next theorem, for an arbitrary integer n, we find a set of odd primes p such that the existence of a standard or a regular G_p action of Type II_0 on M^{2n} implies that the Pontrjagin class of M^{2n} is standard.

THEOREM C. *Suppose that M^{2n} is a cohomology complex projective n-space and that p is an odd prime. If $n < p+3$ and M^{2n} admits a standard G_p action of* Type II_0, *then the Pontrjagin class of M^{2n} is standard. If $n < p+1$ and M^{2n} admits a regular G_p action of* Type II_0, *then the action is a standard action and the Pontrjagin class of M^{2n} is standard.*

Theorem C is sharp in the sense that if $n \geq p+3$, then there are infinitely many PL homotopy complex projective n-spaces, X^{2n}, such that each X^{2n} admits a standard locally linear PL G_p action of Type II_0 and the Pontrjagin class of X^{2n} is not standard and if $n \geq 2p+9$, then there are infinitely many smooth homotopy complex projective n-spaces, M^{2n}, such that each M^{2n} admits a standard smooth G_p action of Type II_0 and the Pontrjagin class of M^{2n} is not standard ([3], Proposition 0.3 and Theorem 0.3).

Theorem C can be combined with Theorem A to yield information about cohomology $\mathbb{C}P^n$ with a regular G_p action of Type II_0 if n is odd and $n < p+1$. This is new information in the case $n = 5$ and $p > 3$ which can be extended to include the case $p = 3$ because G_3 actions of Type II_0 on M^{2n} are standard if $n \leq 7$ ([8], Theorem 1.7).

THEOREM D. *Suppose that M^{2n} is a cohomology complex projective n-space and that p is an odd prime. If n is odd and $n < p+1$ and M^{2n} admits a regular G_p action of* Type II_0, *then every* Type II_0 *G_p action on M^{2n} is a standard action and the Pontrjagin class of M^{2n} is standard.*

THEOREM E. *Suppose that M^{10} is a cohomology complex projective 5-space and that p is an odd prime. If M^{10} admits a regular G_p action of* Type II_0, *then every* Type II_0 *G_p action on M^{10} is standard and the Pontrjagin class of M^{10} is standard.*

There is a constant which depends only on the Pontrjagin class of M^{2n}, $c_{M^{2n}}$, such that if M^{2n} admits a G_p action of Type II_0 and $p \geq c_{M^{2n}}$, then the action and the Pontrjagin class of M^{2n} are standard ([4], Theorem A). It is known that $c_{M^{2n}} \geq n+2$ ([4], Corollary 2.4). Theorem C says that if M^{2n} admits a *regular* G_p action of Type II_0, then the action and the Pontrjagin class of M^{2n} are standard if $p > n-1$. Theorem D states that if n is odd and M^{2n} admits a *regular* G_p action of Type II_0, then *every* G_p action of Type II_0 is standard and the Pontrjagin class of M^{2n} is standard if $p \geq n$.

It is possible to prove theorems like Theorem A which include some even values of n. If $n \not\equiv 0, 8$ or $14 \pmod{16}$ and M^{2n} is a homotopy $\mathbb{C}P^n$ with a standard Type II_0 G_p action, then every regular Type II_0 G_p action is standard (Theorem A and [6], Theorem 1.4). It follows that if $n \not\equiv 0, 8$ or $14 \pmod{16}$ and M^{2n} is a homotopy $\mathbb{C}P^n$ with a standard G_3 action of Type II_0, then every Type II_0 G_3 action on M^{2n} is standard and so $n \not\equiv 0, 8$, or $14 \pmod{16}$ means every G_3 action of Type II_0 on $\mathbb{C}P^n$ is standard. Our last two results are stronger theorems about G_3 actions of Type II_0.

THEOREM F. *Suppose that M^{2n} is a homotopy complex projective n-space. If $n \not\equiv 0$, 16, or 30 (mod 32) and M^{2n} admits a standard G_3 action of* Type II_0, *then every* Type II_0 G_3 *action on M^{2n} is standard.*

THEOREM G. *If $n \not\equiv 0$, 16, or 30 (mod 32) or $n < 30$, then $\mathbb{C}P^n$ admits a G_3 action of* Type II_0 *if and only if the action is standard.*

This paper is organized as follows. Section 2 contains a discussion of the Atiyah-Singer g-Signature Formula for G_p actions of Type II_0 (Theorems 2.1 and 2.6) as well as the proofs of Theorem A (Theorem 2.12), Theorem B (Corollary 2.13), Theorem C (Theorem 2.21), and Theorems D and E (Corollary 2.23). Section 3 contains more discussion of the Atiyah-Singer g-Signature Formula for G_p actions of Type II_0 (Table 3.5) as well as a numerical congruence (formula (3.13)) which is useful in the study of G_3 actions of Type II_0. Section 4 contains some combinatorial results which will be used with the numerical congruence developed in Section 3. Section 5 contains the applications of the materials in Sections 3 and 4 to the proofs of Theorem F (Corollary 5.7) and G (Corollary 5.8).

2. Degrees, eigenvalues, and an Atiyah-Singer g-signature formula

Berend and Katz have formulated the Atiyah-Singer g-Signature Formula in a way which separates topology and number theory ([1], Theorem 2.2). Topology appears in the formula as integer valued quasi-signatures and number theory is represented in the formula by the complex numbers $\alpha_j = (\lambda^j + 1)(\lambda^j - 1)^{-1}$, $1 \le j \le \mu$. The quasi-signatures are signatures of self-intersections in the special case of a component of the fixed point set of codimension-2. If $K^{2n-2} \subset M^{2n}$, let $K^{(s)}$ be the s-fold transverse self-intersection of K^{2n-2} with itself in M^{2n}, $0 \le s \le n$. Recall that the dimension of $K^{(s)}$ is $2(n-s)$, $K^{(0)} = M^{2n}$ and $K^{(1)} = K^{2n-2}$. If $K^{2n-2} \subset M^{2n}$ is dual to a class $y \in H^2(M; \mathbb{Z})$, we will write $K^{2n-2} = K_y^{2n-2}$.

