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MORPHIC RINGS AS TRIVIAL EXTENSIONS

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Abstract. A ring R is called *left morphic* if, for every $a \in R$, $R/Ra \cong \mathbf{l}(a)$ where $\mathbf{l}(a)$ denotes the left annihilator of a in R. Right morphic rings are defined analogously. In this paper, we investigate when the trivial extension $R \propto M$ of a ring R and a bimodule M over R is (left) morphic. Several new families of (left) morphic rings are identified through the construction of trivial extensions. For example, it is shown here that if R is strongly regular or semisimple, then $R \propto R$ is morphic; for an integer $n > 1$, $\mathbb{Z}_n \propto \mathbb{Z}_n$ is morphic if and only if n is a product of distinct prime numbers; if R is a principal ideal domain with classical quotient ring Q, then the trivial extension $R \propto Q/R$ is morphic; for a bimodule M over $\mathbb{Z}$, $\mathbb{Z} \propto M$ is morphic if and only if $M \cong \mathbb{Q}/\mathbb{Z}$. Thus, $\mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$ is a morphic ring which is not clean. This example settled two questions both in the negative raised by Nicholson and Sánchez Campos, and by Nicholson, respectively.

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§**1.** All rings here are associative rings with identity. An element a in a ring R is called *left morphic* if $R/Ra \cong \mathbf{l}(a)$, where $\mathbf{l}(a)$ denotes the left annihilator of a in R; equivalently, $a \in R$ is left morphic if and only if there exists $b \in R$ such that $Ra = \mathbf{l}(b)$ and $Rb = \mathbf{l}(a)$. See [**4**, Lemma 1]. The ring R is called *left morphic* if every element of R is left morphic. Right morphic rings are defined analogously. A left and right morphic ring is simply called a *morphic ring*. Left morphic rings were first introduced by Nicholson and Sánchez Campos [**4**] and were discussed in great detail in [**4**], [**5**] and [**6**]. In this paper, we investigate when the trivial extension $R \propto M$ of a ring R and a bimodule M over R is left morphic. The following new families of (left) morphic rings are identified.

If R is a strongly regular ring and $\sigma : R \to R$ is an endomorphism of R with $\sigma(e) = e$ for every $e^2 = e \in R$, then $R[x;\sigma]/(x^2)$ is a left morphic ring [Theorem 1]; for any semisimple ring R, every matrix ring $\mathbb{M}_n(R \propto R)$ is morphic [Theorem 7]; for any positive integers $d > 1$ and m, $\mathbb{Z}_{md} \propto \mathbb{Z}_d$ is morphic if and only if d and m are relatively prime and d is a product of distinct prime numbers [Theorem 8]; if R is a principal ideal domain with classical quotient ring Q, then $R \propto Q/R$ is morphic [Theorem 13]; for a bimodule M over $\mathbb{Z}$, $\mathbb{Z} \propto M$ is morphic if and only if $M \cong \mathbb{Q}/\mathbb{Z}$ [Theorem 14]. Thus, $\mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$ is a morphic ring which is not clean (a ring is *clean* if every element is the sum of an idempotent and a unit) and does not have stable range 1 (a ring R

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has *stable range* 1 if, whenever $aR + bR = R$ with $a, b \in R$, $a + by$ is a unit for some $y \in R$). This example settled two questions both in the negative raised by Nicholson and Sánchez Campos [**4**], and by Nicholson, respectively.

For a ring R and $a \in R$, we let $\mathbf{l}_R(a) = \{r \in R : ra = 0\}$. Right annihilators are defined analogously. Sometimes, we simply write $\mathbf{l}(a)$ for $\mathbf{l}_R(a)$ and $\mathbf{r}(a)$ for $\mathbf{r}_R(a)$. The $n \times n$ matrix ring over R is denoted by $\mathbb{M}_n(R)$. We write $\mathbb{Z}$ for the ring of integers, $\mathbb{Q}$ for rational numbers, and $\mathbb{Z}_n$ for integers modulo n, respectively. Regular rings here mean von Neumann regular rings.

§**2.** Let R be a ring and M a bimodule over R. The trivial extension of R and M is $R \propto M = \{(a, x) : a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by

$$(a, x)(b, y) = (ab, ay + xb).$$

In fact, $R \propto M$ is isomorphic to the subring $\{\left(\begin{smallmatrix} a & x \\ 0 & a \end{smallmatrix}\right) : a \in R, x \in M\}$ of the formal 2×2 matrix ring $\left(\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right)$, and $R \propto R \cong R[x]/(x^2)$. For convenience, we let $I \propto X = \{(a, x) : a \in I, x \in X\}$ where I is a subset of R and X is a subset of M.

If R is a ring and $\sigma : R \to R$ is a ring endomorphism, let $R[x; \sigma]$ denote the ring of skew polynomials over R; that is all formal polynomials in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$. Note that if $R(\sigma)$ is the (R, R)-bimodule defined by ${}_RR(\sigma) = {}_RR$ and $m \circ r = m\sigma(r)$, for all $m \in R(\sigma)$ and $r \in R$, then $R[x; \sigma]/(x^2) \cong R \propto R(\sigma)$.

A ring R is called *strongly regular* if $a \in a^2R$, for every $a \in R$. It is well known that a ring R is strongly regular if and only if R is von Neumann regular and every idempotent in R is central. If R is strongly regular, then for any $a \in R$, $a = ue$ where u is a unit and e is an idempotent, so that $aR = ueR = uRe = Re = Rue = Ra$. Thus every one-sided ideal of a strongly regular ring is an ideal.

