A NOTE ON THE HARDY-HILLE AND MEHLER FORMULAS

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1. Let $L_n^{(\alpha)}(x)$ and $H_n(x)$ be the nth Laguerre and Hermite polynomials, respectively. Two well-known bilinear generating formulas are the Hardy-Hille formula [**1**, p. 101]

$$\sum_{n=0}^{\infty} \frac{n!}{(1+\alpha)_n} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) t^n = (1-t)^{-\alpha-1} e^{-t(x+y)/(1-t)} {}_0F_1\left(-;\ 1+\alpha;\ \frac{xyt}{(1-t)^2}\right) \tag{1.1}$$

and the Mehler formula [**1**, p. 377]

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{t^n}{n!} = (1-4t^2)^{-\frac{1}{2}} \exp\left\{-\frac{4t^2(x^2+y^2)}{1-4t^2} + \frac{4xyt}{1-4t^2}\right\}. \tag{1.2}$$

This suggests the following problem. Consider the equation

$$\sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n = f(t) e^{(x^k+y^k)a(t)} g\{xyc(t)\}, \tag{1.3}$$

where $f_n(x)$ is a polynomial in x of degree n with highest coefficient equal to 1,

$$a(t) = \sum_{n=k}^{\infty} a_n t^n, \quad c(t) = \sum_{n=1}^{\infty} c_n t^n, \tag{1.4}$$

$$f(t) = \sum_{n=0}^{\infty} A_n t^n, \quad g(t) = \sum_{0}^{\infty} B_n t^n, \tag{1.5}$$

$A_0 = B_0 = 1$. We shall also assume that $a_k = 1$ and $\gamma_0\gamma_1\gamma_2 \cdots \gamma_{k-1} \neq 0$. We seek all sets of polynomials $\{f_n(x)\}$ which satisfy (1.3), (1.4) and (1.5).

We shall prove the following

THEOREM. *The only solution of the functional equation* (1.3), *such that* (1.4) *and* (1.5) *hold, is given by*

$$f_{s+nk}(x) = n! A^n x^s L_n^{(\alpha+2s/k)}(x^k/A) \quad (s = 0, 1, \ldots, k-1),$$

$$f(t) = (1+At^k)^{-\alpha-1},$$

$$c(t) = c_1 t(1+At^k)^{-2/k},$$

$$a(t) = \frac{t^k}{1+At^k}$$

and

$$g(t) = \sum_{s=0}^{k-1} \frac{\gamma_s t^s}{c_1^s} {}_0F_1\left(-;\ \alpha+1+2s/k;\ -\frac{t^k}{Ac_1^k}\right),$$

where α, A *are arbitrary constants.*

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2. Proof of the theorem. If we replace y by $1/y$ and t by ty in (1.3) and then put $y = 0$, we get

$$\sum_0^\infty \gamma_n f_n(x) t^n = e^{t^k} g(xtc_1). \tag{2.1}$$

This in the same way leads to

$$\sum_0^\infty \gamma_n t^n = g(c_1 t). \tag{2.2}$$

Formulas (2.1) and (2.2) give

$$\gamma_n f_n(x) = \sum_r \frac{\gamma_{n-kr}}{r!} x^{n-kr}. \tag{2.3}$$

By differentiating (2.3) s times $(0 \leqq s < k)$, we see that

$$\gamma_{nk+s} f^{(s)}_{nk+s}(0) = \frac{s!\,\gamma_s}{n!} \quad (0 \leqq s < k), \tag{2.4}$$

so that $\gamma_{nk+s} \neq 0$ for $s = 0, 1, \ldots, k-1$. This obviously implies that $\gamma_n \neq 0$ for all n.

Putting $y = 0$ in (1.3) we get

$$\sum_{n=0}^\infty f_{kn}(x) \frac{t^{nk}}{n!} = f(t) e^{x^k a(t)}, \tag{2.5}$$

which yields, on putting $x = 0$,

$$\sum_{n=0}^\infty \frac{t^{kn}}{\gamma_{kn}(n!)^2} = f(t). \tag{2.6}$$

From (2.6) we get

$$tf'(t) = k \sum_{n=0}^\infty \frac{t^{k(n+1)}}{n!(n+1)!\,\gamma_{k(n+1)}}. \tag{2.7}$$

On the other hand, if we differentiate (2.5) k times with respect to x, we get

$$\sum_{n=0}^\infty f^{(k)}_{kn}(0) \frac{t^{kn}}{n!} = k!\, a(t) f(t). \tag{2.8}$$

