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# VERTEX-PRIMITIVE HALF-TRANSITIVE GRAPHS

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**Abstract**

Given an infinite family of finite primitive groups, conditions are found which ensure that almost all the orbitals are not self-paired. If $p$ is a prime number congruent to $\pm 1 (\mathrm{mod} 10)$, these conditions apply to the groups $PSL(2, p)$ acting on the cosets of a subgroup isomorphic to $A_5$. In this way, infinitely many vertex-primitive $\frac{1}{2}$-transitive graphs which are not metacirculants are obtained.

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## 1. Introduction

Let $X = (V(X), E(X))$ be a simple undirected graph. We call an ordered pair of adjacent vertices an *arc* of $X$. Let $G$ be a subgroup of Aut $X$. The graph $X$ is said to be $G$*-vertex-transitive*, $G$*-edge-transitive*, or $G$*-arc-transitive* if $G$ acts transitively on the set of vertices, edges, or arcs of $X$, respectively. Furthermore, $X$ is said to be *vertex-transitive*, *edge-transitive*, or *arc-transitive*, if it is Aut $X$-vertex-transitive, Aut $X$-edge-transitive, or Aut $X$-arc-transitive, respectively. We call a graph $\frac{1}{2}$*-transitive*, if it is vertex-transitive, edge-transitive, but not arc-transitive.

The first examples of $\frac{1}{2}$-transitive graphs were found by I. Z. Bouwer [5] in 1970. He found an infinite family of them. In 1981 D. F. Holt [7] found an example with 27 vertices. Very recently, B. Alspach, D. Marušič and L. Nowitz [1] found several infinite families of these graphs of degree 4: Holt's graph occurs as the smallest example in one of their families. B. Alspach and the second author [3] determined all such graphs of order $3p$ with $p$ a prime number greater than 3. All these graphs have automorphism groups acting imprimitively on their vertices, and all of them are



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so-called metacirculants, defined by Alspach and Parsons [2]. In [8] D. Holton asked if the automorphism group of a $\frac{1}{2}$-transitive graph is necessarily imprimitive, and we may ask if a $\frac{1}{2}$-transitive graph is necessarily metacirculant. These two questions have been answered by Praeger and the second author [11] in the negative. They found several examples of $\frac{1}{2}$-transitive graphs with order a product of two distinct primes. In each case the automorphism group is $PSL(2, p)$ acting on the cosets of a maximal subgroup isomorphic to $A_5$ or $S_4$, and some of them are metacirculants, some are not.

In this paper we exhibit infinitely many vertex-primitive $\frac{1}{2}$-transitive graphs, most of which are not metacirculants. This also answers the two questions mentioned above. More precisely, the main result of this paper is Theorem 3.1 in Section 3, which shows that for some primitive groups almost all orbitals are not self-paired and their underlying undirected graphs are $\frac{1}{2}$-transitive. As an example, we consider the (primitive) action of the group $PSL(2, p)$ on the cosets of a maximal subgroup isomorphic to $A_5$. For each prime $p$ for which $A_5$ is maximal in $PSL(2, p)$ we calculate the exact number of non-self-paired orbitals. The result is presented in §4. To do this we use the same ideas and methods used in [11]. We quote some technical lemmas from [11] in the next section.

The link between groups and graphs that we use is the concept of the orbital graph of a permutation group.

Let $G$ be a transitive permutation group on $\Omega$. Consider the natural action of $G$ on $\Omega \times \Omega$. Assume that $\Gamma_0, \Gamma_1, \ldots, \Gamma_{r-1}$ are the orbits of $G$ on $\Omega \times \Omega$, where

$$\Gamma_0 = \{ (\alpha, \alpha) \mid \alpha \in \Omega \}.$$

We call these orbits the *orbitals* of $G$ and call $\Gamma_0$ the *trivial* orbital. The number $r$ is called the *rank* of $G$. Each nontrivial orbital $\Gamma_i$ can be viewed as a $G$-arc-transitive directed graph, and if $(\alpha, \beta) \in \Gamma_i$ is equivalent to $(\beta, \alpha) \in \Gamma_i$ for any $\alpha, \beta \in \Omega$, then $\Gamma_i$ can be viewed as an undirected graph, identifying two directed edges $(\alpha, \beta)$ and $(\beta, \alpha)$ with one undirected edge $\alpha\beta$. We call this (directed or undirected) graph $\Gamma_i$ an *orbital graph* of $G$. If the graph is undirected, the orbital is said to be *self-paired*. Thus, given $G$, all $G$-arc-transitive (undirected) graphs can be found by finding all self-paired orbitals of $G$. If the full automorphism group of the underlying undirected graph of a non-self-paired orbital digraph is $G$, then this graph is $\frac{1}{2}$-transitive.

Take a point $\alpha \in \Omega$. The orbits of the stabilizer $G_\alpha$ on $\Omega$ are called *suborbits* of $G$. There is a one-to-one correspondence between the suborbits and the orbitals of $G$. For each orbital $\Gamma_i$,

$$\Delta_i = \{ \beta \in \Omega \mid (\alpha, \beta) \in \Gamma_i \}$$

is a suborbit, and for each suborbit $\Delta_i$,

$$\Gamma_i = \{ (\alpha, \beta)^g \mid g \in G,\ \beta \in \Delta_i \}$$

is an orbital of $G$. Thus the number of suborbits is the rank of $G$.

