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Part 6. Heavy tails

HIDDEN REGULAR VARIATION OF MOVING AVERAGE PROCESSES WITH HEAVY-TAILED INNOVATIONS

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HIDDEN REGULAR VARIATION OF MOVING AVERAGE PROCESSES WITH HEAVY-TAILED INNOVATIONS

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Abstract

We look at joint regular variation properties of MA(∞) processes of the form $\boldsymbol{X} = (X_k,\ k \in \mathbb{Z})$, where $X_k = \sum_{j=0}^{\infty} \psi_j Z_{k-j}$ and the sequence of random variables $(Z_i,\ i \in \mathbb{Z})$ are independent and identically distributed with regularly varying tails. We use the setup of $\mathbb{M}_{\mathbb{O}}$-convergence and obtain hidden regular variation properties for $\boldsymbol{X}$ under summability conditions on the constant coefficients $(\psi_j : j \geq 0)$. Our approach emphasizes continuity properties of mappings and produces regular variation in sequence space.

Keywords: Regular variation; multivariate heavy tail; hidden regular variation; moving average process

2010 Mathematics Subject Classification: Primary 28A33; 60G70
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1. Introduction

The purpose of this paper is to obtain joint regular variation properties of moving average processes of the form

$$X_k = \sum_{j=0}^{\infty} \psi_j Z_{k-j}, \qquad k \in \mathbb{Z},$$

where the Z_i are independent and identically distributed (i.i.d.), nonnegative heavy-tailed random variables and the ψ_j are constant nonnegative coefficients. The study of the tail behavior of such processes has a long history. Early studies of the one-dimensional case with constant coefficients include [4, 8, 22, 23]; see also the accounts in [3, 20]. The d-dimensional results as well as results for moving average processes with random coefficients can be found in [12, 16, 21]. Joint regular variation properties of the MA(∞) process were obtained in [8]. Many of these studies emphasized finding proper summability conditions for the coefficient sequence which forces the extremal properties of the process to be determined by the tail behavior of the innovation sequence. In this paper we use a fairly strong summability assumption on the coefficient sequence and concentrate on using continuity arguments to obtain joint regular variation properties of the entire sequence as a random element of the space of double-sided sequences.

Traditionally, multivariate regular variation properties of d-dimensional random vectors have been expressed using the vague convergence of measures whose limit measures are finite on compact sets. To make extremal sets which contain neighborhoods of infinities and are unbounded above compact, the approach has been to compactify a locally compact space such as $[0, \infty)^2$ by adding lines through ∞ to obtain $[0, \infty]^2$ and then to restrict the class of sets on which the limit measure has to be finite by removing a point such as $(0, 0)$ to obtain

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$[0,\infty]^2 \setminus \{(0,0)\}$. See [13, 17, 19]. There are systemic problems inherent in this use of vague convergence theory, as, for example, in dealing with lines through ∞ and points of uncompactification when addressing continuous mapping arguments. Furthermore, the theory is limited to locally compact spaces.

The theory of $\mathbb{M}_{\mathbb{O}}$-convergence developed in [10] provides an alternative setting for dealing with the tail behavior of general random elements. In this approach, the theory of $w_{\#}$-convergence (see [6]) is used to obtain a framework which lends itself well to dealing with regular variation on any complete, separable metric space with a point removed. $\mathbb{M}_{\mathbb{O}}$-convergence theory was further extended in [14] to allow the consideration of spaces with a general closed cone removed. The main attraction of any such theory lies in the powerful mapping theorems and their use to obtain results about transformations and functionals (see [9, 11]). In this paper we use these mapping results to prove that $\boldsymbol{X} = (X_k,\ k \in \mathbb{Z})$ is regularly varying as an element of $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}$ (see Section 3 for this last notation).

Another aspect of multivariate regular variation that is relevant to our paper is the concept of hidden regular variation; this was first developed in [15] and [18]. For a simple example, consider two concurrent regular variation properties of an i.i.d. Pareto(1) pair of random variables (X_1, X_2). For such random variables and $x_1, x_2 \geq 0$,

$$t\,\mathbb{P}\{X_1 > tx_1,\ X_2 > tx_2\} \to \begin{cases} (x_1 \vee x_2)^{-1} & \text{if } x_1 \wedge x_2 = 0, \\ 0 & \text{otherwise,} \end{cases}$$

while

$$t\,\mathbb{P}\{X_1 > t^{1/2}x_1,\ X_2 > t^{1/2}x_2\} \to (x_1x_2)^{-1} \quad \text{if } x_1 \wedge x_2 > 0.$$

Here the second regular variation property of the pair (X_1, X_2) is only applicable on a part of the state space obtained by removing the support of the limit measure in the first regular variation property and then using a scaling function that goes to ∞ more slowly than t. The second property was hidden by the coarse scaling used to obtain convergence to a nonzero measure in the first case. The theory of $\mathbb{M}_{\mathbb{O}}$-convergence has already been fruitfully applied to prove the existence of hidden regular variation (see [7]). In this paper we obtain an infinite sequence of hidden regular variation properties for the finite moving average process as an element of $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$.

In Section 2 we define $\mathbb{M}_{\mathbb{O}}$-convergence and collect relevant results about the theory as well as the definition of regular variation of a random variable in this framework. In Section 3.1 we restate results obtained in [14] about the regular variation of i.i.d. heavy-tailed sequences; these results form the basis for proving the results in this paper. In Section 3.3 we prove the existence of hidden regular variation for the MA(m) process before proving our main theorem in Section 3.4. Owing to technical considerations, proving a hidden regular variation property for the MA(∞) sequence has not yet been possible, but the authors are working towards achieving that end; instead, we prove hidden regular variation for finite-order moving averages. Still, our main result about the joint regular variation of the entire sequence does serve as a good demonstration of the power of continuous mapping theorems in the $\mathbb{M}_{\mathbb{O}}$ framework, and may suggest applications based on other functionals.

2. Basics of $\mathbb{M}_{\mathbb{O}}$-convergence and regular variation of measures

In this section we define the framework for $\mathbb{M}_{\mathbb{O}}$-convergence and collect basic results that will be useful later. For more details and proofs, see Sections 2 and 3 of [14].

2.1. $\mathbb{M}_{\mathbb{O}}$-convergence

Let $(\mathbb{S}, d)$ be a complete separable metric space with Borel σ-algebra $\mathcal{S}$ generated by open sets. Fix a closed set $\mathbb{C} \subset \mathbb{S}$, and set $\mathbb{O} = \mathbb{S} \setminus \mathbb{C}$. The subspace $\mathbb{O}$ is a metric subspace of $\mathbb{S}$ in the relative topology with σ-algebra $\mathcal{S}(\mathbb{O}) = \{A : A \subset \mathbb{O}, \ A \in \mathcal{S}\}$.

