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THE ORDERING OF SPEC R

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Introduction. Let Spec R denote the set of prime ideals of a commutative ring with identity R, ordered by inclusion; and call a partially ordered set *spectral* if it is order isomorphic to Spec R for some R. What are some conditions, necessary or sufficient, for a partially ordered set X to be spectral? The most desirable answer would be the type of result that would allow one to stare at the diagram of a given X and then be able to say whether or not X is spectral. For example, it is known that finite partially ordered sets are spectral (see [**2**] or [**5**]). However, even in the 1-dimensional case a complete characterization of spectral sets still seems very far off. On the other hand, the corresponding topological question was completely answered by Hochster in his remarkable thesis [**2**], and most of our work here uses Hochster's topological characterization as an intermediate step.

We begin in § 2 with two examples. The first shows that the previously known necessary conditions for a partially ordered set to be spectral are not sufficient and thus, in a sense, yields a new necessary condition. The second example shows that the property "all finitely generated flat R-modules are projective" is not determined by the ordering of Spec R (even though it is determined by the Zariski topology of Spec R [**4**]). This example involves defining two suitably distinct order compatible spectral topologies (see § 1 for terminology) on the same partially ordered set.

In § 3 we enlarge on the construction of the second example to prove that a partially ordered set X is spectral if it contains an element m such that any $x \in X$ having infinitely many elements below (resp. above) x necessarily lies above (resp. below) m. Additional, less superficial, sufficient conditions for a partially ordered set to be spectral are given in § 5, where the basic situation is related to the first example of § 2. This involves an X which is decomposed into an upper part X_1 and a lower part X_2, e.g. a 1-dimensional X which is written as the union of its maximal elements and its minimal elements that are not maximal.

In § 4 we prove that the ordered disjoint union of spectral sets is spectral, a result which is used later in one of the constructions of § 5.

Finally, § 6 is devoted to describing all the spectral topologies on the simplest kind of 1-dimensional X, namely a countable, 1-dimensional X with unique minimal element.

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1. Terminology. A set X may come equipped with a partial ordering $\mathscr{O}$ or a a topology $\mathscr{T}$. Given $(X, \mathscr{O})$ and $x \in X$, let $G_{X,\mathscr{O}}(x) = \{y \in X | y \leqq x\}$ (the generization of x in X), and let $S_{X,\mathscr{O}}(x) = \{y \in X | y \geqq x\}$ (the specialization of x in X). More generally, if V is a subset of X, $G_{X,\mathscr{O}}(V) = \{y \in X | y \leqq v$ for some $v \in V\}$ and $S_{X,\mathscr{O}}(V) = \{y \in X | y \geqq v$ for some $v \in V\}$. Given $(X, \mathscr{T})$, we write $\mathrm{cl}_{X,\mathscr{T}}\{x\}$ for the $\mathscr{T}$-closure of $\{x\}$ in X. When our notation is clear from the context, we shall drop the subscripts.

Let X have a topology $\mathscr{T}$ and a partial ordering $\mathscr{O}$. We say that $\mathscr{T}$ is *compatible* with $\mathscr{O}$ if $\mathrm{cl}\{x\} = S(x)$ for all $x \in X$. Note that $\mathscr{T}$ is compatible with $\mathscr{O}$ if and only if

i) $S(x)$ is closed for all $x \in X$, and

ii) closed sets are closed under $\geqq$.

Recall that if R is a ring, the Zariski topology for Spec R is defined by letting $C \subseteq \operatorname{Spec} R$ be closed if and only if there exists an ideal $\mathfrak{A}$ of R such that $C = \{\mathfrak{p} \in \operatorname{Spec} R | \mathfrak{p} \supseteq \mathfrak{A}\}$. Spec R with Zariski topology and inclusion ordering is an example where the topology and ordering are compatible. We will always consider Spec R as a topological space with Zariski topology and as an ordered set with inclusion ordering.

If $Y \subseteq X$, $(Y, \mathscr{O}|_Y)$ will denote Y with the induced ordering and $(Y, \mathscr{T}|_Y)$ will denote Y with the subspace topology. If $\mathscr{T}$ and $\mathscr{O}$ are compatible, then so are $\mathscr{T}|_Y$ and $\mathscr{O}|_Y$.

Let $(X, \mathscr{T})$ be a T_0 space. Then X has a partial ordering, $O(\mathscr{T})$, induced by $\mathscr{T}$ by defining $x \leqq y$ if and only if $y \in \mathrm{cl}\{x\}$. Conversely, let $(X, \mathscr{O})$ be a partially ordered set. The ordering induces a topology, $T(\mathscr{O})$, by defining a subbasis for the closed sets of $T(\mathscr{O})$ to be $\{S(x) | x \in X\}$. We shall call this topology the *closures of points* (*COP*) topology. The *COP* topology is T_0 and is compatible with $\mathscr{O}$. Another topology is $\mathscr{O}$-compatible if and only if it is finer than the *COP* topology and has its closed sets closed under $\geqq$. (Recall that $\mathscr{T}$ is *finer* than $\mathscr{T}'$ if every $\mathscr{T}'$-open set is also $\mathscr{T}$-open.) Clearly, $\mathscr{T}$ is compatible with $\mathscr{O}$ if and only if $O(\mathscr{T}) = \mathscr{O}$. Thus, it is possible to recover the ordering from a given order compatible topology. Also, if $(X, \mathscr{T}) \cong (X', \mathscr{T}')$, then $(X, O(\mathscr{T})) \cong (X', O(\mathscr{T}'))$. The converse, however, is false; so the ordering contains less information than the topology. (We use $\cong$ to denote homeomorphism when topological spaces are involved and order isomorphism when partially ordered sets are involved.)

We define

$$\dim(X, \mathscr{O}) = \sup\{n | \text{there is a chain } x_0 < x_1 < \ldots < x_n;\ x_i \in X\}.$$

Similarly, if $x \in X$,

$$\mathrm{ht}(x) = \sup\{n | \text{there is a chain } x_0 < x_1 < \ldots < x_n = x;\ x_i \in X\}.$$

In [**2**], a space $(X, \mathscr{T})$ is called *spectral* if it has the following properties:

i) X is T_0.
ii) X is quasi-compact.
iii) The quasi-compact open subsets are closed under finite intersections and form an open basis.
iv) Every non-empty closed subspace which is irreducible (i.e. not the union of two proper closed subsets) is the closure of one of its points (i.e. has a generic point).

Spec R is well known to be spectral. Conversely, in [**2**] Hochster proves that any spectral space is homeomorphic to Spec R for some ring R. We shall also need the fact that a closed subset of a spectral space is spectral in the induced topology. We have previously defined a partially ordered set to be spectral if it is order isomorphic to Spec R for some R. Obviously, $(X, \mathscr{O})$ is spectral if and only if there exists an order compatible spectral topology.

