Canad. Math. Bull. Vol. **67** (2), 2024, pp. 265–278
http://dx.doi.org/10.4153/S0008439523000656

Problems for generalized Monge–Ampère equations

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Dedicated to Prof. Gérard A. Philippin on the occasion of his 80th birthday

Abstract. This paper deals with some Monge–Ampère type equations involving the gradient that are elliptic in the framework of convex functions. First, we show that such equations may be obtained by minimizing a suitable functional. Moreover, we investigate a P-function associated with the solution to a boundary value problem of our generalized Monge–Ampère equation in a bounded convex domain. It will be shown that this P-function attains its maximum value on the boundary of the underlying domain. Furthermore, we show that such a P-function is actually identically constant when the underlying domain is a ball. Therefore, our result provides a best possible maximum principles in the sense of L. E. Payne. Finally, in case of dimension 2, we prove that this P-function also attains its minimum value on the boundary of the underlying domain. As an application, we will show that the solvability of a Serrin's type overdetermined problem for our generalized Monge–Ampère type equation forces the underlying domain to be a ball.

1 Introduction

Throughout this paper, we assume $\Omega \subset \mathbb{R}^n$ to be a bounded strictly convex domain with a smooth boundary $\partial\Omega$. For $x \in \Omega$, we write $x = (x^1, \dots, x^n)$. We use subscripts to denote partial differentiation. For example, we write $u_i = \frac{\partial u}{\partial x^i}$, $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$, etc. We consider smooth strictly convex functions u defined in Ω. The Monge–Ampère operator can be written as

$$\det(D^2 u) = \frac{1}{n}\left(T^{ij}_{(n-1)}(D^2 u) u_i\right)_j,$$

where $D^2 u$ denotes the Hessian matrix of the function u, $\det(D^2 u)$ is the determinant of $D^2 u$, and $T_{(n-1)} = T_{(n-1)}(D^2 u)$ is the adjoint matrix of $D^2 u$ (i.e., the cofactor matrix of $D^2 u$). Here and in what follows, the summation convention from 1 to n over repeated indices is in effect.

A useful equation is the following:

$$T^{ij}_{(n-1)}(D^2 u) = \frac{\partial \det(D^2 u)}{\partial u_{ij}}, \quad i, j = 1, \dots, n.$$

Received by the editors May 20, 2023; revised August 16, 2023; accepted August 18, 2023.
Published online on Cambridge Core September 11, 2023.
AMS subject classification: **35B50, 35N25.**
Keywords: Generalized Monge–Ampère equations, best possible maximum principle, overdetermined problems.

Moreover, the tensor $\left[T^{ij}_{(n-1)}(D^2u)\right]$ is symmetric and divergence-free, that is,

$$\left(T^{ij}_{(n-1)}(D^2u)\right)_j = 0, \quad i = 1, \dots, n. \tag{1.1}$$

If I denotes the $n \times n$ identity matrix, we have

$$T_{(n-1)}(D^2u)D^2u = I\det(D^2u). \tag{1.2}$$

The proof of these results can be found in [14, 15].

Let $g : [0, \infty) \to (0, \infty)$ be a smooth real function satisfying

$$G(s^2) := g(s^2) + 2s^2g'(s^2) > 0. \tag{1.3}$$

We also suppose that $g(0) = G(0) > 0$ (i.e., positive and finite). Note that

$$G(s^2) = \frac{d}{ds}\Big(g(s^2)s\Big).$$

Therefore, the function $g(s^2)s$ is positive and strictly increasing for $s > 0$. A typical example is

$$g(s^2) = (1+s^2)^{-\frac{1}{2}}, \quad G(s^2) = (1+s^2)^{-\frac{3}{2}}.$$

We define the g-Monge–Ampère operator as

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j,$$

where Du denotes the gradient vector of the function u, whereas $|\cdot|$ represents the euclidian norm, so that we have $|Du|^2 = u_iu_i$.

By using (1.1)–(1.3), we find

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j = g^{n-1}(|Du|^2)G(|Du|^2)\det(D^2u). \tag{1.4}$$

Since the operator $\det(D^2u)$ (in the framework of strictly convex functions) is elliptic, then our g-Monge–Ampère operator is also elliptic.

