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INTERSECTION THEORY FOR TWISTED COHOMOLOGIES AND TWISTED RIEMANN'S PERIOD RELATIONS I

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To the memory of Professor Michitake Kita

Introduction

The beta function $B(\alpha, \beta)$ is defined by the following integral

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,$$

where $\arg t = \arg(1-t) = 0$, $\Re\alpha$, $\Re\beta > 0$, and the gamma function $\Gamma(\alpha)$ by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,$$

where $\arg t = 0$, $\Re\alpha > 0$. By the use of the well known formulae

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha},$$

we get the following formula:

$$B(\alpha, \beta)B(-\alpha, -\beta) = 2\pi i\Big(\frac{1}{\alpha} + \frac{1}{\beta}\Big)\Big(-\frac{\exp(2\pi i(\alpha+\beta)) - 1}{(\exp(2\pi i\alpha) - 1)(\exp(2\pi i\beta) - 1)}\Big).$$

If we regard the interval (0,1) of integration as a twisted cycle defined by the multi-valued function $t^\alpha(1-t)^\beta$, the factor

$$-\frac{\exp(2\pi i(\alpha+\beta)) - 1}{(\exp(2\pi i\alpha) - 1)(\exp(2\pi i\beta) - 1)}$$

is nothing but the twisted self-intersection number ([KY1]) of the cycle (0,1). It is quite natural to think that the factor

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$$2\pi i\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$$

should be the "twisted self-intersection number" of the 1-form

$$\frac{dt}{t}+\frac{dt}{1-t},$$

so that the above formula should be thought of a twisted version of Riemann's period relation.

This paper establishes the intersection theory for twisted cocycles and the twisted Riemann's period relation connecting the intersection theories for twisted cycles [KY1] and for twisted cocycles.

In the following we explain the results of this paper using as plain language as possible; the notion and notation used are rigorously fixed in the text. Let $x_0, \ldots, x_n$ be $n+1$ distinct points on $\boldsymbol{P}^1$, and

$$\omega=\sum_{j=0}^{n}\alpha_j\frac{dt}{t-x_j}, \quad \left(\sum_{j=0}^{n}\alpha_j=0,\ \alpha_j\notin \mathbf{N}-\{0\}\right)$$

a connection form. The first twisted cohomology group

$$H^1(U, L)\simeq \mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D), \nabla)),\ U:=\boldsymbol{P}^1-D$$

with respect to the connection $\nabla = d+\omega\wedge$ is known to be isomorphic to

$$\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega, \quad D:=x_0+\cdots+x_n,$$

where

$$L:=\ker(\nabla|_U:\mathcal{O}_U\to\Omega^1_{\boldsymbol{P}^1}(\log D)|_U)$$

is a local system on U defined by ∇.

The dual of the cohomology group $H^1(U, L)$ is given by the cohomology group with compact support $H^1_c(U, L^\vee)$, where $L^\vee$ is the local system defined by the connection $\nabla^\vee := d-\omega\wedge$ dual to ∇. We show that the dual cohomology group is isomorphic to $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$. Since there is a natural dual pairing between the two cohomology groups $H^1(U, L)$ and $H^1_c(U, L^\vee)$, there should exist the induced bilinear form on the spaces $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega$ and $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$. By using elements

$$\varphi_j=\frac{dt}{t-x_j}-\frac{dt}{t-x_{j+1}}\in\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)), \quad 1\le j\le n-1,$$

we give bases for the spaces above by

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$\varphi_j^+ \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega$, $\varphi_j^- \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$, $1 \leq j \leq n-1$,

where φ_j^+ and φ_j^- are the images of φ_j by the natural projections from $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$. Our first main theorem gives explicitly the bilinear form, which turns out to be symmetric and will be called the *intersection form*:

$$\langle \varphi_j^+, \varphi_j^- \rangle = 2\pi i\left(\frac{1}{\alpha_j} + \frac{1}{\alpha_{j+1}}\right),$$

$$\langle \varphi_j^+, \varphi_{j+1}^- \rangle = \langle \varphi_{j+1}^+, \varphi_j^- \rangle = -\frac{2\pi i}{\alpha_{j+1}},$$

$$\langle \varphi_j^+, \varphi_k^- \rangle = 0 \quad \text{if } |j-k| \geq 2.$$

Our second main theorem states the relation between the three pairings: the intersection form for twisted cohomologies, that for twisted homologies, and the pairing of twisted homologies and twisted cohomologies, i.e. integrals. Let

$$\gamma_j^+ \in H_1(U, L^\vee), \quad \delta_j^- \in H_1(U, L), \quad j = 1, \ldots, n-1$$

be any bases of twisted cycles (the notation is slightly different from that in [KY1]) and

$$\xi_j^+ \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega, \quad j = 1, \ldots, n-1,$$
$$\eta_j^- \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot(-\omega), \quad j = 1, \ldots, n-1,$$

be any bases of twisted cocycles; let I_h and I_{ch} be the intersection matrices:

$$I_h = \begin{pmatrix} \langle \gamma_1^+, \delta_1^- \rangle & \cdots & \langle \gamma_1^+, \delta_{n-1}^- \rangle \\ \vdots & & \vdots \\ \langle \gamma_{n-1}^+, \delta_1^- \rangle & \cdots & \langle \gamma_{n-1}^+, \delta_{n-1}^- \rangle \end{pmatrix}, \quad I_{ch} = \begin{pmatrix} \langle \xi_1^+, \eta_1^- \rangle & \cdots & \langle \xi_1^+, \eta_{n-1}^- \rangle \\ \vdots & & \vdots \\ \langle \xi_{n-1}^+, \eta_1^- \rangle & \cdots & \langle \xi_{n-1}^+, \eta_{n-1}^- \rangle \end{pmatrix}.$$