But we have from (2.3)

$$f^{(k)}_{kn}(0) = \frac{k!\,\gamma_k}{(n-1)!\,\gamma_{kn}} \quad (n \geqq 1),$$

so that (2.8) becomes

$$a(t) f(t) = \gamma_k \sum_{n=0}^\infty \frac{t^{k(n+1)}}{n!\,(n+1)!\,\gamma_{k(n+1)}}. \tag{2.9}$$

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Comparing (2.9) with (2.7) we get

$$\gamma_k t f'(t) = k a(t) f(t). \tag{2.10}$$

In the same way, we obtain from (2.5),

$$\sum_0^\infty f_{kn}^{(2k)}(0) \frac{t^{kn}}{n!} = \frac{(2k)!}{2} (a(t))^2 f(t), \tag{2.11}$$

which is rewritten as

$$\{a(t)\}^2 f(t) = 2\gamma_{2k} \sum_{n=0}^\infty \frac{t^{k(n+2)}}{n!\,(n+2)!\,\gamma_{k(n+2)}}. \tag{2.12}$$

On the other hand, we see from (2.6) that

$$t^{k+1}\{t^{1-k} f'(t)\}' = k^2 \sum_0^\infty \frac{t^{n(k+2)}}{n!\,(n+1)!\,\gamma_{k(n+2)}}. \tag{2.13}$$

Now (2.12) and (2.13) give

$$\gamma_k^2 t \{f'(t)\}^2 = 2\gamma_{2k}\{t f''(t) - (k-1) f'(t)\} f(t). \tag{2.14}$$

Hence

$$f(t) = (1 + At^k)^{-\alpha - 1}, \tag{2.15}$$

where A and α are constants.

From (2.15) and (2.10) we get

$$a(t) = \frac{t^k}{1 + At^k}. \tag{2.16}$$

Let us next differentiate (1.3) with respect to y and put $y = 0$. We get

$$c_1 \sum_0^\infty f_{nk+1}(x) \frac{t^{nk+1}}{n!} = x f(t) c(t) e^{x^k a(t)}. \tag{2.17}$$

Now if we differentiate (2.17) once with respect to x and put $x = 0$, we obtain

$$f(t) c(t) = \gamma_1 c_1 \sum_{n=0}^\infty \frac{t^{nk+1}}{(n!)^2 \gamma_{nk+1}}; \tag{2.18}$$

on the other hand, if we differentiate (2.17) $k+1$ times with respect to x and put $x = 0$, we get

$$f(t) c(t) a(t) = c_1 \gamma_{k+1} \sum_{n=0}^\infty \frac{t^{1+k(n+1)}}{n!\,(n+1)!\,\gamma_{1+k(n+1)}}. \tag{2.19}$$

Comparing (2.19) and (2.18) we get

$$t^2 \left\{\frac{f(t)c(t)}{t}\right\}' = \frac{k\gamma_1}{\gamma_{k+1}} f(t) c(t) a(t).$$

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Hence

$$c(t) = c_1 t(1+At^k)^{\mu}, \tag{2.20}$$

where μ is a constant.

If we now differentiate (2.9) with respect to t, we get

$$t\{a'(t)f(t)+a(t)f'(t)\} = k\gamma_k \sum_0^{\infty} \frac{t^{k(n+1)}}{(n!)^2 \gamma_{k(n+1)}}. \tag{2.21}$$

If we take the kth derivative with respect to x and with respect to y and then put $x = y = 0$, we get

$$c_1^k \gamma_k \sum_{n=0}^{\infty} \frac{t^{k(n+1)}}{\gamma_{k(n+1)}(n!)^2} = f(t)\{c(t)\}^k + \frac{c_1^k}{\gamma_k}\{a(t)\}^2 f(t).$$

Comparing this formula with (2.21), we obtain

$$\gamma_k c_1^k t\{a'(t)f(t)+a(t)f'(t)\} = k\gamma_k f(t)\{c(t)\}^k + kc_1^k\{a(t)\}^2 f(t).$$

This, together with (2.10), yields

$$c_1^k t a'(t) = k\{c(t)\}^k. \tag{2.22}$$

Formulas (2.22), (2.20) and (2.16) require that

$$\mu k = -2. \tag{2.23}$$

To determine $g(t)$ we differentiate (1.3) s times with respect to y and put $y = 0$ to get

$$c_1^s \sum_0^{\infty} f_{s+kn}(x) \frac{t^{s+kn}}{n!} = x^s f(t)\{c(t)\}^s e^{x^k a(t)} \quad (0 \leqq s < k),$$

which itself leads to

$$\gamma_s c_1^s \sum_0^{\infty} \frac{t^{s+kn}}{\gamma_{s+kn}(n!)^2} = f(t)\{c(t)\}^s. \tag{2.24}$$