A motivation for the definition of the g-Monge–Ampère operator (1.4) is the following. Using the Kronecker delta $\delta^{i\ell}$, define the $n \times n$ matrix $\mathcal{A} = [\mathcal{A}^{ij}]$, where

$$\mathcal{A}^{ij} = (g(|Du|^2)u_i)_j = \Big(g(|Du|^2)\delta^{i\ell} + 2g'(|Du|^2)u_iu_\ell\Big)u_{\ell j}.$$

The trace of the matrix $\mathcal{A}$ is the familiar operator $(g(|Du|^2)u_i)_i$. We claim that the determinant of the matrix $\mathcal{A}$ coincides with our operator (1.4). Indeed, the eigenvalues $\Lambda^1, \dots, \Lambda^n$ of the $n \times n$ matrix

$$\mathcal{B} = \Big[g(|Du|^2)\delta^{i\ell} + 2g'(|Du|^2)u_iu_\ell\Big]$$

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are the following:

$$\Lambda^1 = \cdots = \Lambda^{n-1} = g(|Du|^2), \ \Lambda^n = G(|Du|^2).$$

Since det$\mathcal{A}$=det$\mathcal{B}$·det(D^2u), we find

$$\det\mathcal{A} = g^{n-1}(|Du|^2)G(|Du|^2)\det(D^2u).$$

The claim follows from the latter equation and (1.4).

Note that $\mathcal{A}$ is not symmetric, in general. However, since $\mathcal{A}$ is the product of two symmetric matrices, it is similar to a diagonal matrix (see [9, p. 487, Theorem 7.6.4]).

This paper is organized as follows. In Section 2, we show that the solution u of the g-Monge–Ampère problem

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j = f(-u) \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega,$$

is the minimum of a suitable functional (depending on g). For $g = 1$, this result is well known (see, for example, [5]).

In Section 3, we consider a boundary value problem involving our g-Monge–Ampère operator in a bounded convex domain and introduce a P-function depending on the solution and its derivatives. We will show that this P-function attains its maximum value on the boundary of the underline domain. Furthermore, we will also show that such a P-function is identically constant when the underlying domain is a ball. Therefore, our P-function satisfies a best possible maximum principle in the sense of L. E. Payne [4, 6, 11, 12].

In Section 4, we consider the case when $n = 2$. In this case, we prove a best possible minimum principle. As a corollary, we solve a Serrin's type overdetermined boundary value problem (see [2, 16, 18]) for the corresponding g-Monge–Ampère equation. Similar problems are discussed in [1, 8, 10, 13] and the references therein.

Results of existence, uniqueness, and regularity for Monge–Ampère equations can be found in [3, 17].

2 Minimizing a functional

Define

$$\Psi(\Omega) := \{u \in C^2(\Omega) \cap C^{0,1}(\overline{\Omega}) : \ u \text{ is strictly convex in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

Recall from [5] that a minimizer $u \in \Psi(\Omega)$ of the functional

$$\inf_{v\in\Psi(\Omega)} \int_\Omega \left[\frac{1}{n(n+1)} T^{ij}_{(n-1)}(D^2v)v_i v_j + \int_0^v f(-\tau)d\tau\right]dx$$

satisfies the equation

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)u_i\right)_j = f(-u).$$

We extend the above result to our g-Monge–Ampère equation.

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Theorem 2.1 *Let*

$$h^n(t^2) = t^{-n-1}\int_0^t \tau^n g^n(\tau^2)\,d\tau. \tag{2.1}$$

A minimizer $u \in \Psi(\Omega)$ *of the functional*

$$\inf_{v\in\Psi(\Omega)} \int_\Omega \left[\frac{1}{n} T^{ij}_{(n-1)}(D^2 v)h^n(|Dv|^2)v_i v_j + \int_0^v f(-\tau)d\tau\right]dx \tag{2.2}$$

satisfies

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2 u)g^n(|Du|^2)u_i\right)_j = f(-u).$$