A 0-dimensional partially ordered set is easily seen to be spectral. $(X, \mathscr{T})$ is spectral and $\mathscr{T}$ is compatible with a 0-dimensional ordering of X if and only if $(X, \mathscr{T})$ is a totally disconnected, compact (i.e. quasi-compact, Hausdorff) space. These spaces are called *Boolean* spaces and the one-point compactification of an infinite discrete space is an easy example. If X is countable, $(X, \mathscr{T})$ is a Boolean space if and only if it is homeomorphic to the space of ordinals less than or equal to some countable ordinal with interval topology (see [**7**]).

2. Two examples.

(2.1) *Example.* We know of three conditions on a partially ordered set X that are necessary for X to be order isomorphic to the spectrum of some ring:

($K1$) Every totally ordered subset of X has a supremum and an infimum in X.
($K2$) Between any two distinct related elements there are two immediately adjacent elements (i.e. if $x, y \in X$ and $x < y$, then there exist elements $x_1, y_1 \in X$ such that $x \leqq x_1 < y_1 \leqq y$ and there does not exist $z \in X$ such that $x_1 < z < y_1$).
(H) Let $\mathscr{S} = \{S(x)|x \in X\}$, $\mathscr{G} = \{G(x)|x \in X\}$. If $\mathscr{F}$ is a collection of subsets of X such that $\mathscr{F} \subseteq \mathscr{S}$ or $\mathscr{F} \subseteq \mathscr{G}$, then $\bigcap \{F|F \in \mathscr{F}\} = \emptyset$ implies there is a finite collection of sets from $\mathscr{F}$ whose intersection is empty.

In addition, Hochster [**2**] has proved that, given a ring R, there is a ring whose prime ideals have exactly the reverse order of the primes of R. Thus one also knows that any necessary condition should also be symmetric with respect to reversing the order.

Properties ($K1$) and ($K2$) were discussed by Kaplansky [**3**], while property [H] derives from an example of Hochster given in [**5**]. Property (H) reflects the fact that Spec R is quasi-compact in the Zariski topology and that the sets in $\mathscr{S}$ are necessarily closed sets of Spec R.

We shall now give an example of a partially ordered set which satisfies ($K1$),

$(K2)$, and (H) but is not spectral, thus showing that these conditions are not sufficient for a partially ordered set to be spectral.

Let Y be a compact (Hausdorff) space which is not spectral; for example, the closed unit interval will do. Because Y is compact, there does not exist a properly finer compact topology for Y. Let $Z = \{x_C | C \subseteq Y$ and C is closed in $Y\}$, let $X = Y \cup Z$, and order X by specifying that if $y \in Y$, $y \geqq x_C$ if and only if $y \in C$. Thus, for each closed set $C \subseteq Y$ we create an element x_C and "place" it below each element of C. The set X is one-dimensional, Y being the set of maximal elements and Z the set of minimal elements. Let $\mathscr{T}$ be any order compatible topology for X. Choose C_1, C_2 proper closed subsets of Y such that $Y = C_1 \cup C_2$. Now $Y = S(x_Y) \cap (S(x_{C_1}) \cup S(x_{C_2}))$, so Y is closed in $\mathscr{T}$. Similarly, if C is any closed subset of Y, then $C = Y \cap S(x_C)$ is also $\mathscr{T}$-closed. If $\mathscr{T}$ were a spectral topology, then the topology $\mathscr{T}|_Y$ would also be spectral and would be finer than the original topology for Y. Our choice for Y makes this impossible, so X is not spectral.

It remains to show X satisfies $(K1)$, $(K2)$, and (H). Because X is one-dimensional, $(K1)$ and $(K2)$ are satisfied trivially. Let $\mathscr{F}$ be a collection of subsets of X such that $\bigcap \{F | F \in \mathscr{F}\} = \emptyset$. If $\mathscr{F} \subseteq \mathscr{S}$, note that for $x \neq y \in X$, $S(x) \cap S(y) \subseteq Y$. Since each $F \cap Y$ is closed in Y and $\bigcap \{F \cap Y | F \in \mathscr{F}\} = \emptyset$ the compactness of Y allows the choice of a finite set of F's whose intersection is empty. If $\mathscr{F} \subseteq \mathscr{G}$, we note that $x_Y \in G(y)$ for each $y \in Y$, so there must be an $F \in \mathscr{F}$ and a $z \in Z$ such that $F = G(z) = \{z\}$. Choose any $F' \in \mathscr{F}$ such that $z \notin F'$ and we have $F \cap F' = \emptyset$.

(2.2) *Example.* The second example involves the property "all finitely generated flat R-modules are projective." Let us say that a ring R is $A(0)$ if it has this property (the terminology stems from [**1**]). D. Lazard [**4**] has shown that whether or not R is $A(0)$ depends only on the Zariski topology of Spec R. We shall give here an example of a partially ordered set having two spectral topologies, one of which yields an $A(0)$ ring and the other does not. This shows that the partial ordering of Spec R is not sufficient by itself to determine whether or not R is $A(0)$.

First let us review some facts from Lazard's paper. Let X be a partially ordered set. The *D-component* of an element $x \in X$ is defined to be the intersection of all sets containing x that are closed under $\geqq$ and $\leqq$. Thus y is in the D-component of x if and only if there exist elements $x_1, \ldots, x_n \in X$ such that $x \leqq x_1 \geqq x_2 \leqq x_3 \geqq \ldots \leqq x_n \geqq y$. Moreover, if $\mathscr{T}$ is an order compatible topology for X, a subset of X is defined to be *D-closed* if it is closed in $\mathscr{T}$ and is a union of D-components. Lazard has proved that a ring R has the $A(0)$ property if and only if the D-closed sets of Spec R are open. Since the D-components partition Spec R, it follows that R is $A(0)$ whenever Spec R has only finitely many D-components each of which is closed. In particular, if R has only finitely many minimal primes, then R is $A(0)$. It is also true, but for a different reason, that if R has only finitely many maximal ideals, then R is $A(0)$. In any case, it often suffices to merely glance at the ordering of the prime ideals of R in order to conclude that R is $A(0)$. The following example, however, shows that the $A(0)$ property cannot be characterized in terms of the ordering of Spec R alone.

Let $X = \{m_i | i = 0, 1, 2, \ldots, \infty\} \cup \{p_j | j = 1, 2, \ldots, \infty\}$, and order X by defining $p_j \leqq m_{j-1}, m_j$ if $j < \infty$ and $p_\infty \leqq m_\infty$ (see Figure 1 below).

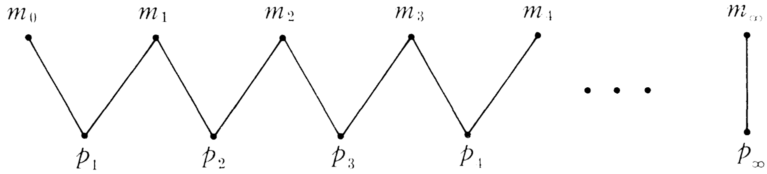

FIGURE 1

X has exactly two D-components, namely $D_\infty = \{m_\infty, p_\infty\}$ and $X \backslash D_\infty$.