THEOREM 2.1. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$ and that p is an odd prime. If M^{2n} admits a G_p action of* Type II_0 *fixing a submanifold, $F^{2n-2} \subset M^{2n}$, and having* *eigenvalue multiplicities $m_1, m_2, \dots, m_\mu$ at the isolated fixed point, then*

$$(2.2)\qquad \pm\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_\mu^{m_\mu} = \begin{cases} 1 + (\alpha_1^2 - 1)\sum\limits_{k=1}^{m} \operatorname{Sign} F^{(2k)}\alpha_1^{2k-2}, & n = 2m, \\ \alpha_1 + (\alpha_1^3 - \alpha_1)\sum\limits_{k=1}^{m} \operatorname{Sign} F^{(2k+1)}\alpha_1^{2k-2}, & n = 2m+1. \end{cases}$$

PROOF. To begin with, suppose that M^{2n} is an *arbitrary* smooth, closed, orientable $2n$-manifold which admits a G_p action with fixed point set $M^{G_p} = F_y^{2n-2} \cup pt$. If the action of G_p on the normal bundle of F is such that the eigenvalue of the generator g is λ, then the g-signature $\operatorname{Sign}(g, M)$ ([1], formula (2.1)) is given by

$$(2.3)\qquad \operatorname{Sign}(g, M) = \frac{\lambda e^{2\overline{y}} + 1}{\lambda e^{2\overline{y}} - 1} L_*(F)[F] \pm \alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_\mu^{m_\mu},$$

where $\overline{y} \in H^2(F; \mathbb{Z})$ is the image of $y \in H^2(M; \mathbb{Z})$ and $L_*(F)$ is the Hirzebruch L-class of F ([10], p. 224). Berend and Katz ([1], formula (8.1)) have shown that

$$(2.4)\qquad \frac{\lambda e^{2\overline{y}} + 1}{\lambda e^{2\overline{y}} - 1} L_*(F)[F] = \begin{cases} -(\alpha_1^2 - 1)\sum\limits_{k=1}^{m} \operatorname{Sign} F^{(2k)}\alpha_1^{2k-2}, & n = 2m, \\ \alpha_1 \operatorname{Sign} F + (\alpha_1^3 - \alpha_1)\sum\limits_{k=1}^{m} \operatorname{Sign} F^{(2k+1)}\alpha_1^{2k-2}, & n = 2m+1. \end{cases}$$

Formulas (2.3) and (2.4) together imply that

$$(2.5)\qquad \pm\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_\mu^{m_\mu} - \operatorname{Sign}(g, M) = \begin{cases} (\alpha_1^2 - 1)\sum\limits_{k=1}^{m} \operatorname{Sign} F^{(2k)}\alpha_1^{2k-2}, & n = 2m, \\ -\alpha_1 \operatorname{Sign} F - (\alpha_1^3 - \alpha_1)\sum\limits_{k=1}^{m} \operatorname{Sign} F^{(2k+1)}\alpha_1^{2k-2}, & n = 2m+1. \end{cases}$$

If M^{2n} is a cohomology $\mathbb{C}P^n$, then we know that if the orientations are chosen in the way described in the introduction, then $\operatorname{Sign}(g, M) = \operatorname{Sign} M = +1, n = 2m$, and $\operatorname{Sign}(g, M) = 0$, $\operatorname{Sign} F = +1, n = 2m+1$ ([5], Lemma 4.1), and so (2.5) is (2.2). □

If M^{2n} is a cohomology $\mathbb{C}P^n$, let $DE_p(M^{2n})$ be the set of $(\mu+1)$-tuples $(d; m_1, m_2, \dots, m_\mu)$ defined by the condition that $(d; m_1, m_2, \dots, m_\mu) \in DE_p(M^{2n})$ if M^{2n} admits a G_p action of Type II_0 fixing a codimension-2 submanifold of degree d and having eigenvalue multiplicities $m_1, m_2, \dots, m_\mu$ at the isolated fixed point. Let

$D_p(M^{2n})$ be the projection of $DE_p(M^{2n})$ on the first factor, that is, $d \in D_p(M^{2n})$ if M^{2n} admits a G_p action of Type II_0 fixing a codimension-2 submanifold of degree d. Note that if $d \in D_p(M^{2n})$, then $d \not\equiv 0 \pmod p$ ([2], pp. 378–383) and we agree that $d > 0$. There are polynomial functions of a complex variable z with *integer* coefficients, $P(z)$, and $Q_d(z)$ ([6], Definition 4.1), which can be used to expose the degree of F in (2.2). If n is a positive integer, let $f(n)$ be $n!$ divided by a maximal power of 2.

THEOREM 2.6. ([6], Theorem 4.4) *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$ and that p is an odd prime. If $(d; m_1, m_2, \ldots, m_\mu) \in DE_p(M^{2n})$, then*

(2.7)
$$\pm f(n)\alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_\mu^{m_\mu} = \begin{cases} f(n)+f(n)d^2(\alpha_1^2-1)P(d\alpha_1)+d^2(1-d^2)(\alpha_1^2-1)Q_d(d\alpha_1), & n \text{ even}, \\ f(n)\alpha_1+f(n)d^3(\alpha_1^3-\alpha_1)P(d\alpha_1)+d^3(1-d^2)(\alpha_1^3-\alpha_1)Q_d(d\alpha_1), & n \text{ odd}. \end{cases}$$

We will not need an explicit formula for $P(z)$ (formula (2.14)) until we prove Theorem C and we will not need an explicit formula for $Q_d(z)$ (formula (3.4)) until we prove Theorems F and G. We record (2.7) in the case $p = 3$ where $\mu = 1$ and $\alpha = -i/\sqrt{3}$. Note that the plus sign in (2.7) holds in the case $p = 3$ because we will see in the proof of Lemma 2.15 that the plus sign in (2.7) holds when the action is regular and G_3 actions of Type II_0 are automatically regular. If t is a positive integer, let $\varepsilon(t) = (3^t + (-1)^{t-1})/4$, and put $a(n) = f(n)\varepsilon([n/2])$. Note that $DE_3(M^{2n}) = D_3(M^{2n})$.

