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# CHARACTERIZING $n$-ISOCLINIC CLASSES OF CROSSED MODULES

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**Abstract.** In this paper, we introduce the notion of the equivalence relation, called $n$-isoclinism, between crossed modules of groups, and give some basic properties of this notion. In particular, we obtain some criteria under which crossed modules are $n$-isoclinic. Also, we present the notion of $n$-stem crossed module and, under some conditions, determine them within an $n$-isoclinism class.

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**1. Introduction.** Let $G$ and $H$ be two groups and $n$ be a non-negative integer. Then, $G$ and $H$ are said to be *$n$-isoclinic*, $G \underset{n}{\sim} H$, if there exist isomorphisms $\alpha : G/Z_n(G) \longrightarrow H/Z_n(H)$ and $\beta : \gamma_{n+1}(G) \longrightarrow \gamma_{n+1}(H)$ in such a way that $\beta$ is compatible with $\alpha$, that is, the $(n+1)$-fold commutator $[\ldots[[h_1, h_2], h_3], \ldots, h_{n+1}]$ equals $\beta([\ldots[[g_1, g_2], g_3], \ldots, g_{n+1}])$ for any $h_i \in \alpha(g_iZ_n(G))$ and $g_i \in G$ for $i = 1, \ldots, n+1$. The pair $(\alpha, \beta)$ is called an *$n$-isoclinism* between $G$ and $H$. It is obvious that $n$-isoclinism is an equivalence relation and, hence, produces a partition on the class of groups such that all nilpotent groups of class at most $n$ fall into an equivalence class. Historically, the 1-isoclinism relation was introduced by Hall [**11**], with the purpose of classifying all finite $p$-groups (a beautiful outcome of the principle of 1-isoclinism is the book [**10**] in which all isomorphism types of groups of order $2^n$ ($n \leq 6$) are described, together with many similar results. See also [**4, 5, 13, 17, 19**] for more applications). Later, Hall [**12**] generalized the notion of 1-isoclinism to the notion of $\mathcal{V}$-isologism, with respect to a given variety of groups $\mathcal{V}$ such that if $\mathcal{V}$ is the variety of all the trivial groups or the variety of all abelian groups, then $\mathcal{V}$-isologism coincides with isomorphism and 1-isoclinism, respectively. For the variety $\mathcal{V}$ of nilpotent groups of class at most $n$, the notion of $\mathcal{V}$-isologism is nothing but the notion of $n$-isoclinism. This concept plays an important role in group theory and has since been further investigated by several authors (see [**2, 3, 6, 14, 15, 22, 30**]). In particular, Bioch [**3**] determined all groups occurring in an $n$-isoclinism class of a given group, and Hekster [**14**], under some conditions, proved that each $n$-isoclinism class of groups contains at least a group with the property that its centre is contained in $(n+1)$-th term of its lower central series, which are called *$n$-stem groups*. Also, it is shown in [**29**] that $n$-isoclinic groups can be $n$-isoclinically embedded into one group.

The algebraic study of the category of crossed modules was initiated by Norrie [**24**] and has led to a substantial algebraic theory contained essentially in the papers

[**1**, **7–9**, **18**, **23**, **25–28**]. In particular, the second author in the joint paper [**26**] presented the analogous notion of the 1-isoclinism for crossed modules and extended some known properties of the 1-isoclinism between groups to the 1-isoclinism between crossed modules. Also, Odabaş et al. [**25**] gave an algorithm for checking 1-isoclinism between crossed modules and applied the algorithm to classify certain crossed modules.

In this paper, we introduce the notion of $n$-isoclinism between crossed modules, which is a vast generalization of $n$-isoclinism of groups and the 1-isoclinism of crossed modules. Similar to the works of Bioch [**3**] and van der Waall [**30**] in the group case, we give some equivalent conditions under which crossed modules are $n$-isoclinic. Also, we present an analogue to the $n$-stem groups in the $n$-isoclinic case of crossed modules and characterize them inside an $n$-isoclinism class in an impotent special case.

**2. Preliminaries on crossed modules.** This section is devoted to recall some basic definitions in the category of crossed modules and give some results related to the terms of upper and lower central series of a given crossed module, which will be needed in the sequel.

A *crossed module* $(T, G, \partial)$ is a group homomorphism $\partial : T \longrightarrow G$ together with an action of $G$ on $T$, written ${}^{g}t$ for $t \in T$ and $g \in G$, satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ and ${}^{\partial(t)}t' = tt't^{-1}$, for all $t, t' \in T$, $g \in G$. It is worth noting that for any crossed module $(T, G, \partial)$, $\mathrm{Im}\partial$ is a normal subgroup of $G$ and $\ker\partial$ is a $G$-invariant subgroup in the centre of $T$. Evidently, for any normal subgroup $N$ of a group $G$, $(N, G, i)$ is a crossed module, where $i$ is the inclusion and $G$ acts on $N$ by conjugation. This way, every group $G$ can be seen as a crossed module in two obvious ways: $(1, G, i)$ or $(G, G, id)$.

A *morphism of crossed modules* $(\gamma_1, \gamma_2) : (T, G, \partial) \longrightarrow (T', G', \partial')$ is a pair of homomorphisms $\gamma_1 : T \longrightarrow T'$ and $\gamma_2 : G \longrightarrow G'$ such that $\partial'\gamma_1 = \gamma_2\partial$ and $\gamma_1({}^{g}t) = {}^{\gamma_2(g)}\gamma_1(t)$ for all $g \in G$, $t \in T$.

Taking objects and morphisms as defined above, we obtain the category $\mathfrak{CM}$ of crossed modules. In this category, one can find the familiar notions of injection, surjection, (normal) subobject, kernel, cokernel, exact sequence, etc.; most of them can be found in detail in [**8**, **24**].

Let $(T, G, \partial)$ be a crossed module with normal crossed submodules $(S, H, \partial)$ and $(L, K, \partial)$. The following is a list of notations that will be used:

- $Z(T, G, \partial) = (T^G, Z(G) \cap st_G(T), \partial)$ is the *centre* of $(T, G, \partial)$, where $Z(G)$ is the centre of $G$, $T^G = \{t \in T | {}^{g}t = t \text{ for all } g \in G\}$, and $st_G(T) = \{g \in G | {}^{g}t = t \text{ for all } t \in T\}$.
- $(T, G, \partial)' = ([G, T], G', \partial)$ is the *commutator crossed submodule* of $(T, G, \partial)$, where $G' = [G, G]$ and $[G, T] = \langle {}^{g}tt^{-1} \mid t \in T, g \in G\rangle$ is the displacement subgroup of $T$ relative to $G$.
- $[(S, H, \partial), (L, K, \partial)]$ is the normal crossed submodule $([K, S][H, L], [H, K], \partial)$ of $(T, G, \partial)$.
- $\gamma_n(T, G, \partial)$ denotes the $n$th term of lower central series of $(T, G, \partial)$ defined inductively by $\gamma_1(T, G, \partial) = (T, G, \partial)$ and $\gamma_{n+1}(T, G, \partial) = [\gamma_n(T, G, \partial), (T, G, \partial)]$, for $n \geq 1$.
- $Z_n(T, G, \partial)$ denotes the $n$th term of the upper central series of $(T, G, \partial)$ defined inductively by $Z_0(T, G, \partial) = 1$ and $Z_{n+1}(T, G, \partial)/Z_n(T, G, \partial)$ is the centre of $(T, G, \partial)/Z_n(T, G, \partial)$, for $n \geq 0$.

Furthermore, for any $n \geq 1$, we define

- $\zeta_n(T) = \{t \in T | [_nG, t] = 1\}$, where $[_1G, t] = \langle {}^g t t^{-1} | g \in G\rangle$ and inductively $[_{n+1}G, t] = [G, [_nG, t]]$;
- $\kappa_n(G) = Z_n(G) \cap \{g \in G | [_iG, [[_{n-1-i}G, g], T]] = 1 \quad \text{for all} \quad 0 \le i \le n-1\}$, where $[_0G, T_1] = T_1$ for each subgroup $T_1$ of $T$, $[g, T] = \langle {}^g t t^{-1} | t \in T\rangle$, $[_0G, g] = g$ and inductively $[_nG, g] = [G, [_{n-1}G, g]]$;
- $\Gamma_n(T, G) = [_{n-1}G, T]$, where $[_0G, T] = T$ and inductively $[_nG, T] = [G, [_{n-1}G, T]]$.

Note that a crossed module $(T, G, \partial)$ is said to be *finite* (respectively, *finitely generated*) if the groups $T$ and $G$ are both finite (respectively, finitely generated). In the case of finite, we define $|(T, G, \partial)|$ to be the ordered pair $(|T|, |G|)$. Clearly, a total order is defined on the class of all finite crossed modules by means of $|(T, G, \partial)| < |(T', G', \partial')|$ if and only if $|T| < |T'|$, or $|T| = |T'|$ and $|G| < |G'|$. Also, the crossed module $(T, G, \partial)$ is called *strongly finitely generated* if it is a homomorphic image of a finitely generated projective crossed module. Plainly, any finitely generated simply connected crossed module is strongly finitely generated. In particular, a group $G$ is finitely generated if and only if it is strongly finitely generated as a crossed module in any of the usual ways. Finally, the crossed module $(T, G, \partial)$ is *nilpotent* of class at most $n \ge 1$ when $\gamma_{n+1}(T, G, \partial) = 1$.

The following results are very helpful in our further proofs.

LEMMA 2.1. *Let $(T, G, \partial)$ be any crossed module. Then,*
($i$) *for all $t, t' \in T$ and $g, g' \in G$, the following identities hold:*

$$[gg', t] = [g, [g', t]][g', t][g, t], \tag{1}$$

$$[g, tt'] = [g, t][t, [g, t']][g, t']; \tag{2}$$

($ii$) *for any $L \trianglelefteq T$ and $H, K \trianglelefteq G$,*
($a$) $[H, [K, L]] \subseteq [K, [H, L]][[H, K], L]$,
($b$) $[[H, K], L] \subseteq [H, [K, L]][K, [H, L]]$, *if $H$ is contained in $K$.*

*Proof.*
($i$) We have

$$[gg', t] = {}^{gg'}tt^{-1} = {}^g({}^{g'}tt^{-1})({}^g t)t^{-1} = {}^g[g', t][g', t]^{-1}[g', t][g, t] = [g, [g', t]][g', t][g, t],$$

$$[g, tt'] = {}^g(tt')(tt')^{-1} = ({}^g tt^{-1})t({}^g t' t'^{-1})t^{-1} = [g, t]t[g, t']t^{-1} = [g, t][t, [g, t']][g, t'].$$

($ii$) One first notes that $[H, L] \trianglelefteq T$ because

$${}^t({}^h l l^{-1}) = {}^{\partial(t)}({}^h l)({}^t l^{-1}) = {}^{\partial(t)h}({}^{\partial(t)}l)({}^t l^{-1}) = {}^{\partial(t)h}({}^t l)({}^t l)^{-1} \in [H, L],$$

for all $t \in T, h \in H, l \in L$. Similarly, $[H, [K, L]]$, $[K, [H, L]]$ and $[[H, K], L]$ are normal in $T$.

($a$) Put $X = [K, [H, L]][[H, K], L]$. Then, for all $h \in H, k \in K, l \in L$, we have

$$\begin{aligned} {}^h[k, l] &= {}^h({}^k l l^{-1}) = {}^{[h,k]k}({}^h l)({}^h l)^{-1} = [[h, k]k, {}^h l] \\ &\overset{\text{by}(1)}{=} [[h, k], [k, {}^h l]][k, {}^h l][[h, k], {}^h l] \equiv [k, {}^h l] \pmod X \\ &= [k, [h, l]l] \overset{\text{by}(2)}{=} [k, [h, l]][[h, l], [k, l]][k, l] \\ &\equiv [k, l] \pmod X), \end{aligned}$$

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and consequently, $[h,[k,l]] = {}^{h}[k,l][k,l]^{-1} \equiv 1 \pmod{X}$. It thus follows that $[H,[K,L]] \subseteq X$.