Let $\mathcal{C}_b$ denote the class of real-valued, nonnegative, bounded, and continuous functions on $\mathbb{S}$, and let $\mathbb{M}_b$ denote the class of finite Borel measures on $\mathcal{S}$. A basic neighborhood of $\mu \in \mathbb{M}_b$ is a set of the form $\{\nu \in \mathbb{M}_b : |\int f_i \, \mathrm{d}\nu - \int f_i \, \mathrm{d}\mu| < \varepsilon, \ i = 1, \dots, k\}$, where $\varepsilon > 0$ and $f_i \in \mathcal{C}_b$ for $i = 1, \dots, k$. This equips $\mathbb{M}_b$ with the weak topology, and the convergence $\mu_n \to \mu$ in $\mathbb{M}_b$ means that $\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$ for all $f \in \mathcal{C}_b$. See, for example, Sections 2 and 6 of [1] for details.

Let $\mathcal{C}(\mathbb{O})$ denote the real-valued, nonnegative, bounded, and continuous functions f on $\mathbb{O}$ such that, for each f, there exists $r > 0$ such that f vanishes on $\mathbb{C}^r$; we use the notation $\mathbb{C}^r = \{x \in \mathbb{S} : d(x, \mathbb{C}) < r\}$, where $d(x, \mathbb{C}) = \inf_{y \in \mathbb{C}} d(x, y)$. Similarly, write $d(A, \mathbb{C}) = \inf_{x \in A, \, y \in \mathbb{C}} d(x, y)$ for $A \subset \mathbb{S}$. Say that a set $A \in \mathcal{S}(\mathbb{O})$ is bounded away from $\mathbb{C}$ if $A \subset \mathbb{S} \setminus \mathbb{C}^r$ for some $r > 0$, or, equivalently, $d(A, \mathbb{C}) > 0$. Then $\mathcal{C}(\mathbb{O})$ consists of nonnegative continuous functions whose supports are bounded away from $\mathbb{C}$.

Let $\mathbb{M}_{\mathbb{O}}$ be the class of Borel measures on $\mathbb{O} = \mathbb{S} \setminus \mathbb{C}$ whose restriction to $\mathbb{S} \setminus \mathbb{C}^r$ is finite for each $r > 0$. When convenient, we also write $\mathbb{M}(\mathbb{O})$ or $\mathbb{M}(\mathbb{S} \setminus \mathbb{C})$ for $\mathbb{M}_{\mathbb{O}}$. A basic neighborhood of $\mu \in \mathbb{M}_{\mathbb{O}}$ is a set of the form $\{\nu \in \mathbb{M}_{\mathbb{O}} : |\int f_i \, \mathrm{d}\nu - \int f_i \, \mathrm{d}\mu| < \varepsilon, \ i = 1, \dots, k\}$, where $\varepsilon > 0$ and $f_i \in \mathcal{C}(\mathbb{O})$ for $i = 1, \dots, k$. The convergence $\mu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ is convergence in the topology defined by this base. As the next theorem shows (see [14, Theorem 2.1]), it actually suffices to consider the class of uniformly continuous functions in $\mathcal{C}(\mathbb{O})$.

Theorem 2.1. *Let $\mu, \mu_n \in \mathbb{M}_{\mathbb{O}}$. Then the following statements are equivalent:*

(i) *$\mu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ as $n \to \infty$;*

(ii) *$\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$ for each $f \in \mathcal{C}(\mathbb{O})$ which is also uniformly continuous on $\mathbb{S}$;*

(iii) *$\mu_n^{(r)} \to \mu^{(r)}$ in $\mathbb{M}_b(\mathbb{S} \setminus \mathbb{C}^r)$ for all $r > 0$ such that $\mu(\partial \, \mathbb{S} \setminus \mathbb{C}^r) = 0$, where $\mu^{(r)}$ denotes the restriction of μ to $\mathbb{S} \setminus \mathbb{C}^r$.*

Continuous mapping theorems play an important role in extending the regular variation property of the innovation sequence to that of the actual moving average sequence. Here we state one version that is useful to us. Consider another separable and complete metric space $\mathbb{S}'$, and let $\mathbb{O}'$, $\mathcal{S}_{\mathbb{O}'}$, $\mathbb{C}'$, and $\mathbb{M}_{\mathbb{O}'}$ have the same meaning relative to the space $\mathbb{S}'$ as do $\mathbb{O}$, $\mathcal{S}_{\mathbb{O}}$, $\mathbb{C}$, and $\mathbb{M}_{\mathbb{O}}$ relative to $\mathbb{S}$.

Theorem 2.2. *Suppose that $h : \mathbb{S} \mapsto \mathbb{S}'$ is uniformly continuous and that $\mathbb{C}' := h(\mathbb{C})$ is closed in $\mathbb{S}'$. Then $\hat{h} : \mathbb{M}_{\mathbb{O}} \mapsto \mathbb{M}_{\mathbb{O}'}$ defined by $\hat{h}(\mu) = \mu \circ h^{-1}$ is continuous.*

2.2. Regular variation of measures

The usual notion of regular variation involves comparisons along a ray and requires a concept of scaling or multiplication. Given any real number $\lambda > 0$ and any $x \in \mathbb{S}$, we assume that there exists a mapping $(\lambda, x) \mapsto \lambda x$ from $(0, \infty) \times \mathbb{S}$ into $\mathbb{S}$ satisfying

- the mapping $(\lambda, x) \mapsto \lambda x$ is continuous; and
- $1x = x$ and $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2) x$.

These two assumptions allow us to define a cone $\mathbb{C} \subset \mathbb{S}$ as a set such that $x \in \mathbb{C}$ implies that $\lambda x \in \mathbb{C}$ for any $\lambda > 0$. For this section, fix a closed cone $\mathbb{C} \subset \mathbb{S}$, so that $\mathbb{O} := \mathbb{S} \setminus \mathbb{C}$ is then an open cone. Also, assume that

- $d(x, \mathbb{C}) < d(\lambda x, \mathbb{C})$ if $\lambda > 1$ and $x \in \mathbb{O}$.

Recall (e.g. from [2]) that a positive measurable function c defined on $(0, \infty)$ is regularly varying with index $\rho \in \mathbb{R}$ if $\lim_{t\to\infty} c(\lambda t)/c(t) = \lambda^\rho$ for all $\lambda > 0$. Similarly, a sequence $\{c_n\}_{n\ge1}$ of positive numbers is regularly varying with index $\rho \in \mathbb{R}$ if $\lim_{n\to\infty} c_{[\lambda n]}/c_n = \lambda^\rho$ for all $\lambda > 0$. Here $[\lambda n]$ denotes the integer part of λn.

Definition 2.1. A sequence $\{\nu_n\}_{n\ge1}$ of measures in $\mathbb{M}_{\mathbb{O}}$ is regularly varying if there exists an increasing sequence $\{c_n\}_{n\ge1}$ of positive numbers for which $\{c_n\}$ is regularly varying and some nonzero $\mu \in \mathbb{M}_{\mathbb{O}}$ such that $c_n\nu_n \to \mu$ in $\mathbb{M}_{\mathbb{O}}$ as $n \to \infty$.