It is well known that $G$ is primitive if and only if all orbital graphs of $G$ are connected. This ensure that all the graphs that we find are connected. The group- and graph-theoretic notation and terminology used in this paper are standard; the reader can refer to [9] and [4] when necessary.

In the next section we give some group-theoretic lemmas which will be used later. In Section 3, the main theorem of the paper is proved. An example is given in Section 4.

## 2. Preliminary lemmas

Since the 2-dimesional projective linear groups $PSL(2, p)$ and $PGL(2, p)$ over the field with $p$ elements are the background of our investigation in this paper, we need information about the subgroup structure of these groups. The following theorem is due to Dickson; the reader may refer to [9, II, §8] or [6] for a proof.

THEOREM 2.1. *Assume that $p$ is a prime number greater than* 11. *Then*

(a) *the maximal subgroups of $PSL(2, p)$ are:*

- (i) *One class of subgroups isomorphic to $Z_p : Z_{\frac{p-1}{2}}$, where $A : B$ denotes the semidirect product of $A$ and $B$ and $Z_n$ denotes the cyclic group of order $n$.*
- (ii) *One class of subgroups isomorphic to $D_{p-1}$ and one class isomorphic to $D_{p+1}$, where $D_n$ denotes the dihedral group of order $n$.*
- (iii) *Two classes of subgroups isomorphic to $A_5$, if $p \equiv \pm 1 \pmod{10}$. (They are conjugate in $PGL(2, p)$.)*
- (iv) *Two classes of subgroups isomorphic to $S_4$, if $p \equiv \pm 1 \pmod{8}$. (They are conjugate in $PGL(2, p)$.)*
- (v) *One class of subgroups isomorphic to $A_4$, if $p \equiv 3, 13, 27$, or $37 \pmod{40}$.*

(b) *the maximal subgroups of $PGL(2, p)$ are:*

- (i) *One class of subgroups isomorphic to $Z_p : Z_{p-1}$.*
- (ii) *One class of subgroups isomorphic to $D_{2(p-1)}$ and one class isomorphic to $D_{2(p+1)}$.*
- (iii) *One class of of subgroups isomorphic to $S_4$.*
- (iv) *$PSL(2, p)$.*

To determine suborbits for primitive groups we need the following result of Manning ([11, Lemma 2.1]).

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LEMMA 2.2. *Let $G$ be a transitive group on $\Omega$, let $H = G_\alpha$ for some $\alpha \in \Omega$, and let $K \leq H$. Suppose that the set of $G$-conjugates of $K$ which are contained in $H$ form $t$ conjugacy classes of $H$, with representatives $K_1, K_2, \ldots, K_t$. Then $K$ fixes*

$$\sum_{i=1}^{t} |N_G(K_i) : N_H(K_i)|$$

*points of $\Omega$. In particular, if $t = 1$, that is if every $G$-conjugate of $K$ in $H$ is conjugate to $K$ in $H$, then $K$ fixes $|N_G(K) : N_H(K)|$ points of $\Omega$.*

Given a subgroup $K$ of a group $H$, there is a natural action of $H$ on the set $[H : K]$ of right cosets of $K$. If $H$ acts on $\Omega$, let $\mu_K(\Omega)$ be the number of $H$-orbits of $\Omega$ isomorphic to $[H : K]$ and let $\mathrm{Fix}_\Omega(K)$ be the set of fixed points of $K$ in $\Omega$. Then $\Omega = \bigcup_L \mu_L(\Omega)\,[H : L]$, and therefore

$$|\,\mathrm{Fix}_\Omega(K)| = \sum_L \mu_L(\Omega)|\,\mathrm{Fix}_{[H:L]}(K)|, \tag{2.1}$$

where the union and summation range over a set of representatives for the conjugacy classes of subgroups of $H$.

We have $\mathrm{Fix}_{[H:L]}(K) = \{\, Lh \mid hKh^{-1} \subseteq L \,\}$ and therefore $|\,\mathrm{Fix}_{[H:L]}(K)| = 0$ unless $L$ contains a conjugate of $K$. Thus ((2.1)) is essentially an upper triangular system of equations and it is easily solved for the $\mu_L(\Omega)$. The coefficients $|\,\mathrm{Fix}_{[H:L]}(K)|$ are given by Lemma 2.2, namely

$$|\,\mathrm{Fix}_{[H:L]}(K)| = \sum_{i=1}^{t} |N_H(K_i) : N_L(K_i)|,$$

where $\{K_1, K_2, \ldots, K_t\}$ is a set of representatives for the conjugacy classes in $L$ of the $H$-conjugates of $K$ that are contained in $L$. In particular, $|\,\mathrm{Fix}_{[H:K]}(K)| = |N_H(K)/K|$.

In the case of a group acting on the cosets of a subgroup isomorphic to $A_5$, these general considerations yield the following lemma (proved in [11]).