Proof By integration by parts, we can write the integral in (2.2) as

$$\int_\Omega \left[\frac{-v}{n}\left(T^{ij}_{(n-1)}(D^2 v)h^n(|Dv|^2)v_i\right)_j + \int_0^v f(-\tau)d\tau\right]dx. \tag{2.3}$$

Arguing as in the proof of (1.4), we find

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2 v)h^n(|Dv|^2)v_i\right)_j = h^{n-1}(|Dv|^2)H(|Dv|^2)\det(D^2 v), \tag{2.4}$$

where

$$H(s^2) = h(s^2) + 2s^2 h'(s^2).$$

In view of (2.4), the expression in (2.3) reads as

$$\int_\Omega \left[(-v)h^{n-1}(|Dv|^2)H(|Dv|^2)\det(D^2 v) + \int_0^v f(-\tau)d\tau\right]dx. \tag{2.5}$$

If u is a minimizer of (2.5), we have

$$\frac{d}{dt}\int_\Omega \Big[(-u-tv)h^{n-1}(|Du+tDv|^2)H(|Du+tDv|^2)\det(D^2u+tD^2v) + \int_0^{u+tv} f(-\tau)d\tau\Big]dx\Big|_{t=0} = 0.$$

By computation, we find

$$\begin{aligned}
&\int_\Omega (-v)h^{n-1}(|Du|^2)H(|Du|^2)\det(D^2 u)\,dx \\
&+ \int_\Omega (-u)\left(h^{n-1}(|Du|^2)H(|Du|^2)\right)' 2Du\cdot Dv\det(D^2 u)\,dx \\
&+ \int_\Omega (-u)h^{n-1}(|Du|^2)H(|Du|^2)\operatorname{trace}\left(T_{(n-1)}(D^2 u)D^2 v\right)dx \\
&+ \int_\Omega f(-u)v\,dx = 0.
\end{aligned} \tag{2.6}$$

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Let us compute

$$\begin{aligned}
&\int_\Omega (-u)h^{n-1}(|Du|^2)H(|Du|^2)\mathrm{trace}\Big(T_{(n-1)}(D^2u)D^2v\Big)\,dx\\
&= \int_\Omega (-u)h^{n-1}(|Du|^2)H(|Du|^2)T^{ij}_{(n-1)}(D^2u)v_{ij}\,dx\\
&= \int_\Omega \Big(u\,h^{n-1}(|Du|^2)H(|Du|^2)T^{ij}_{(n-1)}(D^2u)\Big)_j v_i\,dx\\
&= \int_\Omega h^{n-1}(|Du|^2)H(|Du|^2)T^{ij}_{(n-1)}(D^2u)v_iu_j\,dx\\
&\quad + \int_\Omega u\Big(h^{n-1}(|Du|^2)H(|Du|^2)\Big)' 2u_{jh}u_hT^{ij}_{(n-1)}(D^2u)v_i\,dx.
\end{aligned}$$

Integrating by parts and recalling (1.2), from the latter equation, we find

$$\begin{aligned}
&\int_\Omega (-u)h^{n-1}(|Du|^2)H(|Du|^2)\mathrm{trace}\Big(T_{(n-1)}(D^2u)D^2v\Big)\,dx\\
&= \int_\Omega (-v)\Big(h^{n-1}(|Du|^2)H(|Du|^2)T^{ij}_{(n-1)}(D^2u)u_j\Big)_i\,dx\\
&\quad + \int_\Omega u\Big(h^{n-1}(|Du|^2)H(|Du|^2)\Big)' 2Du\cdot Dv\det(D^2u)\,dx.
\end{aligned} \tag{2.7}$$

Insertion of (2.7) into (2.6) yields

$$\begin{aligned}
&\int_\Omega (-v)h^{n-1}(|Du|^2)H(|Du|^2)\det(D^2u)\,dx\\
&+ \int_\Omega (-v)\Big(h^{n-1}(|Du|^2)H(|Du|^2)T^{ij}_{(n-1)}(D^2u)u_j\Big)_i\,dx = \int_\Omega (-v)f(-u)\,dx.
\end{aligned}$$