($b$) Put $Y = [H,[K,L]][K,[H,L]]$. Then, for all $h \in H$, $k \in K$, $l \in L$, we have

$$[h^{-1}k^{-1},l] \overset{\text{by}(1)}{=} [h^{-1},[k^{-1},l]][k^{-1},l][h^{-1},l] \equiv [k^{-1},l][h^{-1},l] \pmod{Y}$$

and so,

$$\begin{aligned}[k,[h^{-1}k^{-1},l]] &\equiv [k,[k^{-1},l][h^{-1},l]] \pmod{Y}\\ &\overset{\text{by}(2)}{=} [k,[k^{-1},l]][[k^{-1},l],[k,[h^{-1},l]]][k,[h^{-1},l]]\\ &\equiv [k,[k^{-1},l]] \pmod{Y}.\end{aligned}$$

Using the above results, we obtain

$$\begin{aligned}[{}^{k}h^{-1},l] &\overset{\text{by}(1)}{=} [k,[h^{-1}k^{-1},l]][h^{-1}k^{-1},l][k,l] \equiv [k,[k^{-1},l]][k^{-1},l][h^{-1},l][k,l] \pmod{Y}\\ &= [k,l]^{-1}[k^{-1},l]^{-1}[k^{-1},l][h^{-1},l][k,l] = [[k,l]^{-1},[h^{-1},l]][h^{-1},l][k,l]^{-1}[k,l]\\ &\equiv [h^{-1},l] \pmod{Y},\end{aligned}$$

and then, using the assumption that $H \subseteq K$, it follows that

$$\begin{aligned}[[h,k],l] &= [h\,{}^{k}h^{-1},l] = [h,[{}^{k}h^{-1},l]][{}^{k}h^{-1},l][h,l] \equiv [{}^{k}h^{-1},l][h,l] \pmod{Y}\\ &\equiv [h^{-1},l][h,l] \pmod{Y}.\end{aligned}$$

On the other hand, $1 = [hh^{-1},l] \overset{\text{by}(1)}{=} [h,[h^{-1},l]][h^{-1},l][h,l]$ and hence $[h^{-1},l] \equiv [h,l]^{-1} \pmod{Y}$. We therefore conclude that $[[h,k],l] \equiv 1 \pmod{Y}$, whence $[[H,K],L] \subseteq Y$. □

The following proposition provides an explicit description for the terms of upper and lower central series of a crossed module.

Proposition 2.2. *Let $(T,G,\partial)$ be any crossed module and $n \geq 1$. Then,*
($i$) $Z_n(T,G,\partial) = (\zeta_n(T),\kappa_n(G),\partial)$,
($ii$) $\gamma_n(T,G,\partial) = (\Gamma_n(T,G),\gamma_n(G),\partial)$.

*Proof.* Part ($i$) is immediate from [**27**, Lemma 2.1($i$),($iii$)]. As for ($ii$), applying [**27**, Lemma 2.1($ii$),($iv$)], we have $\gamma_n(T,G,\partial) = ([_{n-1}G,T]\prod_{i=2}^{n-1}[_{n-1-i}G,[\gamma_i(G),T]],\gamma_n(G),\partial)$. So, it is enough to prove that for any $i \geq 1$, $[\gamma_i(G),T] \subseteq [_iG,T]$. But, this follows by induction on $i$, and using the fact that $[\gamma_{i+1}(G),T] \subseteq [\gamma_i(G),[G,T]][G,[\gamma_i(G),T]]$ (the latter inclusion follows by Lemma 2.1($ii$)). □

We see from the proof of Proposition 2.2 that the subgroup $\Gamma_n(T,G)$ of $T$ introduced in [**27**] is equal to $[_{n-1}G,T]$.

Proposition 2.3. *Let $(T,G,\partial)$ be any crossed module, and i,j be positive integers with $j \geq i$. Then,*
($i$) $[\gamma_i(G),\kappa_j(G)] \leq \kappa_{j-i}(G)$,
($ii$) $[\gamma_i(G),\zeta_j(T)] \leq \zeta_{j-i}(G)$,
($iii$) $[\kappa_j(G),\Gamma_i(T,G)] \leq \zeta_{j-i}(G)$.

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*Proof.*

($i$) Using induction on $i$, the case $i = 1$ being clear. The three subgroups lemma shows that $[\gamma_{i+1}(G), \kappa_j(G)] = [[\gamma_i(G), G], \kappa_j(G)]$ is contained in the product $[[G, \kappa_j(G)], \gamma_i(G)][[\kappa_j(G), \gamma_i(G)], G]$; by induction, the latter is contained in $\kappa_{j-i-1}(G)$.

($ii$) It is proved by using Lemma 2.1($ii$) and an argument similar to part ($i$).

($iii$) Induct on $i$. By the definition of $\kappa_j(G)$, we have $[_{j-1}G, [\kappa_j(G), T]] = 1$, implying that $[\kappa_j(G), \Gamma_1(T, G)] \leq \zeta_{j-1}(G)$. Now, assume that the result holds for $i \geq 1$. Then, it follows from Lemma 2.1($ii$), the induction hypothesis and parts ($i$), ($ii$) that

$$[\kappa_j(G), \Gamma_{i+1}(T, G)] = [\kappa_j(G), [G, \Gamma_i(T, G)]] \leq [G, [\kappa_j(G), \Gamma_i(T, G)]][[G, \kappa_j(G)], \Gamma_i(T, G)]$$
$$\leq [G, \zeta_{j-i}(T)][\kappa_{j-1}(G), \Gamma_i(T, G)] \leq \zeta_{j-i-1}(T),$$

as wished to show.

□

The following corollary is an interesting consequence of the above.

Corollary 2.4. Let $(T, G, \partial)$ be any crossed module. Then, for all positive integers $i, j$ with $j \geq i$, $[Z_j(T, G, \partial), \gamma_i(T, G, \partial)] \leq Z_{j-i}(T, G, \partial)$.

Lemma 2.5. *Let $(T, G, \partial)$ be any crossed module, $z \in \zeta_n(T)$ and $k \in \kappa_n(G)$. Then, for all $t \in T$ and $g_1, \ldots, g_{n+1} \in G$, we have*

(i) $[g_n, \ldots, [g_2, [g_1, zt]] \ldots] = [g_n, \ldots, [g_2, [g_1, t]] \ldots]$,

(ii) *for all* $1 \leq i \leq n$, $[g_n, \ldots, [g_ik, \ldots, [g_1, t] \ldots] \ldots] = [g_n, \ldots, [g_i, \ldots, [g_1, t] \ldots] \ldots]$,

(iii) *for all* $1 \leq i \leq n+1$, $[\ldots[\ldots[g_1, g_2], \ldots, g_ik] \ldots, g_{n+1}] = [\ldots[g_1, g_2], \ldots, g_{n+1}]$,

(iv) $[g_n, \ldots, g_{n-i+1}, [[g_{n-i}, \ldots, [g_2, g_1k] \ldots], t] \ldots] = [g_n, \ldots, g_{n-i+1}, [[g_{n-i}, \ldots, [g_2, g_1] \ldots], t] \ldots]$.

*Proof.*

($i$) Set $z_0 = z$ and recursively, for $1 \leq i \leq n$, $z_i = [g_i, z_{i-1}][z_{i-1}, [g_i, \ldots, [g_1, t] \ldots]]$. Using Lemma 2.1($i$), Proposition 2.3($iii$) and an induction argument on $i \geq 1$, one gets that $z_i \in \zeta_{n-i}(T)$ and

$$[g_i, \ldots, [g_2, [g_1, zt]] \ldots] = z_i[g_i, \ldots, [g_2, [g_1, t]] \ldots].$$

The result now follows by taking $i = n$.

($ii$) Fix $1 \leq i \leq n$, and put $x_i = [g_i, [k, [g_{i-1}, \ldots, [g_1, t] \ldots]]][k, [g_{i-1}, \ldots, [g_1, t] \ldots]]$ (for $i = 1$, $x_1 = [g_1, [k, t]][k, t]$) and recursively, $x_j = [g_j, x_{j-1}][x_{j-1}, [g_j, \ldots, [g_1, t] \ldots]]$ for $j = i+1, \ldots, n$. By applying Lemma 2.1($ii$) and Proposition 2.3, and using induction on $j \geq i$, it is not hard to see that $x_j \in \zeta_{n-j}(T)$ and $[g_j, \ldots, [g_ik, \ldots, [g_1, t] \ldots] \ldots] = x_j[g_j, [g_{j-1}, \ldots, [g_1, t] \ldots]]$. Now, taking $j = n$, we obtain the result.

($iii$) It is straightforward, since $k \in Z_n(G)$.

($iv$) Putting $k_1 = k$ and recursively, $k_j = [[g_{j-1}, \ldots, g_1], [g_j, k_{j-1}]][g_j, k_{j-1}]$ for $j = 2, \ldots, n$, one can show, by induction on $j$, that $k_j \in \kappa_{n-j+1}(G)$ and $[g_j \ldots, [g_2, g_1k] \ldots] = [g_j \ldots, [g_2, g_1] \ldots]k_j$. So, if we set $y_i = [[g_{n-i} \ldots, [g_2, g_1] \ldots], [k_{n-i}, t]][k_{n-i}, t]$ for $0 \leq i \leq n-1$, it is deduced from

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Lemma 2.1$(i)$ that $[[g_{n-i}, \dots, [g_2, g_1k]\dots], t] = [[g_{n-i}, \dots, [g_2, g_1]\dots]k_{n-i}, t] = y_i[[g_{n-i}, \dots, [g_2, g_1]\dots], t]$. Taking into account that $y_i \in \zeta_i(T)$, we thus conclude from part $(i)$ that

$$\begin{aligned}
&[g_n, \dots, g_{n-i+1}, [[g_{n-i}, \dots, [g_2, g_1k]\dots], t]\dots] \\
&\quad = [g_n, \dots, g_{n-i+1}, y_i[[g_{n-i}, \dots, [g_2, g_1]\dots], t]\dots] \\
&\quad = [g_n, \dots, g_{n-i+1}, [[g_{n-i}, \dots, [g_2, g_1]\dots], t]\dots],
\end{aligned}$$

which proves the result.

□

As an immediate consequence of the above lemma, we deduce that for any crossed module $(T, G, \partial)$ and $n \geq 1$, there exist well-defined maps

$$\eta^{n+1}_{(T,G,\partial)} : \frac{T}{\zeta_n(T)} \times \underbrace{\frac{G}{\kappa_n(G)} \times \dots \times \frac{G}{\kappa_n(G)}}_{n\text{-copies}} \longrightarrow \Gamma_{n+1}(T, G),$$

$$\theta^{n+1}_{(T,G,\partial)} : \underbrace{\frac{G}{\kappa_n(G)} \times \dots \times \frac{G}{\kappa_n(G)}}_{(n+1)\text{-copies}} \longrightarrow \gamma_{n+1}(G)$$

given by

$$\eta^{n+1}_{(T,G,\partial)}(t\zeta_n(T), g_1\kappa_n(G), \dots, g_n\kappa_n(G)) = [g_n, \dots, [g_2, [g_1, t]]\dots],$$
$$\theta^{n+1}_{(T,G,\partial)}(g_1\kappa_n(G), \dots, g_{n+1}\kappa_{n+1}(G)) = [\dots[[g_1, g_2], g_3], \dots, g_{n+1}].$$

If we consider a group $G$ as a crossed module in any of the two usual ways, then one easily sees that $\eta^{n+1}_{(1,G,i)} = 1$, $\eta^{n+1}_{(G,G,id)}$ is defined by $\eta^{n+1}_{(G,G,id)}(g_1\kappa_n(G), \dots, g_{n+1}\kappa_n(G)) = [g_{n+1}, \dots, [g_2, g_1]\dots]$, and $\theta^{n+1}_{(1,G,i)} = \theta^{n+1}_{(G,G,id)}$ is the map $\gamma(n, G)$ defined in [**14**, Definition 3.1].