For any $m \in X$, let $C(m)$ be the topology for X having closed sets

i) finite sets closed under $\geqq$; and

ii) sets containing m and closed under $\geqq$.

It is immediate that $C(m)$ is a T_0, order compatible, quasi-compact topology for which closed irreducible sets have generic points. Moreover, the cofinite (i.e. finite complement) sets containing m and closed under $\leqq$, together with the finite sets not containing m and closed under $\leqq$, form a basis of quasi-compact open sets which is closed under finite intersections. Thus, $(X, C(m))$ is a spectral space.

The D-component D_∞ is closed in the $C(m)$ topology for any choice of m, and hence $(X, C(m))$ has the property that D-closed sets are open if and only if D_∞ is open. But D_∞ is open if and only if $m \in D_0$. Thus, by choosing rings with Spec isomorphic to $(X, C(m_0))$ and $(X, C(m_\infty))$, we get one ring which has the $A(0)$ property and another which does not, yet both rings have Spec order isomorphic to X.

The example above is the simplest available in that it is one-dimensional and has only two D-components, for a 0-dimensional spectral space is $A(0)$, i.e. has the property that D-closed sets are open, if and only if the space is finite.

3. The $C(m)$ topology. In this section we generalize the topology given in Example 2.2 to an arbitrary partially ordered set. Given a partially ordered set X we choose an element $m \in X$. We define a topology, called the $C(m)$ topology, by choosing the following collection of sets as a basis for the closed sets of the topology:

i) finite sets not containing m and closed under $\geqq$ (including $\emptyset$), and

ii) cofinite sets containing m and closed under $\geqq$ (including X).

(3.1) LEMMA. *Let X be a partially ordered set and let $m \in X$. The $C(m)$ topology is compatible with the order of X if and only if the following conditions hold:*

a) *if $x \in X$ and $\{y | y \geqq x\}$ is infinite, then $x \leqq m$; and*

b) *if $x \in X$ and $\{y | y \leqq x\}$ is infinite, then $x \geqq m$.*

Proof. For all $x \not\leqq m$, $\mathrm{cl}\{x\} = \{y|y \geqq x\}$ if and only if condition a) holds. Similarly, for all $x \leqq m$, $\mathrm{cl}\{x\} = \{y|y \geqq x\}$ if and only if condition b) holds.

(3.2) THEOREM. *Let X be a partially ordered set with the $C(m)$ topology for some $m \in X$. If the topology is compatible with the order of X, then X with the $C(m)$ topology is a spectral space.*

Proof. Since the topology is compatible with the order, X is T_0. X is quasi-compact since any open set containing m is cofinite. Corresponding to the closed basis for the topology, we have the following open basis:

i) cofinite sets containing m and closed under $\leqq$; and

ii) finite sets not containing m and closed under $\leqq$.

Clearly these sets must be quasi-compact, and they are closed under finite intersections. Now let V be a closed irreducible set in X. If every element of V is $\geqq m$, then either $V = \mathrm{cl}\{m\}$, or V is finite and a generic point is easy to find; so we may assume V contains an $x \not\geqq m$. Since $\{y|y \leqq x\}$ is finite, we may choose x to be a minimal element of V. Then $\mathrm{cl}\{x\}$ and $X\backslash\{y|y \leqq x\}$ are closed sets whose union contains V. Since $x \in V$ and V is irreducible, we get $V = \mathrm{cl}\{x\}$.

As a special case of Theorem 3.2 we have the following generalization of the fact that any finite partially ordered set is spectral.

(3.3) COROLLARY. *If X is a partially ordered set with the property that $S(x) \cup G(x)$ is finite for all $x \in X$, then X is spectral.*

4. Ordered disjoint unions. If a partially ordered set X is the disjoint union of partially ordered sets $\{X_\alpha\}$, we shall say that X is the *ordered* disjoint union of the X_α's if

$$x \leqq_X y \text{ if and only if there is an } \alpha \text{ such that } x, y \in X_\alpha \text{ and } x \leqq_{X_\alpha} y.$$

Let Λ be an indexing set containing an element o, and let $\Lambda' = \Lambda\backslash o$. Suppose we are given a collection of rings $\{R_\lambda|\lambda \in \Lambda\}$ such that each R_λ, $\lambda \in \Lambda'$, is an R_o-algebra via a homomorphism $\phi_\lambda : R_o \to R_\lambda$; and let R be the subring of $\prod_{\lambda\in\Lambda} R_\lambda$ defined by

$$(r_\lambda) \in R \text{ if and only if } \phi_\lambda(r_o) = r_\lambda \text{ for all but a finite number of } \lambda \in \Lambda'.$$

Let us now examine Spec R. For each $\alpha \in \Lambda$ let $A_\alpha = \{(a_\lambda) \in R|a_\alpha = 0\}$. The A_α are ideals of R; and if $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ and $\alpha \neq o$, then $(0, 0, \ldots, 0, 1_\alpha, 0, \ldots) \in A_\beta$ and $(1, 1, \ldots, 1, 0_\alpha, 1, \ldots) \in A_\alpha$. It follows that $A_\beta + A_\alpha = R$. Let P be any prime ideal of R such that $P \not\supseteq A_o$. Choose $z = (a_\lambda) \in A_o\backslash P$. Then $a_\lambda = 0$ for all but a finite number of $\lambda \in \Lambda'$, say $\alpha_1, \ldots, \alpha_n$. Thus $zA_{\alpha_1}\cdot \ldots \cdot A_{\alpha_n} = 0$. As $z \notin P$, it follows that $A_{\alpha_i} \subseteq P$ for some α_i. Next note that $R/A_\alpha \cong R_\alpha$, $\alpha \in \Lambda$, since A_α is the kernel of the projection homomorphism onto R_α. Thus, if $X_\alpha = \{P \in \text{Spec } R|P \supseteq A_\alpha\}$, then X_α is order isomorphic to Spec R_α. It follows that Spec R is the ordered disjoint union of the sets X_α, $\alpha \in \Lambda$, where X_α is order isomorphic to Spec R_α.

Applications.

(4.1) THEOREM. *Let* $\{X_\lambda | \lambda \in \Lambda\}$ *be a collection of spectral partially ordered sets. Let* X *be the ordered disjoint union of the* X_λ. *Then there is a ring* R *such that* Spec $R \cong X$.

Proof. Choose one element $o \in \Lambda$, and let $\Lambda' = \Lambda \backslash o$. Use [**2**, Theorem 6] to choose a ring R_o such that Spec $R_o \cong X_o$. Let $\mathscr{M}$ be a maximal ideal of R_o and $k = R_o/\mathscr{M}$. Again using [**2**], for each $\lambda \in \Lambda'$ we can choose a ring R_λ such that Spec $R_\lambda \cong X_\lambda$ and R_λ is a k-algebra. Thus, we have composite ring homomorphisms $\phi_\lambda : R_o \rightarrow k \rightarrow R_\lambda$, $\lambda \in \Lambda'$. The theorem now follows from the above remarks.