Since v is arbitrary, we find

$$\begin{aligned}
&h^{n-1}(|Du|^2)H(|Du|^2)\det(D^2u)\\
&+ \Big(h^{n-1}(|Du|^2)H(|Du|^2)T^{ij}_{(n-1)}(D^2u)u_j\Big)_i = f(-u).
\end{aligned} \tag{2.8}$$

Arguing as in the proof of (1.4), one proves that

$$h^{n-1}(|Du|^2)H(|Du|^2)\det(D^2u) = \frac{1}{n}\Big(T^{ij}_{(n-1)}(D^2u)h^n(|Du|^2)u_i\Big)_j.$$

On using the latter equation and the symmetry of $T^{ij}_{(n-1)}(D^2u)$, from (2.8), we find

$$\frac{1}{n}\Big(\Big[h^n(|Du|^2) + nh^{n-1}(|Du|^2)H(|Du|^2)\Big]T^{ij}_{(n-1)}(D^2u)u_i\Big)_j = f(-u).$$

Finally, recalling that $H(s^2) = h(s^2) + 2s^2h'(s^2)$, by (2.1), we find

$$h^n(|Du|^2) + nh^{n-1}(|Du|^2)H(|Du|^2) = g^n(|Du|^2).$$

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Hence,

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j = f(-u).$$

The theorem follows. ∎

3 A best possible maximum principle

Let u be a solution to some boundary value problem in a domain Ω. Following Payne [11], we say that a function $P(x)$, depending on u and its derivatives, satisfies a best possible maximum principle if it satisfies a maximum principle for every convex domain Ω and, in addition, it is a constant for some special domain Ω (a ball in our case).

For a discussion on the best possible maximum principles related to second-order linear (or quasi-linear) elliptic equations, we refer to [12]. Concerning Monge–Ampère equations, we recall a special case of Theorem 2.3 of [8]. Let u be a strictly convex smooth solution to the problem

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)u_i\right)_j = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

and let

$$P(x) = \frac{1}{2}|Du|^2 - u.$$

By Theorem 2.3 of [8], $P(x)$ attains its maximum value on $\partial\Omega$; furthermore, in case Ω is a ball, $P(x)$ is a constant.

We are going to extend this result to our g-Monge–Ampère equation. Consider the problem

$$\frac{1}{n}\left(T^{ij}_{(n-1)}(D^2u)g^n(|Du|^2)u_i\right)_j = 1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.1}$$

and define the P-function

$$P(x) = \int_0^{|Du|} G(t^2)t\,dt - u, \tag{3.2}$$

where G is defined as in (1.3). We note that our result is already proved in [13] by using a quite complicate argument. We give here a different and more clean proof. Moreover, our method allows us to prove that if $P(x)$ is identically constant, then Ω must be a ball.

Theorem 3.1 *Let u be a strictly convex smooth solution to Problem (3.1). If $P(x)$ is defined as in (3.2), we have the following.*

(i) *If Ω is a ball, then $P(x)$ is identically constant.*
(ii) *For any convex Ω, $P(x)$ attains its maximum value on $\partial\Omega$.*
(iii) *If $P(x)$ is identically constant in Ω, then Ω must be a ball.*

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Proof (i) If Ω is a ball, $u(x)$ is radial. If $v(r) = u(x)$ for $|x| = r$, we have

$$P(r) = \int_0^{v'} G(\tau^2)\tau\, d\tau - v.$$

Differentiation yields

$$P'(r) = G((v')^2)v'v'' - v'. \tag{3.3}$$

On using (1.4), we can write the equation in (3.1) (in the radial case) as

$$g^{n-1}((v')^2)G((v')^2)\det(D^2 v) = 1.$$

Since

$$\det(D^2 v) = v''\left(\frac{v'}{r}\right)^{n-1},$$

we find

$$g^{n-1}((v')^2)G((v')^2)(v')^{n-1}v'' = r^{n-1}.$$

Since

$$G(s^2) = \frac{d}{ds}\Big(g(s^2)s\Big),$$

we can write the previous equation as

$$\Big(g((v')^2)v'\Big)^{n-1}\frac{d}{dr}\Big(g((v')^2)v'\Big) = r^{n-1},$$

or, equivalently,

$$\frac{1}{n}\frac{d}{dr}\Big(g((v')^2)v'\Big)^n = r^{n-1}.$$

Recalling that g is continuous on $[0, r)$ and that $v'(0) = 0$, we integrate the above identity over $(0, r)$, to find