THEOREM 1. *If R is a strongly regular ring and $\sigma : R \to R$ is a ring endomorphism such that $\sigma(e) = e$ for all $e^2 = e \in R$, then $R[x; \sigma]/(x^2)$ is a left morphic ring.*

Proof. Let $S = R[x; \sigma]/(x^2)$. Then $S = \{r + sx : r, s \in R\}$ with $x^2 = 0$ and $xt = \sigma(t)x$ for all $t \in R$.

Claim 1. If I is a left or right ideal of S and $r, s \in R$, then $r + sx \in I$ implies that $r \in I$ and $sx \in I$.

In fact, $Rr = Re$, where $e^2 = e \in R$, so that $r = re$ and $e = tr$ for some $t \in R$. If I is a left ideal, then $e = e^2 = (e - tsx)(e + tsx) = (e - tsx)t(r + sx) \in I$ and so $r = re \in I$. Hence $sx \in I$. Let I be a right ideal. Since $rR = eR$ and $e = rt_0$ with $t_0 \in R$, then $e = e^2 = [e + s\sigma(t_0)x][e - s\sigma(t_0)x] = (r + sx)t_0[e - s\sigma(t_0)x] \in I$, so that $r = re = er \in I$ and thus $sx \in I$. Hence the Claim holds.

Now let $a + bx \in S$. We need to show that $a + bx$ is left morphic in S. Write $Ra = Re$, where $e^2 = e \in R$.

Claim 2. There exists $g^2 = g \in R$ with $eg = ge = 0$ such that $S(a + bx) = S(e + gx)$.

In fact, by Claim 1, $S(a + bx) = Sa + Sbx = Se + Sbx = Se + Sb(1 - e)x$. The last equality holds because $bx = (bx)e + b(1 - e)x$. Let $b_1 = b(1 - e)$ and write $Rb_1 = Rf$ where $f^2 = f \in R$. Thus, $Sb_1 = Sf$, so $Se + Sb(1 - e)x = Se + Sb_1x = Se + Sfx$. Since $fe = 0$, we see that $g := (1 - e)f$ is an idempotent and $ge = eg = 0$. Moreover,

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$fg = f(1-e)f = f^2 = f$, showing that $Rf = Rg$. Hence $Sf = Sg$. It follows that we have $S(a+bx) = Se + Sfx = Se + Sgx = S(e+gx)$.

Claim 3. $S(e+gx) = \mathbf{l}((1-e)(1-g) + (1-e)x)$, where e and g are as in Claim 2. For any

$$\begin{aligned} c + dx &\in \mathbf{l}((1-e)(1-g) + (1-e)x), \\ 0 &= (c+dx)[(1-e)(1-g) + (1-e)x] \\ &= c(1-e)(1-g) + [c(1-e) + d(1-e)(1-g)]x. \end{aligned}$$

Thus,

$$c(1-e)(1-g) = 0, \tag{1}$$

$$c(1-e) + d(1-e)(1-g) = 0. \tag{2}$$

Adding (2) times g to (1) gives $c(1-e) = 0$; i.e., $c = ce$. Multiplying (2) by g yields $cg = 0$. Moreover, it follows from (2) that $d(1-e)(1-g) = 0$ and so $d = de + dg$. Now let $u = c + dg$ and $v = d$. Then $(u+vx)(e+gx) = ue + (ug+ve)x = (c+dg)e + [(c+dg)g + de]x = ce + (dg+de)x = c + dx$. Hence, $c + dx \in S(e+gx)$, so that $\mathbf{l}((1-e)(1-g) + (1-e)x) \subseteq S(e+gx)$. However, clearly we have

$$S(e+gx) \subseteq \mathbf{l}((1-e)(1-g) + (1-e)x).$$

Claim 4. $\mathbf{l}(a+bx) = S((1-e)(1-g) + (1-e)x)$. Let b_1 and f be as in the proof of Claim 2. Since $Ra = Re$, we have $aR = eR$. So $aS = eS$. Thus, by Claim 1, $(a+bx)S = aS + bxS = eS + bxS = eS + [e(bx) + b(1-e)x]S = eS + b(1-e)xS = eS + b_1xS = (e+b_1x)S$. Hence, it suffices to show that

$$\mathbf{l}(e+b_1x) = S((1-e)(1-g) + (1-e)x).$$

Since $b_1 = b_1f$ and $f = fg$, we have $[(1-e)(1-g) + (1-e)x](e+b_1x) = (1-e)(1-g)b_1x = (1-e)(1-g)b_1fx = (1-e)(1-g)b_1fgx = (1-e)(1-g)gb_1fx = 0$. Hence $S((1-e)(1-g) + (1-e)x) \subseteq \mathbf{l}(e+b_1x)$.

If $r + sx \in \mathbf{l}(e+b_1x)$, then

$$0 = (r+sx)(e+b_1x) = re + (rb_1 + se)x.$$

Hence $re = 0$ and $rb_1 + se = 0$. Since $b_1 = b_1f = b_1fg$, we have $b_1g = b_1$. Thus, $0 = (rb_1 + se)g = rb_1g = rb_1$, and hence $se = 0$. Since $Rf = Rb_1 = b_1R$, we write $f = b_1t$ with $t \in R$. Then $rg = r(1-e)f = rf = rb_1t = 0$. Hence $r = r(1-e)(1-g)$, $s = s(1-e)$. Therefore, $r + sx \in S(1-e)(1-g) + S(1-e)x = S((1-e)(1-g) + (1-e)x)$. The proof is complete. □

Corollary 2. [**4**, Example 8] *If D is a division ring with an endomorphism σ, then $D[x;\sigma]/(x^2)$ is a left morphic ring.*

Corollary 3. *If R is a strongly regular ring, then $R \propto R$ is a morphic ring.*

Note that there exist strongly regular rings R that are not division rings with an endomorphism $\sigma : R \to R$ such that $\sigma \neq 1_R$ and $\sigma(e) = e$, for every idempotent $e \in R$. For example, let $R = D \times D$, where D is a division ring and $f_i (i = 1, 2) : D \to D$ are endomorphisms that are not all identity maps. Let $\sigma : R \to R$ be given by $(d_1, d_2) \mapsto (f_1(d_1), f_2(d_2))$.