We can also use the above construction to improve a theorem of Lewis [**5**], who proved that any tree X satisfying ($K1$) and ($K2$) and having a unique minimal element is of the form Spec R, where R is a Bezout domain. (Here a partially ordered set X is a *tree* if for each $x \in X$, $G(x)$ is totally ordered; and a ring is called *Bezout* if every finitely generated ideal is principal.)

(4.2) THEOREM. *A partially ordered set* X *is a tree satisfying* ($K1$) *and* ($K2$) (*if and*) *only if* $X \cong$ Spec R *for some Bezout ring* R.

Proof. For any Bezout ring R, Spec R is well known to be a tree; so let us assume X is a tree satisfying (K1) and (K2). If x, y are two distinct minimal elements of X, then $S(x) \cap S(y) = \emptyset$. Thus, X can be written as the ordered disjoint union of trees X_λ, $\lambda \in \Lambda$, where each X_λ has a unique minimal element. As X satisfies (K1) and (K2), so does each X_λ. Pick an element $o \in \Lambda$, let $\Lambda' = \Lambda \backslash o$, and use [**5**, Theorem 3.1] to choose a Bezout domain R_o such that Spec $R_o \cong X_o$. Now let K be the quotient field of R_o. Again use Theorem 3.1 of [**5**] as well as Ohm's proof of Jaffard's theorem [**6**, page 589] to choose R_λ for each $\lambda \in \Lambda'$ such that Spec $R_\lambda \cong X_\lambda$ and $K \subseteq R_\lambda$. Thus, for $\lambda \in \Lambda'$, we have composite ring homomorphisms $\phi_\lambda : R_o \rightarrow K \rightarrow R_\lambda$. The construction at the start of this section provides a ring R such that Spec $R \cong X$.

It remains to show that R is Bezout. Let $z^1 = (a_\lambda{}^1), \ldots, z^n = (a_\lambda{}^n) \in R$. Then, since R_o is Bezout, there is a $y_o \in R_o$ such that $y_o R_o = (a_o{}^1, \ldots, a_o{}^n)R_o$. For all but a finite number of $\lambda \in \Lambda'$, $\phi_\lambda(a_o{}^i) = a_\lambda{}^i$ for all $i = 1, 2, \ldots, n$. At each of the non-exceptional coordinates let $y_\lambda = \phi(y_o)$, and note that the equations expressing the equality $y_o R_o = (a_o{}^1, \ldots, a_o{}^n)R_o$ hold for each such λ when ϕ_λ is applied. Let $\lambda_1, \ldots, \lambda_t \in \Lambda'$ be the coordinates for which y_λ has not been chosen. Now choose $y_{\lambda_1}, \ldots, y_{\lambda_t}$ so that $y_{\lambda_i} R_{\lambda_i} = (a_{\lambda_i}{}^1, \ldots, a_{\lambda_i}{}^n)R_{\lambda_i}$ for $i = 1, \ldots, t$. If $y = (y_t)$, then clearly $y \in R$ and $yR = (z^1, \ldots, z^n)R$.

(4.3) *Remarks on the topology of* Spec R. Let us now go back and examine the topology of Spec R, where R is the ring constructed at the beginning of this section. We have seen that Spec $R = \bigcup X_\alpha$, where $X_\alpha = \{P \in \text{Spec } R | P \supseteq A_\alpha\}$. Thus the X_α are closed disjoint subspaces of Spec R; so it follows that any closed

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set C of Spec R is a disjoint union of closed sets, $C = \bigcup C_\alpha$, where $C_\alpha = C \cap X_\alpha$. Conversely, what sets of the form $\bigcup C_\alpha$ are closed in Spec R?

This question seems difficult to answer in general; so let us answer it in the situation used in proving Theorem 4.1 above, namely, the homomorphisms ϕ_λ, $\lambda \in \Lambda'$, are of the type $\phi_\lambda : R_o \to R_o/\mathscr{M} \to R_\lambda$, where $\mathscr{M}$ is a fixed maximal ideal of R_o. We claim that if C is closed in Spec R, then either

i) $\mathscr{M} \in C_o$, (the C_α, $\alpha \in \Lambda'$, may be arbitrary closed subsets of the X_α), or

ii) $\mathscr{M} \notin C_o$ and $C_\alpha = \emptyset$ for all but a finite number of $\alpha \in \Lambda'$.

To see this, we shall use the fact that the sets of the form

$$V(z) = \{P \in \text{Spec } R | z \in P\}$$

for $z = (a_\lambda) \in R$ are a basis for the closed sets of Spec R. We first describe the possible $V(z)$.

i) If $z = (a_\lambda)$ and $a_o \in \mathscr{M}$, then $C_o = V_{R_o}(a_o)$ is a closed subset of X_o containing $\mathscr{M}$. Moreover, $\phi_\lambda(a_o) = a_\lambda = 0$ for all but a finite number of $\lambda \in \Lambda'$, so $C_\lambda = X_\lambda$ for these λ. For at most a finite number of $\lambda \in \Lambda'$, $a_\lambda \neq \phi_\lambda(a_o)$, so for these coordinates $C_\lambda = V_{R_\lambda}(a_\lambda)$.

ii) If $a_o \notin \mathscr{M}$, then $\mathscr{M} \notin C_o$; and since $a_\lambda = \phi_\lambda(a_o)$ is a unit for all but a finite number of $\lambda \in \Lambda'$, $C_\lambda = \emptyset$ for all but a finite number of $\lambda \in \Lambda'$.

By intersecting sets of the above type, one thus verifies the claim.

Note that there is a similarity between this topology on Spec R and the topology described in § 3.

In connection with (4.1) we would like to raise the following:

(4.4) *Question.* If X is the ordered disjoint union of partially ordered sets X_λ, $\lambda \in \Lambda$, and if X is spectral, then are the X_λ also spectral?

5. Constructing spectral topologies. Let X be a partially ordered set that is the disjoint union of two proper subsets X_1 and X_2, where X_1 is closed under $\geqq$ in X. Let X_1 and X_2 be given the induced order from X. Note that here, in contrast with the ordered disjoint union in § 4, we can have an element of X_2 less than an element of X_1. Suppose that for $i = 1, 2$, X_i has an order compatible topology $\mathscr{T}_i$ which is also compatible with the order of X in the sense that $S_X(x) \cap X_i$ is closed in X_i, for all $x \in X$. (This places an additional restriction on X_1, but not on X_2.) With these assumptions, we define a topology $\mathscr{T}$ for X by:

$C \subseteq X$ is $\mathscr{T}$-closed if and only if

a) $C = C_1 \cup C_2$, C_i closed in X_i, and

b) C is closed under $\geqq$ in X.