Proposition 2.6. *Let* $(T, G, \partial)$ *be any crossed module with a crossed submodule* $(S, H, \partial)$. *Then,*

(i) $Z_n((S, H, \partial)Z_n(T, G, \partial)) = Z_n(S, H, \partial)Z_n(T, G, \partial)$,
(ii) $\gamma_{n+1}((S, H, \partial)Z_n(T, G, \partial)) = \gamma_{n+1}(S, H, \partial)$,
(iii) $Z_n(S, H, \partial) = (S, H, \partial) \cap Z_n((S, H, \partial)Z_n(T, G, \partial))$,
(iv) $\gamma_{n+1}((S, H, \partial)Z_n(T, G, \partial)) \cap Z_n((S, H, \partial)Z_n(T, G, \partial)) = \gamma_{n+1}(S, H, \partial) \cap Z_n(S, H, \partial)$.

*Proof.*

$(i)$ By virtue of Lemma 2.5$(i), (ii)$, it is easy to verify that $\zeta_n(S\zeta_n(T)) = \zeta_n(S)\zeta_n(T)$. So, we have only to show that $\kappa_n(H\kappa_n(G)) = \kappa_n(H)\kappa_n(G)$. Since $[\kappa_n(H)\kappa_n(G), {}_nH\kappa_n(G)] \subseteq [Z_n(HZ_n(G)), {}_nHZ_n(G)] = 1$, it follows that $\kappa_n(H)\kappa_n(G) \subseteq Z_n(H\kappa_n(G))$. On the other hand, for given $1 \leq i \leq n-1$, we have the following consequences:

(1) An easy inductive proof shows that $[{}_{n-i-1}H\kappa_n(G), \kappa_n(H)\kappa_n(G)] \subseteq [{}_{n-i-1}H, \kappa_n(H)]\kappa_{i+1}(G)$.
(2) Invoking Proposition 2.3$(ii), (iii)$, $[[{}_{n-i-1}H, \kappa_n(H)], \zeta_n(T)] \subseteq [\gamma_{n-i}(G), \zeta_n(T)] \subseteq \zeta_i(T)$ and $[\kappa_{i+1}(G), S\zeta_n(T)] \subseteq [\kappa_{i+1}(G), T] \subseteq \zeta_i(T)$.

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(3) By the definition of $\kappa_n(H)$, $[{}_iH, [[{}_{n-i-1}H, \kappa_n(H)], S]] = 1$, implying that $[[{}_{n-i-1}H, \kappa_n(H)], S]$ is contained in $\zeta_i(S)$.

(4) As $S\zeta_n(T) = S\zeta_n(T)\zeta_i(S\zeta_i(T))$, it forces that $\zeta_i(S\zeta_n(T)) = \zeta_i(S\zeta_n(T))\zeta_i(S\zeta_i(T))$ and then $\zeta_i(S\zeta_i(T)) \subseteq \zeta_i(S\zeta_n(T))$.

Now, by using the above information, we obtain

$$
\begin{aligned}
&[{}_iH\kappa_n(G), [[{}_{n-i-1}H\kappa_n(G), \kappa_n(H)\kappa_n(G)], S\zeta_n(T)]]\\
&\quad\overset{by(1)}{\subseteq} [{}_iH\kappa_n(G), [[{}_{n-i-1}H, \kappa_n(H)]\kappa_{i+1}(G), S\zeta_n(T)]]\\
&\quad= [{}_iH\kappa_n(G), [[{}_{n-i-1}H, \kappa_n(H)], S][[{}_{n-i-1}H, \kappa_n(H)], \zeta_n(T)][\kappa_{i+1}(G), S\zeta_n(T)]]\\
&\quad\overset{by(2,3)}{\subseteq} [{}_iH\kappa_n(G), \zeta_i(S)\zeta_i(T)] = [{}_iH\kappa_n(G), \zeta_i(S\zeta_i(T))]\\
&\quad\overset{by(4)}{\subseteq} [{}_iH\kappa_n(G), \zeta_i(S\zeta_n(T))] = 1.
\end{aligned}
$$

We therefore deduce that $\kappa_n(H)\kappa_n(G) \subseteq \kappa_n(H\kappa_n(G))$. To prove the reverse inclusion, we fix an arbitrary $hg \in \kappa_n(H\kappa_n(G))$. We must prove that $h \in \kappa_n(H)$. It is obvious that $[{}_nH\kappa_n(G), hg] = 1$ and consequently, owing to Lemma 2.5(*iii*), $[{}_nH, h] = 1$ and $h \in Z_n(H)$. Also, by the assumption,

$$[{}_iH, [[{}_{n-i-1}H, hg], S]] \subseteq [{}_iH\kappa_n(G), [[{}_{n-i-1}H\kappa_n(G), hg], S\zeta_n(T)]] = 1,$$

for $i = 1, \ldots, n-1$. But, this result together with Lemma 2.5(*iv*) yields that $[{}_iH, [[{}_{n-i-1}H, h], S]] = 1$ for all $i$ and then $h \in \kappa_n(H)$, as desired.

(*ii*) Combining Proposition 2.2(*ii*) with Lemma 2.5, we have

$$
\begin{aligned}
\gamma_{n+1}((S, H, \partial)Z_n(T, G, \partial)) &= (\Gamma_{n+1}(S\zeta_n(T), H\kappa_n(G)), \gamma_{n+1}(H\kappa_n(G)), \partial)\\
&= (\Gamma_{n+1}(S, H), \gamma_{n+1}(H), \partial) = \gamma_{n+1}(S, H, \partial),
\end{aligned}
$$

proving the result.

(*iii*) It easily follows by part (*i*).

(*iv*) This is immediate from parts (*ii*) and (*iii*).

□

**3. *n*-isoclinism of crossed modules.** This section is devoted to present the notion of $n$-isoclinism on the class of all crossed modules and to give some criteria for crossed modules to be $n$-isoclinic.

DEFINITION. The crossed modules $(T_1, G_1, \partial_1)$ and $(T_2, G_2, \partial_2)$ are said to be *$n$-isoclinic* $(n \geq 0)$, $(T_1, G_1, \partial_1) \underset{n}{\sim} (T_2, G_2, \partial_2)$, if there exists a pair of isomorphisms of crossed modules

$$\alpha = (\alpha_1, \alpha_2) : \frac{(T_1, G_1, \partial_1)}{Z_n(T_1, G_1, \partial_1)} \longrightarrow \frac{(T_2, G_2, \partial_2)}{Z_n(T_2, G_2, \partial_2)},$$

$$\beta = (\beta_1, \beta_2) : \gamma_{n+1}(T_1, G_1, \partial_1) \longrightarrow \gamma_{n+1}(T_2, G_2, \partial_2),$$

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such that the following diagrams are commutative:

$$\begin{array}{ccc}
\dfrac{T_1}{\zeta_n(T_1)} \times \dfrac{G_1}{\kappa_n(G_1)} \times \cdots \times \dfrac{G_1}{\kappa_n(G_1)} & \xrightarrow{\eta^{n+1}_{(T_1,G_1,\partial_1)}} & \Gamma_{n+1}(T_1, G_1) \\
\Big\downarrow{\scriptstyle \alpha_1 \times \alpha_2^n} & & \Big\downarrow{\scriptstyle \beta_1} \\
\dfrac{T_2}{\zeta_n(T_2)} \times \dfrac{G_2}{\kappa_n(G_2)} \times \cdots \times \dfrac{G_2}{\kappa_n(G_2)} & \xrightarrow{\eta^{n+1}_{(T_2,G_2,\partial_2)}} & \Gamma_{n+1}(T_2, G_2),
\end{array}$$

$$\begin{array}{ccc}
\dfrac{G_1}{\kappa_n(G_1)} \times \cdots \times \dfrac{G_1}{\kappa_n(G_1)} & \xrightarrow{\theta^{n+1}_{(T_1,G_1,\partial_1)}} & \gamma_{n+1}(G_1) \\
\Big\downarrow{\scriptstyle \alpha_2^{n+1}} & & \Big\downarrow{\scriptstyle \beta_2} \\
\dfrac{G_2}{\kappa_n(G_2)} \times \cdots \times \dfrac{G_2}{\kappa_n(G_2)} & \xrightarrow{\theta^{n+1}_{(T_2,G_2,\partial_2)}} & \gamma_{n+1}(G_2).
\end{array}$$

In other words, for all $t_1 \in T_1$ and $g_1, \ldots, g_{n+1} \in G_1$, we have

$$\beta_1([g_n, \ldots, [g_2, [g_1, t_1]] \ldots]) = [g'_n, \ldots, [g'_2, [g'_1, t'_1]] \ldots],$$
$$\beta_2([\ldots[[g_1, g_2], g_3], \ldots, g_{n+1}]) = [\ldots[[g'_1, g'_2], g'_3], \ldots, g'_{n+1}],$$

where $t'_1 \in \alpha_1(t_1\zeta_n(T_1))$ and $g'_i \in \alpha_2(g_i\kappa_n(G_1))$ for $i = 1, \ldots, n+1$. The pair $(\alpha, \beta)$ is called an *n-isoclinism* between $(T_1, G_1, \partial_1)$ and $(T_2, G_2, \partial_2)$.

If we consider groups as crossed modules in any of the two usual ways, we get the definition of $n$-isoclinism between groups given by Hall [**12**]. It is obvious that $n$-isoclinism between crossed modules is an equivalence relation, and so, it divides the class of all crossed modules into $n$-isoclinism equivalence classes.

In the following, we deal with the connection between the $n$-isoclinism of crossed modules and the $n$-isoclinism of groups.

PROPOSITION 3.1. *Let $(T_1, G_1, \partial_1)$ and $(T_2, G_2, \partial_2)$ be two n-isoclinic crossed modules. Then, $T_1 \underset{n}{\sim} T_2$ and $G_1 \underset{n}{\sim} G_2$.*

*Proof.* Let $(\alpha, \beta)$ be an $n$-isoclinism between $(T_1, G_1, \partial_1)$ and $(T_2, G_2, \partial_2)$. As $\gamma_{n+1}(T_i)$ is a subgroup of $\Gamma_{n+1}(T_i, G_i)$ $(i = 1, 2)$, we show that $\beta_1$ maps any generator of $\gamma_{n+1}(T_1)$ to a generator of $\gamma_{n+1}(T_2)$. Suppose $t_1, \ldots, t_{n+1}$ are arbitrary elements of $T_1$ and choose $t'_i \in \alpha_1(t_i\zeta_n(T_1))$ for $1 \le i \le n+1$. Then, $\alpha_2(\partial_1(t_i)\kappa_n(G_1)) = \partial_2(t'_i)\kappa_n(G_2)$ for all $i$. Now, if $n = 1$, then $\beta_1([t_1, t_2]) = \beta_1([\partial_1(t_1), t_2]) = [\partial_2(t'_1), t'_2] = [t'_1, t'_2]$, thanks to the above definition. We hence assume that $n \ge 2$. Setting $x_i = [\ldots[[t_1, t_2], t_3], \ldots, t_i]$ for $i = 2, \ldots, n$, an easy inductive argument establishes that

$$[\ldots[[t_1, t_2], t_3], \ldots, t_{n+1}] = \begin{cases} [{}^{x_n}t_{n+1}, [t_n, [{}^{x_{n-2}}t_{n-1}, \ldots, [{}^{x_2}t_3, [t_2, t_1]] \ldots]]] & \text{when } n \text{ is even} \\ [t_{n+1}, [{}^{x_{n-1}}t_n, [t_{n-1}, \ldots, [{}^{x_2}t_3, [t_2, t_1]] \ldots]]]^{-1} & \text{when } n \text{ is odd.} \end{cases}$$