COROLLARY 2.8. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$. If $d \in D_3(M^{2n})$, then*

(2.9)
$$a(n) = \begin{cases} f(2m)d^2 3^{m-1} P\left(\frac{di}{\sqrt{3}}\right) + d^2(1-d^2)3^{m-1}Q_d\left(\frac{di}{\sqrt{3}}\right), & n = 2m, \\ f(2m+1)d^3 3^{m-1} P\left(\frac{di}{\sqrt{3}}\right) + d^3(1-d^2)3^{m-1}Q_d\left(\frac{di}{\sqrt{3}}\right), & n = 2m+1. \end{cases}$$

Because of the multiplication by $3^{[n/2]-1}$, the polynomials in (2.9) are polynomials in d^2 with integer coefficients, and so (2.9) implies that d^2 divides $a(n)$, if n is even, and d^3 divides $a(n)$, if n is odd. It follows from these divisibility conditions and the fact that elements of $D_3(M^{2n})$ are not divisible by 3 ([2], pp. 378–383) that if $n \le 7$, then $D_3(M^{2n})$ is either empty or $D_3(M^{2n}) = \{1\}$ ([8], Theorems 1.6 and 1.7).

Our next lemma, when used in concert with (2.7), will enable us to show that if M^{2n} admits a regular G_p action of Type II_0 fixing a codimension-2 submanifold of degree d, then every Type II_0 G_p action which fixes a codimension-2 submanifold of degree d is regular, that is, if $(d; n, 0, \ldots, 0)$ and $(d; m_1, m_2, \ldots, m_\mu)$ both belong to $DE_p(M^{2n})$, then $m_1 = n$ and $m_j = 0$, $2 \le j \le \mu$.

LEMMA 2.10. *Suppose that n is a positive integer, p is an odd prime, and that $\mu = (p-1)/2$. If $\sum_{j=1}^{\mu} m_j = n$, $m_j \geq 0$, $1 \leq j \leq \mu$, and $\pm\alpha_1^n = \alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_\mu^{m_\mu}$, then $m_1 = n$, $m_j = 0$, $2 \leq j \leq \mu$.*

PROOF. Note that $\alpha_j = -i\cot(j\pi/p)$, $1 \leq j \leq \mu$. Since $\mu\pi/p < \pi/2$, we have $\{j\pi/p : 1 \leq j \leq \mu\} \subset (0, \pi/2)$, and so $|\alpha_1| > |\alpha_2| > \cdots > |\alpha_\mu|$. The lemma follows immediately from this chain of inequalities. □

LEMMA 2.11. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$. If $(d; n, 0, \ldots, 0)$ and $(d; m_1, m_2, \ldots, m_\mu)$ both belong to $DE_p(M^{2n})$, then $m_1 = n$, $m_j = 0$, $2 \leq j \leq \mu$.*

PROOF. If $(d; n, 0, \ldots, 0)$ and $(d; m_1, m_2, \ldots, m_\mu)$ both belong to $DE_p(M^{2n})$, then it follows from (2.7) used twice that $\pm\alpha_1^n = \alpha_1^{m_1}\alpha_2^{m_2}\cdots\alpha_\mu^{m_\mu}$, since the right hand side of (2.7) depends only on d and α_1 and M^{2n} because $P(z)$ and $Q_d(z)$ depend only on d and M^{2n} ([6], Definition 4.1 or formulas (2.14) and (3.4) in this paper). Lemma 2.11 now follows immediately from Lemma 2.10. □

Lemma 2.11 shows that if there exists a regular G_p action of Type II_0 on M^{2n} fixing a codimension-2 submanifold of degree d, then every Type II_0 G_p action on M^{2n} which fixes a codimension-2 submanifold of degree d is regular. Our next result is Theorem A stated in terms of $DE_p(M^{2n})$.

THEOREM 2.12. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$ and that p is an odd prime. If n is odd and $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$, then $DE_p(M^{2n}) = \{(1; n, 0, \ldots, 0)\}$.*

PROOF. Suppose that $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$. This means that $1 \in D_p(M^{2n})$ and so, if n is odd, $D_p(M^{2n}) = \{1\}$ ([6], Theorem 1.1(2)). In other words, if $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$ and n is odd, then every element in $DE_p(M^{2n})$ has the form $(1; m_1, m_2, \ldots, m_\mu)$. If $(1; m_1, m_2, \ldots, m_\mu) \in DE_p(M^{2n})$ and $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$, it follows from Lemma 2.11 that $m_1 = n$ and $m_j = 0, 2 \leq j \leq \mu$. This completes the proof. □

COROLLARY 2.13. *If n is odd, then $DE_p(\mathbb{C}P^n) = \{(1; n, 0, \ldots, 0)\}$.*

PROOF. This follows immediately from Theorem 2.12 since for arbitrary n, $(1; n, 0, \ldots, 0) \in DE_p(\mathbb{C}P^n)$. □

Note that Theorem A is equivalent to Theorem 2.12 and Theorem B is equivalent to Corollary 2.13 since we have chosen our generator of G_p in such a way that $(d; m_1, m_2, \ldots, m_\mu)$ is determined by a regular action if and only if $m_1 = n$, $m_j = 0$,

$2 \le j \le \mu$, and $(d; m_1, m_2, \dots, m_\mu)$ comes from a standard action if and only if $d = 1$ and $m_1 = n$ and $m_j = 0$, $2 \le j \le \mu$. It has been conjectured that either $DE_p(M^{2n})$ is empty or $DE_p(M^{2n}) = \{(1; n, 0, \dots, 0)\}$ ([6], Conjecture 1.2). Corollary 2.13 affirms this conjecture in the special case $M^{2n} = \mathbb{C}P^n$, n odd.