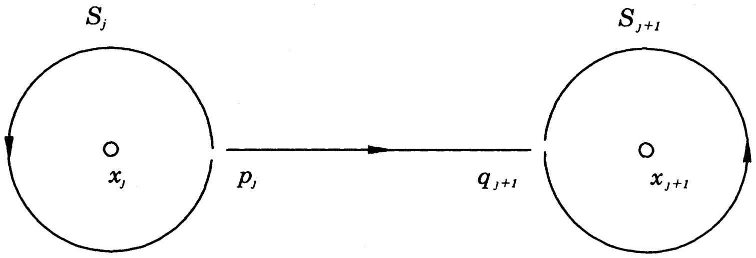
The intersection matrix I_h can be explicitly computed [KY1]; take for instance bases γ_j^+ and $\delta_j^- := \varphi_j^-$ as follows: let us assume for simplicity that the x_j's are all real and are arranged as $x_0 < x_1 < \cdots < x_n$, and u_0 a branch of the multi-valued function $u = \Pi(t - x_j)^{\alpha_j}$ defined on the lower half t-plane. We define special cycles by

$$\gamma_j^+ = \overrightarrow{(p_j, q_{j+1})} \otimes u_0 + \frac{1}{c_j - 1} S_j \otimes u_0 - \frac{1}{c_{j+1} - 1} S_{j+1} \otimes u_0,$$

$$\gamma_j^- = \overrightarrow{(p_j, q_{j+1})} \otimes u_0^{-1} - \frac{c_j}{c_j - 1} S_j \otimes u_0^{-1} + \frac{c_{j+1}}{c_{j+1} - 1} S_{j+1} \otimes u_0^{-1}, \; c_j = \exp 2\pi i\alpha_j,$$

where S_k is a positively oriented circle with center x_k and with starting point p_k

or q_k; see Figure.



Figure

Then the intersection matrix for these special bases turns out to be

$$I_h = \begin{pmatrix} -d_{12}/d_1d_2 & 1/d_2 & 0 & \cdots & 0 & 0 \\ c_2/d_2 & -d_{23}/d_2d_3 & \cdots & & 0 & 0 \\ 0 & \vdots & & & & \vdots \\ \vdots & & & & \vdots & 0 \\ 0 & 0 & & \cdots & -d_{n-2,n-1}/d_{n-2}d_{n-1} & 1/d_{n-1} \\ 0 & 0 & \cdots & 0 & c_{n-1}/d_{n-1} & -d_{n-1,n}/d_{n-1}d_n \end{pmatrix},$$

where $d_j = c_j - 1$, $d_{jk} = c_jc_k - 1$. It is easy to see that

$${}^tI_h^{-1} = \frac{-1}{d_{1\cdots n}} \begin{pmatrix} d_1d_{2\cdots n} & d_1c_2d_{3\cdots n} & d_1c_{23}d_{4\cdots n} & \cdots & d_1c_{2\cdots n-1}d_n \\ d_1d_{3\cdots n} & d_{12}d_{3\cdots n} & d_{12}c_3d_{4\cdots n} & \cdots & d_{12}c_{3\cdots n-1}d_n \\ d_1d_{4\cdots n} & d_{12}d_{4\cdots n} & d_{123}d_{4\cdots n} & \cdots & d_{123}c_{4\cdots n-1}d_n \\ \vdots & \vdots & \vdots & & \vdots \\ d_1d_n & d_{12}d_n & d_{123}d_n & \cdots & d_{1\cdots n-1}d_n \end{pmatrix},$$

where $c_{jk\cdots} = c_jc_k\cdots$, $d_{jk\cdots} = c_jc_k\cdots - 1$. Let us arrange the integrals (periods) as follows:

$$P^+ = \begin{pmatrix} \int_{\gamma_1^+} \xi_1^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_1^+ \\ \vdots & & \vdots \\ \int_{\gamma_1^+} \xi_{n-1}^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_{n-1}^+ \end{pmatrix}, \quad P^- = \begin{pmatrix} \int_{\delta_1^-} \eta_1^- & \cdots & \int_{\delta_{n-1}^-} \eta_1^- \\ \vdots & & \vdots \\ \int_{\delta_1^-} \eta_{n-1}^- & \cdots & \int_{\delta_{n-1}^-} \eta_{n-1}^- \end{pmatrix}.$$

Here the integral $\int_{\gamma_1^+} \xi^+$ (resp. $\int_{\delta^-} \eta^-$) of a twisted cocycle ξ^+ (resp. η^-) over a twisted cycle $\gamma^+ \in H_1(U, L^\vee)$ (resp. $\delta^- \in H_1(U, L)$) is defined as follows: for a

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twisted cocycle ξ^+ (resp. η^-) take a representing form ξ (resp. η) of $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$ and for a twisted cycle γ^+ (resp. δ^-) take a representing twisted chain $\sum_i g_i \otimes u_i$ (resp. $\sum_i g_i' \otimes u_i^{-1}$), where g_i (resp. g_i') is a topological chain and u_i (resp. u^{-1}) is a branch of the multi-valued function

$$u = \prod_{j=0}^{n} (t - x_j)^{\alpha_j} \quad (\text{resp. } u^{-1})$$

along g_i (resp. g_i'); then

$$\int_{\gamma^+} \xi^+ := \sum_i \int_{g_i} u_i \xi, \quad \int_{\delta^-} \eta^- := \sum_i \int_{g_i'} u_i^{-1} \eta,$$

which are independent of the choice of representatives. Our theorem reads

$$P^+ \, {}^t I_h^{-1} \, {}^t P^- = I_{ch}, \quad \text{i.e.} \quad {}^t P^- \, I_{ch}^{-1} \, p^+ = {}^t I_h.$$

We would like to call these identities twisted Riemann's period relations because it resembles Riemann's period relation for a basis of holomorphic 1-forms $\omega_1, \ldots, \omega_g$ and a $\boldsymbol{Z}$-basis of cycles $\gamma_1, \ldots, \gamma_{2g}$ on a compact Riemann surface of genus g. The period matrix P and the intersection matrix I_h of cycles are

$$P = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}, \quad I_h = \begin{pmatrix} \langle \gamma_1, \gamma_1 \rangle & \cdots & \langle \gamma_1, \gamma_{2g} \rangle \\ \vdots & & \vdots \\ \langle \gamma_{2g}, \gamma_1 \rangle & \cdots & \langle \gamma_{2g}, \gamma_{2g} \rangle \end{pmatrix};$$

then Riemann's period relations are given as follows:

$$\begin{pmatrix} P \\ \bar{P} \end{pmatrix} {}^t I_h^{-1} \, ({}^t P \, {}^t \bar{P}) = \begin{pmatrix} \int \omega_j \wedge \omega_k & \int \omega_j \wedge \bar{\omega}_k \\ \int \bar{\omega}_j \wedge \omega_k & \int \bar{\omega}_j \wedge \bar{\omega}_k \end{pmatrix} = i \begin{pmatrix} 0 & H \\ -\bar{H} & 0 \end{pmatrix},$$

where H is positive definite. We remarked it not only because of the resemblance but also because we shall in [Cho1] establish a theory including both Riemann's period relations.

