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A NOTE ON NORMAL ATTRACTION TO A STABLE LAW

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Let $X_1, X_2, \ldots,$ be a sequence of independent and identically distributed random variables, with the common distribution function $F(x)$. The sequence is said to be normally attracted to a stable law V with characteristic exponent α, if for some a_n $S_n/n^{1/\alpha} - a_n \xrightarrow{D} V$ (converges in distribution to V). Necessary and sufficient conditions for normal attraction are known (cf [1, p. 181]). We prove a theorem that relates the limiting behaviour of the distribution of $S_{k_n}/k_n^{1/r}$ to that of $S_n/n^{1/\alpha}$. Distributions are assumed throughout to be nondegenerate.

THEOREM. *Let k_n be a sequence of positive integers converging to ∞, and such that k_{n+1}/k_n is bounded. Let r be a real nonzero number. In order that $S_{k_n}/k_n^{1/r}$ converge in distribution to a stable law with characteristic exponent α, it is necessary that $r=\alpha$. Convergence to the normal law can take place iff $\{X_i\}$ is normally attracted to the normal law. If $k_n/k_{n+1} \to 1$, $S_{k_n}/k_n^{1/r}$ can converge in distribution only to a stable law, and this convergence takes place iff $\{X_i\}$ is normally attracted.*

Proof. We assume, without any loss of generality, that the X_i's are symmetric. For each $x>0$, let $G(x)=P(|X_i|)>x)$.

Now, $S_{k_n}/k_n^{1/r}$ converges in distribution iff there exists $\sigma^2 \geq 0$, and a function $L(x)$, such that (1) and (2) given below, hold [1, p. 124, Theorem 4]:

$$k_n G(k_n^{1/r}x) \to L(x), \quad x > 0 \tag{1}$$

$$\lim_{\epsilon \to 0} \overline{\lim}\, k_n^{(r-2)/r} \int_{|y|<k_n^{1/r}\epsilon} y^2\, dF(y) = \lim_{\epsilon \to 0} \underline{\lim}\, k_n^{(r-2)/r} \int_{|y|<k_n^{1/r}\epsilon} y^2\, dF(y) = \sigma^2. \tag{2}$$

Hence, because the limiting distribution is assumed to be nondegenerate, it follows that

$$0 < r \leq 2. \tag{3}$$

Should the limit law be stable we would have [1, p. 164, p. 128] for some $c>0$.

$$\begin{cases} L(x) = c/x^\alpha, & \alpha < 2 \\ = 0, & \alpha = 2 \end{cases} \tag{4}$$

and

$$\begin{cases} \text{(2) holds with } \sigma^2 = 0 \quad \text{for } \alpha < 2, \text{ and} \\ \lim_{n\to\infty} k_n^{(r-2)/r} \int_{|y|<k_n^{1/r}\epsilon} y^2\, dF(y) = \sigma^2 > 0, \quad \text{if } \alpha = 2. \end{cases} \tag{5}$$

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Next, for any $y \geq k_1$, there exists n such that $k_n \leq y \leq k_{n+1}$, and hence, for any $x > 0$,

$$k_n G(k_{n+1}^{1/r} x) \leq y G(y^{1/r} x) \leq k_{n+1} G(k_n^{1/r} x) \tag{6}$$

Let $S_{k_n}/k_n^{1/r} \xrightarrow{D} V$, where V is a stable law with characteristic exponent α. Assume, at first, that $\alpha < 2$. Since k_n/k_{n+1} is bounded, by hypothesis, we have from (1), (4), and (6), that $y^r G(y)$ is bounded. But, by (1) and (4), $k_n x^r G(k_n^{1/r} x) \to c x^{r-\alpha}$. This shows that $y^r G(y)$ can be bounded only if $r = \alpha$.

Assume, next $\alpha = 2$. It follows from (1), (4), and (6) that, for all $x > 0$,

$$y G(y^{1/r} x) \to 0 \quad \text{as } y \to \infty. \tag{7}$$

But, from (1) and (4),

$$\lim_{n\to\infty} k_n^{(r-2)/r} \int_{|y| < k_n^{1/r}\epsilon} y^2 \, dF(y) = 2 \lim_{n\to\infty} k_n^{(r-2)/r} \int_0^{k_n^{1/r}\epsilon} y G(y) \, dy.$$

Making use of (7), one obtains easily that the last limit equals zero unless $r = 2$. But the limit cannot be zero because of (5). Hence $r = 2$.

Therefore, again making use of (5), $E(X^2) < \infty$, which is the necessary and sufficient condition that $\{X_i\}$ be normally attracted to the normal law (cf. [1, p. 181]). This conclusion is also a direct consequence of (7), and the fact that (1), (4), and (5) with k_n replaced by n, and taking $\alpha = 2$, provide the necessary and sufficient conditions for the convergence in distribution of $S_n/n^{1/r}$ to the normal law.

Finally, suppose that $k_n/k_{n+1} \to 1$. Then, by (6) and (1), $y x^r G(y^{1/r} x) \to L(x) x^r$. Therefore, in particular, $y G(y^{1/r}) \to L(1)$.

Thus, $L(x) = L(1)/x^r$. This, together with (1), (2), (3), (4), and (5), completes the proof of the theorem.

Reference

1. B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Addison-Wesley, Reading, Mass., 1954.

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