For the remainder of this section, any reference to X, X_1, and X_2 assumes the situation described above. Once topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ are defined, or assumed to exist, the topology $\mathscr{T}$ is defined in the manner above.

(5.1) *Observations.*

a) $\mathscr{T}$ induces the topology $\mathscr{T}_i$ on X_i, $i = 1, 2$, and X_1 is closed in $\mathscr{T}$. Also, $\mathscr{T}$ is order compatible and hence T_0.

b) If X_i is quasi-compact in the $\mathscr{T}_i$ topology, $i = 1, 2$, then X is quasi-compact.

Proof. Suppose $\bigcap C_\alpha = \emptyset$ where $C_\alpha = C_\alpha{}^1 \cup C_\alpha{}^2$, $C_\alpha{}^i$ closed in X_i. Then $\emptyset = \bigcap (C_\alpha{}^1 \cup C_\alpha{}^2) \supseteq (\bigcap C_\alpha{}^1) \cup (\bigcap C_\alpha{}^2)$ implies $\bigcap C_\alpha{}^1 = \emptyset$ and $\bigcap C_\alpha{}^2 = \emptyset$. As each X_i is quasi-compact, there exist $\alpha_1, \ldots, \alpha_t$ such that $C_{\alpha_1}{}^i \cap \ldots \cap C_{\alpha_t}{}^i = \emptyset$ for $i = 1, 2$. Obviously, $C_{\alpha_1} \cap \ldots \cap C_{\alpha_t} = \emptyset$.

b′) If S_1 is a quasi-compact subset of $(X_1, \mathscr{T}_1)$, then $G_X(S_1)$ is quasi-compact in $(X, \mathscr{T})$.

Proof. Begin with an open cover of $G_X(S_1)$ and choose a finite subset that covers S_1. Because $\mathscr{T}$-open sets are closed under $\leqq$, this finite subset covers $G_X(S_1)$.

c) If $C \subseteq X$ is an irreducible closed set, then $C \cap X_2$ is either the empty set or it is an irreducible closed subset of X_2.

Proof. Let $C = C_1 \cup C_2$, C_i closed in X_i, and suppose $C_2 \neq \emptyset$. If C_2 is not irreducible in X_2, then write $C_2 = A \cup B$, where A, B are proper closed subsets of X_2. It follows that $C = (C_1 \cup A) \cup (C_1 \cup B)$ with $(C_1 \cup A)$, $(C_1 \cup B)$ closed in X. This contradicts the fact that C is irreducible in X.

d) If every irreducible closed set of $(X_i, \mathscr{T}_i)$, $i = 1, 2$, has a generic point, then every irreducible closed set of $(X, \mathscr{T})$ does also.

Proof. Let C be an irreducible closed set of X. If $C \cap X_2 = \emptyset$, then C is an irreducible closed set of X_1; and then the generic point for C in X_1 is also a generic point for C in X. On the other hand, if $C \cap X_2 \neq \emptyset$, then by c), $C \cap X_2$ is an irreducible closed set in X_2 and hence has a generic point x in X_2. But then $C = (C \cap X_1) \cup S_X(x)$ implies $C = S_X(x)$ by the irreducibility of C in X. Thus, x is a generic point for C in X.

e) If $\mathscr{B}_i$ is an open basis for $\mathscr{T}_i$, $i = 1, 2$, then let $\mathscr{B} = \{G_X(U_1) \cup U_2 | U_i \in \mathscr{B}_i$ and $G_X(U_1) \cup U_2$ is $\mathscr{T}$-open$\}$. If for each non-empty $U_1 \in \mathscr{B}_1$ and for each $\mathscr{T}_2$-open set W_2 such that $W_2 \supseteq G_X(U_1) \cap X_2$, there is a $U_2 \in \mathscr{B}_2$ such that $U_2 \subseteq W_2$ and $U_2 \cup (G_X(U_1) \cap X_2)$ is $\mathscr{T}_2$-open, then $\mathscr{B}$ is an open basis for $\mathscr{T}$.

Proof. Any open set in X is of the form $W = W_1 \cup W_2$ where W_i is $\mathscr{T}_i$-open, $i = 1, 2$. If $y \in W_1$, choose $U_1 \in \mathscr{B}_1$ such that $y \in U_1 \subseteq W_1$. W is closed under $\leqq$, so $G_X(U_1) \cap X_2 \subseteq W_2$. Our assumption in e) provides a $U_2 \in \mathscr{B}_2$ such that $U_2 \subseteq W_2$ and $(G_X(U_1) \cap X_2) \cup U_2$ is open in X_2. Thus $y \in G_X(U_1) \cup U_2 \subseteq W$ and $G_X(U_1) \cup U_2 \in \mathscr{B}$. For $y \in W_2$, choose $U_2 \in \mathscr{B}_2$ such that $y \in U_2 \subseteq W_2$. Now the choice $U_1 = \emptyset$ gives $U_2 \in \mathscr{B}$.

We shall use 5.1 (e) in three places. In the proof of Theorem 5.2, if $W = W_1 \cup W_2$ and $W_1 \neq \emptyset$, we will have $W_2 = X_2$. In the proof of Theorem 5.3, if $W_1 \neq \emptyset$, we will be able to say $W_2 \in \mathscr{B}_2$; and in the proof of Theorem 5.8 every $G_X(U_1)$ will be open in X.

We shall now prove some theorems which give sufficient conditions for the topology $\mathscr{T}$ to be spectral.

(5.2) THEOREM. *Suppose $(X_1, \mathscr{T}_1)$ is spectral, and suppose $(X_2, \mathscr{T}_2)$ satisfies the axioms for a spectral space, with the exception that X_2 need not be quasi-compact. If there is a $\mathfrak{p} \in X_1$ such that $x_2 < \mathfrak{p} \leqq x_1$ for all $x_i \in X_i$, $i = 1, 2$, then $(X, \mathscr{T})$ is spectral.*

Proof In view of 5.1(a), 5.1(b′), and 5.1(d), we need only verify that $\mathscr{T}$ has a basis of quasi-compact open sets which is closed under finite intersections. For $\mathscr{B}_1$ and $\mathscr{B}_2$ we choose the quasi-compact open sets of $\mathscr{T}_1$ and $\mathscr{T}_2$. If $U_1 \neq \emptyset$ is quasi-compact open in X_1, then $X_2 \subseteq G_X(U_1)$. Thus the basis $\mathscr{B}$ defined in 5.1(e) is an open basis. By 5.1(b′), $G_X(U_1)$ is quasi-compact if U_1 is quasi-compact. Thus the sets in $\mathscr{B}$ are quasi-compact. That $\mathscr{B}$ is closed under finite intersections follows from the fact that sets in $\mathscr{B}$ are either of the form U_2, where $U_2 \in \mathscr{B}_2$, or $U_1 \cup X_2$, where $U_1 \in \mathscr{B}_1$ and $\mathfrak{p} \in U_1$.