The simplest case, i.e. $n = 2$ is nothing but the formulae for $B(\alpha, \beta)B(-\alpha, -\beta)$ given in the beginning; the next simplest case, i.e. $n = 3$ yields (§4 Example 1) the famous formula

$$\begin{aligned} &F(\alpha, \beta, \gamma; x)F(1-\alpha, 1-\beta, 2-\gamma; x) \\ &\qquad = F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; x)F(\gamma-\alpha, \gamma-\beta, \gamma; x), \end{aligned}$$

where

$$F(\alpha, \beta, \gamma ; x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} x^n \quad (\alpha)_n := \alpha(\alpha+1) \cdots (\alpha+n-1).$$

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§1. Preliminaries

In the following, notation is so chosen that generalizations to Riemann surfaces of higher genus [Cho1] and to varieties of higher dimension [Cho2] would be smooth. Let $x_0, \ldots, x_n$ be $n+1$ distinct points on $\boldsymbol{P}^1$; put

$$D := x_0 + \cdots + x_n, \quad U := \boldsymbol{P}^1 - D, \quad j : U \hookrightarrow \boldsymbol{P}^1.$$

Let ω be a logarithmic 1-form on $\boldsymbol{P}^1$ with poles at D with residue α_j at x_j; note that

$$\sum_{j=0}^{n} \alpha_j = 0.$$

Consider the connection ∇ with connection form ω:

$$\nabla = d + \omega \wedge : \mathcal{O}_{\boldsymbol{P}^1} \to \Omega^1_{\boldsymbol{P}^1} (\log D),$$

where $\mathcal{O}_{\boldsymbol{P}^1}$ is the sheaf of holomorphic functions on $\boldsymbol{P}^1$, $\Omega^1_{\boldsymbol{P}^1}$ the sheaf of holomorphic 1-forms on $\boldsymbol{P}^1$ and $\Omega^1_{\boldsymbol{P}^1} (\log D)$ the sheaf of meromorphic 1-forms with logarithmic singularities only on D. Let L be a local system on U defined by

$$L := \ker(\nabla|_U : \mathcal{O}_U \to \Omega^1_{\boldsymbol{P}^1} (\log D)|_U),$$

where $\mathcal{O}_U$ is the sheaf of holomorphic functions on U.

We are going to present several isomorphisms for two hypercohomologies; they shall be made explicit in the next section; the definition of hypercohomology shall be also given in §2.2. If $\alpha_j \notin \mathbf{N} - \{0\}$ then the following quasi-isomorphism [Del1] holds

$$\boldsymbol{R}j_* L \underset{qis}{\simeq} (\Omega^{\cdot}(\log D), \nabla)$$
$$:= \cdots 0 \to \mathcal{O}_{\boldsymbol{P}^1} \xrightarrow{\nabla} \Omega^1_{\boldsymbol{P}^1} (\log D) \to 0 \cdots,$$

which leads to

$$H^1(U, L) \simeq \mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D), \nabla))$$
$$\simeq \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))/\mathbf{C}\cdot\omega,$$

where the last isomorphism is derived by the (Hodge-to-logarithmic de Rham) spectral sequence:

$$E_1^{pq} \simeq H^q(\boldsymbol{P}^1, \Omega^p(\log D)) \Rightarrow \mathbf{H}^{p+q}(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D), \nabla)),$$

and $E_1^{pq} = 0$ if $q > 0$.

On the other hand by the Poincaré-Verdier duality [EV1], (i.e. by performing $\boldsymbol{R}\mathcal{H}om(\cdot, \mathbf{C}_{\boldsymbol{P}^1})$) we have:

$$j_! L^\vee \underset{qis}{\simeq} (\Omega^{\cdot}(\log D)(-D), \nabla^\vee)$$
$$:= \cdots 0 \to \mathcal{O}_{\boldsymbol{P}^1}(-D) \overset{\nabla^\vee}{\to} \Omega^1_{\boldsymbol{P}^1}(\log D)(-D) \simeq \Omega^1_{\boldsymbol{P}^1} \to 0 \cdots,$$

where $\nabla^\vee := d - \omega$, and $!$ means the zero-extension; this leads to

$$H_c^1(U, L^\vee) \simeq \mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D)(-D), \nabla^\vee))$$
$$\simeq \ker(\nabla^\vee : H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \to H^1(P^1, \Omega^1_{\boldsymbol{P}^1}))$$
$$= \ker(-\omega : H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \to H^1(P^1, \Omega^1_{\boldsymbol{P}^1})),$$

where H_c means cohomology with compact support, and the second isomorphism is derived by the spectral sequence:

$$E_1^{pq} = H^q(\boldsymbol{P}^1, \Omega^p(\log D)(-D)) \Rightarrow \mathbf{H}^{p+q}(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D)(-D), \nabla^\vee)),$$

and $E_1^{pq} = 0$ if $q = 0$. Notice that the duality between $(\Omega^{\cdot}(\log D), \nabla)$ and $(\Omega^{\cdot}(\log D)(-D), \nabla^\vee)$ holds without any condition for α_j [EV2]. Notice also that the duality above between $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)/\mathbf{C}\cdot\omega$ and $\ker(-\omega : H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \to H^1(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1}))$ is induced by the Serre duality. We denote by φ^+ (resp. φ^-) the image of $\varphi \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$ under the natural projection to $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)/\mathbf{C}\cdot\omega$ (resp. $\Gamma(\boldsymbol{P}^1, \Omega^1(\log D)/\mathbf{C}\cdot(-\omega))$.

§2. Intersection theory for twisted cocycles

Consider the following exact sequence of complexes, which will be referred to as the *basic sequence*:

$$0 \to (\Omega^{\cdot}(\log D)(-D), \nabla^\vee) \overset{\iota}{\to} (\Omega^{\cdot}(\log D), \nabla^\vee) \to (\oplus_{j=0}^n \mathbf{C}_{x_j} \overset{\times \mathrm{res}}{\to} \oplus_{j=0}^n \mathbf{C}_{x_j}) \to 0;$$

that is

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$$(1)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\iota} & \mathcal{O} & \longrightarrow & \bigoplus_{j=0}^{n}\mathbf{C}_{x_j} & \longrightarrow & 0 \\ & & {\scriptstyle \nabla^{\vee}}\downarrow & & {\scriptstyle \nabla^{\vee}}\downarrow & & \downarrow{\scriptstyle \times\mathrm{res}} & & \\ 0 & \longrightarrow & \Omega^1 & \xrightarrow{\iota} & \Omega^1(\log D) & \xrightarrow{\mathrm{Res}} & \bigoplus_{j=0}^{n}\mathbf{C}_{x_j} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where

$$\times\ \mathrm{res}:(c_0,\dots,c_n)\to(-\alpha_0c_0,\dots,-\alpha_nc_n).$$

If $\alpha_j\neq 0$ then $\times$ res is isomorphic, so we have the following isomorphism

$$\iota:\mathbf{H}^{\cdot}(P^1,(\Omega^{\cdot}(\log D)(-D),\ \nabla^{\vee}))\ \tilde{\to}\ \mathbf{H}^{\cdot}(P^1,(\Omega^{\cdot}(\log D),\ \nabla^{\vee})),$$

in particular,

$$\iota:\ker(-\omega:H^1(\boldsymbol{P}^1,\mathcal{O}_{\boldsymbol{P}^1}(-D))\to H^1(\boldsymbol{P}^1,\Omega^1_{\boldsymbol{P}^1}))\ \tilde{\to}\ \Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot(-\omega).$$