Also, setting $y_i = [\ldots[[t'_1, t'_2], t'_3], \ldots, t'_i]$ for $i = 2, \ldots, n$, a similar result holds for $[\ldots[[t'_1, t'_2], t'_3], \ldots, t'_n]$. It is easily verified that $\alpha_1(x_i\zeta_n(T_1)) = y_i\zeta_n(T_2)$ and

$\alpha_2(\partial_1({}^{x_i}t_{i+1})\kappa_n(G_1)) = \partial_2({}^{y_i}t'_{i+1})\kappa_n(G_2)$ $(2 \leq i \leq n)$. Consequently, we have

$$\begin{aligned}
&\beta_1([\ldots[[t_1, t_2], t_3], \ldots, t_{n+1}])\\
&\quad = \beta_1([{}^{x_n}t_{n+1}, [t_n, [{}^{x_{n-2}}t_{n-1}, \ldots, [{}^{x_2}t_3, [t_2, t_1]]\ldots]]])\\
&\quad = \beta_1([\partial_1({}^{x_n}t_{n+1}), [\partial_1(t_n), [\partial_1({}^{x_{n-2}}t_{n-1}), \ldots, [\partial_1({}^{x_2}t_3), [\partial_1(t_2), t_1]]\ldots]]])\\
&\quad = [\partial_2({}^{y_n}t'_{n+1}), [\partial_2(t'_n), [\partial_2({}^{y_{n-2}}t'_{n-1}), \ldots, [\partial_2({}^{y_2}t'_3), [\partial_2(t'_2), t'_1]]\ldots]]]\\
&\quad = [{}^{y_n}t'_{n+1}, [t'_n, [{}^{y_{n-2}}t'_{n-1}, \ldots, [{}^{y_2}t'_3, [t'_2, t'_1]]\ldots]]]\\
&\quad = [\ldots[[t'_1, t'_2], t'_3], \ldots, t'_{n+1}],
\end{aligned}$$

whenever $n$ is even and analogously, the above equality holds when $n$ is odd. Now, one readily sees that the restriction of $\beta_1$ to $\gamma_{n+1}(T_1)$ is an isomorphism of $\gamma_{n+1}(T_1)$ onto $\gamma_{n+1}(T_2)$, and $\alpha_1$ induces an isomorphism $\bar{\alpha}_1 : T_1/Z_n(T_1) \longrightarrow T_2/Z_n(T_2)$ given by $\bar{\alpha}_1(t_1Z_n(T_1)) = t'_1Z_n(T_2)$. Also, using the isomorphism $\beta_2$, the map $\bar{\alpha}_2 : G_1/Z_n(G_1) \longrightarrow G_2/Z_2(G_2)$ defined by $\bar{\alpha}_2(gZ_n(G_1)) = g'Z_2(G_2)$, where $g \in G_1$ and $g' \in \alpha_2(gZ_n(G_1))$, is an isomorphism. It is straightforward to check that the pair $(\bar{\alpha}_1, \beta_1|_{\gamma_{n+1}(T_1)})$ is an $n$-isoclinism between the groups $T_1$ and $T_2$, and the pair $(\bar{\alpha}_2, \beta_2)$ is an $n$-isoclinism between the groups $G_1$ and $G_2$, as required. □

When $n = 1$, Proposition 3.1 improves [**25**, Proposition 6]. Also, it follows from the above proposition that for any two group $G$ and $H$, if $(1, G, i) \underset{n}{\sim} (1, H, i)$ or $(G, G, id) \underset{n}{\sim} (H, H, id)$, then $G \underset{n}{\sim} H$.

We remark that the converse of Proposition 3.1 is not true in general. For a simple counterexample, suppose for $m \geq 1$, $C_{2^m} = \langle a_m \rangle$ denotes the cyclic group of order $2^m$. Then, $(C_{2^m}, C_{2^1}, \partial)$ is a crossed module, where $\partial$ is the trivial homomorphism and $C_{2^1}$ acts on $C_{2^m}$ by ${}^{a_1}a_m = a_m^3$. An easy induction shows that for any $1 \leq n < m$, $\Gamma_{n+1}(C_{2^m}, C_{2^1}) = \langle a_m^{2^n} \rangle \cong C_{2^{m-n}}$, and so $\gamma_{n+1}(C_{2^m}, C_{2^1}, \partial) \neq 1$, that is, $(C_{2^m}, C_{2^1}, \partial) \underset{n}{\not\sim} 1$. However, $C_{2^m} \underset{n}{\sim} 1 \underset{n}{\sim} C_{2^1}$.

The following proposition is a routine extension of [**26**, Lemma 3.2].

Proposition 3.2. *Let $(T, G, \partial)$ be a crossed module with a crossed submodule $(S, H, \partial)$ and a normal crossed submodule $(L, K, \partial)$. Then,*

(i) *$(S, H, \partial) \underset{n}{\sim} (S, H, \partial)Z_n(T, G, \partial)$. In particular, if $(T, G, \partial) = (S, H, \partial)Z_n(T, G, \partial)$, then $(T, G, \partial) \underset{n}{\sim} (S, H, \partial)$. Conversely, if $(T, G, \partial)/Z_n(T, G, \partial)$ is finite and $(T, G, \partial) \underset{n}{\sim} (S, H, \partial)$, then $(T, G, \partial) = (S, H, \partial)Z_n(T, G, \partial)$.*

(ii) *$(T, G, \partial)/(L, K, \partial) \underset{n}{\sim} (T, G, \partial)/((L, K, \partial) \cap \gamma_{n+1}(T, G, \partial))$. In particular, if the crossed submodule $(L, K, \partial) \cap \gamma_{n+1}(T, G, \partial) = 1$, then $(T, G, \partial) \underset{n}{\sim} (T, G, \partial)/(L, K, \partial)$. Conversely, if $\gamma_{n+1}(T, G, \partial)$ is finite and $(T, G, \partial) \underset{n}{\sim} (T, G, \partial)/(L, K, \partial)$, then $(L, K, \partial) \cap \gamma_{n+1}(T, G, \partial) = 1$.*

A result of Bioch [**3**] states that two $n$-isoclinic groups $G_1$ and $G_2$ have a common $n$-isoclinic ancestor $G$, that is, $G_1$ and $G_2$ can be realized as quotients of a group $G$, while $G, G_1, G_2$ are $n$-isoclinic to each other. By arguments rather similar to those used in [**26**, Proposition 3.4], we can extend this result by proving the following.

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THEOREM 3.3. *Let $(T_i, G_i, \partial_i)$, $i = 1, 2$ be two crossed modules. Then, $(T_1, G_1, \partial_1) \underset{n}{\sim} (T_2, G_2, \partial_2)$ if and only if there exists a crossed module $(T, G, \partial)$ containing normal crossed modules $(L_1, K_1, \partial_1)$ and $(L_2, K_2, \partial_2)$, such that*

(a) $(T_1, G_1, \partial_1) \cong (T/L_2, G/K_2, \bar{\partial}) \underset{n}{\sim} (T, G, \partial) \underset{n}{\sim} (T/L_1, G/K_1, \bar{\partial}) \cong (T_2, G_2, \partial_2)$;

(b) $(T/L_i \times T/\Gamma_{n+1}(T, G), G/K_i \times G/\gamma_{n+1}(G), \bar{\partial} \times \bar{\partial}) \underset{n}{\sim} (S_i, H_i, \bar{\partial} \times \bar{\partial}) \cong (T, G, \partial)$, *for some crossed submodule* $(S_i, H_i, \bar{\partial} \times \bar{\partial})$ *of* $(T/L_i \times T/\Gamma_{n+1}(T, G), G/K_i \times G/\gamma_{n+1}(G), \bar{\partial} \times \bar{\partial})$, $i = 1, 2$.

*Proof.* The sufficiency holds trivially. We only need to prove the necessity. Assume $(\alpha, \beta)$ is an $n$-isoclinism between $(T_1, G_1, \partial_1)$ and $(T_2, G_2, \partial_2)$. We define the crossed module $(T, G, \partial)$ by a pullback square in the category of crossed modules:

$$\begin{tikzcd}
(T, G, \partial) \arrow[r] \arrow[d] & (T_2, G_2, \partial_2) \arrow[d, "\gamma'"] \\
(T_1, G_1, \partial_1) \arrow[r, "\alpha\gamma"'] & \left(\frac{T_2}{\zeta_n(T_2)}, \frac{G_2}{\kappa_n(G_2)}, \bar{\partial}_2\right),
\end{tikzcd}$$

where $\gamma$ and $\gamma'$ are the obvious surjective morphisms. Here, $T = \{(t_1, t_2) | \alpha_1(t_1\zeta_n(T_1)) = t_2\zeta_n(T_2)\}$, $G = \{(g_1, g_2) | \alpha_2(g_1\kappa_n(G_1)) = g_2\kappa_n(G_2)\}$ and $\partial$ is the restriction of $\partial_1 \times \partial_2$ to $T$. Denote by $(L_i, K_i, \partial)$ the image of $Z_n(T_i, G_i, \partial_i)$ via the canonical morphism $(T_i, G_i, \partial_i) \longrightarrow (T_1 \times T_2, G_1 \times G_2, \partial_1 \times \partial_2)$, $i = 1, 2$. It is routine to check that $(L_i, K_i, \partial)$, $i = 1, 2$, is a normal crossed submodule of $(T, G, \partial)$, $(T_1, G_1, \partial_1) \cong (T/L_2, G/K_2, \bar{\partial})$ and $(T_2, G_2, \partial_2) \cong (T/L_1, G/K_1, \bar{\partial})$.

($i$) Using the definition of $n$-isoclinism, one sees that

$$\Gamma_{n+1}(T, G) = \langle (x, \beta_1(x)) | x \in \Gamma_{n+1}(T_1, G_1) \rangle \text{ and } \gamma_{n+1}(G) = \langle (y, \beta_2(y)) | y \in \gamma_{n+1}(G_1) \rangle$$

from which is deduced that $(L_i, K_i, \partial) \cap \gamma_{n+1}(T, G, \partial) = 1$ for $i = 1, 2$. Therefore, according to Proposition 3.2($ii$), $(T/L_1, G/K_1, \bar{\partial})$ and $(T/L_2, G/K_2, \bar{\partial})$ are both $n$-isoclinic to $(T, G, \partial)$, proving part ($i$).