(5.3) THEOREM. *Suppose $(X_1, \mathscr{T}_1)$ is spectral and that there is an $m \in X_2$ such that*

a) *$m \leqq y$ for all $y \in X_1$, and*

b) *if $y \in X_2$ has infinitely many elements of X_2 below (resp. above) y, then $y \geqq m$ (resp. $y \leqq m$).*

Then X is spectral.

Proof. Let $\mathscr{T}_2$ be the $C(m)$-topology for X_2. We proved in Theorem 3.2 that $(X_2, \mathscr{T}_2)$ is spectral. The definition of the $C(m)$-topology also shows that any open set of $\mathscr{T}_2$ which contains m is quasi-compact. Again, let $\mathscr{B}_1$ and $\mathscr{B}_2$ be the quasi-compact open sets of $\mathscr{T}_1$ and $\mathscr{T}_2$. Using 5.1(a), (b), (d), and (e), we have X is spectral if the basis $\mathscr{B}$ is closed under finite intersections. In this situation $\mathscr{B}$ can also be described as $\{U_1 \cup U_2 | U_i$ is quasi-compact open in $\mathscr{T}_i$ and $U_1 \cup U_2$ is closed under $\leqq\}$. It follows that $\mathscr{B}$ is closed under finite intersections.

(5.4) *Applications.*

a) In terms of a 1-dimensional partially ordered set X, one can apply Theorem 5.3 as follows. Write $X = X_1 \cup X_2$, where $X_1 = \{$ht 1 elements of $X\}$ and $X_2 = \{$ht 0 elements of $X\}$. Theorem 5.3 asserts that if there is a spectral topology for X_1 which includes all sets of the form $S_X(x) \cap X_1$, $x \in X$, among its closed sets, and if there is an $m \in X_2$ such that m lies below every element of X_1, then X is spectral. In Example 2.1, we constructed a set of this type which was not spectral precisely because no spectral topology for X_1 could include the sets $S_X(x) \cap X_1$ as closed sets.

b) Begin with a partially ordered set X_1 and a collection $\mathscr{C}$ of non-empty subsets of X_1 such that $X_1 \in \mathscr{C}$. Define a set $X_2 = \{x_C | C \in \mathscr{C}\}$. Form a larger set $X = \mathscr{C}(X_1) = X_1 \cup X_2$ and order X by specifying that if $y \in X_1$, $y \geqq x_C$ if and only if $y \in C$. In addition, we preserve any order relations that may exist in X_1.

THEOREM. *Let $\mathscr{T}_1$ be an order compatible topology for the partially ordered set X_1. Then $(X_1, \mathscr{T}_1)$ is spectral and every set in $\mathscr{C}$ is $\mathscr{T}_1$-closed if and only if there exists an order compatible spectral topology $\mathscr{T}$ for $X = \mathscr{C}(X_1)$ such that $\mathscr{T}|_{X_1} = \mathscr{T}_1$ and X_1 is $\mathscr{T}$-closed.*

Proof. ($\Rightarrow$) Apply Theorem 5.3.

($\Leftarrow$) X_1 is closed and $(X, \mathscr{T})$ is spectral; hence $\mathscr{T}_1 = \mathscr{T}|_{X_1}$ is spectral. The rest of the assertion is immediate.

(5.5) *Example.* The $\Rightarrow$ direction of the above theorem is false without the initial assumption that $X_1 \in \mathscr{C}$. For, let $X_1 = \{y_1, y_2, \ldots\}$, $X_2 = \{x_1, x_2, \ldots\}$ and $X = X_1 \cup X_2$, with the order on X defined by

$$x_j < y_i \text{ if and only if } j \geqq i.$$

Then $X = \mathscr{C}(X_1)$ where $\mathscr{C} = \{\{y_1\}, \{y_1, y_2\}, \{y_1, y_2, y_3\}, \ldots\}$. (See Figure 2.)

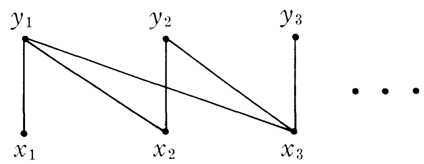

FIGURE 2

This is just the example of Hochster given in [**5**] with the order reversed. Since $\bigcap_i G(y_i) = \emptyset$, property (H) fails and X is not spectral. However, X_1 is spectral, and in fact, the $C(y_1)$-topology (see § 3) is spectral and contains every set in $\mathscr{C}$ among the closed sets.

(5.6) *Example.* The above example leads to another that shows how the ordering of a spectral set can influence an algebraic property of any corresponding ring. Let X be the set defined above and define an ordering on $X' = X \cup \{m\}$ by requiring that, in addition to the ordering of X, $m < x$ for all $x \in X$. It is easily verified that the *COP* topology for X' is spectral, since in this topology all closed sets $\neq X'$ are finite.

Since X' is spectral and has a unique minimal element, there exists a domain D such that Spec $D \cong X'$. However, X itself is not spectral and thus cannot be a closed subset of X'; so it follows that the intersection of the non-zero primes of Spec D must be (0). Thus, in this case the ordering of Spec D implies a very concrete algebraic property of D.

To carry this a bit further, consider what happens when the ordering of X' is reversed. Then any ring R such that Spec R is this new X' has a unique

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maximal prime $\mathscr{M}$ such that $\mathscr{M}$ is the union of the non-maximal primes, because the set of non-maximal primes cannot be spectral.

(5.7) We shall now describe a second set of conditions which enable us to define topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ in such a way that $(X, \mathscr{T})$ will be spectral. Define an equivalence relation on X_1 as follows:

For $x, x' \in X_1$, $x \sim x'$ if and only if $G_X(x) \cap X_2 = G_X(x') \cap X_2$

(i.e. the elements in X_2 below x coincide with the elements in X_2 below x'). Similarly, define an equivalence relation on X_2 by:

For $x, x' \in X_2$, $x \sim x'$ if and only if $S_X(x) \cap X_1 = S_X(x') \cap X_1$.

We will denote the equivalence class of $x \in X$ by $[x]$.

Consider the following conditions on the order of X:

i) For $i = 1, 2$, if $x, x' \in X_i$ and $x < x'$, then $x \sim x'$, and

ii) For each $x \in X_2$, there exist $y_1, \ldots, y_t \in X_1$ such that $S_X(x) \cap X_1 = \bigcup_{j=1}^{t} [y_j]$; and similarly, for each $x \in X_1$, there exist $y_1, \ldots, y_t \in X_2$ such that $G_X(x) \cap X_2 = \bigcup_{j=1}^{t} [y_j]$.

Note that (5.7.i) implies that for $i = 1, 2$, X_i is the ordered disjoint union of the equivalence classes in X_i. For each $x \in X$, we give $[x]$ the order induced from X.