Following [**6**], a ring R is called *strongly left morphic* if every matrix ring $\mathbb{M}_n(R)$ is left morphic. Strongly right morphic rings are defined analogously. A strongly left and

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strongly right morphic ring is called a *strongly morphic ring*. Next, we show that, for a semisimple ring R, $R \propto R$ is strongly morphic. We need a few lemmas.

For a bimodule V over a ring R, let $\mathbb{M}_n(V)$ be the set of $n \times n$ formal matrices with entries in V. Then $\mathbb{M}_n(V)$ is a bimodule over $\mathbb{M}_n(R)$ with the usual multiplication of matrices. If V_i is a bimodule over the ring R_i for $i = 1, \ldots, n$, then $V_1 \oplus \cdots \oplus V_n$ is a bimodule over $R_1 \oplus \cdots \oplus R_n$ in a natural way.

Lemma 4. *If V_i is a bimodule over the ring R_i, for $i = 1, \ldots, n$, then*

$$(R_1 \oplus \cdots \oplus R_n) \propto (V_1 \oplus \cdots \oplus V_n) \cong (R_1 \propto V_1) \oplus \cdots \oplus (R_n \propto V_n).$$

Proof. The map $\theta : (R_1 \oplus \cdots \oplus R_n) \propto (V_1 \oplus \cdots \oplus V_n) \to (R_1 \propto V_1) \oplus \cdots \oplus (R_n \propto V_n)$ defined by

$$((a_1, \cdots, a_n), (v_1, \cdots, v_n)) \mapsto ((a_1, v_1), \cdots, (a_n, v_n))$$

is the required isomorphism. □

Lemma 5. *If V is a bimodule over R, then $\mathbb{M}_n(R \propto V) \cong \mathbb{M}_n(R) \propto \mathbb{M}_n(V)$.*

Proof. The map $\theta : \mathbb{M}_n(R \propto V) \to \mathbb{M}_n(R) \propto \mathbb{M}_n(V)$ defined by

$$((a_{ij}, v_{ij})) \mapsto ((a_{ij}), (v_{ij})).$$

is the required isomorphism. □

Lemma 6. *If D is a division ring, then $D \propto D$ is strongly morphic.*

Proof. It suffices to show that $\mathbb{M}_n(D \propto D)$ is morphic. Let $X \neq 0$ be given by $X = (x_{ij}) \in \mathbb{M}_n(D \propto D)$, where $x_{ij} = (a_{ij}, b_{ij}) \in D \propto D$.

If some $x_{ij} \in D \propto D$ is a unit, then one can move x_{ij} to the $(1, 1)$-entry and further change all the $(1, k)$- and $(k, 1)$-entries for $k > 1$ to 0, using a series of elementary row and column operations.

If all x_{ij} are not units of $D \propto D$, then all $a_{ij} = 0$, but $b_{lm} \neq 0$ for some l and m. Again, by a series of elementary operations, x_{lm} can be moved to the $(1, 1)$-entry, and all entries in the first row and the first column, except the $(1, 1)$-entry, can be reduced to 0. Hence X can be reduced to

$$Y = \begin{bmatrix} y_{11} & 0 & \cdots & 0 \\ 0 & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & y_{n2} & \cdots & y_{nn} \end{bmatrix}.$$

Continuing in this way, we can reduce Y to a diagonal matrix. Therefore, there exist units U and V of $\mathbb{M}_n(D \propto D)$ such that

$$UXV = \begin{bmatrix} z_1 & 0 & \cdots & 0 \\ 0 & z_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z_n \end{bmatrix}.$$

Since $D \propto D$ is morphic, by Corollary 3, each z_i is morphic in $D \propto D$. Therefore, UXV is clearly morphic in $\mathbb{M}_n(D \propto D)$. Since U, V are units, X is morphic in $\mathbb{M}_n(D \propto D)$ by [**4**, Lemma 3]. □

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THEOREM 7. *If R is a semisimple ring, then $R \propto R$ is strongly morphic.*

Proof. Write $R = R_1 \oplus \cdots \oplus R_s$, where $R_i \cong \mathbb{M}_{n_i}(D_i)$, for some division ring D_i, for $i = 1, \ldots, s$. By Lemma 4, $R \propto R \cong \bigoplus_{i=1}^{s}(R_i \propto R_i)$, so that by Lemma 5

$$\mathbb{M}_n(R \propto R) \cong \bigoplus_{i=1}^{s} \mathbb{M}_n(R_i \propto R_i) \cong \bigoplus_{i=1}^{s} \mathbb{M}_n(\mathbb{M}_{n_i}(D_i) \propto \mathbb{M}_{n_i}(D_i))$$
$$\cong \bigoplus_{i=1}^{s} \mathbb{M}_n(\mathbb{M}_{n_i}(D_i \propto D_i)) \cong \bigoplus_{i=1}^{s} \mathbb{M}_{nn_i}(D_i \propto D_i),$$

which is morphic, because each $\mathbb{M}_{nn_i}(D_i \propto D_i)$ is morphic by Lemma 6. □

Next, we consider the morphic property of $R \propto M$, where $R = \mathbb{Z}_n$ or $\mathbb{Z}$.