($ii$) For $i = 1, 2$, set $S_i = \{(tL_i, t\Gamma_{n+1}(T, G)) \mid t \in T\}$ and $H_i = \{(gK_i, g\gamma_{n+1}(G)) \mid g \in G\}$. Then, $(S_i, H_i, \bar{\partial} \times \bar{\partial})$ is a crossed submodule of $(T/L_i \times T/\Gamma_{n+1}(T, G), G/K_i \times G/\gamma_{n+1}(G), \bar{\partial} \times \bar{\partial})$ and the pair $(\delta_i, \theta_i) : (T, G, \partial) \longrightarrow (S_i, H_i, \bar{\partial} \times \bar{\partial})$, given by $\delta_i(t) = (tL_i, t\Gamma_{n+1}(T, G))$ and $\theta_i(g) = (gK_i, g\gamma_{n+1}(G))$, is an isomorphism ($i = 1, 2$). Now, in view of Proposition 3.2($i$), to show

$$(T/L_i \times T/\Gamma_{n+1}(T, G), G/K_i \times G/\gamma_{n+1}(G), \bar{\partial} \times \bar{\partial}) \underset{n}{\sim} (S_i, H_i, \bar{\partial} \times \bar{\partial}),$$

it is enough to prove that

$$\frac{T}{L_i} \times \frac{T}{\Gamma_{n+1}(T, G)} = S_i\zeta_n\left(\frac{T}{L_i} \times \frac{T}{\Gamma_{n+1}(T, G)}\right) \text{ and } \frac{G}{K_i} \times \frac{G}{\gamma_{n+1}(G)} = H_i\kappa_n\left(\frac{G}{K_i} \times \frac{G}{\gamma_{n+1}(G)}\right). \tag{3}$$

Since for all $t, t' \in T$ and $g, g', g_1, \ldots, g_n, g'_1, \ldots, g'_n \in G$, we have

$$\left[{}_n\frac{G}{K_i} \times \frac{G}{\gamma_{n+1}(G)}, (L_i, t^{-1}t'\Gamma_{n+1}(T, G))\right] = \left(\left[{}_n\frac{G}{K_i}, L_i\right], \left[{}_n\frac{G}{\gamma_{n+1}(G)}, t^{-1}t'\Gamma_{n+1}(T, G)\right]\right) = 1,$$

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$$\left[{}_i\frac{G}{K_i}\times\frac{G}{\gamma_{n+1}(G)},\left[\left[{}_{n-1-i}\frac{G}{K_i}\times\frac{G}{\gamma_{n+1}(G)},(K_i,g^{-1}g'\gamma_{n+1}(G))\right],\frac{T}{L_i}\times\frac{T}{\Gamma_{n+1}(T,G)}\right]\right]=1,$$

$$\begin{aligned}[(K_i, g^{-1}g'\gamma_{n+1}(G)), (g_1K_i, g_1'\gamma_{n+1}(G)), \ldots, (g_nK_i, g_n'\gamma_{n+1}(G))]\\ = ([1, g_1, \ldots, g_n]K_i, [g^{-1}g', g_1', \ldots, g_n']\gamma_{n+1}(G)) = 1,\end{aligned}$$

it follows that

$$(tL_i, t'\Gamma_{n+1}(T,G)) = (tL_i, t\Gamma_{n+1}(T,G))(L_i, t^{-1}t'\Gamma_{n+1}(T,G)) \in S_i\zeta_n\left(\frac{T}{L_i}\times\frac{T}{\Gamma_{n+1}(T,G)}\right),$$
$$(gK_i, g'\gamma_{n+1}(G)) = (gK_i, g\gamma_{n+1}(G))(K_i, g^{-1}g'\gamma_{n+1}(G)) \in H_i\kappa_n\left(\frac{G}{K_i}\times\frac{G}{\gamma_{n+1}(G)}\right).$$

So, the equalities hold in (3) and the proof is complete. □

From the above theorem, we conclude that the crossed module $(T_2, G_2, \partial_2)$ lies in the $n$-isoclinism family $\{(T_1, G_1, \partial_1)\}$ if and only if it satisfies one of following conditions:

($i$) $(T_2, G_2, \partial_2)$ is isomorphic to a direct product of $(T_1, G_1, \partial_1)$ with a nilpotent crossed module of class at most $n$.

($ii$) $(T_2, G_2, \partial_2)$ is a crossed submodule of a crossed module $(L, K, \delta)$ in $\{(T_1, G_1, \partial_1)\}$ with $(L, K, \delta) = (T_2, G_2, \partial_2)Z_n(L, K, \delta)$.

($iii$) There exists a surjective morphism $(\theta_1, \theta_2)$ from a crossed module $(L, K, \delta) \in \{(T_1, G_1, \partial_1)\}$ onto $(T_2, G_2, \partial_2)$ such that $\ker(\theta_1, \theta_2) \cap \gamma_{n+1}(L, K, \delta) = 1$.

The next corollary establishes that an $n$-isoclinism between two crossed modules induces some certain $m$-isoclinisms between their upper central factor crossed modules and lower commutator crossed submodules.

Corollary 3.4. Let $(T_1, G_1, \partial_1)$ and $(T_2, G_2, \partial_2)$ be two $n$-isoclinic crossed modules. Then,

(i) for any $0 \le i \le n$, $(T_1, G_1, \partial_1)/Z_i(T_1, G_1, \partial_1) \underset{n-i}{\sim} (T_2, G_2, \partial_2)/Z_i(T_2, G_2, \partial_2)$;

(ii) for any $0 \le i \le n$, $\gamma_{i+1}(T_1, G_1, \partial_1) \underset{n-i}{\sim} \gamma_{i+1}(T_2, G_2, \partial_2)$;

(iii) for any $m \ge n$, $(T_1, G_1, \partial_1) \underset{m}{\sim} (T_2, G_2, \partial_2)$.

*Proof.* By virtue of Theorem 3.3, we may assume that $(T_2, G_2, \partial_2) \cong (T_1/L_1, G_1/K_1, \bar{\partial}_1)$ for some normal crossed submodule $(L_1, K_1, \partial_1)$ of $(T_1, G_1, \partial_1)$ with $(L_1, K_1, \partial_1) \cap \gamma_{n+1}(T_1, G_1, \partial_1) = 1$.

($i$) Put $Z_i(T_1/L_1, G_1/K_1, \bar{\partial}_1) = (S_1, H_1, \partial_1)/(L_1, K_1, \partial_1)$ for some normal crossed submodule $(S_1, H_1, \partial_1)$ of $(T_1, G_1, \partial_1)$. If we define $[(S_1, H_1, \partial_1), {}_0(T_1, G_1, \partial_1)] = (S_1, H_1, \partial_1)$ and recursively $[(S_1, H_1, \partial_1), {}_i(T_1, G_1, \partial_1)] = [[(S_1, H_1, \partial_1), {}_{i-1}(T_1, G_1, \partial_1)], (T_1, G_1, \partial_1)]$ for $i \ge 1$, then it follows from the assumption that $[(S_1, H_1, \partial_1), {}_i(T_1, G_1, \partial_1)] \subseteq (L_1, K_1, \partial_1)$, and consequently

$$[(S_1, H_1, \partial_1) \cap \gamma_{n-i+1}(T_1, G_1, \partial_1), {}_i(T_1, G_1, \partial_1)] \subseteq (L_1, K_1, \partial_1) \cap \gamma_{n+1}(T_1, G_1, \partial_1) = 1.$$

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Therefore, $(S_1, H_1, \partial_1) \cap \gamma_{n-i+1}(T_1, G_1, \partial_1)$ is a crossed submodule of $Z_i(T_1, G_1, \partial_1)$, and then,

$$(S_1, H_1, \partial_1)/Z_i(T_1, G_1, \partial_1) \cap \gamma_{n-i+1}((T_1, G_1, \partial_1)/Z_i(T_1, G_1, \partial_1)) = 1.$$

Now, the result follows from Proposition 3.2(*ii*).

(*ii*) We only need to prove that

$$\gamma_{i+1}(T_1, G_1, \partial_1) \underset{n-i}{\sim} \gamma_{i+1}(\frac{T_1}{L_1}, \frac{G_1}{K_1}, \bar{\partial}_1) \cong \frac{\gamma_{i+1}(T_1, G_1, \partial_1)}{\gamma_{i+1}(T_1, G_1, \partial_1) \cap (L_1, K_1, \partial_1)}.$$

It suffices to show from Proposition 3.2(*ii*) that

$$\begin{aligned}\gamma_{i+1}(T_1, G_1, \partial_1) \cap (L_1, K_1, \partial_1) \cap \gamma_{n-i+1}(\gamma_{i+1}(T_1, G_1, \partial_1))\\ = (L_1, K_1, \partial_1) \cap \gamma_{n-i+1}(\gamma_{i+1}(T_1, G_1, \partial_1)) = 1.\end{aligned}$$

But, the latter equality always holds because

$$(L_1, K_1, \partial_1) \cap \gamma_{n-i+1}(\gamma_{i+1}(T_1, G_1, \partial_1)) \subseteq (L_1, K_1, \partial_1) \cap \gamma_{n+1}(T_1, G_1, \partial_1) = 1.$$

(*iii*) Noticing that for any $m \geq n$,

$$(L_1, K_1, \partial_1) \cap \gamma_{m+1}(T_1, G_1, \partial_1) \subseteq (L_1, K_1, \partial_1) \cap \gamma_{n+1}(T_1, G_1, \partial_1) = 1,$$

Proposition 3.2(*ii*) yields that $(T_1, G_1, \partial_1) \underset{m}{\sim} (T_1/L_1, G_1/K_1, \bar{\partial}_1)$, as desired.

□

The $(n-i)$-isoclinism obtained between the quotient crossed modules $(T_1, G_1, \partial_1)/Z_i(T_1, G_1, \partial_1)$ and $(T_2, G_2, \partial_2)/Z_i(T_2, G_2, \partial_2)$ in Corollary 3.4(*i*) is the best possible, in the sense that $n-i$ is the smallest non-negative integer $j$ such that $(T_1, G_1, \partial_1)/Z_i(T_1, G_1, \partial_1) \underset{j}{\sim} (T_2, G_2, \partial_2)/Z_i(T_2, G_2, \partial_2)$. Because suppose that $(T_1, G_1, \partial_1)$ is a finite nilpotent crossed module of class $n$ and $(T_2, G_2, \partial_2) = (T_1 \times T_1, G_1 \times G_1, \partial_1 \times \partial_1)$. Then, $(T_1, G_1, \partial_1) \underset{n}{\sim} (T_2, G_2, \partial_2) \underset{n}{\sim} 1$. Now, if $(T_1, G_1, \partial_1)/Z_i(T_1, G_1, \partial_1) \underset{j}{\sim} (T_2, G_2, \partial_2)/Z_i(T_2, G_2, \partial_2)$ for some $j \geq 0$, then

$$\frac{(T_1, G_1, \partial_1)}{Z_{i+j}(T_1, G_1, \partial_1)} \cong \frac{(T_2, G_2, \partial_2)}{Z_{i+j}(T_2, G_2, \partial_2)} \cong \frac{(T_1, G_1, \partial_1)}{Z_{i+j}(T_1, G_1, \partial_1)} \times \frac{(T_1, G_1, \partial_1)}{Z_{i+j}(T_1, G_1, \partial_1)}.$$

Thus, the finiteness of $(T_1, G_1, \partial_1)$ implies that $(T_1, G_1, \partial_1) = Z_{i+j}(T_1, G_1, \partial_1)$ and so, $j \geq n-i$.

In [**29**], it was established that two $n$-isoclinic groups $G_1$ and $G_2$ have a common $n$-isoclinic descendant $G$, that is, $G_1$ and $G_2$ can be isomorphically embedded into a group $G$, whereas $G, G_1, G_2$ are $n$-isoclinic to each other. In case $n = 1$, this result was generalized to crossed modules, thanks to [**26**, Proposition 3.5]. It is not clear whether the result holds for $n \geq 2$. However, the following theorem supplies a partial answer.

THEOREM 3.5. *Let* $(\alpha, \beta) : (T_1, G_1, \partial_1) \longrightarrow (T_2, G_2, \partial_2)$ *be an n-isoclinism of crossed modules. Then, there exists a crossed module* $(\widetilde{T}, \widetilde{G}, \widetilde{\partial})$ *with crossed submodules* $(\widetilde{S}_i, \widetilde{H}_i, \widetilde{\partial})$, $i = 1, 2$, *such that*

(i) $(\widetilde{S}_1, \widetilde{H}_1, \widetilde{\partial}) \underset{n}{\sim} (\widetilde{T}, \widetilde{G}, \widetilde{\partial}) \underset{n}{\sim} (\widetilde{S}_2, \widetilde{H}_2, \widetilde{\partial})$,

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(ii) $(T_1, G_1, \partial_1)/[(T_1, G_1, \partial_1), Z_n(T_1, G_1, \partial_1)] \cong (\widetilde{S}_1, \widetilde{H}_1, \widetilde{\partial})$,
(iii) $(T_2, G_2, \partial_2)/\beta([(T_1, G_1, \partial_1), Z_n(T_1, G_1, \partial_1)] \cap \gamma_{n+1}(T_1, G_1, \partial_1)) \cong (\widetilde{S}_2, \widetilde{H}_2, \widetilde{\partial})$.

To prove this, we first need the following observation.