In our next lemma, we prepare for the proof of Theorem C by studying the effect of the existence of a standard action on the signatures Sign $K_x^{(s)}$, $1 \le s \le n$, where $K_x^{2n-2} \subset M^{2n}$ is a submanifold of degree 1, that is, dual to the generator $x \in H^2(M; \mathbb{Z})$. Recall ([6], formula (4.2)) that the polynomial $P(z)$ satisfies the equation below.

$$(2.14) \qquad P(z) = \begin{cases} \sum_{k=1}^{m} \operatorname{Sign} K_x^{(2k)} z^{2k-2}, & n = 2m, \\ \sum_{k=1}^{m} \operatorname{Sign} K_x^{(2k+1)} z^{2k-2}, & n = 2m+1. \end{cases}$$

LEMMA 2.15. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$. If $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$, then if n is odd,* Sign $K_x = +1$, *and*

$$(2.16) \qquad \begin{cases} \sum_{k=1}^{m-1} (\operatorname{Sign} K_x^{(2k)} - 1)\alpha_1^{2k-2} = 0, & n = 2m, \\ \sum_{k=1}^{m-1} (\operatorname{Sign} K_x^{(2k+1)} - 1)\alpha_1^{2k-2} = 0, & n = 2m+1. \end{cases}$$

PROOF. If $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$, then $1 \in D_p(M^{2n})$ and so if n is odd, Sign $K_x = \pm 1$ ([6], Corollary 2.4(1)). We can conclude that Sign $K_x = 1$ if M^{2n} and K_x^{2n-2} are oriented in the fashion described in the second paragraph of the introduction. The next step is to observe that if the action is regular, then (2.2) and (2.7) must hold with the *plus* sign. To see this, recall that there are ring homomorphisms $\eta : \mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_\mu] \longrightarrow \mathbb{Z}/4\mathbb{Z}$ with the properties that $\eta(1) = 1$ and $\eta(\alpha_j) = \pm 1$, $1 \le j \le \mu$ ([1], Lemma 7.8). It follows that there is such an η with the properties that $\eta(1) = 1$ and $\eta(\alpha_1) = \pm 1$. If this η is applied to (2.2) or (2.7) with $m_1 = n$, $m_j = 0$, $2 \le j \le \mu$ and with the *minus* sign, a contradiction mod 4 is obtained since $\eta(\alpha_1^2 - 1) = 0$. Therefore, if n is arbitrary and $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$, then (2.7) holds with $d = 1$ and *plus* sign and so $P(\alpha_1) = (\alpha_1^{2[n/2]} - 1)(\alpha_1^2 - 1)^{-1}$. Formula (2.16) follows from this last observation about $P(\alpha_1)$ together with (2.14) and the fact that $P(z)$ is a monic polynomial since Sign $K_x^{(n)} = 1$ because $K_x^{(n)}$ is a single point and M^{2n} and K_x^{2n-2} are oriented in the way described in the introduction. □

Before proceeding we record the version of Lemma 2.15 which holds in the special case where M^{2n} is a homotopy $\mathbb{C}P^n$. If M^{2n} is a homotopy $\mathbb{C}P^n$ and $n \ge 3$, then the PL homeomorphism type of M^{2n} is determined by an n-2-tuple of splitting invariants $(\sigma_2, \sigma_3, \dots, \sigma_{n-1})$ [11]. The splitting invariants are integers and mod 2 integers and

σ_k often appears as s_{2k} ([12], p. 191). The splitting invariants with even subscript $\sigma_2, \sigma_4, \dots, \sigma_{2[(n-1)/2]}$, are integers which determine the Pontrjagin class of M^{2n} ([11], [7] Theorem 3.1). If $n - s$ is even, then Sign $K_x^{(s)} = 1 + 8\sigma_{n-s}$ and $\sigma_0 = 0$ because $K_x^{(n)}$ is a point and Sign $K_x^{(n)} = 1$ [11]. The next corollary follows immediately from these remarks and Lemma 2.15.

COROLLARY 2.17. *Suppose that $n \geq 3$ and M^{2n} is a homotopy $\mathbb{C}P^n$ with integral splitting invariants $\sigma_2, \sigma_4, \dots, \sigma_{2[(n-1)/2]}$. If $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$, then if n is odd, $\sigma_{n-1} = 0$, and for any n,*

$$\sum_{k=1}^{[n/2]-1} \sigma_{2([n/2]-k)}\alpha_1^{2k-2} = 0. \tag{2.18}$$

Equation (2.18) is a sufficient condition for the existence of a standard G_p action of Type II_0 on a PL homotopy $\mathbb{C}P^n$ with integral splitting invariants $\sigma_2, \sigma_4, \dots, \sigma_{2([n/2]-1)}$ if $n \geq p + 3$ ([3], Theorem 0.3 Note that $\sigma_k = s_{2k}$ and the equation here is in terms of α_1^{-2}). Our next lemma studies the effect of the existence of a standard G_p action of Type II_0 if $n < p + 3$.

LEMMA 2.19. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$. If $n < p + 3$ and $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$, then* Sign $K_x^{(s)} = 1$ *for every s such that $n - s$ is even and $0 < s < n$.*

PROOF. The degree of the minimal polynomial of α_1^2 is $(p-1)/2$ ([12], pp. 220–221) and so $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$ and $n < p + 3$ imply that Sign $K_x^{(2k)} = 1$, $1 \leq k < m$, if $n = 2m$, and Sign $K_x^{(2k+1)} = 1$, $1 \leq k < m$, if $n = 2m + 1$ in view of (2.16). If $(1; n, 0, \dots, 0) \in DE_p(M^{2n})$ and n is odd, then Sign $K_x = +1$ by Lemma 2.15 and the fact that Sign $K_x^{(2k+1)} = +1$, $1 \leq k < m$. □

LEMMA 2.20. *If M^{2n} is a cohomology $\mathbb{C}P^n$, then the Pontrjagin class of M^{2n} is standard if and only if* Sign $K_x^{(s)} = 1$ *for all s such that $n - s$ is even and $0 < s < n$.*

PROOF. If $L_*(M^{2n}) = 1 + L_1(M^{2n}) + \cdots + L_{[n/2]}(M^{2n})$ is the Hirzebruch L-class of M^{2n}, then Lemma 2.20 is equivalent to the assertion that $L_*(M^{2n})$ is standard, that is, $L_*(M^{2n}) = x^{n+1} \coth^{n+1} x$, if and only if Sign $K_x^{(s)} = 1$ for all s such that $n - s$ is even and $0 < s < n$. It follows from Hirzebruch Index Theorem, the uniqueness properties of the L-class ([10], p. 224), the fact that $K_x^{(s)}$ is dual to x^s, and induction on $n - s$, that the coefficient of x^{n-s} in $L_{(n-s)/2}(M^{2n})$ in $H^{2(n-s)}(M; \mathbb{Z})$ is the same as the coefficient of x^{n-s} in $x^{n+1} \coth^{n+1} x$ if and only if Sign $K_x^{(s)} = 1$ for all s such that $n - s$ is even and $0 < s < n$. Note that if n is even, we automatically have Sign $K_x^{(0)} =$ Sign $M^{2n} = 1$ since M^{2n} is a cohomology $\mathbb{C}P^n$ and this determines the coefficient of x^n in $L_{n/2}(M^{2n})$. □