THEOREM 8. *Let d, m be positive integers with $d > 1$ and $n = dm$. Then $\mathbb{Z}_n \propto \mathbb{Z}_d$ is a morphic ring if and only if d and m are relatively prime and d is a product of distinct primes.*

Proof. Let $S = \mathbb{Z}_n \propto \mathbb{Z}_d$. For $k \in \mathbb{Z}$, we write $\bar{k} \in \mathbb{Z}_n$ to mean $\bar{k} = k + n\mathbb{Z}$ and $\bar{k} \in \mathbb{Z}_d$ to mean $\bar{k} = k + d\mathbb{Z}$. This will not cause problems. If $\bar{k} \in \mathbb{Z}_n$ and $\bar{l} \in \mathbb{Z}_d$, then $\overline{lk} = \bar{k}\bar{l} = \overline{kl} \in \mathbb{Z}_d$. Throughout the proof, the greatest common divisor of integers m and n is denoted by $gcd(m, n)$.

"$\Rightarrow$". Suppose $\mathbb{Z}_n \propto \mathbb{Z}_d$ is morphic. We first show that $gcd(d, m) = 1$.

Since $(\bar{d}, 0) \in S$ is morphic, there exists $(\bar{a}, \bar{b}) \in S$ such that $\mathbf{l}(\bar{a}, \bar{b}) = S(\bar{d}, 0)$ and $\mathbf{l}(\bar{d}, 0) = S(\bar{a}, \bar{b})$. So $(\bar{d}, 0)(\bar{a}, \bar{b}) = (\bar{d}\bar{a}, \bar{d}\bar{b}) = 0$. Thus $n|da$; i.e., $dm|da$ showing that $m|a$. Write $a = mm_1$.

Since $(\bar{m}, 0) \in \mathbf{l}(\bar{d}, 0) = S(\bar{a}, \bar{b})$, $(\bar{m}, 0) = (\bar{r}, \bar{s})(\bar{a}, \bar{b}) = (\overline{ra}, \bar{r}\bar{b} + \bar{s}\bar{a})$, where $(\bar{r}, \bar{s}) \in S$. Thus, $\bar{m} = \overline{ra}$, showing that $n|(m - ra)$ and so $dm|(m - rmm_1)$. This gives $d|(1 - rm_1)$, so that $gcd(d, m_1) = 1$.

Since $(0, \bar{1}) \in \mathbf{l}(\bar{d}, 0) = S(\bar{a}, \bar{b})$, $(0, \bar{1}) = (\bar{x}, \bar{y})(\bar{a}, \bar{b}) = (\bar{x}\bar{a}, \bar{x}\bar{b} + \bar{y}\bar{a})$, where $(\bar{x}, \bar{y}) \in S$. Thus $n|xa$; i.e., $dm|xmm_1$ and so $d|xm_1$. Since $gcd(d, m_1) = 1$, we have $d|x$. Write $x = dd_1$. Also, $\bar{1} = \bar{x}\bar{b} + \bar{y}\bar{a} = \bar{d}\bar{d}_1\bar{b} + \bar{y}\bar{m}\bar{m}_1 = \bar{y}\bar{m}\bar{m}_1$. Thus, $d|(1 - ymm_1)$, showing that $gcd(d, m) = 1$.

Next we show that d is a product of distinct primes. Since $d > 1$, we can write $d = pd_1$, so that $n = dm = pd_1m$.

Since $(0, \bar{p}) \in S$ is morphic, there exists $(\bar{a}, \bar{b}) \in S$ such that $\mathbf{l}(\bar{a}, \bar{b}) = S(\bar{0}, \bar{p})$ and $\mathbf{l}(0, \bar{p}) = S(\bar{a}, \bar{b})$. Thus, $0 = (\bar{a}, \bar{b})(0, \bar{p}) = (0, \overline{ap})$. Hence $d|ap$; i.e., $pd_1|pa$. Thus $d_1|a$. Write $a = d_1a_1$.

Since $(\bar{d}_1, 0) \in \mathbf{l}(0, \bar{p}) = S(\bar{a}, \bar{b})$, $(\bar{d}_1, 0) = (\bar{x}, \bar{y})(\bar{a}, \bar{b}) = (\bar{x}\bar{a}, \bar{x}\bar{b} + \bar{y}\bar{a})$, where $(\bar{x}, \bar{y}) \in S$. Hence $n|(d_1 - xa)$; i.e., $pd_1m|(d_1 - xd_1a_1)$. Thus, $pm|(1 - xa_1)$, showing that $gcd(a_1, pm) = 1$.

Since $(0, \bar{1}) \in \mathbf{l}(0, \bar{p}) = S(\bar{a}, \bar{b})$, $(0, \bar{1}) = (\bar{x}_1, \bar{y}_1)(\bar{a}, \bar{b}) = (\bar{x}_1\bar{a}, \bar{x}_1\bar{b} + \bar{y}_1\bar{a})$, where $(\bar{x}_1, \bar{y}_1) \in S$. It follows that $n|x_1a$; i.e., $pd_1m|x_1d_1a_1$. Hence $pm|x_1a_1$. Because $gcd(pm, a_1) = 1$, we have $pm|x_1$. Write $x_1 = pmt_1$. Also $\bar{1} = \bar{x}_1\bar{b} + \bar{y}_1\bar{a} = \bar{p}\bar{m}\bar{t}_1\bar{b} + \bar{y}_1\bar{d}_1\bar{a}_1$, showing that $\bar{p} = \bar{p}^2\bar{m}\bar{t}_1\bar{b} + \bar{p}\bar{y}_1\bar{d}_1\bar{a}_1 = \bar{p}^2\bar{m}\bar{t}_1\bar{b}$. Thus, $d|p(1 - pmt_1b)$, so that $pd_1|p(1 - pmt_1b)$. Hence $gcd(d_1, p) = 1$. Therefore, d is a product of distinct primes.