(5.8) THEOREM. *If* $X = X_1 \cup X_2$ *satisfies* (5.7(i) *and* (ii)) *and for each* $x \in X$, $[x]$ *is spectral, then* X *is spectral.*

Proof. Let us assume that we begin with a spectral topology for each $[x]$, $x \in X$. We can define an order compatible spectral topology $\mathscr{T}_1$ for X_1 in the manner of (4.3). Fix an element $m \in X_1$, and define a set $C \subseteq X_1$ to be $\mathscr{T}_1$-closed if and only if $C \cap [x]$ is closed in $[x]$ for all $x \in X_1$, and either

i) $m \in C$, or

ii) $m \notin C$ and $C \cap [x] = \emptyset$ for all but a finite number of equivalence classes in X_1.

If m is maximal, this is the spectral topology described in § 4. It is easy to check that for any m this gives a spectral topology for X_1. Now if $x \in X_2$, then $S_X(x) \cap X_1$ is a finite union of equivalence classes and is therefore $\mathscr{T}_1$-closed. Thus $\mathscr{T}_1$ is compatible with the order of X in the sense described at the start of § 5.

Now define an order compatible topology (usually not spectral) $\mathscr{T}_2$ on X_2 by

$C \subseteq X_2$ is $\mathscr{T}_2$-closed if and only if $C \cap [x]$ is closed in $[x]$, for all $x \in X_2$.

The topology on each $[x] \subseteq X_2$ is spectral; so we can choose a quasi-compact open basis for $\mathscr{T}_2$ using finite unions of sets each of which is a quasi-compact open subset of some $[x]$. This basis is closed under finite intersections. Note that an irreducible closed subset of X_2 must be contained in some $[x]$, and thus has a generic point.

Let $\mathscr{T}$ be the topology for X defined as before. By (5.1.a), $\mathscr{T}$ is T_0, and by (5.1.d), $\mathscr{T}$ has the generic point property. If each $x \in X_2$ is less than some

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element of X_1, then we use (5.1.b′) to show X is quasi-compact. On the other hand, if there is an $x \in X_2$ such that $S_X(x) \cap X_1 = \emptyset$, then $X = G_X(X_1) \cup [x]$. Since $[x]$ is closed in X, it is quasi-compact in X by the assumption that it is spectral. Since $G_X(X_1)$ is also quasi-compact, by (5.1.b′), it follows that X is quasi-compact.

It remains to verify that $\mathscr{T}$ has a basis of quasi-compact open sets which is closed under finite intersections. Let $\mathscr{B}$ be all sets of the form $G_X(U_1) \cup U_2$, where U_i is a quasi-compact open set in X_i. First note that if $x \in G_X(U_1) \cap X_2$, then $[x] \subseteq G_X(U_1)$. It follows that $G_X(U_1)$ is $\mathscr{T}$-open, because $G_X(U_1) \cap X_2$ is a union of equivalence classes of X_2 and hence is open in X_2. Any $\mathscr{T}$-open cover for U_1 covers $G_X(U_1)$, so U_1 quasi-compact implies $G_X(U_1)$ is quasi-compact. If U_2 is a quasi-compact open set in X_2, then it is also a quasi-compact $\mathscr{T}$-open set. Thus, $\mathscr{B}$ consists of quasi-compact open sets. By 5.1.e, $\mathscr{B}$ is then a quasi-compact open basis for $\mathscr{T}$.

Now $\mathscr{B}$ is closed under finite unions; so to show it is closed under finite intersections, it suffices to show that an intersection of two elements of $\mathscr{B}$ is a finite union of elements of $\mathscr{B}$. Let $G_X(U_1) \cup U_2$ and $G_X(U_1') \cup U_2'$ be two elements of $\mathscr{B}$.

First consider $U_2 \cap U_2'$. As U_2, U_2' are quasi-compact open in $\mathscr{T}_2$ as well as $\mathscr{T}$, so is their intersection. Next consider $G_X(U_1) \cap U_2'(G_X(U_1') \cap U_2$ is, of course, similar). To be quasi-compact in the $\mathscr{T}_2$ topology, U_2' must be contained in a finite union of equivalence classes. Let us say $U_2' = W_1 \cup \ldots \cup W_t$, W_j quasi-compact open and $W_j \subseteq [x_j]$ for some $x_j \in X_2$. If $W_j \cap G_X(U_1) \neq \emptyset$, then $W_j \subseteq [x_j] \subseteq G_X(U_1)$. Thus, $G_X(U_1) \cap U_2' = \cup \{W_j | W_j \subseteq G_X(U_1)\} \in \mathscr{B}$.

Finally, consider $G_X(U_1) \cap G_X(U_1')$. Note that $G_X(U_1) \cap G_X(U_1') = G_X(U_1 \cap U_1') \cup (\cup_\alpha \{[y_\alpha] | y_\alpha \in X_2$, and $[y_\alpha] \subseteq (G_X(U_1) \cap G_X(U_1'))\backslash G_X(U_1 \cap U_1')\}$. It will suffice to show that this collection of $[y_\alpha]$'s is finite, since then their union is a quasi-compact open set in the $\mathscr{T}_2$ topology. Consider, therefore, two cases:

i) $m \notin U_1 \cap U_1'$ (where m is the defining point for the $\mathscr{T}_1$ topology). We assume $m \notin U_1$; so U_1 quasi-compact implies U_1 is covered by a finite number of equivalence classes, say $[x_1], \ldots, [x_t]$, where $x_i \in X_1$. Thus, $G_X(U_1) \cap X_2 \subseteq \bigcup_{i=1}^t (G_X(x_i) \cap X_2)$. But each $G_X(x_i) \cap X_2$ is a finite union of equivalence classes. Thus the $[y_\alpha]$'s are chosen from a finite set.

ii) $m \in U_1 \cap U_1'$. From the definition of $\mathscr{T}_1$ we see that all but a finite number of equivalence classes of X_1 are contained in $U_1 \cap U_1'$. Thus, as in i) above, the $[y_\alpha]$'s will be chosen from a finite set of equivalence classes.

Our main object in proving the above theorem is to obtain the following corollary. However, we feel that the generality of the setting makes the pieces of the proof fit better than would be the case if we merely proved the corollary directly.

(5.9) COROLLARY. *Let X be a* 1-*dimensional partially ordered set, let X_1 be the* ht 1 *elements and X_2 be the* ht 0 *elements. If for every $x \in X_2$, $S_X(x) \cap X_1$ is a* *finite union of equivalence classes and for every* $x \in X_1$, $G_X(x) \cap X_2$ *is a finite union of equivalence classes, then* X *is spectral.*

Proof. We have assumed (5.7.ii), and (5.7.i) is trivially satisfied. Moreover, each $[x]$ is 0-dimensional; and as we noted in § 1, such sets are spectral.