LEMMA 2.3. ([11, Lemma 2.2]) *Let $G$ be a primitive permutation group on $\Omega$, and let $H = G_\alpha$ for some $\alpha \in \Omega$. Suppose that $H = A_5$ and let $K_1, \ldots, K_7$ be seven subgroups of $H$ satisfying $K_1 \cong A_4$, $K_2 \cong D_{10}$, $K_3 \cong D_6$, $K_4 \cong Z_5$, $K_5 \cong Z_3$, $K_6 \cong D_4$ and $K_7 \cong Z_2$. Let $k_i$ be the number of points in $\Omega$ fixed by $K_i$, for $i = 1, 2, \ldots, 7$. Then $G$ has* 1 *suborbit of length* 1, $k_1 - 1$ *suborbits of length* 5, $k_2 - 1$ *suborbits of length* 6, $k_3 - 1$ *suborbits of length* 10, $\frac{1}{2}(k_4 - k_2)$ *suborbits of length* 12, $\frac{1}{2}(k_5 - 2k_1 - k_3 + 2)$ *suborbits of length* 20, $\frac{1}{3}(k_6 - k_1)$ *suborbits of length* 15, $\frac{1}{2}(k_7 - 2k_2 - 2k_3 - k_6 + 4)$ *suborbits of length* 30, *and all the other suborbits have length* 60.

To determine which suborbits are self-paired, we need two other lemmas from [11].

LEMMA 2.4. ([11, Lemma 2.2]) *Let $D = \langle a, b\rangle \cong D_{2n}$, $n \geq 2$, be a permutation group on $\Omega = \{1, 2, \dots, n\}$, where*

$$a = (1, 2, \cdots, n) \quad \textit{and} \quad b = (1)(2, n)(3, n-1)\cdots(i, n+2-i)\cdots$$

*Then the nontrivial orbitals of $D$ are $\Delta_i = (1, i)^D = (1, n+2-i)^D$, for $2 \leq i \leq \frac{1}{2}(n+2)$. Each of these orbitals is self-paired. Moreover, for all points $i$, $j$, with $i \neq j$, there is an involution in $D$ which interchanges $i$ and $j$.*

The next lemma is a generalization of [11, Lemma 2.4].

LEMMA 2.5. *Let $G$ be a transitive group on $\Omega$ and let $H = G_\alpha$ for some $\alpha \in \Omega$. Assume that $G$ has $t$ conjugacy classes of involutions, say $C_1, \dots, C_t$. Take a representative $u_j$ in $C_j$. Assume that $u_j$ has $N_j$ cycles of length 2. For a nontrivial self-paired orbital $\Delta$ and a pair $(\alpha, \beta) \in \Delta$, let* $\mathrm{inv}(\Delta)$ *be the number of involutions in $G$ with a* 2*-cycle $(\alpha, \beta)$. Then*

$$\sum_{j=1}^{t} \frac{N_j}{c_j} = \frac{1}{2|H|} \sum_{\Delta=\Delta'} |\Delta(\alpha)|\,\mathrm{inv}(\Delta),$$

*where $c_j$ is the order of the centralizer of $u_j$.*

PROOF. We count the elements of the set

$$M = \{\, (u, \{\alpha, \beta\}) \mid u \text{ is an involution and } (\alpha, \beta) \text{ is a 2-cycle of } u \,\}.$$

On the one hand

$$|M| = \sum_{j=1}^{t} N_j \frac{|G|}{c_j}.$$

On the other hand we can classify the pairs $\{\alpha, \beta\}$ according to the (self-paired) orbital to which they belong. For each self-paired orbital $\Delta$ there are

$$\frac{1}{2}|\Delta(\alpha)|\,|\Omega| = \frac{1}{2}|\Delta(\alpha)|\,|G|/|H|$$

pairs $\{\alpha, \beta\}$ such that $(\alpha, \beta) \in \Delta$ and each of these has $\mathrm{inv}(\Delta)$ associated involutions. Thus

$$|M| = \sum_{\Delta=\Delta'} \frac{1}{2}|\Delta(\alpha)|\frac{|G|}{|H|}\,\mathrm{inv}(\Delta),$$

and the result follows.

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## 3. Main theorem

THEOREM 3.1. *Let $H$ be a finite group, and let $\mathcal{P}$ be an infinite set of positive integers. Assume that $\{G(p) \mid p \in \mathcal{P}\}$ is an infinite family of finite primitive groups acting on a finite set $\Omega(p)$, and that for every $p$, the stabilizer of a point $\alpha \in \Omega(p)$ is a maximal subgroup of $G(p)$ identified with $H$. Then the degree $d(p)$ of $G(p)$ is \G(p)\|/\H\. Let $m(p)$ be the maximum order of the normalizers of nontrivial subgroups of $H$ in $G(p)$, and let $c(p)$ be the minimun order of the centralizers of involutions in $G(p)$. Let $r(p)$ be the rank of $G(p)$, and let $h(p)$ be the number of non-self-paired orbitals of $G(p)$. Assume that $G(p)$ has $t(p)$ classes of involutions, and that $t(p) < t$ for every $p$. Suppose that* $\lim_{p\to\infty}(|G(p)|/m(p)) = \infty$ *and* $\lim_{p\to\infty} c(p) = \infty$. *Then almost all orbitals of $G(p)$ are not self-paired, that is,*

$$\lim_{p\to\infty} \frac{h(p)}{r(p)} = 1.$$

*Furthermore, if $G(p)$ is a maximal subgroup of the symmetric group* $\mathrm{Sym}(\Omega(p))$ *for every $p$, then the underlying undirected graphs of the non-self-paired orbital digraphs are $\frac{1}{2}$-transitive and pairwise non-isomorphic.*