"$\Leftarrow$". Suppose that $gcd(d, m) = 1$ and $d = p_1 \cdots p_s$ is a product of distinct primes. The $\mathbb{Z}_d \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_s}$ is semisimple and $\mathbb{Z}_n \cong \mathbb{Z}_d \oplus \mathbb{Z}_m$. Because $\mathbb{Z}_d \propto \mathbb{Z}_d$ is morphic (Theorem 7) and $\mathbb{Z}_m \propto 0 \cong \mathbb{Z}_m$ is morphic [**4**, Example 12], $\mathbb{Z}_n \propto \mathbb{Z}_d \cong (\mathbb{Z}_d \oplus \mathbb{Z}_m) \propto (\mathbb{Z}_d \oplus 0) \cong (\mathbb{Z}_d \propto \mathbb{Z}_d) \oplus (\mathbb{Z}_m \propto 0)$ is morphic. □

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COROLLARY 9. *For* $n \geq 2$, $\mathbb{Z}_n \propto \mathbb{Z}_n$ *is morphic if and only if n is a product of distinct primes.*

REMARK 10. Corollary 9 shows that $\mathbb{Z}_4 \propto \mathbb{Z}_4$ is not morphic. Thus, a trivial extension of a morphic ring by itself is not morphic. Since $\mathbb{Z}_4$ is strongly π-regular (a ring R is strongly π-regular if, for any $a \in R$, the chain $aR \supseteq a^2R \supseteq \cdots$ terminates), this example also shows that Theorem 1 cannot be extended to a strongly π-regular ring.

If R is a commutative domain and Q is the classical quotient ring of R, then $Q = \{\frac{s}{t} : s, 0 \neq t \in R\}$ and so every element of the R-module Q/R can be expressed as $\overline{\frac{s}{t}} = \frac{s}{t} + R$.

LEMMA 11. *Let R be a commutative domain with classical quotient ring Q and let* $T = R \propto Q/R$. *Then every* $(r, x) \in T$ *with* $r \neq 0$ *is morphic in* T.

Proof. By direct computation, we have

$$\mathbf{l}_T\left(0, \overline{\frac{1}{r}}\right) = rR \propto \frac{Q}{R} \text{ and } \mathbf{l}_T(r, x) = \{(0, y) \in T : ry = 0\}.$$

It is clear that

$$T(r, x) \subseteq \mathbf{l}_T\left(0, \overline{\frac{1}{r}}\right) \text{ and } T\left(0, \overline{\frac{1}{r}}\right) \subseteq \mathbf{l}_T(r, x).$$

Write $x = \overline{\frac{s}{t}}$. For any $(rm, \overline{\frac{c}{d}}) \in \mathbf{l}_T(0, \overline{\frac{1}{r}})$, $(rm, \overline{\frac{c}{d}}) = (m, \overline{\frac{c}{rd}} - \overline{\frac{ms}{rt}})(r, \overline{\frac{s}{t}}) \in T(r, x)$. Hence $T(r, x) = \mathbf{l}_T(0, \overline{\frac{1}{r}})$.

If $y \in \frac{Q}{R}$ with $ry = 0$, we write $y = \overline{\frac{c}{d}}$, so that $rc \in dR$. Hence $rc = dm$, for some $m \in R$. Then $y = \overline{\frac{c}{d}} = \overline{\frac{rc}{rd}} = \overline{\frac{dm}{rd}} = \overline{\frac{m}{r}}$, and so $(0, y) = (m, 0)(0, \overline{\frac{1}{r}}) \in T(0, \overline{\frac{1}{r}})$. Hence $T(0, \overline{\frac{1}{r}}) = \mathbf{l}_T(r, x)$. Therefore, (r, x) is morphic in T. □

For an ideal I of a ring R and $s \in R$, let $s^{-1}I = \{r \in R : sr \in I\}$.

LEMMA 12. *Let R be a commutative domain with classical quotient ring Q and let* $T = R \propto \frac{Q}{R}$. *For nonzero elements s, t in R,* $(0, \overline{\frac{s}{t}})$ *is morphic in T if and only if there exists* $k \in R$ *such that* $s^{-1}(tR) = kR$ *and, for any* $c, d \in R$, $ck \in dR$ *implies* $ct \in d(sR + tR)$.

Proof. Let $a = (0, \overline{\frac{s}{t}})$ and $b = (k, y) \in T$. Then $\mathbf{l}_T(a) = I \propto Q/R$, where $I = s^{-1}(tR)$, and $Tb = \{(km, kz + my) : m \in R, z \in Q/R\}$. If $\mathbf{l}_T(a) = Tb$, then $s^{-1}(tR) = kR$. Conversely, if $s^{-1}(tR) = kR$, then $k \neq 0$ because $t \in kR$, so that

$$Tb = \left\{(km, kz + my) : m \in R, z \in \frac{Q}{R}\right\} = kR \propto \frac{Q}{R}.$$

Thus, $Tb = \mathbf{l}_T(a)$ and so

$$\mathbf{l}_T(a) = Tb \Leftrightarrow s^{-1}(tR) = kR.$$

We now assume that $\mathbf{l}_T(a) = Tb$; i.e., $s^{-1}(tR) = kR$. Then $k \neq 0$ and so $\mathbf{l}_T(b) = \{(0, z) \in T : kz = 0\} = \{(0, \overline{\frac{c}{d}}) : c, 0 \neq d \in R, ck \in dR\}$. On the other hand, $Ta = \{(0, \overline{\frac{ms}{t}}) : m \in R\}$. Since $s^{-1}(tR) = kR$, we have $Ta \subseteq \mathbf{l}_T(b)$. Note that $\overline{\frac{c}{d}} = \overline{\frac{ms}{t}}$, for some $m \in R$, if and

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only if $ct \in d(sR + tR)$. Therefore, $\mathbf{l}_T(a) = Tb$ and $\mathbf{l}_T(b) = Ta$ if and only if $s^{-1}(tR) = kR$ and, for any $c, d \in R$, $ck \in dR$ implies that $ct \in d(sR + tR)$. The proof is complete. □

THEOREM 13. *Let R be a principal ideal domain and Q the classical quotient ring of R. Then $R \propto Q/R$ is a commutative morphic ring.*

Proof. Let $T = R \propto Q/R$ and let $0 \neq (r, x) \in T$. If $r \neq 0$, then (r, x) is morphic in T by Lemma 11.