THEOREM 2.21. *Suppose that M^{2n} is a cohomology $\mathbb{C}P^n$ and that p is an odd prime. If $n < p+3$ and $(1; n, 0, \ldots, 0) \in DE_p(M^{2n})$, then the Pontrjagin class of M^{2n} is standard. If $n < p+1$ and $(d; n, 0, \ldots, 0) \in DE_p(M^{2n})$, then $d = 1$ and the Pontrjagin class of M^{2n} is standard.*

PROOF. The first sentence in Theorem 2.21 is the sum of Lemmas 2.19 and 2.20. To establish the second assertion, we need only show that $n < p + 1$ and $(d; n, 0, \ldots, 0) \in DE_p(M^{2n})$ imply that $d = 1$. Suppose that $(d; n, 0, \ldots, 0) \in DE_p(M^{2n})$. It follows from (2.2) with + sign because the action is regular that

$$\begin{cases} \sum_{k=1}^{m} (\operatorname{Sign} F^{(2k)} - 1)\alpha_1^{2k-2} = 0, & n = 2m, \\ \sum_{k=1}^{m} (\operatorname{Sign} F^{(2k+1)} - 1)\alpha_1^{2k-2} = 0, & n = 2m+1. \end{cases} \tag{2.22}$$

If $n < p+1$, then it follows from (2.22) and the fact that the degree of the minimal polynomial of α_1^2 is $(p-1)/2$ ([12], pp. 220–221) that $\operatorname{Sign} F^{(n)} = 1$. But $\operatorname{Sign} F^{(n)} = d^n$ since F is dual to dx ([8], formula (2.9)) and so $d = 1$. □

Theorem 2.21 is Theorem C phrased in terms of $DE_p(M^{2n})$. Corollary 2.23 below is Theorem D phrased in terms of $DE_p(M^{2n})$ and it is an immediate consequence of Theorems 2.12 and 2.21. We observed that Theorem E is an immediate consequence of Theorem D in the case $n = 5$ and $p > 3$ and the fact that every Type II_0 G_3 action on M^{2n} is standard if $n \leq 7$ ([8], Theorem 1.7).

COROLLARY 2.23. *Suppose M^{2n} is a cohomology $\mathbb{C}P^n$. If n is odd and $n < p+1$ and $(d_1; n, 0, \ldots, 0)$ and $(d_2; m_1, m_2, \ldots, m_\mu)$ both belong to $DE_p(M^{2n})$, then $d_1 = d_2 = 1$ and $m_1 = n$, $m_j = 0$, $2 \leq j \leq \mu$, and the Pontrjagin class of M^{2n} is standard.*

3. The polynomial $Q_d(z)$

In order to prove Theorems F and G, we need the full g-signature formula for $p = 3$ (2.9) and so we review the construction of $Q_d(z)$ ([6], formula (4.3)). If M^{2n} is a cohomology $\mathbb{C}P^n$ and $K_{dx}^{2n-2} \subset M^{2n}$ is dual to $dx \in H^2(M; \mathbb{Z})$, then $\operatorname{Sign} K_{dx}^{(s)}$ can be expanded ([8], formula (2.9)) in terms of $\operatorname{Sign} K_x^{(2k+s)}$, $0 \leq k \leq (n-s)/2$, in such a way that $\operatorname{Sign} K_x^{(2k+s)}$, $1 \leq k \leq (n-s)/2$, appears in the expansion multiplied by a polynomial $R_{k,s}(d)$ with rational coefficients with the property that $f(n)R_{k,s}(d)$ is divisible by $d^s(1-d^2)$ and the quotient is a polynomial with *integer* coefficients, $\widehat{c}_{k,s}(d^2)$ ([6], formulas (3.5) and (3.8)). These polynomials depend only on k, s, and n and are the main ingredients in the recipe for $Q_d(z)$.

TABLE 3.5.

n	$Q_d(z)$
4	2
5	15
6	$30S^{(4)} + 23 - 17d^2 + 60z^2$
7	$315S^{(5)} + 294 - 231d^2 + 525z^2$
8	$210S^{(4)} + (161 - 119d^2)S^{(6)} + 132 - 176d^2 + 62d^4$ $+(420S^{(6)} + 462 - 378d^2)z^2 + 630z^4$

DEFINITION 3.1. If M^{2n} is a cohomology $\mathbb{C}P^n$ and $n - s$ is a positive even integer, then

$$\delta_s(d^2, x) = \sum_{l=1}^{(n-s)/2} \widehat{c}_{l,s}(d^2) \operatorname{Sign} K_x^{(2l+s)}. \tag{3.2}$$

DEFINITION 3.3. If M^{2n} is a cohomology $\mathbb{C}P^n$ and d is a positive integer, then

$$Q_d(z) = \begin{cases} \sum_{k=1}^{m-1} \delta_{2k}(d^2, x) z^{2k-2}, & n = 2m, \\ \sum_{k=1}^{m-1} \delta_{2k+1}(d^2, x) z^{2k-2}, & n = 2m+1. \end{cases} \tag{3.4}$$

The polynomials $Q_d(z)$ can be quite complicated. The table above contains $Q_d(z)$ for $4 \le n \le 8$. These values can be used in concert with (2.7) and (2.14) to write the full signature formula in terms of the signatures $S^{(s)} = \operatorname{Sign} K_x^{(s)}$ if $4 \le n \le 8$.

Table 3.5 shows why it is desirable to verify the conjecture that d is always 1 in (2.7) ([6], Conjecture 1.0). The advantage of (2.7) over (2.2), short of computing $Q_d(z)$, is that in (2.7), the degree d is exposed and the coefficients of $P(z)$ and $Q_d(z)$ are integers, making it easy to study the image of (2.7) in various finite rings if $d \ne 1$. If M^{2n} is a homotopy $\mathbb{C}P^n$, then $\operatorname{Sign} K_x^{(s)} = 1 + 8\sigma_{n-s} \equiv 1 \pmod 8$, if $n - s$ is even, and so it is useful to reduce (2.7) mod 8, at least if $p = 3$ (2.9). We now define an integer valued function to be used in this reduction.