We now define regular variation for a single measure in $\mathbb{M}_{\mathbb{O}}$ as well as an equivalent formulation that is more pleasing to handle algebraically.

Definition 2.2. A measure $\nu \in \mathbb{M}_{\mathbb{O}}$ is regularly varying if the sequence $\{\nu(n\cdot)\}_{n\ge1}$ in $\mathbb{M}_{\mathbb{O}}$ is regularly varying, or, equivalently, there exists a nonzero $\mu \in \mathbb{M}_{\mathbb{O}}$ and an increasing function b such that $t\,\nu(b(t)\,\cdot) \to \mu(\cdot)$ in $\mathbb{M}_{\mathbb{O}}$ as $t \to \infty$. Similarly, an $\mathbb{S}$-valued random variable $\boldsymbol{Y}$ is regularly varying if the associated probability measure is regularly varying, i.e. if $\mathbb{P}\{\boldsymbol{Y} \in \cdot\}$ is regularly varying.

We refer to the function b in Definition 2.2 as the scaling function corresponding to the regularly varying measure ν on $\mathbb{M}_{\mathbb{O}}$.

3. Main results

3.1. Hidden regular variation for i.i.d. heavy-tailed sequences

From here on we take $\mathbb{S} = \mathbb{R}^\infty_{+,\mathbb{Z}}$, where $\mathbb{R}^\infty_{+,\mathbb{Z}}$ is defined to be the space of all double-sided sequences of nonnegative real numbers, i.e. $\mathbb{R}^\infty_{+,\mathbb{Z}} = \{\boldsymbol{x} = (x_i,\ i \in \mathbb{Z})\colon x_i \ge 0\}$, equipped with the metric $d_{\infty,\mathbb{Z}}$ defined by

$$d_{\infty,\mathbb{Z}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i| \wedge 1}{2^{|i|+1}}; \tag{3.1}$$

the concept of multiplication here is given by the standard pointwise multiplication of a sequence by a real number.

Observe that convergence in this metric is equivalent to convergence of all finite-dimensional sequences, i.e. $d_{\infty,\mathbb{Z}}(\boldsymbol{x^n}, \boldsymbol{x}) \to 0$, if and only if, for any $M \in \mathbb{Z}_+$, the sequences $(x_i^n,\ |i| \le M)$ converge pointwise to $(x_i,\ |i| \le M)$ in $\mathbb{R}^{2M+1}$. Furthermore, observe that $(\mathbb{R}^\infty_{+,\mathbb{Z}}, d_{\infty,\mathbb{Z}})$ as a metric space is homeomorphic to $(\mathbb{R}^\infty_+, d_\infty)$, where $\mathbb{R}^\infty_+ = \{\boldsymbol{x} = (x_i,\ i \in \mathbb{N})\colon x_i \ge 0\}$ and $d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^\infty (|x_i - y_i| \wedge 1)/2^i$.

Define $\boldsymbol{0}_\infty$ to be the sequence with all components 0 in $\mathbb{R}^\infty_{+,\mathbb{Z}}$, and $\boldsymbol{e}_i$ to be the sequence in $\mathbb{R}^\infty_{+,\mathbb{Z}}$ with the ith component 1 and all other components 0. Furthermore, define $\boldsymbol{\epsilon}_x(A) = 1$ if $x \in A$ and 0 otherwise. We use this to define

$$\begin{aligned} \mathbb{C}_{=j} &= \Big\{\boldsymbol{x} \in \mathbb{R}^\infty_{+,\mathbb{Z}}\colon \sum_{i=-\infty}^{\infty} \boldsymbol{\epsilon}_{x_i}((0,\infty)) = j\Big\} \quad \text{for all } j \ge 1, \\ \mathbb{C}_{\le j} &= \Big\{\boldsymbol{x} \in \mathbb{R}^\infty_{+,\mathbb{Z}}\colon \sum_{i=-\infty}^{\infty} \boldsymbol{\epsilon}_{x_i}((0,\infty)) \le j\Big\} \quad \text{for all } j \ge 0. \end{aligned} \tag{3.2}$$

Define $\mathbb{O}_j = \mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbb{C}_{\le j-1}$ for $j \ge 1$.

Let $\boldsymbol{Z} = (Z_i,\ i \in \mathbb{Z})$ be i.i.d. random variables in $\mathbb{R}_+$ with regularly varying tails with index $\alpha > 0$, i.e.

$$\lim_{t\to\infty} \frac{\mathbb{P}\{Z_0 > tz\}}{\mathbb{P}\{Z_0 > t\}} = z^{-\alpha} \quad \text{for all } z > 0, \tag{3.3}$$

or, equivalently, for some regularly varying function $b(\cdot)$,

$$\lim_{t\to\infty} t\,\mathbb{P}\{Z_0 > b(t)z\} = z^{-\alpha} \quad \text{for all } z > 0. \tag{3.4}$$

With this setup, we can restate Theorem 4.2 of [14] as a statement about the space $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$ and the sequence of i.i.d. random variables $\boldsymbol{Z} \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$. Define, for each $j \geq 0$,

$$\mu_t^{(j)}(\cdot) = t\,\mathbb{P}\left\{\frac{\boldsymbol{Z}}{b(t^{1/(j+1)})} \in \cdot\right\},$$

$$\mu^{(j)}(\cdot) = \sum_{\{i_1,\dots,i_{j+1}\}} \int \mathbb{I}\left\{\sum_{k=1}^{j+1} z_k \boldsymbol{e}_{i_k} \in \cdot\right\} \nu_\alpha(\mathrm{d}z_1)\cdots \nu_\alpha(\mathrm{d}z_{j+1}),$$

where $\nu_\alpha(x,\infty) = x^{-\alpha}$ and the indices of $\{i_1,\dots,i_{j+1}\}$ run through the ordered subsets of size $j+1$ of $\mathbb{Z}$.

Theorem 3.1. *For every $j \geq 0$, $\mu_t^{(j)} \to \mu^{(j)}$ in $\mathbb{M}(\mathbb{O}_j)$. The measure $\mu^{(j)}$ has support $\mathbb{C}_{\leq j+1} \setminus \mathbb{C}_{\leq j} = \mathbb{C}_{=j+1}$ and admits the alternative form*

$$\mu^{(j)}(\dots,\mathrm{d}z_1,\mathrm{d}z_0,\mathrm{d}z_{-1},\dots) = \sum_{\{i_1,\dots,i_{j+1}\}} \left(\prod_{k\notin\{i_1,\dots,i_{j+1}\}} \epsilon_0(\mathrm{d}z_k)\right)\left(\prod_{k\in\{i_1,\dots,i_{j+1}\}} \nu_\alpha(\mathrm{d}z_k)\right). \tag{3.5}$$

3.2. Definition of the MA(∞) process and the framework for the proof of the main result

Let $(\psi_j,\ j \geq 0)$ be a sequence of nonnegative constants with

(A1) $\psi_0 > 0$; and

(A2) for some $\delta < \alpha \wedge 1$, $\sum_{j=0}^{\infty} \psi_j^{\delta} < \infty$.