$$\Big(g((v')^2)v'\Big)^n = n\int_0^r t^{n-1} = r^n,$$

or, equivalently,

$$g((v')^2)v' = r.$$

Differentiation yields

$$G((v')^2)v'' = 1.$$

By (3.3) and the latter equation, we find

$$P'(r) = v'\Big[G((v')^2)v'' - 1\Big] = 0.$$

It follows that $P(r)$ is identically constant.

(ii) Let Ω be a bounded convex domain. Arguing by contradiction, let $\tilde{x} \in \Omega$ be a point such that

$$P(\tilde{x}) = \int_0^{|Du(\tilde{x})|} G(t^2)t\,dt - u(\tilde{x}) > \max_{x\in\partial\Omega} \int_0^{|Du(x)|} G(t^2)t\,dt.$$

Choose $0 < \tau < 1$ close enough to 1 so that

$$\int_0^{|Du(\tilde{x})|} G(t^2)t\,dt - \tau u(\tilde{x}) > \max_{x\in\partial\Omega} \int_0^{|Du(x)|} G(t^2)t\,dt.$$

Then, also the function

$$\bar{P}(x) = \int_0^{|Du(x)|} G(t^2)t\,dt - \tau u(x)$$

attains its maximum value at some point $\bar{x} \in \Omega$. At the point $\bar{x}$, we have either $Du = 0$ or $|Du| > 0$. Consider first the case $Du = 0$. Then,

$$\bar{P}_i = G(|Du|^2)u_{ih}u_h - \tau u_i.$$

Further differentiation and computation at $Du = 0$ yields

$$\bar{P}_{ii} = G(0)u_{ih}u_{ih} - \tau u_{ii}, \quad i = 1, \dots, n.$$

Let us make a rigid rotation around the point $\bar{x}$ so that

$$D^2u = \operatorname{diag}\{u_{11}, \dots, u_{nn}\}. \tag{3.4}$$

Then,

$$\bar{P}_{ii} = G(0)u_{ii}u_{ii} - \tau u_{ii}, \quad i = i, \dots, n. \tag{3.5}$$

Clearly, if $(D^2u)^{-1}$ is the inverse of D^2u, also $(D^2u)^{-1}$ will be diagonal, and

$$(D^2u)^{-1} = \operatorname{diag}\{u^{11}, \dots, u^{nn}\},$$

where u^{ij} is the (i, j)-entry of the matrix $(D^2u)^{-1}$. Multiplying (3.5) by u^{ii} and adding from $i = 1$ to $i = n$, we find

$$u^{ii}\bar{P}_{ii} = G(0)\Delta u - \tau n. \tag{3.6}$$

On the other hand, from equations (3.1) and (1.4), we find (recall that $g(0) = G(0)$)

$$\det(D^2u) = \frac{1}{G^n(0)}.$$

By using this equation, from (3.6), we find

$$u^{ii}\bar{P}_{ii} = \frac{\Delta u}{\Big(\det(D^2u)\Big)^{\frac{1}{n}}} - \tau n. \tag{3.7}$$

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Finally, since the matrix D^2u is diagonal and positive definite, we have (we also use the arithmetic–geometric mean inequality)

$$\left(\det(D^2u)\right)^{\frac{1}{n}} = (u_{11}\cdots u_{nn})^{\frac{1}{n}} \le \frac{\Delta u}{n}.$$

By the latter inequality and (3.7), we find

$$u^{ii}\bar{P}_{ii} \ge n(1-\tau) > 0.$$

Hence, $\bar{P}$ cannot have a maximum point at $\bar{x}$ with $Du(\bar{x}) = 0$.