REMARK. The conditions of Theorem 3.1 are satisfied in the following cases:

(a) $H = A_5$, $\mathcal{P}$ is the set of all prime numbers congruent to $\pm 1 \pmod{10}$, and $G(p) = PSL(2, p)$;

(b) $H = S_4$, $\mathcal{P}$ is the set of all prime numbers congruent to $\pm 1 \pmod{8}$, and $G(p) = PSL(2, p)$;

(c) $H = S_4$, $\mathcal{P}$ is the set of primes greater than 5, and $G(p) = PGL(2, p)$.

PROOF OF THEOREM 3.1. Let $K_1, K_2, \ldots, K_s$ be representatives for the conjugacy classes of nontrivial subgroups of $H$, let $\mu_i(p)$ be the number of $H$-orbits in which the stabilizer of a point is conjugate to $K_i$, and let $\mu(p)$ be the number of regular $H$-orbits (i.e., for which the stabilizer of a point is trivial).

Then $\mu_i(p) \le |\operatorname{Fix}_{\Omega(p)}(K_i)|$ and by Lemma 2.2, $|\operatorname{Fix}_{\Omega(p)}(K_i)| \le sm(p)$, whence

$$\mu_i(p) \le sm(p).$$

We have

$$|H|\mu(p) = d(p) - \sum_{i=1}^{s} \mu_i(p)\frac{|H|}{K_i} \ge \frac{G(p)}{|H|} - s^2 m(p),$$

and as $\lim_{p\to\infty} |G(p)|/m(p) = \infty$, it follows that

$$\lim_{p\to\infty} \frac{\mu_i(p)}{\mu(p)} = 0 \quad \text{for every } i. \tag{3.1}$$

In addition,

$$r(p) = \mu(p) + \sum_{i=1}^{s} \mu_i(p) \tag{3.2}$$

and therefore

$$\lim_{p\to\infty} \frac{\mu(p)}{r(p)} = 1. \tag{3.3}$$

Now suppose that there are $\sigma(p)$ self-paired regular $H$-orbits. Then from Lemma 2.5 we have

$$\frac{1}{2}\sigma(p) \le \sum_j \frac{N_j}{c_j} \le \frac{td(p)}{2c(p)},$$

and so

$$\frac{\sigma(p)}{r(p)} \le \frac{d(p)}{r(p)} \frac{t}{c(p)}.$$

On the other hand, it follows from equations (3.1), (3.2) and (3.3) that $\lim_{p\to\infty} d(p)/r(p) = |H|$. Therefore $\lim_{p\to\infty} \sigma(p)/r(p) = 0$ because, by assumption, $\lim_{p\to\infty} c(p) = \infty$.

Since $r(p) \ge h(p) \ge \mu(p) - \sigma(p)$ we have

$$1 \ge \lim_{p\to\infty} \frac{h(p)}{r(p)} \ge \lim_{p\to\infty} \frac{\mu(p)}{r(p)} - \lim_{p\to\infty} \frac{\sigma(p)}{r(p)} = 1.$$

and so

$$\lim_{p\to\infty} \frac{h(p)}{r(p)} = 1.$$

We are assuming that $G(p)$ is a maximal subgroup of the symmetric group $\mathrm{Sym}(\Omega(p))$ and therefore it is the full automorphism group of every orbital graph. In particular, it is the full automorphism group of every non-self-paired orbital digraph and of the underlying undirected graph. Thus these underlying undirected graphs are $\frac{1}{2}$-transitive.

Finally, we shall show that the orbital graphs and digraphs are pairwise non-isomorphic. If $\sigma$ is an isomorphism between two of the orbital graphs, $\Delta_1$ and $\Delta_2$, then $\sigma$ lies in the normalizer in $\mathrm{Sym}(\Omega(p))$ of their common automorphism group $G(p)$. Since $G(p)$ is maximal in $\mathrm{Sym}(\Omega(p))$, it is self-normalizing, and so $\sigma \in G(p)$, whence $\Delta_1 = \Delta_2$, a contradiction. This completes the proof of the theorem.