We shall explicitly give the inverse of the isomorphism ι. We first define a homomorphism: $\tau:\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega)\to\ker(-\omega:H^1(\mathcal{O}(-D))\to H^1(\Omega^1))$ and secondly prove that this gives the inverse of the natural isomorphism ι.

§2.1. Definition of τ

The corresponding long exact sequences of (1) read

$$\begin{array}{ccccccc} \longrightarrow & H^0(\mathcal{O}) & \longrightarrow & \bigoplus_{j=0}^{n}\mathbf{C}_{x_j} & \xrightarrow{\delta} & H^1(\mathcal{O}(-D)) \\ & & & \downarrow{\scriptstyle \times\mathrm{res}} & & \\ \longrightarrow & H^0(\Omega^1(\log D)) & \xrightarrow{\mathrm{Res}} & \bigoplus_{j=0}^{n}\mathbf{C}_{x_j} & \longrightarrow & H^1(\Omega^1) \end{array}$$

where δ is the connecting homomorphism. Tracing the above commutative diagram, we have

$$\delta\circ(\times\ \mathrm{res})^{-1}\circ\mathrm{Res}:H^0(\Omega^1(\log D))\to H^1(\mathcal{O}(-D))\ ;$$

it is immediate that this induces the isomorphism

$$\tau:\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega)\to\ker(-\omega:H^1(\mathcal{O}(-D))\to H^1(\Omega^1)).$$

§2.2. Naturality of τ

LEMMA. $\tau = \iota^{-1}$.

Proof. Let us honestly see the homomorphism ι, i.e. following the definition of hypercohomologies. A fine resolution of the complex $(\Omega^{\cdot}(\log D)(-D), \nabla^{\vee})$ is given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\nabla^{\vee}} & \Omega^1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00}(-D) & \xrightarrow{\partial-\omega} & \mathcal{E}^{10} & \longrightarrow & 0 \\ & & \bar{\partial}\downarrow & & \downarrow\bar{\partial} & & \\ 0 & \longrightarrow & \mathcal{E}^{01}(-D) & \xrightarrow{\partial-\omega} & \mathcal{E}^{11} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where $\mathcal{E}^{pq}$ stands for the sheaf of smooth (p, q)-forms on $\boldsymbol{P}^1$ and $\mathcal{E}^{pq}(-D)$ the sheaf of (p, q)-forms g on $\boldsymbol{P}^1$ such that g/t_j is smooth for a local parameter t_j around x_j. The associated single complex is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{\nabla^{\vee}} & \Omega^1 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00}(-D) & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{01}(-D) \oplus \mathcal{E}^{10} & \xrightarrow{\nabla^{\vee}} & \mathcal{E}^{11} & \longrightarrow & 0. \end{array}$$

Thus we have

$$\mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D)(-D), \nabla^{\vee})) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}(-D)) \oplus \Gamma(\mathcal{E}^{10}) \to \Gamma(\mathcal{E}^{11})\}}{\nabla^{\vee}\Gamma(\mathcal{E}^{00}(-D))};$$

for $\eta \in \Gamma(\boldsymbol{P}^1, \Omega^1(\log D))$, we denote by $\eta^{\vee}$ the image of $\eta^{-} \in \Gamma(\Omega^1(\log D))/\mathbf{C} \cdot (-\omega)$ under τ. Since the Dolbeault resolution implies

$$H^1(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(-D)) \simeq \frac{\Gamma(\mathcal{E}^{01}(-D))}{\bar{\partial}\Gamma(\mathcal{E}^{00}(-D))}, \quad H^1(\boldsymbol{P}^1, \Omega^1_{\boldsymbol{P}^1})) \simeq \frac{\Gamma(\mathcal{E}^{11})}{\bar{\partial}\Gamma(\mathcal{E}^{10})},$$

$\nabla^{\vee} = d - \omega$ annihilates $\eta^{\vee}$ means that there exists $\mu \in \Gamma(\mathcal{E}^{10})$ such that

$$(d-\omega)\eta^{\vee} = \bar{\partial}\mu,$$

namely,

$$\nabla^{\vee}(\eta^{\vee} + \mu) = 0.$$

This gives an explicit expression of the isomorphism

$$\ker(\nabla^\vee : H^1(\mathcal{O}(-D)) \to H^1(\Omega^1)) \xrightarrow{\sim} \mathbf{H}^1((\Omega^\cdot(\log D)(-D), \nabla^\vee)) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}(-D)) \oplus \Gamma(\mathcal{E}^{10}) \to \Gamma(\mathcal{E}^{11})\}}{\nabla^\vee \Gamma(\mathcal{E}^{00}(-D))}.$$

Similarly a single fine resolution of $(\Omega^\cdot(\log D), \nabla^\vee)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \xrightarrow{\nabla^\vee} & \Omega^1(\log D) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00} & \xrightarrow{\nabla^\vee} & \mathcal{E}^{01} \oplus \mathcal{E}^{10}(\log D) & \xrightarrow{\nabla^\vee} & \mathcal{E}^{11}(\log D) & \longrightarrow & 0 \end{array}$$

gives

$$\mathbf{H}^1(\boldsymbol{P}^1, (\Omega^\cdot(\log D), \nabla^\vee)) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}) \oplus \Gamma(\mathcal{E}^{10}(\log D)) \to \Gamma(\mathcal{E}^{11}(\log D))\}}{\nabla^\vee \Gamma(\mathcal{E}^{00})}.$$

An explicit expression of the isomorphism

$$\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega) \xrightarrow{\sim} \mathbf{H}^1(\boldsymbol{P}^1, (\Omega^\cdot(\log D), \nabla^\vee)) \simeq \frac{\ker\{\Gamma(\mathcal{E}^{01}) \oplus \Gamma(\mathcal{E}^{10}(\log D)) \to \Gamma(\mathcal{E}^{11}(\log D))\}}{\nabla^\vee \Gamma(\mathcal{E}^{00})}$$

is given by

$$\eta^- \mapsto 0 \oplus \eta.$$

Summing up, a fine resolution of the basic sequence is given as follows (pay attention that rows and columns are reversed):