Let $\bar{x} \in \Omega$ be a point of maximum for $\bar{P}$, and let $|Du| > 0$ at $\bar{x}$. We have

$$\bar{P}_i = G(|Du|^2)u_{ih}u_h - \tau u_i, \tag{3.8}$$

and

$$\bar{P}_{ii} = 2G'(u_{ih}u_h)^2 + Gu_{iih}u_h + Gu_{ih}u_{ih} - \tau u_{ii}, \quad i = 1, \dots, n.$$

Let us make a rigid rotation around the point $\bar{x}$ so that (3.4) holds. Then (for i fixed), we have

$$\bar{P}_{ii} = 2G'u_{ii}^2u_i^2 + Gu_{iih}u_h + Gu_{ii}^2 - \tau u_{ii}.$$

Multiplying by u^{ii} and adding from $i = 1$ to $i = n$, we find

$$u^{ii}\bar{P}_{ii} = 2G'u_{ii}u_i^2 + Gu^{ii}u_{iih}u_h + G\Delta u - n\tau. \tag{3.9}$$

By using (1.4), let us write the equation in (3.1) as

$$\det(D^2u) = \frac{1}{g^{n-1}(|Du|^2)G(|Du|^2)}. \tag{3.10}$$

Differentiation with respect to x^h yields

$$T^{ij}_{(n-1)}(D^2u)u_{ijh} = -\frac{1}{\left(g^{n-1}G\right)^2}\left[(n-1)g^{n-2}g'G + g^{n-1}G'\right]2u_{hk}u_k. \tag{3.11}$$

Since $T_{(n-1)}(D^2u)D^2u = \det(D^2u)I$, on using (3.10) and recalling that u^{ij} is the (i, j)-entry of the matrix $(D^2u)^{-1}$, we get

$$T^{ij}_{(n-1)}(D^2u) = \frac{u^{ij}}{g^{n-1}G}, \quad i, j = 1, \dots, n.$$

Therefore, recalling that D^2u has a diagonal form, from (3.11), we find

$$u^{ii}u_{iih} = -\left[(n-1)\frac{g'}{g} + \frac{G'}{G}\right]2u_{hh}u_h.$$

Insertion of this equation into (3.9) leads to

$$u^{ii}\bar{P}_{ii} = 2G'u_{ii}u_i^2 - \left[(n-1)\frac{g'G}{g} + G'\right]2u_{hh}u_h^2 + G\Delta u - n\tau.$$

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Simplifying, we find

$$u^{ii}\bar{P}_{ii} = -2(n-1)\frac{g'G}{g}u_{hh}u_h^2 + G\Delta u - n\tau, \tag{3.12}$$

Since $\bar{x}$ is assumed to be a point of maximum, we have $\bar{P}_i = 0$, and from (3.8), we find

$$Gu_{hh}u_h^2 = \tau|Du|^2. \tag{3.13}$$

Insertion of (3.13) into (3.12) yields

$$u^{ii}\bar{P}_{ii} = 2(1-n)\frac{g'}{g}\tau|Du|^2 + G\Delta u - n\tau. \tag{3.14}$$

If $\mathcal{A} = [\mathcal{A}^{ij}]$ with $\mathcal{A}^{ij} = \Big(g(|Du|^2)u_i\Big)_j$, we know that

$$\det\mathcal{A} = g^{n-1}(|Du|^2)G(|Du|^2)\det(D^2u).$$

Therefore, by (1.4), the equation in (3.1) can be written as

$$\det\mathcal{A} = 1.$$

On the other hand, since $\mathcal{A}$ is positive definite, by the Hadamard inequality (see Theorem 7.8.1 of [9]) and the arithmetic–geometric mean inequality, we have

$$1 = \Big(\det\mathcal{A}\Big)^{\frac{1}{n}} \le \Big(\mathcal{A}^{11}\cdots\mathcal{A}^{nn}\Big)^{\frac{1}{n}} \le \frac{\mathcal{A}^{11}+\cdots+\mathcal{A}^{nn}}{n},$$

with equality sign if and only if

$$\mathcal{A}^{11} = \cdots = \mathcal{A}^{nn}, \ \text{ and } \ \mathcal{A}^{ij} = 0 \ \ \forall i \neq j. \tag{3.15}$$