If $r = 0$, then $x \neq 0$. Write $x = \overline{\frac{s}{t}}$ where $0 \neq s, 0 \neq t \in R$. Since R is a principal ideal domain, we can assume that the greatest common factor of s and t is 1, and hence $R = sR + tR$. Let $k = t$. Then $s^{-1}(tR) = kR$ and, for any $c, d \in R$, $ck \in dR$ automatically implies that $ct \in d(sR + tR)$. Thus, by Lemma 12, $(0, x) = (0, \overline{\frac{s}{t}})$ is morphic in T, so that T is a morphic ring. □

THEOREM 14. *Let M be a bimodule over $\mathbb{Z}$. Then $\mathbb{Z} \propto M$ is a morphic ring if and only if $M \cong \mathbb{Q}/\mathbb{Z}$.*

Proof. If $M \cong \mathbb{Q}/\mathbb{Z}$, then $\mathbb{Z} \propto M$ is a morphic ring, by the previous theorem.

Conversely, suppose that $R = \mathbb{Z} \propto M$ is a morphic ring. We first show that M is torsion. If $x \in M$ is a nonzero torsionfree element, let $a = (0, x) \in R$. Then $aR = 0 \propto \mathbb{Z}x$ and $\mathbf{l}(a) = 0 \propto M$, so that $\mathbf{r}(\mathbf{l}(a)) = \mathbf{r}(0 \propto M) = 0 \propto M$. Since a is morphic, $aR = \mathbf{r}(\mathbf{l}(a)) = 0 \propto M$. This implies that $M = \mathbb{Z}x \cong \mathbb{Z}$. Hence $R \cong \mathbb{Z} \propto \mathbb{Z}$. But, $\mathbb{Z} \propto \mathbb{Z}$ is not morphic. Therefore, M is torsion, and so $M = \oplus\{\tau_p(M) : p \text{ is a prime}\}$, where $\tau_p(M)$ is the p-torsion component of M.

If $\tau_p(M) = 0$, for some prime p, let $b = (p, 0) \in R$. Then $\mathbf{l}(b) = 0$ and so $\mathbf{rl}(b) = \mathbf{r}(0) = R \neq bR$. Thus b is not morphic. Thus $\tau_p(M) \neq 0$, for every prime number p.

For any $0 \neq n \in \mathbb{Z}$, let $c = (n, 0) \in R$. Since c is morphic, there exists $d = (m, x) \in R$ such that $Rc = \mathbf{l}(d)$ and $Rd = \mathbf{l}(c)$. We have $\mathbf{l}(c) = \{(0, z) : nz = 0\}$ and $Rc = \{(kn, nz) : k \in \mathbb{Z}, z \in M\}$. Hence $d = (0, x)$ with $nx = 0$. Now $\mathbf{l}(d) = \mathbf{l}(x) \propto M$. From $Rc = \mathbf{l}(d)$, it follows that $nM = M$ and so M is divisible or injective over $\mathbb{Z}$. Since every injective module over a noetherian ring is a direct sum of indecomposable injective modules, $M = \oplus_i M_i$, where each M_i is an indecomposable torsion injective module over $\mathbb{Z}$. But $\{\mathbb{Z}_{p^\infty} : p \text{ is a prime}\}$ are all indecomposable torsion injective modules over $\mathbb{Z}$. Hence every M_i is isomorphic to $\mathbb{Z}_{p^\infty}$, for some prime number p. If $M_i \cong M_j \cong \mathbb{Z}_{p^\infty}$, for some i and j with $i \neq j$, where p is a prime number, then there exist $0 \neq v \in M_i$ and $0 \neq w \in M_j$ such that $pv = pw = 0$. Then $R(0, v) = 0 \propto \mathbb{Z}v$ and $\mathbf{l}(0, v) = p\mathbb{Z} \propto M$. Hence $(0, w) \in \mathbf{rl}(0, v) \setminus R(0, v)$. Thus, $(0, v)$ is not morphic. Therefore, $M_i \cong M_j$ if and only if $i = j$. Furthermore, for every prime p, since $\tau_p(M) \neq 0$, $\mathbb{Z}_{p^\infty} \cong M_j$ for some j. Hence $M \cong \bigoplus\{\mathbb{Z}_{p^\infty} : p \text{ is a prime}\} \cong \mathbb{Q}/\mathbb{Z}$. □

COROLLARY 15. *Let $R = \mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$. Then R is strongly morphic.*

Proof. Let $0 \neq A = (a_{ij}) \in \mathbb{M}_n(R)$. It suffices to show that A is morphic in $\mathbb{M}_n(R)$. Write $a_{ij} = (n_{ij}, \overline{q_{ij}}) \in R$.