## 4. An example

THEOREM 4.1. *Suppose that $p \equiv \pm 1 \pmod{10}$ and that $G = G(p) = PSL(2, p)$ acts on the set $\Omega = \Omega(p) = [G : H]$ of right cosets of a subgroup $H \cong A_5$. Then*

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(a) *We have* $d = d(p) = |\Omega| = (p^3 - p)/120$, *the rank of* $G(p)$ *is* $r = r(p)$, *and* $G$ *has* 1 *suborbit of length* 1, $x_1(p)$ *suborbits of length* 5, $x_2(p)$ *suborbits of length* 6, $x_3(p)$ *suborbits of length* 10, $x_4(p)$ *suborbits of length* 12, $x_5(p)$ *suborbits of length* 20, $x_6(p) = 0$ *suborbits of length* 15, $x_7(p)$ *suborbits of length* 30, *and* $x(p)$ *suborbits of length* 60. *All the suborbits of* $G$ *of length less than* 60 *and* $y(p)$ *of the suborbits of length* 60 *are self-paired. The numbers* $x_i = x_i(p)$ *are given in Table* 4.1 *and the numbers* $r = r(p)$, $x = x(p)$, $y = y(p)$ *and* $h = h(p) = x - y$ *are given in Table* 4.2.

(b) *The underlying undirected graphs of the* $h$ *non-self-paired orbital digraphs are* $\frac{1}{2}$*-transitive graphs. These graphs are pairwise non-isomorphic. All have automorphism group* $PSL(2, p)$.

(c) *Almost all orbital graphs of* $PSL(2, p)$ *give* $\frac{1}{2}$*-transitive graphs. That is,*

$$\lim_{p\to\infty} \frac{h(p)}{r(p)} = 1.$$

PROOF OF THEOREM 2.1. Proof By Theorem 2.1, $PSL(2, p)$ has only one class of involutions and the centralizer of an involution is $D_{p+1}$ or $D_{p-1}$, depending on whether 4 divides $p+1$ or $p-1$. Every maximal subgroup of $PSL(2, p)$ has order at most a quadratic polynomial of $p$, and so the conditions of Theorem 3.1 are satisfied. Thus conclusions *(b)* and *(c)* hold.

Next we shall calculate the exact values of $x_i$, $x$, $y$ and $h$ for $G(p) = PSL(2, p)$ and $H = A_5$. We must have $p \equiv \pm 1 \pmod{10}$ and the calculations fall naturally into 16 cases according to the value of $p \pmod{120}$, namely $p \equiv \pm 1, \pm 11, \pm 31, \pm 41, \pm 61, \pm 71, \pm 91$, or $\pm 101 \pmod{120}$. We give details of the proof for the cases $p \equiv \pm 1 \pmod{120}$. The proofs for the other cases are entirely similar and are omitted.

There are 7 conjugacy classes of nontrivial subgroups of $H = A_5$, with representatives $K_1 = A_4$, $K_2 = D_{10}$, $K_3 = D_6$, $K_4 = Z_5$, $K_5 = Z_3$, $K_6 = D_4$ and $K_7 = Z_2$ as in Lemma 2.3. We have the following table, the last line of which comes from Lemma 2.2.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $K_i$ | $A_4$ | $D_{10}$ | $D_6$ | $Z_5$ | $Z_3$ | $D_4$ | $Z_2$ |
| $N_H(K_i)$ | $A_4$ | $D_{10}$ | $D_6$ | $D_{10}$ | $D_6$ | $A_4$ | $D_4$ |
| $N_G(K_i)$ | $S_4$ | $D_{20}$ | $D_{12}$ | $D_{p\mp1}$ | $D_{p\mp1}$ | $S_4$ | $D_{p\mp1}$ |
| $k_i = \lvert\mathrm{Fix}_\Omega(K_i)\rvert$ | 2 | 2 | 2 | $\frac{1}{10}(p\mp1)$ | $\frac{1}{6}(p\mp1)$ | 2 | $\frac{1}{4}(p\mp1)$ |

By Lemma 2.3, $G$ has 1 suborbit of length 1, 1 of length 5, 1 of length 6, 1 of length 10, $\frac{1}{20}(p\mp1) - 1$ of length 12, $\frac{1}{12}(p\mp1) - 2$ of length 20, none of length 15,

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$\frac{1}{8}(p \mp 1) - 3$ of length 30, and $x$ of length 60, where

$$60x = \frac{1}{120}(p^3 - p) - 1 - 5 - 6 - 10 - 12 \cdot (\frac{1}{20}(p \mp 1) - 1) - 20 \cdot (\frac{1}{12}(p \mp 1) - 2) - 30 \cdot (\frac{1}{8}(p \mp 1) - 3).$$

Therefore $x = \frac{1}{7200}(p^3 - 723p \pm 722) + 2$ and hence $G$ has rank

$$\begin{aligned} r &= 1 + 1 + 1 + 1 + (\frac{1}{20}(p \mp 1) - 1) + (\frac{1}{12}(p \mp 1) - 2) \\ &\quad + (\frac{1}{8}(p \mp 1) - 3) + \frac{1}{7200}(p^3 - 723p \pm 722) + 2 \\ &= \frac{1}{7200}(p^3 + 1137p \mp 1138). \end{aligned}$$

Now we show that all the nontrivial suborbits of length less than 60 are self-paired. The arguments are the same in all cases. For instance, consider a suborbit $\Delta(\alpha)$ of length 12. For $\beta \in \Delta(\alpha)$, we have $G_{\alpha\beta} = Z_5$, and $N_G(G_{\alpha\beta}) \cong D_{p\mp 1}$ acts on $\mathrm{Fix}_\Omega(G_{\alpha\beta})$, a set of size $\frac{1}{10}(p \mp 1)$, as $D_{\frac{p\mp 1}{5}}$. By Lemma 2.4, some element of $N_G(G_{\alpha\beta})$ interchanges $\alpha$ and $\beta$, and so $\Delta(\alpha)$ is self-paired.