A special case of the above corollary that deserves emphasis is the following:

(5.10) COROLLARY. *Let* X *be a* 1*-dimensional partially ordered set, let* X_1 *be the* ht 1 *elements and* X_2 *be the* ht 0 *elements. If for all* $x \in X_2$, $S_X(x) \cap S_X(y) = \emptyset$ *for all but finitely many* $y \in X_2$, *and for all* $x \in X_1$, $G_X(x) \cap G_X(y) = \emptyset$ *for all but finitely many* $y \in X_1$, *then* X *is spectral.*

In pushing this kind of investigation further, one might try next the following question: If X is a 1-dimensional partially ordered set such that for all $x \neq y \in X$, $S(x) \cap S(y)$ and $G(x) \cap G(y)$ are finite, is X spectral?

6. The spectral topologies on a countable 1-dimensional partially ordered set with unique minimal element. For some partially ordered sets there is only one order compatible spectral topology. For example, a finite partially ordered set is always spectral, and the *COP* topology is the only possible order compatible spectral topology. A totally ordered set is spectral if and only if it has properties ($K1$) and ($K2$) [**5**, Theorem 3.1], and since every closed set is irreducible, the *COP* topology is again the only order compatible spectral topology. On the other hand, an order compatible topology for a 0-dimensional set is spectral if and only if it is Boolean. As we said in § 1, for countable sets these topologies have been characterized by Pierce in [**7**].

Let Y be a 0-dimensional set and let $X = Y \cup \{\theta\}$, where $\theta \leqq y$ for all $y \in Y$. We assume that Y is infinite, since the *COP* topology is the only order compatible spectral topology if Y is finite. With the additional assumption that Y is countable, we will be able to describe all order compatible spectral topologies for X (using, of course, Pierce's description of countable Boolean spaces); but for the moment, we can avoid assuming Y is countable. There are two cases to consider, depending on whether Y is or is not a closed subset of X in a given spectral topology. Consider first the spectral topologies for X for which Y is closed.

(6.1) THEOREM. *There is a one to one correspondence between Boolean topologies on* Y *and order compatible spectral topologies on* X *for which* Y *is a closed subset.*

Proof. If Y is closed in a spectral topology for X, then Y is spectral, hence Boolean, when given the induced topology. Conversely, if Y has a Boolean topology, define $C \subseteq X$ to be closed if and only if $C = X$ or C is a closed subset of Y. Obviously, this is a spectral topology.

We now want to consider spectral topologies for X where Y is not a closed subset of X. Suppose Y is written as the ordered disjoint union of subsets

Y_λ, $\lambda \in \Lambda$, where Λ is an infinite set. For each Y_λ, choose a T_1 topology $\mathscr{T}_\lambda$. An order compatible topology for Y is obtained by defining

a) $C \subseteq Y$ is closed if and only if $C = Y$ or
$C = C_{\lambda_1} \cup \ldots \cup C_{\lambda_t}$, C_{λ_i} a closed subset of Y_{λ_i},

and an order compatible topology for X is obtained by defining

b) $C \subseteq X$ is closed if and only if $C = X$ or C is a proper closed subset of Y.

Thus, Y is not a closed subset of X, and one can easily verify that X is spectral if each Y_α is spectral (Boolean). In this situation, Y satisfies all the necessary properties for being spectral, except that Y is irreducible but does not have a generic point.

Now let Y be countable, and let $\mathscr{T}$ be an order compatible spectral topology for X for which Y is not a closed set. We will show that $\mathscr{T}$ arises in the manner above. Obviously, $\mathscr{T}$ arises from $\mathscr{T}|_Y$ by way of b) above. Also, any proper closed subset of Y is Boolean.

It remains to see that Y can be written as a disjoint union of closed subsets Y_λ, $\lambda \in \Lambda$, such that the topology of Y can be built from the induced topologies on the Y_λ using a). Note that Y is T_1, quasi-compact, irreducible, and has a basis of quasi-compact open sets which is closed under finite intersections. Our task is completed by the following two lemmas.

(6.2) LEMMA. *Suppose Y is T_1, quasi-compact, irreducible, and has a quasi-compact open basis. Then $Y = \bigcup_{i=1}^\infty Y_i$, where this is a disjoint union and $Y \setminus Y_i$ is quasi-compact open for all i.*

Proof. Order Y (i.e., let $Y = \{y_1, y_2, \ldots\}$). Choose a quasi-compact open set U_1 such that $y_2 \in U_1$, $y_1 \notin U_1$. Let $Y_1 = Y \setminus U_1$. We have $y_1 \in Y_1$ and $Y \setminus Y_1 = U_1$. Y_1 is closed, so it is quasi-compact. Cover Y_1 with quasi-compact open sets that miss y_2, and take a finite subcover. Since a finite union of quasi-compact open sets is quasi-compact open, we can choose a quasi-compact open set U_2 such that $Y_1 \subseteq U_2$, and $y_2 \notin U_2$. Let $Y_2 = Y \setminus U_2$. We have $y_2 \in Y_2$, $Y \setminus Y_2 = U_2$, and $Y_1 \cap Y_2 = \emptyset$. Note that $Y \neq Y_1 \cup Y_2$ since Y is irreducible. Let y_{i_3} be the first $y_i \notin Y_1 \cup Y_2$. As $Y_1 \cup Y_2$ is closed, we can choose a quasi-compact open set U_3 such that $Y_1 \cup Y_2 \subseteq U_3$, and $y_{i_3} \notin U_3$. Let $Y_3 = Y \setminus U_3$, and we have $y_{i_3} \in Y_3$, $Y \setminus Y_3 = U_3$, and Y_1, Y_2, Y_3 are pairwise disjoint. Y irreducible now gives $Y \neq Y_1 \cup Y_2 \cup Y_3$. Repeating this process, we inductively define the Y_i.

(6.3) LEMMA. *Suppose $Y = \bigcup_{i=1}^\infty Y_i$, where this is a disjoint union and $Y \setminus Y_i$ is quasi-compact open for all i, and that every proper closed subset of Y is Boolean. Then* a) *is satisfied.*

Proof. Since every Y_i is closed in Y, the implication $\Leftarrow$ is immediate. Suppose then $C \neq Y$ is closed in Y. Then $C = \bigcup C_i$, where $C_i = C \cap Y_i$ is closed in Y_i. We must show $C_i = \emptyset$ for all but a finite number of i. Now $Y \setminus Y_i$ is quasi-compact open in Y, so $C \cap (Y \setminus Y_i)$ is quasi-compact open in the relative topology for C. But C is a Boolean space; hence $C \cap (Y \setminus Y_i)$ must also be closed in C. Since this set is the complement of C_i in C, C_i is open in C. Thus $\{C_i\}$ is an open cover of C by pairwise disjoint sets. As C is quasi-compact, $C_i = \emptyset$ for all but a finite number of i.

Remark. Different partitions for Y can still yield the same spectral space X.

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