Case 1. $n_{ij} \neq 0$ for some i and j. Then there exists a positive integer k which is smallest with respect to the property that there exist units U and V of $\mathbb{M}_n(R)$ and $\overline{q} \in \mathbb{Q}/\mathbb{Z}$ such that $(k, \overline{q})$ is the $(1, 1)$-entry of UAV. Since A is morphic if and only if UAV is too, by [**4**, Lemma 3], we may assume that $a_{11} = (k, \overline{q})$. For any j with $1 \leq j \leq n$, $n_{1j} = sk + r$, for some $s, r \in \mathbb{Z}$ with $0 \leq r < k$. Thus, $a_{1j} = a_{11}(s, 0) + (r, \overline{q_{1j}} - s\overline{q})$. Now subtracting the first column times $(s, 0)$ from the jth column and then interchanging

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the 1th and jth columns will bring $(r, \overline{q_{1j}} - s\overline{q})$ to the $(1, 1)$-entry. The minimality of k shows that $r = 0$. Thus, the elementary column operation changes a_{1j} to the new $(1, j)$-entry $(0, \overline{q_{1j}} - s\overline{q})$. Similarly, an elementary row operation changes a_{j1} to the new $(j, 1)$-entry $(0, *)$. Therefore, without loss of generality, we can assume that $n_{1j} = 0 = n_{j1}$ for $j = 2, \ldots, n$. Since $\mathbb{Q}/\mathbb{Z}$ is divisible, $\overline{q_{1j}} = k\overline{s}_j$ for $j = 2, \ldots, n$ with $s_j \in \mathbb{Q}$. Thus, $(0, \overline{q_{1j}}) = (0, \overline{s}_j)a_{11}$. Hence subtracting the first column times $(0, \overline{s}_j)$ from the jth column brings a_{1j} to the new $(1, j)$-entry 0, and similarly an elementary row operation brings a_{j1} to the new $(j, 1)$-entry 0 as well. Hence a series of elementary operations change A to

$$B = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{bmatrix}.$$

Case 2. $n_{ij} = 0$, for all i and j. Write $a_{ij} = (0, \overline{\frac{t_{ij}}{s}})$ with $t_{ij}, 0 < s \in \mathbb{Z}$. Since $A \neq 0$, $\overline{\frac{t_{ij}}{s}} \neq 0$ for some i and j. Then there exists a positive integer t which is smallest with respect to the property that there exist units U and V in $\mathbb{M}_n(R)$ such that $0 \neq (0, \overline{\frac{t}{s}})$ is the $(1, 1)$-entry of UAV. As above, we may assume that $a_{11} = (0, \overline{\frac{t}{s}})$. By the Division Algorithm, for any j with $2 \leq j \leq n$, $t_{1j} = tk + r$ where $k, r \in \mathbb{Z}$ with $0 \leq r < t$. Thus, $a_{1j} = (k, 0)a_{11} + (0, \overline{\frac{r}{s}})$. Hence subtracting the first column times $(k, 0)$ from the jth column yields the new $(1, j)$-entry $(0, \overline{\frac{r}{s}})$; and then interchanging the 1th and jth columns will bring $(0, \overline{\frac{r}{s}})$ to the $(1, 1)$-entry. The minimality of t shows that $\overline{\frac{r}{s}} = 0$. Thus, an elementary column operation will change a_{1j} to the new $(1, j)$-entry 0. Similarly, an elementary row operation changes a_{j1} to the new $(j, 1)$-entry 0 as well. Hence a series of elementary operations will change A to a matrix of the same form as B above.

Thus, continuing in this way, we can change A to a diagonal by elementary transformations. Therefore, there exist units U_1 and V_1 of $\mathbb{M}_n(R)$ such that

$$U_1AV_1 = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix},$$

where $a_i \in R$, for $i = 1, \cdots, n$. (In fact, U_1 and V_1 are products of certain elementary matrices over R.) Since each a_i is morphic, by the previous theorem, U_1AV_1 is morphic. Therefore, A is morphic. □

REMARK 16. A ring is called *clean* if every element is the sum of an idempotent and a unit. Because every unit regular ring is clean by Camillo-Yu [**2**] (or by Camillo-Khurana [**1**]) and because every unit regular ring is morphic (see [**4**, Example 4]), it is asked in [**4**, Question, p. 393] whether every morphic ring is clean. The ring $\mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$ is commutative strongly morphic by Corollary 15, but it is not clean because its image $\mathbb{Z}$ is not clean. This example answers the above question in the negative.

REMARK 17. A ring R is said to have *stable range* 1 if, whenever $aR + bR = R$ with $a, b \in R$, $a + by$ is a unit for some $y \in R$. In the Fourth China-Japan-Korea International Symposium on Ring Theory [June 24–28, 2004, Nanjing], Nicholson

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asked whether a morphic ring always has stable range 1. The answer to this question is "No". Let $R = \mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$. Then R is morphic by Theorem 14. Since $R/J(R) \cong \mathbb{Z}$ does not have stable range 1, R does not have stable range 1.

PROPOSITION 18. *If $S = R \propto V$ with R an integral domain and V a nonzero bimodule over R, then $S \propto S$ is not left morphic.*

Proof. Let $T = S \propto S$. Take $0 \neq v \in V$ and let $x = (0, v) \in S$. We show next that $(0, x)$ is not left morphic in T. Suppose that this is not the case. Then there exists $(b, c) \in T$ such that $\mathbf{l}(0, x) = T(b, c)$. Since $(0, 1) \in \mathbf{l}(0, x)$, $(0, 1) = (u, v)(b, c)$ with $(u, v) \in T$. It follows that

$$0 = ub \text{ and } 1 = uc + vb. \tag{1}$$

Write $u = (u_0, u_1)$, $v = (v_0, v_1)$, $b = (b_0, b_1)$ and $c = (c_0, c_1)$, where $u_0, v_0, b_0, c_0 \in R$ and $u_1, v_1, b_1, c_1 \in V$. It follows from (1) that

$$0 = u_0b_0 \text{ and } 1 = u_0c_0 + v_0b_0. \tag{2}$$

If $b_0 = 0$, then $1 = u_0c_0$, so that u_0 is a unit in R. Thus u is a unit in S, and so $b = 0$ by (1). Hence $(x, 0) \in \mathbf{l}(0, x) = T(0, c)$, showing that $x = 0$, a contradiction, so that $b_0 \neq 0$. Since R is an integral domain, it must be that $u_0 = 0$ and $1 = v_0b_0$, by (2). Hence b_0 is a unit in R. Now from $(b, c) \in \mathbf{l}(0, x)$, it follows that $bx = 0$ in S; i.e., $(b_0, b_1)(0, v) = 0$. We have $b_0v = 0$ and so $v = 0$, a contradiction. □