Let the crossed module $(T, G, \partial)$ with normal crossed submodules $(L_i, K_i, \partial_i)$, $i = 1, 2$, be defined as in the proof of Theorem 3.3. We put

$$(V, W, \delta) = \Big(\frac{T}{L_2} \times \frac{T}{\Gamma_{n+1}(T, G)}, \frac{G}{K_2} \times \frac{G}{\gamma_{n+1}(G)}, \bar{\partial} \times \bar{\partial}\Big),$$
$$U = \{(xL_2, x\Gamma_{n+1}(T, G)) \mid x \in L_1\},$$
$$N = \{(yK_2, y\gamma_{n+1}(G)) \mid y \in K_1\},$$

and define $(V_1, W_1, \delta) = (U[W, U][N, V], N[W, N], \delta)$. Then, we have the following:

LEMMA 3.6.
(i) $(V_1, W_1, \delta)$ *is a normal crossed submodule of* $(V, W, \delta)$.
(ii) $V_1 = U\langle((1, 1)L_2, (x, 1)\Gamma_{n+1}(T, G)) \mid x \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)]\rangle$.
(iii) $W_1 = N\langle((1, 1)K_2, (y, 1)\gamma_{n+1}(G)) \mid y \in [G_1, \kappa_n(G_1)]\rangle$.

*Proof.*
($i$) One immediately verifies that $(V_1, W_1, \delta)$ is a crossed submodule of $(V, W, \delta)$. We now demonstrate that $(V_1, W_1, \delta)$ satisfies the conditions of normality.
(1) $W_1$ is the normal closure of $N$ in $W$ and so is normal in $W$.
(2) $V_1$ is a $W$-invariant subgroup of $V$. For, let $w = ((g_1, g_2)K_2, (g_1', g_2')\gamma_{n+1}(G))$ be an arbitrary element of $W$. Then for any $u \in U$, ${}^{w}uu^{-1} \in [W, U]$ and in consequence, ${}^{w}u \in U[W, U] \subseteq V_1$. This result also implies that the action of the element $w$ on any generator of $[W, U]$ belongs in $V_1$. Now, assume that $u$ is any generator of $[N, V]$. It is easy to see that $u = (({}^{k_1}t_1t_1^{-1}, 1)L_2, ({}^{k_1}t_1't_1'^{-1}, 1)\Gamma_{n+1}(T, G))$ for some $t_1, t_1' \in T_1$ and $k_1 \in \kappa_n(G_1)$. Setting $a = {}^{k_1}t_1t_1^{-1}$ and $a' = {}^{k_1}t_1't_1'^{-1}$, Proposition 2.3($iii$) yields that $a, a' \in \zeta_n(T_1)$ and also the element $((a', 1)L_2, (a, 1)\Gamma_{n+1}(T, G))$ is equal to

$${}^{((k_1,1)K_2,(k_1,1)\gamma_{n+1}(G))}((t_1', t_2')L_2, (t_1, t_2)\Gamma_{n+1}(T, G))((t_1', t_2')L_2, (t_1, t_2)\Gamma_{n+1}(T, G))^{-1} \in [N, V].$$

So,

$$\begin{aligned} {}^{w}u &= (({}^{g_1}a, 1)L_2, ({}^{g_1'}a', 1)\Gamma_{n+1}(T, G)),\\ &= {}^{((g_1,g_2)K_2,\gamma_{n+1}(G))}((a, 1)L_2, (a, 1)\Gamma_{n+1}(T, G))((a'^{-1}, 1)L_2, (a^{-1}, 1)\Gamma_{n+1}(T, G))\\ &\quad {}^{(K_2,(g_1',g_2')\gamma_{n+1}(G))}((a', 1)L_2, (a', 1)\Gamma_{n+1}(T, G)) \in V_1. \end{aligned}$$

These results confirm that $V_1$ is invariant under the action of $W$.
(3) $[W_1, V] \le V_1$. To see this, we must prove that $[nx, v] \in V_1$ for all $n \in N$, $x \in [W, N]$, $v \in V$. Using Lemma 2.1($i$) and the definition of $V_1$, it suffices to show that $[x, v] \in V_1$. Without loss of generality, we may assume that $x = {}^{w}nn^{-1}$ where $w \in W$ and $n \in N$. Then, we have

$$[x, v] = {}^{({}^{w}nn^{-1})}vv^{-1} = {}^{({}^{w}n)}({}^{n^{-1}}vv^{-1}){}^{({}^{w}n)}vv^{-1} = {}^{({}^{w}n)}({}^{n^{-1}}vv^{-1})^{w}({}^{n}({}^{w^{-1}}v)({}^{w^{-1}}v)^{-1}) \in V_1,$$

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since ${}^{n^{-1}}vv^{-1}$, ${}^{n}({}^{w^{-1}}v)({}^{w^{-1}}v)^{-1} \in [N, V] \subseteq V_1$. The proof of part ($i$) is complete.

($ii$) Denote $X = U\langle((1, 1)L_2, (x, 1)\Gamma_{n+1}(T, G)) \mid x \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)]\rangle$. Since

$$((1,1)L_2, ({}^{k_1}t_1t_1^{-1}, 1)\Gamma_{n+1}(T, G)) = {}^{((k_1,1)K_2,(k_1,1)\gamma_{n+1}(G))}(L_2, (t_1, 1)\Gamma_{n+1}(T, G))$$
$$(L_2, (t_1, 1)\Gamma_{n+1}(T, G))^{-1} \in [N, V] \subseteq V_1,$$
$$((1,1)L_2, ({}^{g_1}a_1a_1^{-1}, 1)\Gamma_{n+1}(T, G)) = {}^{(K_2,(g_1,1)\gamma_{n+1}(G))}((a_1, 1)L_2, (a_1, 1)\Gamma_{n+1}(T, G))$$
$$((a_1, 1)L_2, (a_1, 1)\Gamma_{n+1}(T, G))^{-1} \in [W, U] \subseteq V_1,$$

for all $t_1 \in T$, $a_1 \in \zeta_n(T_1)$, $g_1 \in G_1$, $k_1 \in \kappa_n(G_1)$, one deduces that $X \subseteq V_1$. On the other hand,

$$\begin{aligned}&(({}^{k_1}t_1t_1^{-1}, 1)L_2, ({}^{k_1}t_1't_1'^{-1}, 1)\Gamma_{n+1}(T, G))\\ &\quad = (({}^{k_1}t_1t_1^{-1}, 1)L_2, ({}^{k_1}t_1t_1^{-1}, 1)\Gamma_{n+1}(T, G))(L_2, (({}^{k_1}t_1t_1^{-1})^{-1}({}^{k_1}t_1't_1'^{-1}), 1)\Gamma_{n+1}(T, G))\end{aligned}$$

for all $t_1, t_1' \in T_1$ and $k_1 \in \kappa_n(G_1)$, from which we infer that $[N, V] \subseteq X$. Analogously, $[W, U]$ is contained in $X$ and so $V_1 \subseteq X$.

($iii$) It is proved in a similar way to part ($ii$).

□

We are now ready to provide the proof of Theorem 3.5.

*Proof of Theorem 3.5* We consider the crossed module $(V, W, \delta)$ with normal crossed submodule $(V_1, W_1, \delta)$ obtained in Lemma 3.6($i$) and define two morphisms

$$\varphi = (\varphi_1, \varphi_2) : (T_1, G_1, \partial_1) \longrightarrow \left(\frac{V}{V_1}, \frac{W}{W_1}, \bar{\delta}\right) \text{ and}$$
$$\psi = (\psi_1, \psi_2) : (T_2, G_2, \partial_2) \longrightarrow \left(\frac{V}{V_1}, \frac{W}{W_1}, \bar{\delta}\right)$$

as follows:
First, suppose $t_1 \in T_1$, $g_1 \in G_1$ and choose $t_2 \in T_2$, $g_2 \in G_2$ such that $\alpha_1(t_1\zeta_n(T_1)) = t_2\zeta_n(T_2)$ and $\alpha_2(g_1\kappa_n(G_1)) = g_2\kappa_n(G_2)$. Then, we set $\varphi_1(t_1) = ((t_1, t_2)L_2, \Gamma_{n+1}(T, G))V_1$ and $\varphi_2(g_1) = ((g_1, g_2)K_2, \gamma_{n+1}(G))W_1$.
Second, suppose $t_2 \in T_2$, $g_2 \in G_2$ and choose $t_1 \in T_1$, $g_1 \in G_1$ such that $\alpha_1(t_1\zeta_n(T_1)) = t_2\zeta_n(T_2)$ and $\alpha_2(g_1\kappa_n(G_1)) = g_2\kappa_n(G_2)$. Then, we set $\psi_1(t_2) = ((t_1, t_2)L_2, (t_1, t_2)\Gamma_{n+1}(T, G))V_1$ and $\psi_2(g_2) = ((g_1, g_2)K_2, (g_1, g_2)\gamma_{n+1}(G))W_1$.
Note that the above maps are well defined. It is easy to check, but somewhat tedious. For the sake of clarity, we divide the rest of the proof into three steps.
*Step* 1. $\mathrm{Im}\varphi Z_n(V/V_1, W/W_1, \bar{\delta}) = (V/V_1, W/W_1, \bar{\delta}) = \mathrm{Im}\psi Z_n(V/V_1, W/W_1, \bar{\delta})$.
These equalities hold because

$$\begin{aligned}(tL_2, t'\Gamma_{n+1}(T, G))V_1 &= (tL_2, \Gamma_{n+1}(T, G))V_1(L_2, t'\Gamma_{n+1}(T, G))V_1 \in \varphi_1(T_1)\zeta_n(V/V_1),\\ (gK_2, g'\gamma_{n+1}(G))W_1 &= (gK_2, \gamma_{n+1}(G))W_1(K_2, g'\gamma_{n+1}(G))W_1 \in \varphi_2(G_1)\kappa_n(W/W_1),\\ (tL_2, t'\Gamma_{n+1}(T, G))V_1 &= (tL_2, t\Gamma_{n+1}(T, G))V_1(L_2, t^{-1}t'\Gamma_{n+1}(T, G))V_1 \in \psi_1(T_1)\zeta_n(V/V_1),\\ (gK_2, g'\gamma_{n+1}(G))W_1 &= (gK_2, g\gamma_{n+1}(G))W_1(K_2, g^{-1}g'\gamma_{n+1}(G))W_1 \in \varphi_2(G_1)\kappa_n(W/W_1),\end{aligned}$$

for all $t, t' \in T$, $g, g' \in G$.
*Step* 2. $\ker\varphi = [(T_1, G_1, \partial_1), Z_n(T_1, G_1, \partial_1)]$.

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Suppose $t_1 \in \ker\varphi_1$ and choose $t_2 \in T_2$ with $\alpha_1(t_1\zeta_n(T_1)) = t_2\zeta_n(T_2)$. Then, $((t_1, t_2)L_2, \Gamma_{n+1}(T, G)) \in V_1$ and so, using Lemma 3.6(*ii*), one sees that

$$((t_1, t_2)L_2, \Gamma_{n+1}(T, G)) = ((a_1, 1)L_2, (a_1, 1)\Gamma_{n+1}(T, G))(L_2, (c_1, 1)\Gamma_{n+1}(T, G)),$$

for some $a_1 \in \zeta_n(T_1)$ and $c_1 \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)]$. The conclusion is that $t_1 = a_1$ and $(a_1c_1, 1) \in \Gamma_{n+1}(T, G) \cap T_1 = 1$, implying that $t_1 \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)]$. Conversely, suppose $t_1 \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)]$. Then, $t_1 \in \zeta_n(T_1)$ and $\alpha_1(t_1\zeta_n(T_1)) = \zeta_n(T_2)$. Hence, $\varphi_1(t_1) = ((t_1, 1)L_2, \Gamma_{n+1}(T, G))V_1 = ((t_1, 1)L_2, (t_1, 1)\Gamma_{n+1}(T, G))(L_2, (t_1^{-1}, 1)\Gamma_{n+1}(T, G))V_1$. It follows, by applying again Lemma 3.6(*ii*), that $\varphi_1(t_1) = 1$. Therefore, indeed $\ker\varphi_1 = [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)]$. Using Lemma 3.6(*iii*) and applying a similar argument, we can show that $\ker\varphi_2 = [G_1, \kappa_n(G_1)]$.