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00}(-D) & \xrightarrow{\nabla^\vee} & \mathcal{E}^{01}(-D) \oplus \mathcal{E}^{10} & \xrightarrow{\nabla^\vee} & \mathcal{E}^{11} & \longrightarrow & 0 \\ & & \downarrow & & \iota \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}^{00} & \xrightarrow{\nabla^\vee} & \mathcal{E}^{01} \oplus \mathcal{E}^{10}(\log D) & \xrightarrow{\nabla^\vee} & \mathcal{E}^{11}(\log D) & \longrightarrow & 0 \\ & & \text{restr} \downarrow & & (\text{restr}, \downarrow \text{Res}) & & \downarrow & & \\ 0 & \longrightarrow & \oplus \mathbf{C}_{x_j} & \underset{(0, \times \text{res})}{\longrightarrow} & \oplus \mathbf{C}_{x_j} \oplus \oplus \mathbf{C}_{x_j} & \underset{(\times \text{res}) \circ pr_1}{\longrightarrow} & \oplus \mathbf{C}_{x_j} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Now we are going to trace back ι. Let us give $\eta \in \Gamma(\Omega^1(\log D))$. We change the represetative η to

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$$\eta' = \eta + \nabla^\vee h$$

so that $(\mathrm{restr}, \mathrm{Res})\eta' = 0$; this can be achieved by taking $h \in \Gamma(\mathscr{E}^{00})$ so that

$$(0, \times \mathrm{res}) \circ \mathrm{restr}\, h = (\mathrm{restr}, \mathrm{Res})\eta.$$

Then there is a form $\tilde{\eta} + \mu \in \Gamma(\mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10})$ which maps under ι to η'; it can be readily checked that $\tilde{\eta} + \mu$ represents an element of

$$\mathbf{H}^1(\boldsymbol{P}^1, (\Omega^{\cdot}(\log D)(-D), \nabla^\vee)) \simeq \frac{\ker\{\Gamma(\mathscr{E}^{01}(-D)) \oplus \Gamma(\mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11})\}}{\nabla^\vee \Gamma(\mathscr{E}^{00}(-D))}.$$

Recall the connecting homomorphism $\delta : \oplus\, \mathbf{C}_{x_j} \to H^1(\mathscr{O}(-D))$ used when defining τ; it is exactly the same as tracing part of the above diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & * & \longrightarrow & \mathscr{E}^{01}(-D) \oplus \mathscr{E}^{10} & \longrightarrow & * & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle \iota}\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}^{00} & \xrightarrow{\nabla^\vee} & \mathscr{E}^{01} \oplus \mathscr{E}^{10}(\log D) & \longrightarrow & * & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{restr}}\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus\, \mathbf{C}_{x_j} & \longrightarrow & * & \longrightarrow & * & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Therefore we proved that in cohomology level

$$\tilde{\eta} = \eta^\vee \quad \text{in } H^1(\mathscr{O}(-D));$$

and so (it will be the key in §3),

(2) $$\iota(\eta^\vee + \mu) = \eta + \nabla^\vee h, \quad \mu \in \Gamma(\mathscr{E}^{10}), \quad h \in \Gamma(\mathscr{E}^{00}).$$

§2.3. Intersection form for cocycles

We assume $\alpha_j \neq 0$. Let us fix an isomorphism

$$\int : H^1(\Omega^1) \to \mathbf{C}$$

by

$$H^1(\Omega^1) \simeq H^1_{\mathrm{Dol}}(\Omega^1) := \Gamma(\mathscr{E}^{11}) / \bar{\partial}\Gamma(\mathscr{E}^{10}) \ni \zeta \mapsto \int_{\boldsymbol{P}^1} \zeta \in \mathbf{C}.$$

For cocycles ξ^+ and η^- represented by $\xi, \eta \in \Gamma(\Omega^1(\log D))$, we now define the intersection form by the natural bilinear form $\langle *, * \rangle$:

$$\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot\omega \times \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega) \to \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot\omega \times H^1_{\mathrm{Dol}}(\mathcal{O}(-D))$$
$$\xrightarrow{\text{Serre duality}} H^1_{\mathrm{Dol}}(\Omega^1) \xrightarrow{\int} \mathbf{C}$$
$$(\xi^+, \eta^-) \mapsto (\xi^+, \eta^\vee) \mapsto \eta^\vee \wedge \xi \mapsto \int \eta^\vee \wedge \xi.$$

Since $\eta^\vee \in \ker(-\omega : H^1_{\mathrm{Dol}}(\mathcal{O}(-D)) \to H^1_{\mathrm{Dol}}(\Omega^1))$ and $\omega^\vee \sim 0$, it is well defined, and is non-degenerate thanks to non-degeneracy of the Serre duality. We compute the intersection numbers for the following forms:

$$\omega_{ij} = \left(\frac{1}{t-x_i} - \frac{1}{t-x_j}\right)dt \in \Gamma(\Omega^1(\log D)),\ 0 \le i \ne j \le n.$$

Let us first explicitly write the image $\omega_{ij}^\vee \in H^1(\mathcal{O}(-D))$ under τ of ω_{ij}^- in terms of the Čech cohomology $\check{H}^1(\mathcal{U}, \mathcal{O}(-D))$ with respect to the covering $\mathcal{U} = \{U_j\}$

$$U_j := U \cup \{x_j\},\ j = 0, \dots, n.$$

CLAIM. *Let $(\omega_{ij}^\vee)_{\mathrm{Cech}}$ be the expression of $\omega_{ij}^\vee$ in the Čech cohomology, then we have*

$$(\omega_{ij}^\vee)_{\mathrm{Cech}} = \begin{cases} 1/\alpha_i + 1/\alpha_j & \text{on } U_{ij} \\ 1/\alpha_i & \text{on } U_{ik}\ (k \ne i, j) \\ -1/\alpha_j & \text{on } U_{jk}\ (k \ne i, j) \\ 0 & \text{on } U_{kl}\ (k, l \ne i, j), \end{cases}$$

where $U_{ij} := U_i \cap U_j$.

Here we use the convention $s_{ij} = -s_{ji}$ for $\{s_{ij}\} \in C^1(\mathcal{O}(-D))$, where $s_{ij} \in \Gamma(U_{ij}, O(-D))$, $s_{ji} \in \Gamma(U_{ji}, O(-D))$.