Comparing coefficients of t^{s+kn} we get

$$\gamma_{s+kn} = \frac{(-1)^n A^{-n} \gamma_s}{(\alpha+1+2s/k)_n} \quad (0 \leqq s < k). \tag{2.25}$$

Consequently we obtain from (2.2) and (2.26)

$$g(t) = \sum_{s=0}^{k-1} \frac{\gamma_s}{c_1^s} t^s {}_0F_1\left(-;\ \alpha+1+2s/k; \frac{t^k}{c_1^k A}\right). \tag{2.26}$$

Putting (2.26), (2.24), (2.20), (2.16), (2.15) in (1.3) we get

$$\sum \gamma_n f_n(x) f_n(y) t^n$$

$$= (1+At^k)^{-\alpha-1} \exp\left\{\frac{(x^k+y^k)t^k}{1+At^k}\right\} \sum_{s=0}^{k-1} \frac{\gamma_s x^s y^s t^s}{(1+At^k)^{2s/k}} {}_0F_1\left(-;\ \alpha+1+2s/k;\ -\frac{x^k y^k t^k}{A(1+At^k)^2}\right). \tag{2.27}$$

Comparing (2.27) with (1.1) we see that

$$f_{s+kn}(x) = n!\, A^n x^s L_n^{(\alpha+2s/k)}(x^k/A) \quad (0 \leqq s \leqq k-1). \tag{2.28}$$

Note that $\gamma_0 = 1$, $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are arbitrary.

This completes the proof of the theorem.

If $k = 1$ we see that the only solution of the functional equation

$$\sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n = f(t) e^{(x+y)a(t)} g\{xyc(t)\},$$

where $f(t)$, $a(t)$, $g(t)$, $c(t)$ are defined as before, is essentially (1.1).

In case $k = 2$ we see that the solution of the functional equation

$$\sum_{n=0}^{\infty} \gamma_n f_n(x) f_n(y) t^n = f(t) e^{(x^2+y^2)a(t)} g\{xyc(t)\} \tag{2.29}$$

is obtained from (1.1) in the following manner:

Denote the right-hand member of (1.1) by $F_\alpha(x, y, t)$. We then exhibit the general solution of (2.29) by replacing the right member by

$$F_\alpha(x^2y^2t^2) + BxytF_{\alpha+1}(x^2y^2t^2),$$

where B is a non-zero constant.

The special case $\alpha = -\frac{1}{2}$ leads to the Mehler formula (1.2). However it is not necessary to assume $\alpha = -\frac{1}{2}$.

We remark that

$$f_{2n}(x) = L_n^{(\alpha)}(x^2)$$

and

$$f_{2n+1}(x) = xL_n^{(\alpha+1)}(x^2)$$

with α arbitrary.

3. Remarks and generalization. It may be of interest to examine the more general problem of solving the functional equation

$$\sum \gamma_n f_n(x) g_n(x) t^n = f(t) e^{x^k a(t) + y^k b(t)} g\{xyc(t)\}, \tag{3.1}$$

where $f_n(x)$, $g_n(x)$ are polynomials of exact degree n and highest coefficient equal to 1. Hence we require

$$a_0 = a_1 = \ldots = a_{k-1} = 0, \quad b_0 = b_1 = \ldots = b_{k-1} = 0.$$

If $a_k = 0$ we obtain from (3.1)

$$\sum \gamma_n f_n(x) t^n = g\{xc_1 t\},$$

which leads to

$$\sum \gamma_n t^n = g(c_1 t).$$

Hence

$$\sum \gamma_n x^n t^n = \sum \gamma_n f_n(x) t^n.$$

We thus conclude that, if $a_k = 0$, then

$$f_n = x^n.$$

Similar remarks apply in case $b_k = 0$. Since this solution is trivial, we assume that

$$a_k = b_k = 1. \tag{3.2}$$

With (3.2) in mind we get from formula (3.1)

$$\sum \gamma_n f_n(x) t^n = e^{t^k} g(xtc_1)$$

and

$$\sum \gamma_n g_n(y) t^n = e^{t^k} g(ytc_1),$$

which clearly shows that

$$f_n(x) = g_n(x).$$

Hence the problem is reduced to the previous problem which was treated in § 2.

REFERENCE

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