Therefore,

$$\mathcal{A}^{11}+\cdots+\mathcal{A}^{nn} \ge n$$

and

$$\Big(g(|Du|^2)u_i\Big)_i \ge n.$$

Recalling that D^2u has a diagonal form, this inequality can be written as

$$g\Delta u + 2g'u_{ii}u_i^2 \ge n.$$

On using (3.13), the latter inequality reads as

$$g\Delta u + 2\frac{g'}{G}\tau|Du|^2 \ge n,$$

from which we find

$$G\Delta u + 2\frac{g'}{g}\tau|Du|^2 \ge n\frac{G}{g} = n + 2n\frac{g'}{g}|Du|^2.$$

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Hence,

$$G\Delta u + 2(\tau - n)\frac{g'}{g}|Du|^2 \geq n.$$

Inserting this estimate into (3.14), we find

$$\begin{aligned} u^{ii}\bar{P}_{ii} &\geq n(1-\tau) + \frac{g'}{g}|Du|^2 2n(1-\tau) \\ &= n(1-\tau)\left(1 + \frac{g'}{g}2|Du|^2\right) \\ &= n(1-\tau)\frac{G}{g} > 0. \end{aligned}$$

It follows that $\bar{P}$ cannot have a maximum point at $\bar{x}$ with $|Du(\bar{x})| > 0$. We conclude that P must attain its maximum value on the boundary $\partial\Omega$.

(iii) If $P(x)$ is a constant, we have

$$u^{ii}P_{ii} = 0 \quad \text{in} \quad \Omega.$$

Therefore, by the argument used to prove (ii), all equations in (3.15) must hold. This means that

$$\left(g(|Du|^2)u_1\right)_1 = \cdots = \left(g(|Du|^2)u_n\right)_n, \text{ and } \left(g(|Du|^2)u_i\right)_j = 0, \quad \forall i \neq j.$$

Then, for some $x_0 \in \Omega$, we have

$$\begin{aligned} &g(|Du|^2)u_i = x^i - x_0^i, \quad i = 1, \ldots, n, \\ &g^2(|Du|^2)u_i^2 = (x^i - x_0^i)^2, \\ &g^2(|Du|^2)\sum_1^n u_i^2 = \sum_1^n (x^i - x_0^i)^2 = r^2, \\ &g^2(|Du|^2)|Du|^2 = r^2, \\ &g(|Du|^2)|Du| = r. \end{aligned}$$

Since $g(s^2)s$ is strictly increasing, $|Du|$ must be radially symmetric around the point x_0. Finally, since

$$\int_0^{|Du|} G(t^2)t\,dt - u = c,$$

also u will be radially symmetric. Statement (iii) follows.

The theorem is proved. ∎

Remark From Theorem 3.1, we get the following estimate:

$$-u_m \leq \int_0^{|Du|_M} G(t^2)t\,dt,$$

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where

$$u_m = \min_{\Omega} u(x), \quad |Du|_M = \max_{\partial\Omega} |Du|.$$

Note that this estimate is sharp, in the sense that the equality sign holds when Ω is a ball.

4 The case $n = 2$

Here, we prove a minimum principle for our P-function, which extend the result obtained in the particular case $g \equiv 1$ in [7].

Theorem 4.1 *Let u be a strictly convex smooth solution to Problem (3.1) in case $n = 2$, and let $P(x)$ be defined as in (3.2). Then P attains its minimum value on the boundary $\partial\Omega$.*

Proof Arguing by contradiction, let $\tilde{x} \in \Omega$ be a point such that

$$P(\tilde{x}) = \int_0^{|Du(\tilde{x})|} G(t^2)t\,dt - u(\tilde{x}) < \min_{x\in\partial\Omega} \int_0^{|Du(x)|} G(t^2)t\,dt.$$