Observe that assumption (A2) implies the following:

(C1) $\sum_{j=0}^{\infty} \psi_j < \infty$;

(C2) $\sum_{j=0}^{\infty} \psi_j^{\alpha} < \infty$;

(C3) for any $k \in \mathbb{Z}$, the sequence $\sum_{j=0}^{\infty} \psi_j Z_{k-j}$ converges almost surely; and

(C4) for any $x > 0$ and $k \in \mathbb{Z}$,

$$\lim_{N\to\infty} \limsup_{t\to\infty} t\,\mathbb{P}\left\{\sum_{j>N} \psi_j Z_{k-j} > b(t)x\right\} = 0,$$

where $\boldsymbol{Z} = (Z_i,\ i \in \mathbb{Z})$ is defined in (3.4). Conditions (C1) and (C2) are easy to see, while the proofs of (C3) and (C4) can be found in [17, Section 4.5, especially Lemma 4.24] and [4, 5].

Remark 3.1. Assumption (A2) is not the weakest condition known in the literature that implies (C1)–(C4). See [12, 16, 21, 24] for different summability assumptions on the sequence $(\psi_j,\ j \geq 0)$ as well as a treatment of moving average processes with random coefficients and heavy-tailed innovations.

For $k \in \mathbb{Z}$, define

$$X_k = \sum_{j=0}^{\infty} \psi_j Z_{k-j}.$$

Condition (C3) ensures that $\boldsymbol{X} = (X_k,\ k \in \mathbb{Z})$ is a well-defined sequence of random variables in $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$. Defining the map $\boldsymbol{T}^{\infty} \colon \mathbb{R}^{\infty}_{+,\mathbb{Z}} \ni \boldsymbol{z} = (z_i,\ i \in \mathbb{Z}) \mapsto \boldsymbol{T}^{\infty}(\boldsymbol{z}) = (T^{\infty}_k(\boldsymbol{z}),\ k \in \mathbb{Z}) \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$, where

$$T^{\infty}_k(\boldsymbol{z}) = \sum_{j=0}^{\infty} \psi_j z_{k-j}, \tag{3.6}$$

then

$$\boldsymbol{X} = \boldsymbol{T}^{\infty}(\boldsymbol{Z}).$$

This leads us to suspect that regular variation properties can be obtained from Theorem 3.1 using a continuous mapping argument. But, unfortunately, the map $\boldsymbol{T}^{\infty}$, even though well defined $\mathbb{P}$-almost surely, is nowhere continuous on $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$. This forces us to use a truncation argument as detailed in the sequel by using a sequence of maps which map $\boldsymbol{Z}$ to the partial sums of the infinite sums that make up the elements of $\boldsymbol{X}$ and then using a Slutsky-style approximation. The details are technical as we are dealing with infinite measures.

3.3. Hidden regular variation of the MA(m) process

For every $m \geq 0$, define the random variable $\boldsymbol{X}^m = (X^m_k,\ k \in \mathbb{Z}) \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$, where

$$X^m_k = \sum_{j=0}^{m} \psi_j Z_{k-j}.$$

Similarly to (3.6), define, for every $m \geq 0$, the map $\boldsymbol{T}^m \colon \mathbb{R}^{\infty}_{+,\mathbb{Z}} \ni \boldsymbol{z} = (z_i,\ i \in \mathbb{Z}) \mapsto \boldsymbol{T}^m(\boldsymbol{z}) = (T^m_k(\boldsymbol{z}),\ k \in \mathbb{Z}) \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$, where

$$T^m_k(\boldsymbol{z}) = \sum_{j=0}^{m} \psi_j z_{k-j}.$$

Again, we have $\boldsymbol{X}^m = \boldsymbol{T}^m(\boldsymbol{Z})$. However, the map $\boldsymbol{T}^m$ is well behaved enough for us to use Theorem 2.2. We first prove two preliminary lemmas to enable the use of that theorem; these will then lead us to the main result about the MA(m) processes.

Lemma 3.1. *For every $m \geq 0$, the map $\boldsymbol{T}^m$ is uniformly continuous.*

Proof. Fix $m \geq 0$ and $\epsilon > 0$. Let $M > 0$ be such that $2 \times 2^{-M} < \varepsilon/2$. Let $\boldsymbol{x} = (x_i,\ i \in \mathbb{Z})$ and $\boldsymbol{y} = (y_i,\ i \in \mathbb{Z})$, so $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^{\infty}_{+,\mathbb{Z}}$. Then

$$\begin{aligned} d_{\infty,\mathbb{Z}}(\boldsymbol{T}^m(\boldsymbol{x}), \boldsymbol{T}^m(\boldsymbol{y})) &< \sum_{|i|<M} \frac{|\sum_{j=0}^{m} \psi_j x_{i-j} - \sum_{j=0}^{m} \psi_j y_{i-j}| \wedge 1}{2^{|i|+1}} + \frac{\varepsilon}{2} \\ &\leq 2\bigg(\sum_{j=0}^{m} \psi_j\bigg)\bigg(\bigvee_{|i|<M+m} |x_i - y_i|\bigg) + \frac{\varepsilon}{2}. \end{aligned} \tag{3.7}$$

Let $\delta < (\sum_{j=0}^{m} \psi_j)\frac{1}{4}\varepsilon\, 2^{-(M+m)}$, and assume that $d_{\infty,\mathbb{Z}}(\boldsymbol{x}, \boldsymbol{y}) < \delta$. Then, from (3.1),

$$\bigvee_{|i|<M+m} |x_i - y_i| \ < \ 2^{(M+m)}\delta \ < \ \bigg(\sum_{j=0}^{m} \psi_j\bigg)\frac{\varepsilon}{4}.$$

Then, using (3.7),

$$d_{\infty,\mathbb{Z}}(\boldsymbol{T}^m(\boldsymbol{x}), \boldsymbol{T}^m(\boldsymbol{y})) < \varepsilon.$$

Lemma 3.2. *For every $m \geq 0$, $j \geq 0$, and $\mathbb{C}_{\leq j}$ as in (3.2), $\boldsymbol{T}^m(\mathbb{C}_{\leq j})$ is closed.*

Proof. Fix $m \geq 0$, and observe that, for $j = 0$, $\boldsymbol{T}^m(\mathbb{C}_{\leq 0}) = \boldsymbol{T}^m(\{\boldsymbol{0}_\infty\}) = \{\boldsymbol{0}_\infty\}$, which is trivially closed. This settles the base case for a proof of the result by induction. Assume that the result holds for $j < J$. Take $\boldsymbol{z}^n \in \mathbb{C}_{=J}$ such that $\boldsymbol{z}^n \to \boldsymbol{z} \in \mathbb{R}^\infty_{+,\mathbb{Z}}$. It is enough to assume this as $\mathbb{C}_{\leq J} = \mathbb{C}_{\leq J-1} \cup \mathbb{C}_{=J}$. Furthermore, assume that we have sequences $\lambda_1^n, \lambda_2^n, \dots, \lambda_J^n > 0$ and $i_1^n < i_2^n < \cdots < i_J^n \in \mathbb{Z}$ such that

$$\boldsymbol{z}^n = \sum_{k=1}^{J} \lambda_k^n \boldsymbol{e}_{i_k^n} \quad \text{and} \quad \boldsymbol{T}^m(\boldsymbol{z}^n) = \sum_{l=0}^{m} \sum_{k=1}^{J} \psi_l \lambda_k^n \boldsymbol{e}_{i_k^n - j}.$$