For a morphic ring R, $R \propto M$ may not be morphic (e.g., $R = \mathbb{Z}_4$ and $M = \mathbb{Z}_4$). On the other hand, $\mathbb{Z} \propto \mathbb{Q}/\mathbb{Z}$ is morphic, but $\mathbb{Z}$ is not. We have been unable to completely determine when $R \propto M$ is (left) morphic. But, the next result shows that R being left morphic just means that certain elements in $R \propto R$ are left morphic.

THEOREM 19. *Let R be a ring and $a \in R$. Then the following are equivalent.*
1. *$a \in R$ is left morphic.*
2. *$(a, 0) \in R \propto R$ is left morphic.*
3. *$(a, a) \in R \propto R$ is left morphic.*

Proof. Let $S = R \propto R$. Since $(a, a)(1, -1) = (a, 0)$ and $(1, -1)$ is a unit in S, we have that (2) ⇔ (3) by [**4**, Lemma 3].

(1) ⇒ (2). Since $a \in R$ is left morphic, there exists $b \in R$ such that $\mathbf{l}(a) = Rb$ and $\mathbf{l}(b) = Ra$. It can be verified that $\mathbf{l}(a, 0) = S(b, 0)$ and $\mathbf{l}(b, 0) = S(a, 0)$. Hence $(a, 0) \in S$ is left morphic.

(2) ⇒ (1). If $(a, 0) \in S$ is left morphic, then there exists $(b, c) \in S$ such that $\mathbf{l}(a, 0) = S(b, c)$ and $\mathbf{l}(b, c) = S(a, 0)$. It can be verified that $\mathbf{l}(a) = Rb$ and $\mathbf{l}(b) = Ra$, so that $a \in R$ is left morphic. □

PROPOSITION 20. *Let R be a ring and let $S = R \propto R$.*
1. *If $e^2 = e \in R$ and u is a unit of R, then $(0, eu), (0, ue) \in S$ are morphic.*
2. *If $(0, a) \in S$ is left morphic, then a is left morphic in R.*
3. *$\bar{2} \in \mathbb{Z}_4$ is morphic, but $(0, \bar{2})$ is not morphic in $\mathbb{Z}_4 \propto \mathbb{Z}_4$.*

Proof. (1). It can be verified that $S(0, e) = \mathbf{l}(1 - e, 1)$ and $S(1 - e, 1) = \mathbf{l}(0, e)$. Hence $(0, e) \in S$ is morphic. Then $(0, ue) = (u, 0)(0, e)$ and $(0, eu) = (0, e)(u, 0)$. Since $(u, 0)$ is a unit in S, $(0, ue)$ and $(0, eu)$ are morphic in S.

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(2). If $(0, a)$ is left morphic in S, there exists $(b, c) \in S$ such that $S(0, a) = \mathbf{l}(b, c)$ and $S(b, c) = \mathbf{l}(0, a)$. Then it can be verified that $Ra = \mathbf{l}(b)$ and $Rb = \mathbf{l}(a)$.

(3). Let $S = \mathbb{Z}_4 \propto \mathbb{Z}_4$. Suppose that $\mathbf{l}(0, \bar{2}) = S(a, b)$, where $(a, b) \in S$. Since $\mathbf{l}(0, \bar{2}) = 2\mathbb{Z}_4 \propto \mathbb{Z}_4$, we must have $a = \bar{2}$. Since $(0, \bar{1}) \in \mathbf{l}(0, \bar{2})$, there exists $(c, d) \in S$ such that $(0, \bar{1}) = (c, d)(\bar{2}, b) = (\bar{2}c, \bar{2}d + cb)$. Hence $\bar{2}c = 0$ and $\bar{1} = \bar{2}d + cb$. It follows that $\bar{2} = \bar{2}(\bar{2}d + cb) = 0$, a contradiction. □

Corollary 21. *Let R be a ring. If $R \propto R$ is left morphic, then so is R.*

The converse of Corollary 21 is not true since $\mathbb{Z}_4$ is morphic but $\mathbb{Z}_4 \propto \mathbb{Z}_4$ is not.

If $R = \prod R_i$, where each R_i is either strongly regular or semisimple, then the trivial extension $R \propto R = (\prod R_i) \propto (\prod R_i) \cong \prod(R_i \propto R_i)$. The last equality can be proved as in Lemma 4 and so, by Corollary 3 and Theorem 7, $R \propto R$ is morphic. Note that the ring R is unit regular; i.e., for any $a \in R$, $a = aua$, for some unit u of R.

If $R = End(V_D)$, where V is a vector space of countably infinite dimension over a division ring D, then R is regular, right self-injective, but not unit regular. So R is not one-sided morphic by [**4**, Proposition 5], and hence $R \propto R$ is not one-sided morphic by Corollary 21. Thus there exist regular, right self-injective rings R and strongly π-regular rings R (e.g., $R = \mathbb{Z}_4$) such that $R \propto R$ is not left morphic.

By Corollary 21, if R is regular and $R \propto R$ is left morphic, then R is left morphic and hence is unit regular. But we have been unable to answer the question whether R being unit regular always implies that the trivial extension $R \propto R$ is a morphic ring.

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