Choose $\tau > 1$ close enough to 1 so that

$$\int_0^{|Du(\tilde{x})|} G(t^2)t\,dt - \tau u(\tilde{x}) < \min_{x\in\partial\Omega} \int_0^{|Du(x)|} G(t^2)t\,dt.$$

Then, also the function

$$\bar{P}(x) = \int_0^{|Du(x)|} G(t^2)t\,dt - \tau u(x)$$

attains its minimum value at some point $\bar{x} \in \Omega$. We may have either $|Du(\bar{x})| > 0$ or $Du(\bar{x}) = 0$. Consider first the case $|Du(\bar{x})| > 0$. By the same computations as in the proof of Theorem 3.1, we find (3.12) with $n = 2$, that is,

$$u^{ii}\bar{P}_{ii} = -2\frac{g'G}{g}u_{hh}u_h^2 + G\Delta u - 2\tau. \tag{4.1}$$

As in the proof of Theorem 3.1, we assume that (3.4) holds at $\bar{x}$. Since $\bar{x}$ is a point of minimum, we have $\bar{P}_i = 0$, and from (3.8), we find

$$Gu_{hh}u_h^2 = \tau|Du|^2. \tag{4.2}$$

Insertion of (4.2) into (4.1) yields

$$u^{ii}\bar{P}_{ii} = -2\frac{g'}{g}\tau|Du|^2 + G\Delta u - 2\tau. \tag{4.3}$$

Since $|Du| > 0$, we have either $u_1 \neq 0$ or $u_2 \neq 0$. If $u_1 \neq 0$, by (3.8), we have

$$u_{11} = \frac{\tau}{G}.$$

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Since $n = 2$, equation (3.1) at $\bar{x}$ reads as $gGu_{11}u_{22} = 1$, and then, by our last equation, we find

$$u_{22} = \frac{1}{\tau g}.$$

Hence,

$$\Delta u = \frac{\tau}{G} + \frac{1}{\tau g}. \tag{4.4}$$

Note that (4.4) continues to holds if $u_1 = 0$ and $u_2 \neq 0$. Insertion of (4.4) into (4.3) leads to

$$u^{ii}\bar{P}_{ii} = \frac{G}{g}\left(\frac{1}{\tau} - \tau\right) < 0.$$

It follows that $\bar{P}$ cannot have a minimum point at any $x \in \Omega$ with $|Du| > 0$.

Consider now the case $Du(\bar{x}) = 0$. At $\bar{x}$, we have

$$\bar{P}_{kk} = Gu_{kk}^2 - \tau u_{kk} \geq 0, \quad k = 1, 2. \tag{4.5}$$

Since $\bar{x}$ is a point of minimum (also) for u, we have $u_{11} \geq 0$ and $u_{22} \geq 0$. But since

$$u_{11}u_{22} = \frac{1}{g(0)G(0)} > 0,$$

we must have $u_{11} > 0$ and $u_{22} > 0$. Hence, (4.5) implies that

$$Gu_{11} - \tau \geq 0 \ \text{ and } \ Gu_{22} - \tau \geq 0.$$

It follows that

$$G^2u_{11}u_{22} \geq \tau^2. \tag{4.6}$$

On the other hand, our equation at $\bar{x}$ (where $Du = 0$, so $g = G$) reads as

$$G^2u_{11}u_{22} = 1,$$

in contradiction with (4.6) because $\tau > 1$.

We have proved that $\bar{P}$ cannot have a minimum point at $\bar{x}$ with $|Du(\bar{x})| = 0$. We conclude that P must attain its minimum value on the boundary $\partial\Omega$. The theorem is proved. ■

Corollary 4.2 *Let u be a strictly convex smooth solution to Problem (3.1) in case $n = 2$. If u satisfies the additional condition*

$$|Du| = c \quad \text{on } \partial\Omega,$$

then Ω must be a ball.

Proof By Theorems 3.1(ii) and 4.1, the function $P(x)$ defined as in (3.2) is a constant in Ω. Hence, the corollary follows by Theorem 3.1(iii). ■

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https://doi.org/10.4153/S0008439523000656 Published online by Cambridge University Press