DEFINITION 3.6. If $n - s$ is a positive even integer, then

$$\delta_s(d^2) = \sum_{l=1}^{(n-s)/2} \widehat{c}_{l,s}(d^2). \tag{3.7}$$

Note that $\delta_s(d^2)$ is a purely numerical function which does not depend on a particular cohomology $\mathbb{C}P^n$. If M^{2n} has a standard Pontrjagin class, then $\delta_s(d^2) = \delta_s(d^2, x)$ in view of Lemma 2.20. Theorems F and G are extensions of results which hold for odd n to some even values, and so we will assume at this point that $n = 2m$. If $n = 2m$ in (3.7), then $s = 2k$, $1 \le k \le m-1$, and $1 \le l \le m-k$.

DEFINITION 3.8. If $m \ge 2$, then

$$A_m(d^2) = \sum_{k=1}^{m-1} (-1)^{k-1} 3^{m-k} \delta_{2k}(d^2) d^{2k}. \tag{3.9}$$

DEFINITION 3.10. If $m \ge 2$, then

$$B_m(d^2) = \sum_{k=1}^{m} (-1)^{k-1} 3^{m-k} (1 + d^2 + \cdots + d^{2k-2}). \tag{3.11}$$

PROPOSITION 3.12. *Suppose that M^{4m} is a cohomology $\mathbb{C}P^{2m}$ such that* Sign $K_x^{(2j)} \equiv 1 \pmod{\rho}$, $1 \le j \le m$. *If* $1 \in D_3(M^{4m})$ *and* $d \in D_3(M^{4m})$, $d \ne 1$, *then*

$$A_m(d^2) \equiv f(2m) B_m(d^2) (\operatorname{mod} \rho). \tag{3.13}$$

PROOF. Suppose that M^{4m} is an arbitrary cohomology $\mathbb{C}P^{2m}$ and that $1 \in D_3(M^{4m})$. If $d \in D_3(M^{4m})$, $d \ne 1$, then it follows by using (2.9) in the case $n = 2m$ twice, first for d and then with d replaced by 1, subtracting the second equation from the first and then dividing by $1 - d^2$, that

$$\begin{aligned} 3^{m-1} d^2 Q_d\left(\frac{di}{\sqrt{3}}\right) \\ = f(2m) \sum_{k=1}^{m} (-1)^{k-1} 3^{m-k} \operatorname{Sign} K_x^{(2k)} (1 + d^2 + \cdots + d^{2k-2}). \end{aligned} \tag{3.14}$$

The next step is to note that (3.4) implies

$$3^{m-1} d^2 Q_d\left(\frac{di}{\sqrt{3}}\right) = \sum_{k=1}^{m-1} (-1)^{k-1} 3^{m-k} \delta_{2k}(d^2, x) d^{2k}. \tag{3.15}$$

If Sign $K_x^{(2j)} \equiv 1 \pmod{\rho}$, $1 \le j \le m$, then $\delta_{2k}(d^2, x) \equiv \delta_{2k}(d^2) \pmod{\rho}$ and so (3.13) follows from (3.9), (3.11), (3.14) and (3.15). □

Formula (3.13) will be our main tool in the proofs of Theorems F and G. Note that (3.13) holds with $\rho = 8$ if M^{4m} is a homotopy $\mathbb{C}P^{2n}$ satisfying the hypotheses of Proposition 3.12 and that (3.13) holds for any ρ and any $d \in D_3(\mathbb{C}P^{2m})$, $d \ne 1$. In Section 4, we determine $A_m(d^2)$ and $B_m(d^2)$ modulo 8, and in Section 5, we use the results of Section 4 to prove Theorems F and G.

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4. Combinatorics modulo 8

The function $A_m(d^2)$ is computable mod 16 if d is even and mod 8 if d is odd and $B_m(d^2)$ can be computed mod 8.

PROPOSITION 4.1. *If $m \geq 2$, then*

(4.2)
$$A_m(d^2) \equiv \begin{cases} 3^{m-2} f(2m)[(-1)^m(2m+1)+3] \pmod{16}, & d \equiv 2 \pmod 4, \\ 0 \pmod{16}, & d \equiv 0 \pmod 4. \end{cases}$$

$$\text{(4.3)} \qquad A_m(d^2) \equiv \begin{cases} f(2m)m \pmod 8, & m \text{ even and } d \text{ odd}, \\ 0 \pmod 8, & m \text{ odd and } d \text{ odd}. \end{cases}$$

PROPOSITION 4.4. *If $m \geq 2$, then*

$$\text{(4.5)} \qquad B_m(d^2) \equiv \begin{cases} m(m-1)/2 \pmod 8, & m \text{ even}, d \text{ odd}, \\ m-4 \pmod 8, & m \text{ even}, d \equiv 2 \pmod 4, \\ m \pmod 8, & m \text{ even}, d \equiv 0 \pmod 4. \end{cases}$$

$$\text{(4.6)} \qquad B_m(d^2) \equiv \begin{cases} m(m+1)/2 \pmod 8, & m \text{ odd}, d \text{ odd}, \\ -m+2 \pmod 8, & m \text{ odd}, d \text{ even}. \end{cases}$$

The proofs of Propositions 4.1 and 4.4 will be discussed below, but only formulas (4.2), (4.3), (4.5), and (4.6) will be used in Section 5 when we return to G_3 actions of Type II_0. The proof of Proposition 4.1 begins with the observation that the polynomials $\widehat{c}_{k,s}(d^2)$ ([6], formulas (3.5) and (3.8)) mentioned at the beginning of Section 3 can be computed exactly if $d = 2$ or 3.