Observe that if $i_J^n \to -\infty$ along some subsequence n_q then the limit of any finite-dimensional subsequence of $\boldsymbol{T}^m(\boldsymbol{z}^{n_q})$ is the same as the finite-dimensional subsequential limit of $\boldsymbol{T}^m(\sum_{k=1}^{J-1} \lambda_k^{n_q} \boldsymbol{e}_{i_k^{n_q}})$. Since the limit of a sequence in $\mathbb{R}^\infty_{+,\mathbb{Z}}$ is determined by the limits of the finite-dimensional subsequences, the induction hypothesis then implies that $\boldsymbol{z} \in \boldsymbol{T}^m(\mathbb{C}_{\leq J-1}) \subset \boldsymbol{T}^m(\mathbb{C}_{\leq J})$. A similar argument shows that if $i_1^n \to \infty$ along some subsequence then $\boldsymbol{z} \in \boldsymbol{T}^m(\mathbb{C}_{\leq J})$. So $\{i_1^n, i_2^n, \dots, i_J^n\}$ must be contained in some bounded set, and so they must equal some $\{i_1, i_2, \dots, i_J\}$ infinitely often, where $i_1 < i_2 < \cdots < i_J$. Without loss of generality, we may now assume that

$$\boldsymbol{z}^n = \sum_{k=1}^{J} \lambda_k^n \boldsymbol{e}_{i_k} \quad \text{and} \quad \boldsymbol{T}^m(\boldsymbol{z}^n) = \sum_{l=0}^{m} \sum_{k=1}^{J} \psi_l \lambda_k^n \boldsymbol{e}_{i_k - j}.$$

Since $\{\boldsymbol{T}^m(\boldsymbol{z}^n)\}_{i_J - m} = \psi_0 \lambda_J^n$ converges as $n \to \infty$, and $\psi_0 > 0$ by assumption (A1), we must have $\lambda_J^n \to \lambda$ for some $\lambda \geq 0$. This implies that $\boldsymbol{T}^m(\sum_{k=1}^{J-1} \lambda_k^n \boldsymbol{e}_{i_k}) \to \boldsymbol{z} - \boldsymbol{T}^m(\lambda \boldsymbol{e}_{i_J})$. But the induction hypothesis now implies that $\boldsymbol{z} - \boldsymbol{T}^m(\lambda \boldsymbol{e}_{i_J}) \in \boldsymbol{T}^m(\mathbb{C}_{\leq J-1})$. Hence, $\boldsymbol{z} \in \boldsymbol{T}^m(\mathbb{C}_{\leq J})$, proving the induction step.

A quick application of Theorem 2.2 now gives the following result.

Theorem 3.2. *For every $m \geq 0$ and $j \geq 0$, as $t \to \infty$, $\mu_t^{(j)} \circ (\boldsymbol{T}^m)^{-1} \to \mu^{(j)} \circ (\boldsymbol{T}^m)^{-1}$ in $\mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \boldsymbol{T}^m(\mathbb{C}_{\leq j}))$, or, equivalently,*

$$t\,\mathbb{P}\biggl\{\frac{\boldsymbol{X}^m}{b(t^{1/(j+1)})} \in \cdot\biggr\} \to \sum_{\{i_1,\dots,i_{j+1}\}} \int \mathbb{I}\biggl\{\boldsymbol{T}^m\biggl(\sum_{k=1}^{j+1} z_k \boldsymbol{e}_{i_k}\biggr) \in \cdot\biggr\}\, \nu_\alpha(\mathrm{d}z_1) \cdots \nu_\alpha(\mathrm{d}z_{j+1}).$$

Remark 3.2. Observe that Theorem 3.2 implies an infinitude of regular variation properties for $\boldsymbol{X}^m$. For example, for $j = 0$,

$$t\,\mathbb{P}\biggl\{\frac{\boldsymbol{X}^m}{b(t)} \in \cdot\biggr\} \to \nu^{m,(0)}(\cdot) \quad \text{in } \mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \boldsymbol{T}^m(\{\boldsymbol{0}_\infty\})) = \mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \{\boldsymbol{0}_\infty\}),$$

where

$$\nu^{m,(0)}(\cdot) = \sum_{i=-\infty}^{\infty} \int \mathbb{I}\{\boldsymbol{T}^m(z_i \boldsymbol{e}_i) \in \cdot\}\, \nu_\alpha(\mathrm{d}z_i).$$

It is clear from the above that $\nu^{m,(0)}$ is a nonzero measure, finite on subsets of $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$ bounded away from $\mathbf{0}_\infty$, and its support is on $\boldsymbol{T}^m(\mathbb{C}_{=1})$. Thus, $\boldsymbol{X}^m$ is regularly varying on $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}$ with scaling function $b(\cdot)$ and limit measure $\nu^{m,(0)}$. Using (3.5) and assuming that $\psi_j > 0$ for all $j \le m$, we have the following alternative and slightly more illuminating formulation for $\nu^{m,(0)}$:

$$\nu^{m,(0)}(\dots, \mathrm{d}z_1, \mathrm{d}z_0, \mathrm{d}z_{-1}, \dots)$$
$$= \sum_{i=-\infty}^{\infty} \bigg(\prod_{k<i} \boldsymbol{\epsilon}_0(\mathrm{d}z_k)\bigg)\bigg(\prod_{i\le k\le i+m} \nu_\alpha\bigg(\frac{\mathrm{d}z_k}{\psi_{k-i}}\bigg)\bigg)\bigg(\prod_{k>i+m} \boldsymbol{\epsilon}_0(\mathrm{d}z_k)\bigg).$$

Furthermore, for any $k \in \mathbb{Z}$,

$$t\,\mathbb{P}\big\{X_k^m > b(t)x\big\} \to \bigg(\sum_{l=0}^{m} \psi_l^\alpha\bigg)x^{-\alpha} \qquad \text{for } x > 0.$$