Proof. It is easy to see that

$$\omega_{ij} \overset{\mathrm{Res}}{\mapsto} (1 \in \mathbf{C}_{x_i}, -1 \in \mathbf{C}_{x_j}, 0 \in \mathbf{C}_{x_k}\ k \ne i, j)$$
$$\overset{\times\mathrm{Res}}{\mapsto} (-1/\alpha_i \in \mathbf{C}_{x_i}, 1/\alpha_j \in \mathbf{C}_{x_j}, 0 \in \mathbf{C}_{x_k}).$$

The connecting map δ is given by tracing the following commutative diagram from the right-top to the left-bottom:

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(\mathcal{O}(-D)) & \longrightarrow & C^0(\mathcal{O}) & \longrightarrow & C^0(\oplus_{j=0}^{n}\mathbf{C}_{x_j}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1(\mathcal{O}(-D)) & \longrightarrow & C^1(\mathcal{O}) & \longrightarrow & C^1(\oplus_{j=0}^{n}\mathbf{C}_{x_j}) & \longrightarrow & 0, \end{array}$$

where $C^{\cdot}$ denotes a space of cochains. Thus the claim follows. □

THEOREM 1. *The intersection numbers for the twisted forms are*

$$\langle \overset{+}{\omega}_{pq}, \overset{-}{\omega}_{ij} \rangle = 2\pi i \left(\frac{1}{\alpha_i}(\delta_{ip} - \delta_{iq}) - \frac{1}{\alpha_j}(\delta_{jp} - \delta_{jq}) \right),$$

where δ_{ip} is the Kronecker delta. As a result, the intersection form is symmetric.

Proof. In terms of the Čech cohomology the isomorphism $\int : H^1(\Omega^1) \xrightarrow{\sim} \mathbf{C}$ is given as follows: for

$$(\zeta)_{\text{Cech}} = (\zeta_{pq}) \in \Omega^1(U_{pq}) \in \check{H}^1(\mathcal{U}, \Omega^1), \quad \zeta \in H^1_{\text{Dol}}(\Omega^1),$$

find meromorphic 1-forms η_p on U_p such that

$$\eta_q - \eta_p = \zeta_{pq} \quad \text{on} \quad U_{pq}$$

($\{\eta_p\}$ is called a Mittag-Leffler distribution for $(\zeta)_{\text{Cech}}$), then [For] implies

$$\int \zeta = 2\pi i \sum_{x \in \boldsymbol{P}^1} \operatorname{Res}_x\{\eta_p\}. \tag{3}$$

Since

$$(\overset{\vee}{\omega}_{ij})_{\text{Cech}} \in \check{H}^1(\mathcal{O}(-D)), \ \omega_{pq} \in \Gamma(\Omega^1(\log D))$$

and $U_a \cap U_b = U$ $(a \neq b)$, we have

$$(\overset{\vee}{\omega}_{ij})_{\text{Cech}} \cdot \omega_{pq} \in \check{H}^1(\mathcal{U}, \Omega^1).$$

Notice that

$$H^1_{\text{Dol}}(\Omega^1) \ni \overset{\vee}{\omega}_{ij} \wedge \omega_{pq} \leftrightarrow -\overset{\vee}{\omega}_{ij} \cdot \omega_{pq} \in \check{H}^1(\mathcal{U}, \Omega^1).$$

If we define $\xi = \{\xi_l\}$ by

$$\begin{array}{ll} \xi_i := \omega_{pq}/\alpha_i & \text{a meromorphic 1-form on } U_i \\ \xi_j := -\omega_{pq}/\alpha_j & \text{a meromorphic 1-form on } U_j \\ \xi_k := 0 & \text{on } U_k \quad \text{if } k \neq i, j \end{array}$$

it forms a Mittag-Leffler distribution for $-(\omega_{ij}^{\vee})_{\mathrm{Cech}}\cdot\omega_{pq}$. Hence using the formula (3), we get

$$\langle\omega_{pq}^{+},\omega_{ij}^{-}\rangle = 2\pi i \sum_{x\in\boldsymbol{P}^1}\mathrm{Res}_x\,\xi,$$

which completes the proof. □

By using forms

$$\varphi_j=\frac{dt}{t-x_j}-\frac{dt}{t-x_{j+1}}\in\Gamma(\boldsymbol{P}^1,\Omega^1(\log D)),\quad 1\le j\le n-1,$$

we give bases for the spaces $\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot\omega$ and $\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$ by

$$\varphi_j^{+}\in\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot\omega,\quad \varphi_j^{-}\in\Gamma(\boldsymbol{P}^1,\Omega^1(\log D))/\mathbf{C}\cdot(-\omega),\quad 1\le j\le n-1.$$

COROLLARY. *For the bases above, the intersection numbers are given as follows:*

$$\langle\varphi_j^{+},\varphi_j^{-}\rangle = 2\pi i\left(\frac{1}{\alpha_j}+\frac{1}{\alpha_{j+1}}\right),$$

$$\langle\varphi_j^{+},\varphi_{j+1}^{-}\rangle=\langle\varphi_{j+1}^{+},\varphi_j^{-}\rangle=-\frac{2\pi i}{\alpha_{j+1}},$$

$$\langle\varphi_j^{+},\varphi_k^{-}\rangle=0\quad \textit{if}\ |j-k|\ge 2.$$

§3. Twisted Riemann's period relations

In this section we assume $\alpha_j\notin\boldsymbol{Z}$. Let ξ_j (resp. η_j) $1\le j\le n-1$ be elements of $\Gamma(\Omega^1(\log D))$ such that ξ_j^{+} (resp η_j^{-}) forms a basis of $\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot\omega$ (resp. $\Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega)$). Recall the de Rham expression:

$$H_c^1(L^{\vee})\simeq\frac{\ker\{\nabla^{\vee}:\Gamma_c(U,\mathscr{E}^1)\to\Gamma_c(U,\mathscr{E}^2)\}}{\nabla^{\vee}\Gamma_c(U,\mathscr{E}^0)};$$

the natural inclusion

$$\ker\{\nabla^{\vee}:\Gamma_c(U,\mathscr{E}^1)\to\Gamma_c(U,\mathscr{E}^2)\}\hookrightarrow\ker\{\Gamma(\mathscr{E}^{01}(-D))\oplus\Gamma(\mathscr{E}^{10})\to\Gamma(\mathscr{E}^{11})\}$$

induces the isomorphism (here the assumption $\alpha_j\notin\mathbf{N}-\{0\}$ is used)

$$(H_c^1(L^{\vee})\simeq)\ \frac{\ker\{\nabla^{\vee}:\Gamma_c(U,\mathscr{E}^1)\to\Gamma_c(U,\mathscr{E}^2)\}}{\nabla^{\vee}\Gamma_c(U,\mathscr{E}^0)}$$

$$\simeq \frac{\ker\{\Gamma(\mathscr{E}^{01}(-D)) \oplus \Gamma(\mathscr{E}^{10}) \to \Gamma(\mathscr{E}^{11})\}}{\nabla^\vee \Gamma(\mathscr{E}^{00}(-D))} \left(\underset{\iota}{\overset{\tau}{\rightleftarrows}} \Gamma(\Omega^1(\log D))/\mathbf{C}\cdot(-\omega)\right).$$