Next, we use Lemma 2.5 to determine the number $y$ of self-paired suborbits of length 60. In the following table $y_{|\Delta(\alpha)|}$ denotes the number of self-paired suborbits of length $|\Delta(\alpha)|$ and $\mathrm{inv}(\Delta)$ is defined in Lemma 2.5.

| $\lvert\Delta(\alpha)\rvert$ | 5 | 6 | 10 | 12 | 20 | 30 | 60 |
|---|---|---|---|---|---|---|---|
| $y_{\lvert\Delta(\alpha)\rvert}$ | 1 | 1 | 1 | $\frac{1}{20}(p \mp 1) - 1$ | $\frac{1}{12}(p \mp 1) - 2$ | $\frac{1}{8}(p \mp 1) - 3$ | $y$ |
| $G_{\alpha\beta}$ | $A_4$ | $D_{10}$ | $D_6$ | $Z_5$ | $Z_3$ | $Z_2$ | 1 |
| $G_{\{\alpha,\beta\}}$ | $S_4$ | $D_{20}$ | $D_{12}$ | $D_{10}$ | $D_6$ | $D_4$ | $Z_2$ |
| $\mathrm{inv}(\Delta)$ | 6 | 6 | 4 | 5 | 3 | 2 | 1 |

Since every involution has $\frac{1}{4}(p \mp 1)$ fixed points, so every involution has

$$N = \frac{1}{2}(\frac{p^3 - p}{120} - \frac{p \mp 1}{4}) = \frac{1}{240}(p^3 - 31p \pm 30)$$

cycles of length 2. The centralizer of an involution has order $p \mp 1$ and so, by Lemma 2.5,

$$\frac{p^2 \pm p}{2} - 15 = 5 \cdot 1 \cdot 6 + 6 \cdot 1 \cdot 6 + 10 \cdot 1 \cdot 4 + 12 \cdot (\frac{p \mp 1}{20} - 1) \cdot 5 + 20 \cdot (\frac{p \mp 1}{12} - 2) \cdot 3 + 30 \cdot (\frac{p \mp 1}{8} - 3) \cdot 2 + 60 \cdot y \cdot 1,$$

whence

$$y = \begin{cases} (p^2 - 30p + 509)/120 & \text{if } p \equiv 1 \pmod{120}, \\ (p^2 - 32p + 447)/120 & \text{if } p \equiv -1 \pmod{120}. \end{cases}$$

