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ON THE EXISTENCE OF RINGS R WITH R ISOMORPHIC TO RFM(R)

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Abstract

We construct a class of associative rings with the property that, for each ring R in the class, $R \cong \mathrm{End}({}_R R^{(\mathbb{N})})$.

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There are many well-known examples of unital rings R such that $R \cong M_n(R)$ for some positive integer $n > 1$, where $M_n(R)$ denotes the $n \times n$ matrix ring over R; for instance, any ring without WIBN (see [2]) has this property. In general, such rings are constructed by first exhibiting an isomorphism between the left modules ${}_R R$ and ${}_R R^n$, and then inducing an isomorphism between the corresponding endomorphism rings: $R \cong \mathrm{End}({}_R R) \cong \mathrm{End}({}_R R^n) \cong M_n(R)$.

In this article we investigate an infinite analog of this phenomenon. Namely, we ask: do there exist unital rings R such that $R \cong \mathrm{End}({}_R R^{(\mathbb{N})})$? If RFM($R$) denotes the ring of $\mathbb{N} \times \mathbb{N}$ row-finite matrices over R, this question can be rephrased as follows: do there exist unital rings R such that $R \cong \mathrm{RFM}(R)$? We answer this question in the affirmative. Note, of course, that such an isomorphism cannot be induced by an isomorphism of the underlying modules, since ${}_R R$ and ${}_R R^{(\mathbb{N})}$ are never isomorphic (the former is finitely generated, whereas the latter is not).

The affirmative answer to the above question becomes somewhat more intriguing in light of the following "non-solution". It is perhaps tempting to claim that $\mathrm{RFM}(S) \cong \mathrm{RFM}(\mathrm{RFM}(S))$ for any unital S; if so, then $R = \mathrm{RFM}(S)$ would be a solution. However, these rings are NOT isomorphic in general. In fact, by [1] we have $\mathrm{RFM}(S) \cong \mathrm{RFM}(\mathrm{RFM}(S))$ if and only if S and $\mathrm{RFM}(S)$ are Morita equivalent, a property which obviously does not hold for arbitrary S. This observation implies that some modification of the results in [2, Section 6] is required.

PROPOSITION. *There exist unital rings R with $R \cong \mathrm{RFM}(R)$.*

PROOF. Let S be any unital ring. Then, as left S-modules, ${}_S(S^{(\mathbb{N})})^{(\mathbb{N})} \cong {}_S S^{(\mathbb{N})}$; this induces a ring isomorphism $\bar{\mu}$: $\mathrm{End}({}_S(S^{(\mathbb{N})})^{(\mathbb{N})}) \to \mathrm{End}({}_S S^{(\mathbb{N})})$. Using the usual matrix representation for these rings, we have the isomorphism μ: $\mathrm{RCM}(\mathrm{RFM}(S)) \to \mathrm{RFM}(S)$, where RCM denotes "row-convergent $\mathbb{N} \times \mathbb{N}$ matrices" (see [3, Theorem 106.1]). Let μ_1 denote the restriction of μ to $\mathrm{RFM}(\mathrm{RFM}(S))$, and let S_1 denote its image. Let μ_2 denote the restriction of μ_1 to $\mathrm{RFM}(S_1)$, and let S_2 denote its image. Continuing in this way, we produce the following diagram of isomorphisms and inclusion

$$\begin{array}{ccccccccc} \mathrm{RFM}(S) & \supseteq & S_1 & \supseteq & S_2 & \supseteq & S_3 & \supseteq & \cdots \\ \uparrow \mu & & \uparrow \mu_1 & & \uparrow \mu_2 & & \uparrow \mu_3 & & \\ \mathrm{RCM}(\mathrm{RFM}(S)) & \supseteq & \mathrm{RFM}(\mathrm{RFM}(S)) & \supseteq & \mathrm{RFM}(S_1) & \supseteq & \mathrm{RFM}(S_2) & \supseteq & \cdots \end{array}$$

Let T_1 denote $\bigcap_\omega \mathrm{RFM}(S_k)$, and let T_2 denote $\bigcap_\omega S_k$ (where $S_0 = S$). Note that each ring in the diagram contains the identity, whence T_1 and T_2 are unital.

It is easy to show that $T_1 = \mathrm{RFM}(T_2)$; that is, $\bigcap_\omega \mathrm{RFM}(S_k) = \mathrm{RFM}(\bigcap_\omega S_k)$. Now define ψ: $T_1 \to T_2$ by $\psi(X) = \mu_k(X)$ for any k. Then this map is well-defined (since the μ_k are defined by restriction), and it is easily shown to be a ring isomorphism by noting that all the μ_k are isomorphisms and that the horizontal sequences are ordered by inclusion. The construction is now completed by observing that $T_1 \cong T_2$, whence $\mathrm{RFM}(T_1) \cong \mathrm{RFM}(T_2) = T_1$.

At this point one may reasonably hope to obtain a more precise description of the subring T_2 of $\mathrm{RFM}(S)$ constructed above. Unfortunately, those coordinates in which elements of T_2 are allowed to have nonzero entries arise in a very unnatural, recursive way; such a coordinatewise description turns out to be quite unenlightening and is therefore omitted.

We conclude by noting a motivation for this construction. In [2, Section 6] Franzsen and Schultz investigate the following situation: let ${}_R M$ be free, let $\varepsilon = \varepsilon^2 \in \mathrm{End}({}_R M)$ with ${}_R M\varepsilon \cong {}_R R$, and let α be an automorphism of $\mathrm{End}({}_R M)$ such that ${}_R M\alpha(\varepsilon)$ is free, and of countably infinite rank. One may then conclude that $R \cong \mathrm{RFM}(R)$.

Using the aforementioned result of Camillo, we see that the example given in [2] of a ring R with $R \cong \mathrm{RFM}(R)$ is invalid; in particular, $\mathrm{RFM}(\mathbb{Z}) \ncong \mathrm{RFM}(\mathrm{RFM}(\mathbb{Z}))$. However, using the proof of the theorem in [1], one can show that if α is any automorphism of $\mathrm{End}({}_RM)$ with ${}_RM$ free, and if $\varepsilon = \varepsilon^2 \in \mathrm{End}({}_RM)$ with ${}_RM\varepsilon$ finitely generated, then ${}_RM\alpha(\varepsilon)$ is finitely generated as well. Thus the situation described above in fact cannot occur. This then renders moot the need for Franzsen and Schultz to produce a ring with $R \cong \mathrm{RFM}(R)$, while leaving unanswered the question of the existence of such rings; further, it demonstrates that the isomorphism constructed in this article cannot be induced by an automorphism of $\mathrm{RFM}(S)$.

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