Similarly, for $j = 1$,

$$t\,\mathbb{P}\bigg\{\frac{\boldsymbol{X}^m}{b(t^{1/2})} \in \cdot\bigg\} \to \nu^{m,(1)}(\cdot) \qquad \text{in } \mathbb{M}(\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \boldsymbol{T}^m(\mathbb{C}_{\le 1})),$$

where $\nu^{m,(1)}$ is a nonzero measure on $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \boldsymbol{T}^m(\mathbb{C}_{\le 1})$ with support $\boldsymbol{T}(\mathbb{C}_{=2})$. So $\boldsymbol{X}^m$ is also regularly varying on $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \boldsymbol{T}^m(\mathbb{C}_{\le 1})$ with scaling function $b(t^{1/2})$. Observe that, for $j = 0$, we removed just $\boldsymbol{T}^m(\mathbb{C}_{\le 0}) = \{\mathbf{0}_\infty\}$ from $\mathbb{R}^{\infty}_{+,\mathbb{Z}}$ and concluded that $\boldsymbol{X}^m$ was regularly varying with a limit measure concentrating on $\boldsymbol{T}^m(\mathbb{C}_{=1})$ which is a very small part of the entire state space $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}$. Now, on also removing the support of $\nu^{m,(0)}$, i.e. $\boldsymbol{T}^m(\mathbb{C}_{=1})$, from the state space we obtained a new regular variation property for $\boldsymbol{X}^m$ on a smaller state space $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \boldsymbol{T}^m(\mathbb{C}_{\le 1})$ with a finer scaling function $b(t^{1/2})$. This regular variation property was in some sense hidden by the cruder scaling that we used for the larger state space. This is a typical example of hidden regular variation. For a more expository account on such a nested sequence of regular variation properties in the case of i.i.d. heavy-tailed random variables, see [14, Section 4.5].

In fact, we have an increasing sequence of cones,

$$\boldsymbol{T}^m(\mathbb{C}_{\le 0}) \subset \boldsymbol{T}^m(\mathbb{C}_{\le 1}) \subset \cdots \subset \boldsymbol{T}^m(\mathbb{C}_{\le j}) \subset \cdots,$$

a sequence of nonzero measures $\nu^{m,(j)}$, $j \ge 0$, where $\nu^{m,(j)}$ is supported on $\boldsymbol{T}^m(\mathbb{C}_{\le j+1}) \setminus \boldsymbol{T}^m(\mathbb{C}_{\le j})$, and a sequence of decreasing scaling functions

$$b(t) > b(t^{1/2}) > \cdots > b(t^{1/(j+1)}) > \cdots$$

such that $\boldsymbol{X}^m$ is regularly varying on $\mathbb{R}^{\infty}_{+,\mathbb{Z}} \setminus \boldsymbol{T}^m(\mathbb{C}_{\le j})$ with limit measure $\nu^{m,(j)}$ and scaling function $b(t^{1/(j+1)})$. Thus, by removing more and more of the state space and using finer and finer scaling functions, we are able to get a more detailed picture of the extremal properties of $\boldsymbol{X}^m$.

3.4. Regular variation of the MA(∞) process

As mentioned before, the map $\boldsymbol{T}^\infty$ is only well defined $\mathbb{P}$-almost surely. For each $j > 0$, $\boldsymbol{T}^\infty(\mathbb{C}_{\le j})$ is not closed, even though $\boldsymbol{T}^\infty$ is well defined on each $\mathbb{C}_{\le j}$. This prevents us from proving a result implying hidden regular variation of $\boldsymbol{X}$ as in Theorem 3.2 for $\boldsymbol{X}^m$. However, the fact that $\boldsymbol{T}^\infty(\{\mathbf{0}_\infty\}) = \{\mathbf{0}_\infty\}$, and the use of (C4) and interpreting $\boldsymbol{X}$ as the limit of $\boldsymbol{X}^m$ as $m \to \infty$, allows us to prove the following result. The proof, except for technical details, is similar in spirit to [19, Theorem 3.5] or [1, Theorem 3.2].

Theorem 3.3. *It holds that* $\mu_t^{(0)} \circ (\boldsymbol{T}^\infty)^{-1} \to \mu^{(0)} \circ (\boldsymbol{T}^\infty)^{-1} = \nu^{(0)}$ *in* $\mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\}) = \mathbb{M}(\mathbb{O}_0)$*, or, equivalently,*

$$t\,\mathbb{P}\bigg\{\frac{\boldsymbol{X}}{b(t)} \in \cdot\bigg\} \to \sum_{i=\infty}^{\infty} \int \mathbb{I}\{\boldsymbol{T}^\infty(z_i\boldsymbol{e}_i) \in \cdot\}\,\nu_\alpha(\mathrm{d}z_i).$$

Remark 3.3. (i) Theorem 3.3 implies that $\boldsymbol{X}$ is regularly varying on $\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \mathbf{0}_\infty$ with limit measure $\nu^{(0)}$ and scaling function $b(\cdot)$. The limit measure $\nu^{(0)}$ can also be expressed in the following way, emphasizing the fact that its support is on $\boldsymbol{T}^\infty(\mathbb{C}_{=1})$ and it is indeed nonzero:

$$\nu^{(0)}(\ldots, \mathrm{d}z_1, \mathrm{d}z_0, \mathrm{d}z_{-1}, \ldots) = \sum_{i=-\infty}^{\infty}\bigg(\prod_{k<i \text{ or } \psi_{k-i}=0} \boldsymbol{\epsilon}_0(\mathrm{d}z_k)\bigg)\bigg(\prod_{k\geq i \text{ and } \psi_{k-i}>0} \nu_\alpha\bigg(\frac{\mathrm{d}z_k}{\psi_{k-i}}\bigg)\bigg).$$

Also, for any $k \in \mathbb{Z}$,

$$t\,\mathbb{P}\{X_k > b(t)x\} \to \bigg(\sum_{l=0}^{\infty} \psi_l^\alpha\bigg)x^{-\alpha} \qquad \text{for } x > 0.$$

(ii) It is also instructive to compare Theorem 3.3 with Theorem 2.4 of [8] where a point process version of the same result was obtained.

(iii) An application of the continuous mapping theorem (Theorem 2.2) allows us to prove regular variation for sums of MA(∞) processes from Theorem 3.3. For every $m \geq 0$, define the random variable $\boldsymbol{Y}^m = (Y_k^m,\ k \in \mathbb{Z}) \in \mathbb{R}^\infty_{+,\mathbb{Z}}$, where

$$Y_k^m = \sum_{j=0}^{m} X_{k-j}.$$

Observe that $\boldsymbol{Y}^m = \mathrm{SUM}^m(\boldsymbol{X})$ where, for every $m \geq 0$, the map $\mathrm{SUM}^m\colon \mathbb{R}^\infty_{+,\mathbb{Z}} \ni \boldsymbol{x} = (x_i,\ i \in \mathbb{Z}) \mapsto \mathrm{SUM}^m(\boldsymbol{x}) = (\sum_{j=0}^m x_{k-j},\ k \in \mathbb{Z}) \in \mathbb{R}^\infty_{+,\mathbb{Z}}$. The function SUM^m is uniformly continuous by Lemma 3.1, so we can apply Theorem 2.2 to $\nu^{(0)}$ to conclude that, in $\mathbb{M}(\mathbb{R}^\infty_{+,\mathbb{Z}} \setminus \{\mathbf{0}_\infty\})$,

$$t\,\mathbb{P}\bigg\{\frac{\boldsymbol{Y}^m}{b(t)} \in \cdot\bigg\} \to \nu^{(0)} \circ (\mathrm{SUM}^m)^{-1}(\cdot).$$

Application of Theorem 3.3 to obtain regular variation of other functionals of MA(∞), such as sample covariances, is not so straightforward in the sense that proving uniform continuity of the corresponding function, such as SUM^m in the case of the summation functional, is nontrivial.