For each η_j there exist (see §2 (2)) $\mu_j \in \Gamma(\mathscr{E}^{10})$ and $h_j \in \Gamma(\mathscr{E}^{00})$ such that

$$\eta_j^\vee + \mu_j = \eta_j + \nabla^\vee h_j;$$

moreover by the isomorphism above there exist $f_j \in \Gamma(\mathscr{E}^{00}(-D))$ such that

$$\eta_j^{\mathrm{c}} := \eta_j^\vee + \mu_j + \nabla^\vee f_j \in \Gamma_c(U, \mathscr{E}^1),$$

which form a basis of $\Gamma_c(U, \mathscr{E}^1)$. Let

$$\gamma_j^+ \in H_1(L^\vee), \quad \delta_j^- \in H_1(L)$$

be bases of the twisted cycles. We use the following isomorphism called the Poincaré duality (without any condition):

$$\theta_c : H_1(U, L^\vee) \simeq H^1(\Gamma_c(U, \mathscr{E}^\cdot), \nabla^\vee).$$

Let us define the intersection matrices and the period matrices as follows:

$$I_h = \begin{pmatrix} \langle\gamma_1^+, \delta_1^-\rangle & \cdots & \langle\gamma_1^+, \delta_{n-1}^-\rangle \\ \vdots & & \vdots \\ \langle\gamma_{n-1}^+, \delta_1^-\rangle & \cdots & \langle\gamma_{n-1}^+, \delta_{n-1}^-\rangle \end{pmatrix}, \; I_{ch} = \begin{pmatrix} \langle\xi_1^+, \eta_1^-\rangle & \cdots & \langle\xi_1^+, \eta_{n-1}^-\rangle \\ \vdots & & \vdots \\ \langle\xi_{n-1}^+, \eta_1^-\rangle & \cdots & \langle\xi_{n-1}^+, \eta_{n-1}^-\rangle \end{pmatrix}.$$

$$P^+ = \begin{pmatrix} \int_{\gamma_1^+} \xi_1^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_1^+ \\ \vdots & & \vdots \\ \int_{\gamma_1^+} \xi_{n-1}^+ & \cdots & \int_{\gamma_{n-1}^+} \xi_{n-1}^+ \end{pmatrix}, \; P^- = \begin{pmatrix} \int_{\delta_1^-} \eta_1^- & \cdots & \int_{\delta_{n-1}^-} \eta_1^- \\ \vdots & & \vdots \\ \int_{\delta_1^-} \eta_{n-1}^- & \cdots & \int_{\delta_{n-1}^-} \eta_{n-1}^- \end{pmatrix},$$

where the intersection for twisted cycles are defined by

$$\langle\gamma^+, \delta^-\rangle := \int_{\delta^-} \theta_c(\gamma^+), \quad \gamma^+ \in H_1(L^\vee), \quad \delta^- \in H_1(L).$$

Then we have the twisted Riemann's period relation:

THEOREM 2.

$$P^+ \, {}^tI_h^{-1} \, {}^tP^- = I_{ch}, \quad \text{i.e. } {}^tP^- \, {}^tI_{ch}^{-1} \, P^+ = {}^tI_h.$$

Proof. Let $\Theta = (\theta_{ij})$ be the matrix expression of θ_c under the bases above:

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$$\theta_c(\gamma_j^+) = \sum_k \theta_{kj}\eta_k{}^c.$$

The intersection numbers for twisted cycles are computed as follows:

$$\begin{aligned}\langle \gamma_j^+, \delta_k^- \rangle &:= \int_{\delta_k^-} \theta_c(\gamma_j^+) = \int_{\delta_k^-} \sum_a \theta_{aj}\eta_a{}^c \\ &= \sum_a \theta_{aj} \int_{\delta_k^-} \eta_a + \nabla^\vee h_a + \nabla^\vee f_a = \sum_a \theta_{aj} \int_{\delta_k^-} \eta_a^-,\end{aligned}$$

that is

$$I_h = {}^t\Theta P^-.$$

The (k, j)-components θ_{kj} of Θ are computed as follows:

$$\begin{aligned}\int_{\gamma_j^+} \xi_a^+ &= \int \theta_c(\gamma_j^+) \wedge \xi_a^+ = \int \sum_k \theta_{kj}\eta_k{}^c \wedge \xi_a \\ &= \sum_k \theta_{kj} \int (\eta_k^\vee + \mu_k + \nabla^\vee f_k) \wedge \xi_a \\ &= \sum_k \theta_{kj} \int \eta_k^\vee \wedge \xi_a = \sum_k \langle \xi_a^+, \eta_k^- \rangle \theta_{kj},\end{aligned}$$

that is

$$P^+ = I_{ch}\Theta.$$

Eliminating Θ from the two equalities above, we get the relation. □

§4. Examples

EXAMPLE 1. Quadric relations for the Gauss hypergeometric functions. For

$$n = 3,\ x_0 = x_4 = \infty,\ x_1 = 0,\ x_2 = 1,\ x_3 = 1/x\ (0 < x < 1),$$
$$\alpha_1 = \alpha,\ \alpha_2 = \gamma - \alpha,\ \alpha_3 = -\beta,\ \alpha_0 = \beta - \gamma,$$

put

$$u = t^\alpha(1-t)^{\gamma-\alpha}(1-xt)^{-\beta},$$
$$\varphi_1 = \left(\frac{dt}{t-x_1} - \frac{dt}{t-x_2}\right) = \frac{dt}{t(1-t)},\ \varphi_3 = \left(\frac{dt}{t-x_3} - \frac{dt}{t-x_4}\right) = \frac{-xdt}{1-xt},$$

$\gamma_1^+, \gamma_3^+ \in H_1(U, L^\vee)$ and $\gamma_1^-, \gamma_3^- \in H_1(U, L)$, (see Figure), then we have