LEMMA 4.7. *If k, s, and n are positive integers such that $n - s$ is even and $1 \leq k \leq (n-s)/2$, then*

$$\text{(4.8)} \qquad \widehat{c}_{k,s}(4) = (-1)^{k-1} f(n) \sum_{r=1}^{s} \binom{s}{r} \binom{k-1}{r-1} 3^{-1},$$

$$\text{(4.9)} \qquad \widehat{c}_{k,s}(9) = (-1)^{k-1} f(n) \sum_{r=1}^{s} 2^{3r-3} \binom{s}{r} \binom{k-1}{r-1} 3^{k-2r}.$$

PROOF. The polynomials $R_{k,s}(d)$ ([8], Definition (2.5)) satisfy the equation

$$R_{k,s}(d) = \sum_{i_1+i_2+\cdots+i_s=k} r_{i_1}(d) r_{i_2}(d) \cdots r_{i_s}(d), \tag{4.10}$$

where $r_k(d)$ occurs in signature expansions ([8], formula (2.9)) and $r_k(d) = d(1 - d^2)q_k(d^2)$ and $f(2k+1)q_k(d^2)$ is a polynomial with integer coefficients ([8], Lemma 2.14). It follows from the basic facts about $\widehat{c}_{k,s}(d^2)$ ([6], Lemma 3.4 and Definition 3.7) that if $Q_{k,s}(d^2)$ is the analogue of $R_{k,s}(d)$ formed with $q_k(d^2)$ and

$$Q^{+}_{k,s}(d^2) = \sum_{i_1+i_2+\cdots+i_s=k} q_{i_1}(d^2) q_{i_2}(d^2) \cdots q_{i_s}(d^2), \tag{4.11}$$

where the $+$ indicates that each i_j is positive in each partition of k, then

$$\widehat{c}_{k,s}(d^2) = f(n) \sum_{r=1}^{s} \binom{s}{r} (1 - d^2)^{r-1} Q^{+}_{k,r}(d^2). \tag{4.12}$$

Formulas (4.8) and (4.9) now follow from (4.11), (4.12), used together with the facts that $q_k(4) = (-1)^{k-1}3^{-1}$, $k \geq 1$, and $q_k(9) = (-1)^{k-1}3^{k-2}$, $k \geq 1$ ([6], Formula 3.6). □

COROLLARY 4.13. *If $n = 2m$ and $m \geq 2$, then*

$$\widehat{c}_{l,2}(4) = (-1)^{l-1} f(2m)(l+1)3^{-1}, \quad 1 \leq l \leq m-1. \tag{4.14}$$

$$\widehat{c}_{l,2k}(9) \equiv (-1)^{l-1} f(2m)(2k)3^{l-2} (\operatorname{mod} 8), \quad 1 \leq l \leq m-k. \tag{4.15}$$

PROOF. Formulas (4.14) and (4.15) follow immediately from (4.8) and (4.9). □

Formulas (4.14) and (4.15) are the only consequences of Lemma 4.7 that we will use. The next lemma follows immediately from (3.7), (4.14), (4.15) and the formula for the sum of a geometric series.

LEMMA 4.16. *If $n = 2m$ and $m \geq 2$, then*

(4.17)
$$\delta_2(4) = f(2m)\left[\frac{(-1)^m(2m+1)+3}{12}\right].$$

(4.18)
$$\delta_{2k}(9) \equiv k f(2m)\left[\frac{(-1)^{m-k+1}3^{m-k}+1}{6}\right] (\operatorname{mod} 8), \quad 1 \leq k \leq m-1.$$

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PROOF OF PROPOSITION 4.1. Formula (4.2) in the case $d \equiv 2 \pmod 4$ follows from (3.9), (4.17), and the fact that $A_m(d^2) \equiv A_m(4) \pmod{32}$ if $d \equiv 2 \pmod 4$. Formula (4.2) in the case $d \equiv 0 \pmod 4$ follows immediately from (3.9). To establish (4.3), note that (3.9) and (4.18) imply that if d is odd, then $A_m(d^2) \equiv f(2m)G(m) (\bmod 8)$, where $G(m) = \sum_{k=1}^{m-1} kg(m,k)$ and $g(m,k) = [(-1)^m 3^{2(m-k)-1} + (-1)^{k-1}3^{m-k-1}]/2$. Plainly, $g(m,k) \equiv 2(m-k) \pmod 8$, if m is even, and $g(m,k) = 6(m-k) \pmod 8$, if m is odd. It follows that $G(m+8) \equiv G(m) \pmod 8$. Computation of $G(m)$, $2 \leq m \leq 10$, shows that $G(m) \equiv m \pmod 8$, if m is even, and $G(m) \equiv 0 \pmod 8$, if m is odd, and this establishes (4.3). □

Proposition 4.4 follows from (3.11) and elementary considerations. We omit the details.

5. G_3 Actions of Type II_0

This section contains applications of (3.13), (4.2), (4.3), (4.5), and (4.6) to the study of G_3 actions on a cohomology $\mathbb{C}P^{2m}$, M^{4m}. The sum of these applications will equal proofs of Theorems F and G.

PROPOSITION 5.1. *Suppose that M^{4m} is a cohomology $\mathbb{C}P^{2m}$ such that* Sign $K_x^{(2j)} \equiv 1 \pmod 8$, $1 \leq j \leq m$. *If $D_3(M^{4m})$ contains* 1 *and at least one other odd integer, then $m \equiv 0$ or* 15 $\pmod{16}$.

PROOF. Suppose that $D_3(M^{4m})$ contains 1 and one other odd integer $d \neq 1$. If m is even, then it follows from (3.13) with $\rho = 8$, (4.3) and (4.5) that $2m \equiv m(m-1) \pmod{16}$ and so $m \equiv 0 \pmod{16}$. If m is odd, then it follows from (3.13) with $\rho = 8$, (4.3) and (4.6) that $m(m+1) \equiv 0 \pmod{16}$, and so $m \equiv 15 \pmod{16}$. □

A homotopy $\mathbb{C}P^{2m}$ satisfies the condition that Sign $K_x^{(2j)} \equiv 1 \pmod 8$, $1 \leq j \leq m$, and so it follows from Proposition 4.1 that if M^{4m} is a homotopy $\mathbb{C}P^{2m}$ and $D_3(M^{4m})$ contains 1 and one other odd integer, then $m \equiv 0$ or $15 \pmod{16}$. In particular, if $D_3(\mathbb{C}P^{2m})$ contains an odd integer other than 1, then $m \equiv 0$ or $15 \pmod{16}$.

If M^{4m} is any cohomology $\mathbb{C}P^{2m}$, $m \not\equiv 0 \pmod 4$, and $d \in D_3(M^{4m})$, then d is odd. This is because $d \in D_3(M^{4m})$ implies that d^2 divides $a(2m) = f(2m)\varepsilon(m)$ (formula (2.9)) and $\varepsilon(m) \equiv 0 \pmod 4$ if and only if $m \equiv 0 \pmod 4$. This observation plus Proposition 5.1 yields the results below.