Before proving Theorem 3.3 we prove two technical lemmas. Set $\sum_{j=0}^\infty \psi_j = S$, which is finite by (C1).

Lemma 3.3. *For any* $\gamma > 0$ *and setting* $\delta_n = 2^{-(n+1)}/S$*,*

$$\lim_{n\to\infty}\limsup_{t\to\infty} \mu_t^{(0)}\{\boldsymbol{z}\colon d_{\infty,\mathbb{Z}}(\boldsymbol{T}^\infty(\boldsymbol{z}), \mathbf{0}_\infty) > \gamma,\ d_{\infty,\mathbb{Z}}(\boldsymbol{z}, \mathbf{0}_\infty) < \delta_n\} = 0.$$

Proof. Fix $\gamma > 0$, and let $M > 0$ be such that $2\sum_{|i|\geq M} 2^{-(|i|+1)} < \gamma/2$. Then

$$\begin{aligned}&\{\boldsymbol{z}\colon d_{\infty,\mathbb{Z}}(\boldsymbol{T}^\infty(\boldsymbol{z}), \mathbf{0}_\infty) > \gamma,\ d_{\infty,\mathbb{Z}}(\boldsymbol{z}, \mathbf{0}_\infty) < \delta_n\}\\ &\quad\subset \bigg\{\boldsymbol{z}\colon \sum_{|i|<M} \frac{T_i^\infty(\boldsymbol{z}) \wedge 1}{2^{|i|+1}} > \frac{\gamma}{2},\ \sum_{|i|<n} \frac{z_i \wedge 1}{2^{|i|+1}} < \delta_n\bigg\}\end{aligned}$$

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$$
\begin{aligned}
&\subset \Big\{ z \colon \bigvee_{|i|<M} T_i^{\infty}(z) > \gamma,\ \bigvee_{|i|<n} z_i < 2^n \delta_n \Big\} \\
&\subset \bigcup_{|i|<M} \Big\{ z \colon \sum_{l=0}^{\infty} \psi_l z_{i-l} > \gamma,\ \bigvee_{|i|<n} z_i < 2^n \delta_n \Big\} \\
&\subset \bigcup_{|i|<M} \Big\{ z \colon \Big(\sum_{l=0}^{\infty} \psi_l \Big) \Big(\bigvee_{|i|<n} z_i \Big) + \sum_{l>i+n} \psi_l z_{i-l} > \gamma,\ \bigvee_{|i|<n} z_i < 2^n \delta_n \Big\} \\
&\subset \bigcup_{|i|<M} \Big\{ z \colon \sum_{l>i+n} \psi_l z_{i-l} > \gamma - S\, 2^n \delta_n \Big\} \\
&\subset \bigcup_{|i|<M} \Big\{ z \colon \sum_{l>i+n} \psi_l z_{i-l} > \frac{\gamma}{2} \Big\}
\end{aligned}
$$

for large enough n. So we have

$$
\begin{aligned}
&\lim_{n\to\infty} \limsup_{t\to\infty} \mu_t^{(0)} \{ z \colon d_{\infty,\mathbb{Z}}(\boldsymbol{T}^{\infty}(z), \mathbf{0}_{\infty}) > \gamma,\ d_{\infty,\mathbb{Z}}(z, \mathbf{0}_{\infty}) < \delta_n \} \\
&\quad \le \sum_{|i|<M} \lim_{n\to\infty} \limsup_{t\to\infty} \mu_t^{(0)} \Big\{ z \colon \sum_{l>i+n} \psi_l z_{i-l} > \frac{\gamma}{2} \Big\} \\
&\quad \le \sum_{|i|<M} \lim_{n\to\infty} \limsup_{t\to\infty} t\, \mathbb{P} \Big\{ z \colon \sum_{l>i+n} \psi_l Z_{i-l} > \frac{b(t)\gamma}{2} \Big\} \\
&\quad = 0,
\end{aligned}
$$

where the last equality follows from (C4).

Lemma 3.4. *For any $\beta > 0$,*

$$
\lim_{m\to\infty} \limsup_{t\to\infty} \mu_t^{(0)} \{ z \colon d_{\infty,\mathbb{Z}}(\boldsymbol{T}^{\infty}(z), \boldsymbol{T}^{m}(z)) > \beta \} = 0.
$$

Proof. Fix $\beta > 0$, and let $M > 0$ be such that $2 \sum_{|i|\ge M} 2^{-(|i|+1)} < \beta/2$. Then

$$
\begin{aligned}
&\{ z \colon d_{\infty,\mathbb{Z}}(\boldsymbol{T}^{\infty}(z), \boldsymbol{T}^{m}(z)) > \beta \} \\
&\quad \subset \Big\{ z \colon \sum_{|i|<M} \frac{|\boldsymbol{T}^{\infty}(z) - \boldsymbol{T}^{m}(z)|_i \wedge 1}{2^{|i|+1}} > \frac{\beta}{2} \Big\} \\
&\quad \subset \Big\{ z \colon \sum_{|i|<M} \frac{\sum_{l>m+1} \psi_l z_{i-l} \wedge 1}{2^{|i|+1}} > \frac{\beta}{2} \Big\} \\
&\quad \subset \Big\{ z \colon \bigvee_{|i|<M} \sum_{l>m+1} \psi_l z_{i-l} > \beta \Big\} \\
&\quad \subset \bigcup_{|i|<M} \Big\{ z \colon \sum_{l>m+1} \psi_l z_{i-l} > \beta \Big\}.
\end{aligned}
$$

Now, as in Lemma 3.3, we have

$$\begin{aligned}&\lim_{m\to\infty}\limsup_{t\to\infty}\mu_t^{(0)}\{z\colon d_{\infty,\mathbb{Z}}(\boldsymbol{T}^{\infty}(z),\boldsymbol{T}^{m}(z))>\beta\}\\&\quad\le\lim_{m\to\infty}\limsup_{t\to\infty}\sum_{|i|<M}t\,\mathbb{P}\bigg\{z\colon\sum_{l>m+1}\psi_l Z_{i-l}>b(t)\beta\bigg\},\end{aligned}$$

which is 0 by (C4).