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$$P^{+}=\begin{pmatrix}\int_0^1 u\varphi_1 & \int_{1/x}^{\infty} u\,\varphi_1\\ \int_0^1 u\varphi_3 & \int_{1/x}^{\infty} u\,\varphi_3\end{pmatrix},\ P^{-}=\begin{pmatrix}\int_0^1 u^{-1}\varphi_1 & \int_{1/x}^{\infty} u^{-1}\varphi_1\\ \int_0^1 u^{-1}\varphi_3 & \int_{1/x}^{\infty} u^{-1}\varphi_3\end{pmatrix},$$

$$I_h=-\begin{pmatrix}d_{12}/d_1d_2 & 0\\ 0 & d_{30}/d_3d_0\end{pmatrix},\ I_{ch}=2\pi i\begin{pmatrix}1/\alpha+1/(\gamma-\alpha) & 0\\ 0 & -1/\beta+1/(\beta-\gamma)\end{pmatrix}.$$

By the help of the well-known formulae

$$\int_0^1 u\,\varphi_1=B(\alpha,\gamma-\alpha)F(\alpha,\beta,\gamma;x),$$

$$\int_{1/x}^{\infty} u\,\varphi_1=-(-1)^{\gamma-\alpha-\beta}x^{1-\gamma}B(\beta-\gamma+1,-\beta+1)$$
$$\times F(\beta-\gamma+1,\alpha-\gamma+1,2-\gamma;x),$$

the identity

$$P^{+}\,{}^tI_h^{-1}\,{}^tP^{-}=I_{ch},$$

leads quadratic identities for hypergeometric functions in [SY]: the (1,2)-component yields the formula presented in Introduction

$$F(\alpha,\beta,\gamma;x)F(1-\alpha,1-\beta,2-\gamma;x)$$
$$=F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;x)F(\gamma-\alpha,\gamma-\beta,\gamma;x),$$

and the (1, 1)-component yields

$$F(\alpha,\beta,\gamma;x)F(-a,-\beta,-\gamma;x)-1$$
$$=\frac{\alpha\beta(\gamma-\alpha)(\gamma-\beta)}{\gamma^2(\gamma+1)(\gamma-1)}F(\beta-\gamma+1,\alpha-\gamma+1,-\gamma+2;x)$$
$$\times F(\gamma-\beta+1,\gamma-\alpha+1,\gamma+2;x).$$

EXAMPLE 2. Quadric relations for Lauricella's hypergeometric function. Lauricella's hypergeometric function F_D of m-variable is defined by

$$F_D(\alpha,\beta,\gamma;z)=\sum_{n_1,n_2,\dots,n_m=0}^{\infty}\frac{(\alpha)_{n_1+\dots+n_m}(\beta_1)_{n_1}\cdots(\beta_m)_{n_m}}{(\gamma)_{n_1+\dots+n_m}(1)_{n_1}\cdots(1)_{n_m}}z_1^{n_1}\cdots z_m^{n_m},$$

where

$$z=(z_1,\dots,z_m),\quad \beta=(\beta_1,\dots,\beta_m);$$

the series admits the integral representation

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$$F_D(\alpha, \beta, \gamma ; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-z_1t)^{-\beta_1}\cdots(1-z_mt)^{-\beta_m}dt.$$

Put

$$n = m+2,\ x_0 = \infty,\ x_1 = 0,\ x_2 = 1,\ x_{j+2} = 1/z_j\ (1 \le j \le m),$$

$$\alpha_0 = \alpha_{m+3} = \beta_1 + \cdots + \beta_m - \gamma,\ \alpha_1 = \alpha,\ \alpha_2 = \gamma - \alpha,\ \alpha_{j+2} = -\beta_j\ (1 \le j \le m),$$

$$u = t^{\alpha}(1-t)^{\gamma-\alpha}(1-z_1t)^{-\beta_1}\cdots(1-z_mt)^{-\beta_m},$$

$$\xi_j = \left(\frac{1}{t-x_1} - \frac{1}{t-x_{j+1}}\right)dt,\ \eta_j = \left(\frac{1}{t-x_{j+1}} - \frac{1}{t-x_0}\right)dt\ (1 \le j \le m+1),$$

$$\gamma_j^+,\ H_1(U, L^{\vee}),\ \gamma_j^- \in H_1(U, L)\ (1 \le j \le m),\ \text{(see Figure)}.$$

The (1,1)-component of

$${}^tP^-\, I_{ch}^{-1}\, P^+ = {}^tI_h,$$

reads

$$\left(\int_0^1 u^{-1}\eta_1, \ldots, \int_0^1 u^{-1}\eta_{m+1}\right) I_{ch}^{-1}\, {}^t\!\left(\int_0^1 u\,\xi_1, \ldots, \int_0^1 u\xi_{m+1}\right) = I_h(1,1).$$

Since the (1,1)-component of I_h is $-(e^{2\pi i\gamma}-1)/((e^{2\pi i\alpha}-1)(e^{2\pi i(\gamma-\alpha)}-1))$, and

$$I_{ch}^{-1} = -\frac{1}{2\pi i}\begin{pmatrix} \alpha-\gamma & 0 & 0 & \cdots & 0 \\ 0 & \beta_1 z_1 & 0 & \cdots & 0 \\ 0 & 0 & \beta_2 z_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_m z_m \end{pmatrix},$$

we have the following formula:

$$F_D(\alpha, \beta, \gamma ; z)F_D(1-\alpha, -\beta, -\gamma+1 ; z) - 1$$

$$= \frac{\gamma-\alpha}{\gamma(\gamma-1)} \sum_{j=1}^{m} \beta_j z_j F_D(\alpha, \beta+e_j, \gamma+1 ; z)F_D(-\alpha+1, -\beta+e_j, -\gamma+2 ; z),$$

where

$$e_j = (\ldots, 0, \overset{j\text{-th}}{1}, 0, \ldots).$$

Remark. Once the inversion formula for the beta function is obtained as an example of the twisted Riemann's period relations, the inversion formula for the gamma function can be obtained as a special case of beta's as follows:

$$\Gamma(\alpha)\Gamma(-\alpha) = B(\alpha, -\alpha/2)B(-\alpha, \alpha/2)$$
$$= \frac{-2\pi i}{\alpha}\frac{\exp(\pi i\alpha)}{\exp(2\pi i\alpha)-1} = -\frac{1}{\alpha}\frac{\pi}{\sin\pi\alpha},$$

namely $\Gamma(a)\Gamma(1-\alpha) = \pi/\sin\pi\alpha$. Since the gamma function can be thought of a confluent beta function (see the integral representations of these functions in the beginning of Introduction), this formula suggests a confluent version of our intersection theory.

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