| | $i$ | 1 | 2 | 3 | 4 | 5 | 7 |
|---|---|---|---|---|---|---|---|
| | $K_i$ | $A_4$ | $D_{10}$ | $D_6$ | $Z_5$ | $Z_3$ | $Z_2$ |
| | $N_H(K_i)$ | $A_4$ | $D_{10}$ | $D_6$ | $D_{10}$ | $D_6$ | $D_4$ |
| $p \equiv \pm 1 \pmod{120}$ | $N_G(K_i)$ | $S_4$ | $D_{20}$ | $D_{12}$ | $D_{p\mp1}$ | $D_{p\mp1}$ | $D_{p\mp1}$ |
| | $k_i$ | 2 | 2 | 2 | $\frac{p\mp1}{10}$ | $\frac{p\mp1}{6}$ | $\frac{p\mp1}{4}$ |
| | $x_i$ | 1 | 1 | 1 | $\frac{p\mp1}{20} - 1$ | $\frac{p\mp1}{12} - 2$ | $\frac{p\mp1}{8} - 3$ |
| $p \equiv \pm 11 \pmod{120}$ | $N_G(K_i)$ | $A_4$ | $D_{10}$ | $D_{12}$ | $D_{p\mp1}$ | $D_{p\pm1}$ | $D_{p\pm1}$ |
| | $k_i$ | 1 | 1 | 2 | $\frac{p\mp1}{10}$ | $\frac{p\pm1}{6}$ | $\frac{p\pm1}{4}$ |
| | $x_i$ | 0 | 0 | 1 | $\frac{p\mp1}{20} - \frac{1}{2}$ | $\frac{p\pm1}{12} - 1$ | $\frac{p\pm1}{8} - \frac{3}{2}$ |
| $p \equiv \pm 31 \pmod{120}$ | $N_G(K_i)$ | $S_4$ | $D_{10}$ | $D_6$ | $D_{p\mp1}$ | $D_{p\mp1}$ | $D_{p\pm1}$ |
| | $k_i$ | 2 | 1 | 1 | $\frac{p\mp1}{10}$ | $\frac{p\mp1}{6}$ | $\frac{p\pm1}{4}$ |
| | $x_i$ | 1 | 0 | 0 | $\frac{p\mp1}{20} - \frac{1}{2}$ | $\frac{p\mp1}{12} - \frac{3}{2}$ | $\frac{p\pm1}{8} - 1$ |
| $p \equiv \pm 41 \pmod{120}$ | $N_G(K_i)$ | $S_4$ | $D_{20}$ | $D_6$ | $D_{p\mp1}$ | $D_{p\pm1}$ | $D_{p\mp1}$ |
| | $k_i$ | 2 | 2 | 1 | $\frac{p\mp1}{10}$ | $\frac{p\pm1}{6}$ | $\frac{p\mp1}{4}$ |
| | $x_i$ | 1 | 1 | 0 | $\frac{p\mp1}{20} - 1$ | $\frac{p\pm1}{12} - \frac{3}{2}$ | $\frac{p\mp1}{8} - 2$ |
| $p \equiv \pm 61 \pmod{120}$ | $N_G(K_i)$ | $A_4$ | $D_{20}$ | $D_{12}$ | $D_{p\mp1}$ | $D_{p\mp1}$ | $D_{p\mp1}$ |
| | $k_i$ | 1 | 2 | 2 | $\frac{p\mp1}{10}$ | $\frac{p\mp1}{6}$ | $\frac{p\mp1}{4}$ |
| | $x_i$ | 0 | 1 | 1 | $\frac{p\mp1}{20} - 1$ | $\frac{p\mp1}{12} - 1$ | $\frac{p\mp1}{8} - \frac{5}{2}$ |
| $p \equiv \pm 71 \pmod{120}$ | $N_G(K_i)$ | $S_4$ | $D_{10}$ | $D_{12}$ | $D_{p\mp1}$ | $D_{p\pm1}$ | $D_{p\pm1}$ |
| | $k_i$ | 2 | 1 | 2 | $\frac{p\mp1}{10}$ | $\frac{p\pm1}{6}$ | $\frac{p\pm1}{4}$ |
| | $x_i$ | 1 | 0 | 1 | $\frac{p\mp1}{20} - \frac{1}{2}$ | $\frac{p\pm1}{12} - 2$ | $\frac{p\pm1}{8} - 2$ |
| $p \equiv \pm 91 \pmod{120}$ | $N_G(K_i)$ | $A_4$ | $D_{10}$ | $D_6$ | $D_{p\mp1}$ | $D_{p\mp1}$ | $D_{p\pm1}$ |
| | $k_i$ | 1 | 1 | 1 | $\frac{p\mp1}{10}$ | $\frac{p\mp1}{6}$ | $\frac{p\pm1}{4}$ |
| | $x_i$ | 0 | 0 | 0 | $\frac{p\mp1}{20} - \frac{1}{2}$ | $\frac{p\mp1}{12} - \frac{1}{2}$ | $\frac{p\pm1}{8} - \frac{1}{2}$ |
| $p \equiv \pm 101 \pmod{120}$ | $N_G(K_i)$ | $A_4$ | $D_{20}$ | $D_6$ | $D_{p\mp1}$ | $D_{p\pm1}$ | $D_{p\mp1}$ |
| | $k_i$ | 1 | 2 | 1 | $\frac{p\mp1}{10}$ | $\frac{p\pm1}{6}$ | $\frac{p\mp1}{4}$ |
| | $x_i$ | 0 | 1 | 0 | $\frac{p\mp1}{20} - 1$ | $\frac{p\pm1}{12} - \frac{1}{2}$ | $\frac{p\mp1}{8} - \frac{3}{2}$ |

TABLE 4.1. The number $x_i$ of non-regular suborbits with stabilizer $K_i$.

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Thus the number $h = x - y$ of non-self-paired suborbits of length 60 is

$$h = \begin{cases} (p^3 - 60p^2 + 1077p - 15418)/7200 & \text{if } p \equiv 1 \pmod{120}, \\ (p^3 - 60p^2 + 1197p - 13142)/7200 & \text{if } p \equiv -1 \pmod{120}. \end{cases}$$

Now $PSL(2, p)$ has two conjugacy classes of subgroups isomorphic to $A_5$ and these are interchanged by $PGL(2, p)$. Hence $PGL(2, p)$ has no subgroup of index $d$, and by the result of Liebeck, Praeger and Saxl [10], there are no simply primitive groups properly containing $PSL(2, p)$. From these observations it follows that each of the above orbital graphs and digraphs has automorphism group $PSL(2, p)$.