Proof of Theorem 3.3. By Theorem 2.1, it is enough to show that, for any uniformly continuous $f\in\mathcal{C}(\mathbb{O}_{\mathscr{C}})$, $\int f\,\mathrm{d}\mu_t^{(j)}\circ(\boldsymbol{T}^{\infty})^{-1}\to\int f\,\mathrm{d}\mu^{(j)}\circ(\boldsymbol{T}^{\infty})^{-1}$. Fix any such f, and set $\mathbb{F}=\{z\in\mathbb{R}^{\infty}_{+,\mathbb{Z}}\colon f(z)>0\}$. Since $f\in\mathcal{C}(\mathbb{O}_{\mathscr{C}})$, we may assume that $d_{\infty,\mathbb{Z}}(\mathbb{F},\mathbf{0}_{\infty})>\gamma>0$ and $\sup_{z\in\mathbb{R}^{\infty}_{+,\mathbb{Z}}}f(z)=1$. Let $\omega_f(\cdot)$ be the modulus of continuity of f.

Fix $\varepsilon>0$. By Lemma 3.3 we can find $\mathbb{G}:=\{z\in\mathbb{R}^{\infty}_{+,\mathbb{Z}}\colon d_{\infty,\mathbb{Z}}(z,\mathbf{0}_{\infty})>\delta\}$ for some $\delta>0$ such that $\mu^{(0)}(\partial\mathbb{G})=0$ and $\lim_{t\to\infty}\mu_t^{(0)}(\mathbb{F}\setminus\boldsymbol{T}^{\infty}(\mathbb{G}))=\mu^{(0)}(\mathbb{F}\setminus\boldsymbol{T}^{\infty}(\mathbb{G}))<\varepsilon$. Then

$$\begin{aligned}&\bigg|\int f\,\mathrm{d}\mu_t^{(0)}\circ(\boldsymbol{T}^{\infty})^{-1}-\int f\,\mathrm{d}\mu^{(0)}\circ(\boldsymbol{T}^{\infty})^{-1}\bigg|\\&\quad=\bigg|\int_{\boldsymbol{T}^{\infty}(z)\in\mathbb{F}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu_t^{(0)}(\mathrm{d}z)-\int_{\boldsymbol{T}^{\infty}(z)\in\mathbb{F}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu^{(0)}(\mathrm{d}z)\bigg|\\&\quad\le\bigg|\int_{\mathbb{G}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu_t^{(0)}(\mathrm{d}z)-\int_{\mathbf{G}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu^{(0)}(\mathrm{d}z)\bigg|+\mu_t^{(0)}(\mathbb{F}\setminus\boldsymbol{T}^{\infty}(\mathbb{G}))\\&\qquad+\mu^{(0)}(\mathbb{F}\setminus\boldsymbol{T}^{\infty}(\mathbb{G})).\end{aligned}$$

Since the last two terms are less than ε for large enough t, it suffices to show that the first term in the last line above goes to 0. By a standard triangular inequality argument we have

$$\begin{aligned}&\bigg|\int_{\mathbb{G}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu_t^{(0)}(\mathrm{d}z)-\int_{\mathbf{G}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu^{(0)}(\mathrm{d}z)\bigg|\\&\quad\le\bigg|\int_{\mathbb{G}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu_t^{(0)}(\mathrm{d}z)-\int_{\mathbf{G}}f\circ\boldsymbol{T}^{m}(z)\;\mu_t^{(0)}(\mathrm{d}z)\bigg|\\&\qquad+\bigg|\int_{\mathbb{G}}f\circ\boldsymbol{T}^{m}(z)\;\mu_t^{(0)}(\mathrm{d}z)-\int_{\mathbf{G}}f\circ\boldsymbol{T}^{m}(z)\;\mu^{(0)}(\mathrm{d}z)\bigg|\\&\qquad+\bigg|\int_{\mathbb{G}}f\circ\boldsymbol{T}^{m}(z)\;\mu^{(0)}(\mathrm{d}z)-\int_{\mathbf{G}}f\circ\boldsymbol{T}^{\infty}(z)\;\mu^{(0)}(\mathrm{d}z)\bigg|\\&\quad=:I+II+III,\end{aligned}$$

and we deal with I, II, and III separately. Observe first that

$$\begin{aligned}I&\le\int_{\mathbb{G}}|f\circ\boldsymbol{T}^{\infty}(z)-f\circ\boldsymbol{T}^{m}(z)|\,\mathbb{I}\{d_{\infty,\mathbb{Z}}(\boldsymbol{T}^{\infty}(z),\boldsymbol{T}^{m}(z))\le\beta\}\;\mu_t^{(0)}(\mathrm{d}z)\\&\quad+\int_{\mathbb{G}}|f\circ\boldsymbol{T}^{\infty}(z)-f\circ\boldsymbol{T}^{m}(z)|\,\mathbb{I}\{d_{\infty,\mathbb{Z}}(\boldsymbol{T}^{\infty}(z),\boldsymbol{T}^{m}(z))>\beta\}\;\mu_t^{(0)}(\mathrm{d}z)\\&\le\omega_f(\beta)\mu_t^{(0)}(\mathbb{G})+2\mu_t^{(0)}\{z\colon d_{\infty,\mathbb{Z}}(\boldsymbol{T}^{\infty}(z),\boldsymbol{T}^{m}(z))>\beta\}.\end{aligned}$$

The first term above goes to 0 as $\beta\to\infty$ because $\mu_t^{(0)}(\mathbb{G})$ is finite for all large t, while the second term goes to 0 by Lemma 3.4 by first letting $t\to\infty$ and then letting $m\to\infty$.

For any fixed m, $f\circ\boldsymbol{T}^{m}$ is continuous on $\mathbb{O}_0$ and, hence, on $\mathbb{G}$, so, by part (ii) of Theorem 2.1 and using Theorem 3.2 for $j=0$, we see that $II\to0$ as $t\to\infty$ for every m.

For *III*, first note that, for any $z \in \mathbb{C}_{=1}$, $\lim_{m\to\infty} \boldsymbol{T}^m(z) = \boldsymbol{T}^\infty(z)$. To see this, let $z = \lambda \boldsymbol{e}_i$. Then

$$d_{\infty,\mathbb{Z}}(\boldsymbol{T}^m(z), \boldsymbol{T}^\infty(z)) \le \lambda \sum_{l=m+1}^{\infty} \psi_l,$$

and the right-hand side tends to 0 as $m \to \infty$ because $\sum_{j=0}^{\infty} \psi_j < \infty$ by (C1). Since $\mu^{(0)}$ is finite on $\mathbb{G}$ and concentrates on $\mathbb{C}_{=1}$, and f is continuous and bounded, it follows, by dominated convergence, that $III \to 0$ as $m \to \infty$.

Remark 3.4. The entire exercise in this paper could have been carried out in somewhat more generality by assuming that the i.i.d. sequence $(Z_i,\ i \in \mathbb{Z})$ was real valued, and instead of (3.3) we assumed that $|Z_0|$ was regularly varying with tail index $\alpha > 0$ and

$$\lim_{t\to\infty} \frac{\mathbb{P}\{Z_0 > t\}}{\mathbb{P}\{|Z_0| > t\}} = p \quad \text{and} \quad \lim_{t\to\infty} \frac{\mathbb{P}\{Z_0 < t\}}{\mathbb{P}\{|Z_0| > t\}} = 1 - p.$$

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