*Step* 3. $\ker\psi = \beta([(T_1, G_1, \partial_1), Z_n(T, G, \partial)] \cap \gamma_{n+1}(T, G, \partial))$.

Let $t_2 \in \ker\psi_1$. Choose $t_1 \in T_1$ with $\alpha_1(t_1\zeta_n(T_1)) = t_2\zeta_n(T_2)$. Then, $((t_1, t_2)L_2, (t_1, t_2)\Gamma_{n+1}(T, G)) = ((a_1, 1)L_2, (a_1c_1, 1)\Gamma_{n+1}(T, G))$ for some $a_1 \in \zeta_n(T_1)$ and $c_1 \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)]$. Thus, we must have $t_1 = a_1$ and $(t_1c_1a_1^{-1}, t_2) \in \Gamma_{n+1}(T, G)$, forcing $\beta_1(t_1c_1a_1^{-1}) = t_2$. On the other hand, $t_1c_1a_1^{-1} \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)] \cap \Gamma_{n+1}(T_1, G_1)$. So,

$$\ker\psi_1 \subseteq \beta_1([\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)] \cap \Gamma_{n+1}(T_1, G_1)).$$

Conversely, suppose $t_1 \in [\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)] \cap \Gamma_{n+1}(T_1, G_1)$ and $t_2 = \beta_1(t_1)$. Then, $(t_1, t_2) \in \Gamma_{n+1}(T, G))$ and so, invoking Lemma 3.6(*ii*), we have

$$\begin{aligned}\psi_1(t_2) &= ((t_1, t_2)L_2, \Gamma_{n+1}(T, G))V_1\\ &= ((t_1, 1)L_2, (t_1, 1)\Gamma_{n+1}(T, G))(L_2, (t_1^{-1}, 1)\Gamma_{n+1}(T, G))V_1 = V_1,\end{aligned}$$

whence $t_2 \in \ker\psi_1$. Hence, $\ker\psi_1 = \beta_1([\kappa_n(G_1), T_1][G_1, \zeta_n(T_1)] \cap \Gamma_{n+1}(T_1, G_1))$. Using Lemma 3.6(*iii*) and applying a similar argument, one can show that $\ker\psi_2 = \beta_2([G_1, \kappa_n(G_1)] \cap \gamma_{n+1}(G_1))$.

Now, taking $(\widetilde{T}, \widetilde{G}, \widetilde{\partial}) = (V/V_1, W/W_1, \widetilde{\delta})$, $(\widetilde{S}_1, \widetilde{H}_1, \widetilde{\partial})$ =Im$\varphi$ and $(\widetilde{S}_2, \widetilde{H}_2, \widetilde{\partial})$ =Im$\psi$, the above steps together with Proposition 3.2(*i*) give the required results. □

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.7. If $(T_1, G_1, \partial_1) \underset{n}{\sim} (T_2, G_2, \partial_2)$ and $Z_n(T_1, G_1, \partial_1) = Z(T_1, G_1, \partial_1)$, then there exists a crossed module $(\widetilde{T}, \widetilde{G}, \widetilde{\partial})$ with crossed submodules $(\widetilde{S}_i, \widetilde{H}_i, \widetilde{\partial})$, $i = 1, 2$, such that

$$(T_1, G_1, \partial_1) \cong (\widetilde{S}_1, \widetilde{H}_1, \widetilde{\partial}) \underset{n}{\sim} (\widetilde{T}, \widetilde{G}, \widetilde{\partial}) \underset{n}{\sim} (\widetilde{S}_2, \widetilde{H}_2, \widetilde{\partial}) \cong (T_2, G_2, \partial_2).$$

**4. Construction of $n$-stem crossed modules.** For $n \geq 1$, a crossed module $(T, G, \partial)$ is called *$n$-stem* if satisfies $Z_n(T, G, \partial) \subseteq \gamma_{n+1}(T, G, \partial)$.

In case $n = 1$, Salemkar et al. [**26**] proved the existence of 1-stem crossed module within an arbitrary 1-isoclinism class and determined them inside 1-isoclinism classes containing at least one finite crossed module. These results and a theorem of Hekster

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[**14**, Theorem 6.5] for groups are both extended in the following main result of this section.

THEOREM 4.1. *Let $\mathcal{E}$ be an n-isoclinism family of crossed modules containing at least one strongly finitely generated crossed module $(T, G, \partial)$ with $\gamma_{n+1}(T, G, \partial)$ finite. Then,*

(i) *$\mathcal{E}$ contains at least a finite n-stem crossed module;*

(ii) *the n-stem crossed modules of $\mathcal{E}$ are precisely the crossed modules such as $(S, H, \delta)$ for which*

  (a) *$Z(S, H, \partial) \subseteq \gamma_n(S, H, \delta)$, and*

  (b) *$\gamma_n(S, H, \delta)$ is finite and $|\gamma_n(S, H, \delta)|$ =min$\{|\gamma_n(T, G, \partial)| : (T, G, \partial) \in \mathcal{E}\}$.*

Note that the $n$-isoclinism family of crossed modules containing at least one nilpotent crossed module of class at most $n$ satisfies the requirements of Theorem 4.1. Also, if $G$ is a finitely generated group with $\gamma_{n+1}(G)$ finite, [**14**, Theorem 6.5] shows that the $n$-isoclinism families $\{(1, G, i)\}$ and $\{(G, G, id)\}$ of crossed modules again satisfy the conditions of Theorem 4.1.

The rest of this section will provide a proof.

LEMMA 4.2. *Let $(Y, F, \mu)$ be a projective crossed module. Then, in the category of abelian crossed modules, any crossed submodule of $\gamma_n(Y, F, \mu)/\gamma_{n+1}(Y, F, \mu)$ is projective.*

*Proof.* In view of [**1**, Proposition 3], $\mu$ is monic and the groups $Y$, $F$ and $F/\mu(Y)$ are free. So, we can assume that $\mu$ is the inclusion map. We first claim that the crossed module

$$(\Gamma_n(Y, F)/\Gamma_{n+1}(Y, F), \gamma_n(F)/\gamma_{n+1}(F), \bar{\mu})$$

is aspherical. The freeness of $F/Y$ ensures that the $n$-nilpotent multiplier of $F/Y$ is trivial (see [**6**]) and then $Y \cap \gamma_{n+1}(F) = \Gamma_{n+1}(Y, F)$. Hence, we have

$$\ker \bar{\mu} = \frac{\Gamma_n(Y, F) \cap \gamma_{n+1}(F)}{\Gamma_{n+1}(Y, F)} = \frac{\Gamma_n(Y, F) \cap Y \cap \gamma_{n+1}(F)}{\Gamma_{n+1}(Y, F)} = 1,$$

as claimed. By [**16**, Theorem 11.15($a$)], $\gamma_n(F)/\gamma_{n+1}(F)$ and then $\Gamma_n(Y, F)/\Gamma_{n+1}(Y, F)$ are free abelian groups. So, if $(A, B, \bar{\mu})$ is a crossed submodule of $(\Gamma_n(Y, F)/\Gamma_{n+1}(Y, F), \gamma_n(F)/\gamma_{n+1}(F), \bar{\mu})$, then the groups $A$ and $B$ are free abelian on subsets, say $X$ and $Y$, respectively. We may assume that $X \subseteq Y$. Now, let $\varepsilon = (\varepsilon_1, \varepsilon_2) : (T_1, G_1, \partial_1) \longrightarrow (T_2, G_2, \partial_2)$ and $\delta = (\delta_1, \delta_2) : (A, B, \bar{\mu}) \longrightarrow (T_2, G_2, \partial_2)$ be given morphisms of abelian crossed modules with $\varepsilon$ surjective. Due to the projectivity property of $A$, there is a homomorphism $\theta_1 : A \longrightarrow T_1$ with $\varepsilon_1\theta_1 = \delta_1$. We define the map $h : Y \longrightarrow G_1$ as follows: For any $x \in X$, $h(x) = \partial_1\theta_1(x)$ and for any $x \in Y \setminus X$, $h(x) = g_x$ chosen in the pre-image of $\delta_2(x)$ (by axiom of choice) via $\varepsilon_2$. Then, $h$ extends to a homomorphism $\theta_2 : B \longrightarrow G_1$. It is readily verified that $\theta = (\theta_1, \theta_2) : (A, B, \bar{\mu}) \longrightarrow (T_1, G_1, \partial_1)$ is a morphism such that $\varepsilon\theta = \delta$. □

PROPOSITION 4.3. *Let $(T, G, \partial)$ be any crossed module and $n \geq 1$. Then,*

(i) *if $\gamma_n((T, G, \partial)/Z(T, G, \partial))$ is finite, then so is $\gamma_{n+1}(T, G, \partial)$;*

(ii) *if $(T, G, \partial)$ is finitely generated, then the following statements are equivalent:*

  (a) *$(T, G, \partial)/Z_n(T, G, \partial)$ is finite,*

  (b) *$\gamma_n((T, G, \partial)/Z(T, G, \partial))$ is finite,*

  (c) *$\gamma_{n+1}((T, G, \partial)$ is finite.*

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*Proof.* It is well known that the category $\mathfrak{CM}$ of crossed modules is equivalent to the category $\mathfrak{Cat}^1$ of cat$^1$-groups (see [**20**]). (Recall that a *cat*$^1$*-group* is a triple $(C, \nu, \omega)$, where $C$ is a group and $\nu, \omega : C \longrightarrow C$ are group homomorphisms satisfying the conditions $\nu\omega = \omega$, $\omega\nu = \nu$, and $[\ker\nu, \ker\omega] = 1$.) A functor $\mathfrak{s} : \mathfrak{CM} \longrightarrow \mathfrak{Cat}^1$ defining the equivalence of the categories $\mathfrak{CM}$ and $\mathfrak{Cat}^1$ can be given as

$$\mathfrak{s} : (T, G, \partial) \mapsto (C := T \rtimes G, \nu : (t, g) \mapsto (1, g), \omega : (t, g) \mapsto (1, \partial(t)g)),$$

where $T \rtimes G$ denotes the semi-direct product group of $T$ by $G$. It is routine to show that the subgroups $Z_n(C)$ and $\gamma_{n+1}(C)$ of the group $C$ inherit cat$^1$-structures equivalent to $Z_n(T, G, \partial)$ and $\gamma_{n+1}(T, G, \partial)$, respectively, and the factor group $C/Z_n(C)$ inherits a cat$^1$-structure equivalent to $(T, G, \partial)/Z_n(T, G, \partial)$, for all $n \geq 1$. These conclusions together with [**21**, Theorem 1] and [**14**, Theorem 2.10] give the results, as required. □

LEMMA 4.4. *For each crossed module* $(T, G, \partial)$*, there exists a crossed module* $(L, K, \delta)$ *$n$-isoclinic to* $(T, G, \partial)$ *such that*

(i) $Z(L, K, \delta) \cap \gamma_n(L, K, \delta) \leq \gamma_{n+1}(L, K, \delta)$*;*

(ii) *if the crossed module* $\gamma_n((T, G, \partial)/Z(T, G, \partial))$ *is finite, then so is* $\gamma_n(L, K, \delta)$*;*

(iii) *if* $(T, G, \partial)$ *is strongly finitely generated, then so is* $(L, K, \delta)$*.*

