ON THE NUMBER OF COMPLETE SUBGRAPHS OF A GRAPH

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A graph G_n consists of n nodes some pairs of which are joined by a single edge. A complete k-graph has k nodes and $\binom{k}{2}$ edges. Erdős [1] proved that if a graph G_n has $[\frac{1}{4}n^2] + h$ edges, then it contains at least $[\frac{1}{2}n] + h - 1$ complete 3-graphs if it contains any at all. The main object of this note is to extend this result to complete k-graphs.

If n and k are integers such that $n = t(k-1) + r$, where t is a non-negative integer and $1 \le r \le k-1$, let

$$d_k(n) = \frac{1}{2}\frac{k-2}{k-1}(n^2 - r^2) + \binom{r}{2}$$

and

$$f_k(n) = \frac{n(k-2) + r}{k-1} - (k-1) .$$

These expressions satisfy the identity

$$(1) \quad d_k(n) - d_k(n-k) = f_k(n) + \binom{k}{2} + (k-2)(n-k) - 1 ,$$

if $n \ge k$. (We adopt the convention that $d_k(0) = 0$.)

THEOREM 1. Let n and k be integers such that $3 \le k \le n$. If the graph G_n has $d_k(n) + h$ edges, when h is any integer, then it contains at least $f_k(n) + h$ complete k-graphs if it contains any at all.

<u>Proof.</u> The proof is by induction on h. The theorem is trivially true if $f_k(n) + h \le 1$. Consider a graph G_n with

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$d_k(n) + h$ edges, where $h > 1 - f_k(n)$. We may assume that G_n contains at least one complete k-graph.

If G_n contains exactly one complete k-graph K, then any node not belonging to K can be joined to at most $k - 2$ nodes of K. Hence, the graph obtained from G_n by removing the k nodes of K and all edges incident with these nodes has at least

$$d_k(n) + h - \binom{k}{2} - (k-2)(n-k) > d_k(n) + 1 - f_k(n) - \binom{k}{2} - (k-2)(n-k)$$

$$= d_k(n-k)$$

edges, by (1). But, a theorem due to Turán [4] states that any graph with $n - k$ nodes and more than $d_k(n - k)$ edges contains at least one complete k-graph. Therefore it is impossible for G_n to contain exactly one complete k-graph.

We may now assume that G_n contains more than one complete k-graph. It is not difficult to see that there must exist an edge of G_n that belongs to at least one, but not to all, of the complete k-graphs of G_n. Then the graph G'_n obtained from G_n by removing this edge has $f_k(n) + h - 1$ edges and it still contains at least one complete k-graph. Hence G'_n contains at least $f_k(n) + (h - 1)$ complete k-graphs, by the induction hypothesis. It follows therefore that G_n contains at least $f_k(n) + (h - 1) + 1 = f_k(n) + h$ complete k-graphs. This suffices to complete the proof of the theorem.

Let $\alpha(k)$ denote the number of complete k-graphs contained in the graph G_n and, for convenience, let $e = \alpha(2)$. If $e \leq d_k(n - 1) + (k - 1) + (n - r)/(k - 1)$. Then simple examples can be given to show that the lower bound for $\alpha(k)$ given by Theorem 1 is best possible; for larger values of e the bound is undoubtedly not best possible.

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Nordhaus and Stewart [3] proved that

$$\alpha(3) \geq \frac{4}{3}\frac{e}{n}(e - \frac{1}{4}n^2)$$

for any graph G_n. Moon and Moser [2] proved that if $e \geq \frac{1}{2}\frac{k-2}{k-1}n^2$, where $3 \leq k \leq n$, then

$$k(k-2)\,\alpha(k)/\alpha(k-1) \geq (k-1)^2\,\alpha(k-1)/\alpha(k-2) - n$$

for any graph G_n. This can be iterated to yield the inequality

$$\alpha(k)/\alpha(k-1) \geq \frac{k-1}{k} \cdot \frac{2}{n}(e - \frac{1}{2}\frac{k-2}{k-1}n^2) .$$

This also can be iterated to yield the following result, which for large values of e is stronger than Theorem 1.

THEOREM 2. If $e \geq \frac{1}{2}\frac{k-2}{k-1}n^2$, where $3 \leq k \leq n^2$, then

$$\alpha(k) \geq \frac{2}{k}\left(\frac{2}{n}\right)^{k-2} e(e - \frac{1}{4}n^2)(e - \frac{1}{3}n^2)\ldots(e - \frac{1}{2}\frac{k-2}{k-1}n^2)$$

for any graph G_n.

In closing, we remark that it can be shown that the distribution of the random variable $\alpha(k)$ over the class of all graphs G_n is asymptotically normal with mean

$$\mu' = \binom{n}{k} 2^{\binom{k}{2}}$$

and variance

$$\sigma^2 = 2^{k(1-k)}\binom{n}{k}\sum_{i=2}^{k}\binom{k}{i}\binom{n-k}{k-i}(2^{\binom{i}{2}} - 1) ,$$

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for each fixed value of k.

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