Finally, the same argument as in the proof of Theorem 3 shows that the orbital graphs and digraphs are pairwise non-isomorphic. This completes the proof in this case.

| $p$ (mod 120) | $x$ | $r$ | $y$ | $h$ |
|---|---|---|---|---|
| 1 | $\frac{p^3-723p+15122}{7200}$ | $\frac{p^3+1137p-1138}{7200}$ | $\frac{p^2-30p+509}{120}$ | $\frac{p^3-60p^2+1077p-15418}{7200}$ |
| $-1$ | $\frac{p^3-723p+13678}{7200}$ | $\frac{p^3+1137p+1138}{7200}$ | $\frac{p^2-32p+447}{120}$ | $\frac{p^3-60p^2+1197p-13142}{7200}$ |
| 11 | $\frac{p^3-723p+6622}{7200}$ | $\frac{p^3+1137p+562}{7200}$ | $\frac{p^2-32p+231}{120}$ | $\frac{p^3-60p^2+1197p-7238}{7200}$ |
| $-11$ | $\frac{p^3-723p+7778}{7200}$ | $\frac{p^3+1137p-562}{7200}$ | $\frac{p^2-30p+269}{120}$ | $\frac{p^3-60p^2+1077p-8362}{7200}$ |
| 31 | $\frac{p^3-723p+7022}{7200}$ | $\frac{p^3+1137p-238}{7200}$ | $\frac{p^2-32p+271}{120}$ | $\frac{p^3-60p^2+1197p-9238}{7200}$ |
| $-31$ | $\frac{p^3-723p+7378}{7200}$ | $\frac{p^3+1137p+238}{7200}$ | $\frac{p^2-30p+269}{120}$ | $\frac{p^3-60p^2+1077p-8762}{7200}$ |
| 41 | $\frac{p^3-723p+11122}{7200}$ | $\frac{p^3+1137p-338}{7200}$ | $\frac{p^2-30p+389}{120}$ | $\frac{p^3-60p^2+1077p-12218}{7200}$ |
| $-41$ | $\frac{p^3-723p+10478}{7200}$ | $\frac{p^3+1137p+338}{7200}$ | $\frac{p^2-32p+367}{120}$ | $\frac{p^3-60p^2+1197p-11542}{7200}$ |
| 61 | $\frac{p^3-723p+11522}{7200}$ | $\frac{p^3+1137p-1138}{7200}$ | $\frac{p^2-30p+389}{120}$ | $\frac{p^3-60p^2+1077p-11818}{7200}$ |
| $-61$ | $\frac{p^3-723p+10078}{7200}$ | $\frac{p^3+1137p+1138}{7200}$ | $\frac{p^2-32p+327}{120}$ | $\frac{p^3-60p^2+1197p-9542}{7200}$ |
| 71 | $\frac{p^3-723p+10222}{7200}$ | $\frac{p^3+1137p+562}{7200}$ | $\frac{p^2-32p+351}{120}$ | $\frac{p^3-60p^2+1197p-10838}{7200}$ |
| $-71$ | $\frac{p^3-723p+11378}{7200}$ | $\frac{p^3+1137p-562}{7200}$ | $\frac{p^2-30p+389}{120}$ | $\frac{p^3-60p^2+1077p-11962}{7200}$ |
| 91 | $\frac{p^3-723p+3422}{7200}$ | $\frac{p^3+1137p-238}{7200}$ | $\frac{p^2-32p+151}{120}$ | $\frac{p^3-60p^2+1197p-5638}{7200}$ |
| $-91$ | $\frac{p^3-723p+3778}{7200}$ | $\frac{p^3+1137p+238}{7200}$ | $\frac{p^2-30p+149}{120}$ | $\frac{p^3-60p^2+1077p-5162}{7200}$ |
| 101 | $\frac{p^3-723p+7522}{7200}$ | $\frac{p^3+1137p-338}{7200}$ | $\frac{p^2-30p+269}{120}$ | $\frac{p^3-60p^2+1077p-8618}{7200}$ |
| $-101$ | $\frac{p^3-723p+6878}{7200}$ | $\frac{p^3+1137p+338}{7200}$ | $\frac{p^2-32p+247}{120}$ | $\frac{p^3-60p^2+1197p-7942}{7200}$ |

TABLE 4.2. The number $h$ of non-self-paired regular sub-orbits.

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For the other cases, we list the results in Tables 4.1 and 4.2.

Table 4.1 refers to the action of $PSL(2, p)$ on the cosets of $A_5$. In the table $K_i$ is a subgroup of $A_5$, $k_i = |\operatorname{Fix}_\Omega(K_i)|$, and $x_i$ is the number of the suborbits with a point stabilizer isomorphic to $K_i$. In all cases $x_6 = 0$.

Table 4.2 gives the number $x$ of regular suborbits of $PSL(2, p)$ acting on cosets of $A_5$, the rank $r$, the number $y$ of self-paired regular suborbits, and the number $h = x - y$ of non-self-paired regular suborbits. The number of 2-cycles in each involution is $\frac{1}{240}(p^3 - 31p + 30\varepsilon)$, where $\varepsilon = \pm 1$ and $p \equiv \varepsilon \pmod 4$.

COROLLARY 4.2. *Let $G$ and $H$ be the same as in Theorem* 4.1. *Assume that $\Delta$ is an orbital graph of $G$, or the underlying undirected graph of a non-self-paired orbital digraph. If $p > 61$, then $\Delta$ is not a metacirculant.*

PROOF. If $\Delta$ were a metacirculant, then $G$ would have a metacyclic subgroup $M$, which is transitive. It follows that $d = \frac{1}{120}(p^3 - p)$ would divide the order of $M$. If $p > 61$, Theorem 2.1 tells us that the only such subgroup of $G$ is $G$ itself. But $G$ is not metacyclic, a contradiction.

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