*Proof.* Let $(V, R, \mu) \rightarrowtail (Y, F, \mu) \twoheadrightarrow (T, G, \partial)$ be a projective presentation of $(T, G, \partial)$ and denote $(\bar{Y}, \bar{F}, \bar{\mu}) = (Y, F, \mu)/((V, R, \mu) \cap \gamma_{n+1}(Y, F, \mu))$. By Proposition 3.2(*ii*), $(T, G, \partial) \underset{n}{\sim} (\bar{Y}, \bar{F}, \bar{\mu})$. Since

$$\frac{Z(\bar{Y}, \bar{F}, \bar{\mu}) \cap \gamma_n(\bar{Y}, \bar{F}, \bar{\mu})}{Z(\bar{Y}, \bar{F}, \bar{\mu}) \cap \gamma_{n+1}(\bar{Y}, \bar{F}, \bar{\mu})} \cong \frac{(Z(\bar{Y}, \bar{F}, \bar{\mu}) \cap \gamma_n(\bar{Y}, \bar{F}, \bar{\mu}))\gamma_{n+1}(\bar{Y}, \bar{F}, \bar{\mu})}{\gamma_{n+1}(\bar{Y}, \bar{F}, \bar{\mu})}$$

is isomorphic to a crossed submodule $\gamma_n(\bar{Y}, \bar{F}, \bar{\mu})/\gamma_{n+1}(\bar{Y}, \bar{F}, \bar{\mu}) \cong \gamma_n(Y, F, \mu)/\gamma_{n+1}(Y, F, \mu)$, we observe from Lemma 4.2 that there is a crossed submodule $(\bar{U}, \bar{Q}, \bar{\mu})$ of $Z(\bar{Y}, \bar{F}, \bar{\mu}) \cap \gamma_n(\bar{Y}, \bar{F}, \bar{\mu})$ such that

$$Z(\bar{Y}, \bar{F}, \bar{\mu}) \cap \gamma_n(\bar{Y}, \bar{F}, \bar{\mu}) \cong (Z(\bar{Y}, \bar{F}, \bar{\mu}) \cap \gamma_{n+1}(\bar{Y}, \bar{F}, \bar{\mu})) \times (\bar{U}, \bar{Q}, \bar{\mu}).$$

Note that $(\bar{U}, \bar{Q}, \bar{\mu})$ is a normal crossed submodule of $(\bar{Y}, \bar{F}, \bar{\mu})$ and $(\bar{U}, \bar{Q}, \bar{\mu}) \cap \gamma_{n+1}(\bar{Y}, \bar{F}, \bar{\mu}) = 1$. Taking $(L, K, \delta) = (\bar{Y}, \bar{F}, \bar{\mu})/(\bar{U}, \bar{Q}, \bar{\mu})$, Proposition 3.2(*ii*) indicates that $(T, G, \partial) \underset{n}{\sim} (L, K, \delta)$. We now claim that $(L, K, \delta)$ satisfies the properties mentioned in the lemma. Choose arbitrary elements $\bar{y} \in L^K \cap \Gamma_n(L, K)$ and $\bar{x} \in Z(K) \cap st_K(L) \cap \gamma_n(K)$, in which $y \in \Gamma_n(\bar{Y}, \bar{F})$ and $x \in \gamma_n(\bar{F})$. Then, for each $z \in \bar{F}$, $[z, y] \in \Gamma_{n+1}(\bar{Y}, \bar{F}) \cap \bar{U} = 1$ and $[x, z] \in \gamma_{n+1}(\bar{F}) \cap \bar{Q} = 1$, whence

$$y \in \bar{Y}^{\bar{F}} \cap \Gamma_n(\bar{Y}, \bar{F}) = (\bar{Y}^{\bar{F}} \cap \Gamma_{n+1}(\bar{Y}, \bar{F}))\bar{U} \leq \Gamma_{n+1}(\bar{Y}, \bar{F})\bar{U},$$
$$x \in Z(\bar{F}) \cap st_{\bar{F}}(\bar{Y}) \cap \gamma_n(\bar{F}) = (Z(\bar{F}) \cap st_{\bar{F}}(\bar{Y}) \cap \gamma_{n+1}(\bar{F}))\bar{Q} \leq \gamma_{n+1}(\bar{F}))\bar{Q}.$$

Hence, $\bar{y} \in \Gamma_{n+1}(L, K)$ and $\bar{x} \in \gamma_{n+1}(K)$. It therefore follows that

$$(L^K \cap \Gamma_n(L, K), Z(K) \cap st_K(L) \cap \gamma_n(K), \delta) \leq (\Gamma_{n+1}(L, K), \gamma_{n+1}(K), \delta),$$

proving (*i*).

Suppose that $\gamma_n((T, G, \partial)/Z(T, G, \partial))$ is finite. Then, owing to Proposition 4.3(*i*), $\gamma_{n+1}(T, G, \partial)$ is finite. Bearing in mind that $(T, G, \partial) \underset{n}{\sim} (\bar{Y}, \bar{F}, \bar{\mu})$, and

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using Corollary 3.4($i$), it can be inferred that the crossed modules $\gamma_{n+1}(\bar{Y},\bar{F},\bar{\mu})$ and $\gamma_n(\bar{Y},\bar{F},\bar{\mu}))/(Z(\bar{Y},\bar{F},\bar{\mu})\cap\gamma_n(\bar{Y},\bar{F},\bar{\mu})))$ are also finite. By the foregoing, $(Z(\bar{Y},\bar{F},\bar{\mu})\cap\gamma_n(\bar{Y},\bar{F},\bar{\mu})))/(\bar{U},\bar{Q},\bar{\mu})$ is isomorphic to a crossed submodule of $\gamma_{n+1}(\bar{Y},\bar{F},\bar{\mu})$. We thus derive that $\gamma_n(L,K,\delta)=\gamma_n(\bar{Y},\bar{F},\bar{\mu}))/(\bar{U},\bar{Q},\bar{\mu})$ is finite, proving ($ii$).

Finally, the part ($iii$) holds trivially. This completes the proof of the lemma. □

PROPOSITION 4.5. *Let $(L,K,\delta)$ be any crossed module in which the crossed submodule $\gamma_n(L,K,\delta)$ is finite. Then, $Z(L,K,\delta)\cap\gamma_n(L,K,\delta)\leq\gamma_{n+1}(L,K,\delta)$ if and only if for any crossed module $(T,G,\partial)$ $n$-isoclinic to $(L,K,\delta)$, $|\gamma_n(L,K,\delta)|\leq|\gamma_n(T,G,\partial)|$.*

*Proof.* We begin by proving the 'only if' direction. Assume $(T,G,\partial)$ is a crossed module $n$-isoclinic to $(L,K,\delta)$ with $\gamma_n(T,G,\partial)$ finite. By Corollary 3.4($i$), the factor crossed modules $(T,G,\partial)/Z(T,G,\partial)$ and $(L,K,\delta)/Z(L,K,\delta)$ are $(n-1)$-isoclinic. In particularly, we have $\gamma_{n+1}((T,G,\partial)/Z(T,G,\partial))\cong\gamma_{n+1}((L,K,\delta)/Z(L,K,\delta))$, implying that $|Z(T,G,\partial)\cap\gamma_{n+1}(T,G,\partial)|=|Z(L,K,\delta)\cap\gamma_{n+1}(L,K,\delta)|$. It therefore follows from the hypothesis that

$$\begin{aligned}|\gamma_n(T,G,\partial)|&=\frac{|\gamma_n(L,K,\delta)||Z(T,G,\partial)\cap\gamma_n(T,G,\partial)|}{|Z(L,K,\delta)\cap\gamma_n(L,K,\delta)|}=\frac{|\gamma_n(L,K,\delta)||Z(T,G,\partial)\cap\gamma_n(T,G,\partial)|}{|Z(L,K,\delta)\cap\gamma_{n+1}(L,K,\delta)|}\\&=\frac{|\gamma_n(L,K,\delta)||Z(T,G,\partial)\cap\gamma_n(T,G,\partial)|}{|Z(T,G,\partial)\cap\gamma_{n+1}(T,G,\partial)|}\geq|\gamma_n(L,K,\delta)|.\end{aligned}$$

We now prove the 'if' direction. As $\gamma_n((L,K,\delta)/Z(L,K,\delta))$ is finite, Lemma 4.4 guarantees the existence of a crossed module $(S,H,\sigma)$ with $\gamma_n(S,H,\sigma)$ finite, being $n$-isoclinic to $(L,K,\delta)$ and satisfying the condition $Z(S,H,\sigma)\cap\gamma_n(S,H,\sigma)\leq\gamma_{n+1}(S,H,\sigma)$. As above, one can see that

$$|\gamma_n(L,K,\delta)|=\frac{|\gamma_n(S,H,\sigma)||Z(L,K,\delta)\cap\gamma_n(L,K,\delta)|}{|Z(L,K,\delta)\cap\gamma_{n+1}(L,K,\delta)|}.$$

The minimality of $|\gamma_n(L,K,\delta)|$ so yields that $|Z(L,K,\delta)\cap\gamma_n(L,K,\delta)|=|Z(L,K,\delta)\cap\gamma_{n+1}(L,K,\delta)|$, or equivalently, $Z(L,K,\delta)\cap\gamma_n(L,K,\delta)\leq\gamma_{n+1}(L,K,\delta)$. □

After these preparations, the proof of the main Theorem 4.1 is easy to describe.

*Proof of Theorem 4.1* ($i$) We proceed by induction on $n\geq 1$. The result is true for $n=1$, thanks to Lemma 4.4 and Proposition 4.5. Hence, let $n\geq 2$. By Proposition 4.3($ii$), $\gamma_n((T,G,\partial)/Z(T,G,\partial))$ is finite. Combining Lemma 4.4 with Proposition 4.5, we can find a crossed module $(L,K,\delta)$ such that $|\gamma_n(L,K,\delta)|$ is finite and $|\gamma_n(L,K,\delta)|=\min\{|\gamma_n(T_1,G_1,\partial_1)|:(T_1,G_1,\partial_1)\in\mathcal{E}\}$. By the induction hypothesis, there is a finite crossed module $(S,H,\sigma)$ with $Z(S,H,\sigma)\subseteq\gamma_n(S,H,\sigma)$ and $(L,K,\delta)\underset{n-1}{\sim}(S,H,\sigma)$. Now, we have $(L,K,\delta)\underset{n}{\sim}(S,H,\sigma)$, because of Corollary 3.4($iii$), and $Z(S,H,\sigma)=Z(S,H,\sigma)\cap\gamma_n(S,H,\sigma)\subseteq\gamma_{n+1}(S,H,\sigma)$, thanks to Proposition 4.5. We therefore infer that $(S,H,\sigma)\in\mathcal{E}$ is a finite $n$-stem crossed module.

($ii$) Let $(L,K,\delta)$ be an $n$-stem crossed module in $\mathcal{E}$. Since $\gamma_{n+1}(L,K,\delta)\cong\gamma_{n+1}(T,G,\partial)$, one gets that $\gamma_{n+1}(L,K,\delta)$ and then $Z(L,K,\delta)$ are finite. By part ($i$), $\mathcal{E}$ contains a finite $n$-stem crossed module, $(S,H,\sigma)$ say. Using Corollary 3.4($i$), $(L,K,\delta)/Z(L,K,\delta)\underset{n-1}{\sim}(S,H,\sigma)/Z(S,H,\sigma)$, and consequently, $\gamma_n((L,K,\delta)/Z(L,K,\delta))$ is finite. Therefore, $\gamma_n(L,K,\delta)$ is finite. Now, the proofs of parts ($a$) and ($b$) follow by Proposition 4.5.

Conversely, let $(S, H, \sigma) \in \mathcal{E}$ be a crossed module satisfying the conditions $(a)$ and $(b)$. Then, $Z(S, H, \sigma) = Z(S, H, \sigma) \cap \gamma_n(S, H, \sigma) \subseteq \gamma_{n+1}(S, H, \sigma)$, thanks to Proposition 4.5. Hence, $(S, H, \sigma)$ is an $n$-stem crossed module. This completes the proof. $\square$

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