PROPOSITION 5.2. *Suppose that M^{4m} is a cohomology $\mathbb{C}P^{2m}$ such that* Sign $K_x^{(2j)} \equiv 1 \pmod 8$, $1 \leq j \leq m$. *If $m \not\equiv 0 \pmod 4$ and $m \not\equiv 15 \pmod{16}$ and $1 \in D_3(M^{4m})$, then $D_3(M^{4m}) = \{1\}$.*

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COROLLARY 5.3. *Suppose that M^{4m} is a homotopy $\mathbb{C}P^{2m}$. If $m \not\equiv 0 \pmod 4$ and $m \not\equiv 15 \pmod{16}$ and $1 \in D_3(M^{4m})$, then $D_3(M^{4m}) = \{1\}$.*

COROLLARY 5.4. *If $m \not\equiv 0 \pmod 4$ and $m \not\equiv 15 \pmod{16}$, then $D_3(\mathbb{C}P^{2m}) = \{1\}$.*

Note that Corollaries 5.3 and 5.4 improve Theorem 1.6 and Corollary 1.7 of [6], adding those integers m such that $m \equiv 7 \pmod{16}$ to the set of integers m such that $1 \in D_3(M^{4m})$ implies $D_3(M^{4m}) = \{1\}$. The next step is to improve Proposition 5.2. We know that if M^{4m} is a cohomology $\mathbb{C}P^{2m}$ such that Sign $K_x^{(2j)} \equiv 1 \pmod 8$, $1 \leq j \leq m$, $m \not\equiv 0, 15 \pmod{16}$, then $1 \in D_3(M^{4m})$ implies that any integer in $D_3(M^{4m})$ other than 1 must be even, by Proposition 5.1. We will show that if $m \not\equiv 0, 8, 15 \pmod{16}$, then $1 \in D_3(M^{4m})$ implies that there are no even integers in $D_3(M^{4m})$ and so $D_3(M^{4m}) = \{1\}$.

PROPOSITION 5.5. *Suppose that M^{4m} is a cohomology $\mathbb{C}P^{2m}$ such that* Sign $K_x^{(2j)}$ $\equiv 1 \pmod 8$, $1 \leq j \leq m$. *If $1 \in D_3(M^{4m})$ and $D_3(M^{4m})$ contains an even integer, then $m \equiv 0 \pmod 8$.*

PROOF. If M^{4m} is any cohomology $\mathbb{C}P^{2m}$ and $D_3(M^{4m})$ contains an even integer, then $m \equiv 0 \pmod 4$, because $d \in D_3(M^{4m})$ implies that d^2 divides $a(2m) = f(2m)\varepsilon(m)$ (formula (2.9)) and $\varepsilon(m) \equiv 0 \pmod 4$ if and only if $m \equiv 0 \pmod 4$. If M^{4m} satisfies the hypotheses of Proposition 5.5, then it follows from (3.13), (4.2) and (4.5) that $m \equiv 0 \pmod 8$. □

THEOREM 5.6. *Suppose that M^{4m} is a cohomology $\mathbb{C}P^{2m}$ such that* Sign $K_x^{(2j)} \equiv 1$ $\pmod 8$, $1 \leq j \leq m$, *and that $m \not\equiv 0, 8,$ or $15 \pmod{16}$. If $1 \in D_3(M^{4m})$, then $D_3(M^{4m}) = \{1\}$.*

PROOF. Theorem 5.6 follows immediately from Propositions 5.1 and 5.5. □

COROLLARY 5.7. *Suppose that M^{4m} is a homotopy $\mathbb{C}P^{2m}$ and that $m \not\equiv 0, 8,$ or $15 \pmod{16}$. If $1 \in D_3(M^{4m})$, then $D_3(M^{4m}) = \{1\}$.*

COROLLARY 5.8. *If $m \not\equiv 0, 8,$ or $15 \pmod{16}$, then $D_3(\mathbb{C}P^{2m}) = \{1\}$.*

PROOF PROOFS OF THEOREMS F AND G. Corollary 5.7 says that if M^{4m}, a homotopy $\mathbb{C}P^{2m}$, $m \not\equiv 0, 8, 15 \pmod{16}$, admits a standard G_3 action of Type II_0, then every Type II_0 G_3 action on M^{4m} is a standard action. Therefore, Theorem F is the sum of Theorem A, in the case $p = 3$, and Corollary 5.7. Similarly, it follows from Theorem B, in the case $p = 3$, and Corollary 5.8, that if $n \not\equiv 0, 16, 30 \pmod{32}$ or $n < 30$, $n \neq 16$, then $\mathbb{C}P^n$ admits a G_3 action of Type II_0 if and only if the action is

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standard. This means that the proof of Theorem G will be complete when we show that $\mathbb{C}P^{16}$ admits a G_3 action of Type II_0 if and only if the action is standard. If M^{32} is any cohomology $\mathbb{C}P^{16}$ and $d \in D_3(M^{32})$, then d^2 divides $a(16) = f(16)\varepsilon(8)$ (formula (2.9)). If Sign $K_x^{(2j)} \equiv 1 \pmod 8$, $1 \leq j \leq 8$ and $1 \in D_3(M^{32})$ then it follows from Proposition 5.1. that the elements of $D_3(M^{32})$, other than 1, are all even. It follows from these observations and a look at the divisors of $a(16)$, that if M^{32} is a cohomology $\mathbb{C}P^{16}$ such that Sign $K_x^{(2j)} \equiv 1 \pmod 8$, $1 \leq j \leq 8$ and $1 \in D_3(M^{32})$, then, if $d \in D_3(M^{32})$ and $d \not\equiv 1$, then $d \equiv 2 \pmod 4$. It follows from (4.2) that $A_8(d^2) \equiv 4 \pmod{16}$. It follows from (3.10) that $B_8(d^2) \equiv 12 \pmod{16}$ and so (3.13) with $\rho = 16$ does not hold since $f(16) \equiv 11 \pmod{16}$ and so $D_3(\mathbb{C}P^{16}) = \{1\}$. $\square$

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