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ON ALGEBRAIC GROUPS DEFINED BY JORDAN PAIRS

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Introduction

Let G be an algebraic group over a field k, and let ψ be an action of the multiplicative group k_m of k on G by automorphisms. We say ψ is an *elementary action* if it has only the weights 0, ± 1; more precisely, if there exist subgroups H, U^+, U^- of G such that (i) H is fixed under ψ, (ii) U^+ and U^- are vector groups and $\psi_t(x) = t^{\pm 1}x$ for $t \in k_m$, $x \in U^{\pm}$, (iii) $\Omega = U^- \cdot H \cdot U^+$ is open in G, and (iv) G is generated by H, U^+, U^-. This situation is characteristic for the complexifications of the automorphism groups of bounded symmetric domains (see, e.g., [9, 16]). A typical example is $G = \mathbf{GL}_n$ with (matrices being decomposed into 4 blocks) ψ given by

$$\psi_t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix} .$$

If G is reductive and U^+ and U^- are one-dimensional, then an elementary action is essentially equivalent to an elementary system in the sense of Demazure [7, Exp. XX], with the technical difference that we consider an external torus action instead of a sub-torus of G acting by conjugation.

After some preliminaries, our first goal (§ 4) is to find relations describing the structure of G in terms of the generators H, U^+, U^-. Since H normalizes U^+ and U^- this essentially amounts to a formula expressing products in $U^+ \cdot U^-$ in terms of their components in $\Omega = U^- \cdot H \cdot U^+$. In more detail, let $\mathfrak{V}^{\pm}$ be the Lie algebra of $U^{\pm}$. Then there are ψ-equivariant isomorphism $\exp\colon \mathfrak{V}^{\pm} \to U^{\pm}$, and there is a unique Jordan pair structure on $\mathfrak{V} = (\mathfrak{V}^+, \mathfrak{V}^-)$ such that, for $x \in \mathfrak{V}^+$, $y \in \mathfrak{V}^-$, the product $\exp(x) \cdot \exp(y)$ belongs to Ω if and only if (x, y) is quasi-invertible, and in this case

$$(*) \qquad \exp(x) \cdot \exp(y) = \exp(y^x) \cdot b(x, y) \cdot \exp(x^y) .$$

Here x^y, y^x denotes the quasi-inverse in $\mathfrak{V}$ and b is a morphism from the set of quasi-invertible pairs of $\mathfrak{V}$ into H which has properties analogous to the "Bergmann transformations" $B(x, y)$ of a Jordan pair. The formula

($*$) is the higher-dimensional analogue of Demazure's formula (F) ([7, Exp. XX, Th. 2.1]). In case U^+ and U^- are conjugate by an element of G^0, one can show that the Jordan pair $\mathfrak{V}$ contains invertible elements and hence is the Jordan pair defined by a (not canonically determined) unital quadratic Jordan algebra; see Borel-Tits [4, § 5] and Springer [20, 2.21–2.26].

Next we show how to reconstruct G, given $\mathfrak{V}$ and H. The necessary ingredients for this are an action ρ of H on $\mathfrak{V}$ by automorphisms and a morphism b from the quasi-invertible pairs of $\mathfrak{V}$ into H, satisfying suitable conditions (5.1). Then for every "Jordan system" $(\mathfrak{V}, H, \rho, b)$ there exists an "elementary system" (G, ψ), and this establishes an equivalence of categories. In case H is the automorphism group of $\mathfrak{V}$ and $\rho = \mathrm{Id}$, and under restrictions on k and $\mathfrak{V}$, Koecher proved the existence of G by realizing it as a group of birational transformations; see [11, 12, 13]. The proof given here (5.2–5.9) is more direct but also more computational. The Lie algebra of G was studied extensively by Tits [21], Koecher [10], and Meyberg [17].

In § 6, we prove that the unipotent radical of G and the Jacobson radical of $\mathfrak{V}$ are related by the formula

$$R_u(G) = N^- \cdot R_u(H) \cdot N^+$$

where $N^{\pm} = \exp(\mathrm{Rad}\,\mathfrak{V}^{\pm})$. We also show that the automorphism group of a separable Jordan pair is reductive and its identity component consists of inner automorphisms, a result due to Springer [20] in the Jordan algebra case. Finally, we give a description of the group $G(k)$ of k-rational points by generators and relations.

The desire to replace the base field k by an arbitrary base ring leads naturally to considering group schemes and Jordan pairs which are finitely generated and projective k-modules. In view of recent work of H. P. Petersson [18] on orders in Jordan pairs over quotient fields of Dedekind rings, this degree of generality seems not without interest. It turns out that the formal theory of § 3–§ 5 is easier to handle in the framework of group sheaves (in the flat topology). Representability questions are then treated separately; if the base ring is Dedekind, they have satisfactory answers (5.13). We have therefore adopted this point of view, in particular, since it involves very little extra effort as compared to a more classical approach.

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§1. Notations and Preliminaries

1.1. Let k denote a commutative base ring and k-**alg** the category of commutative k-algebras. We follow the notations and conventions of [5]. In particular, schemes are considered as special k-functors (covariant functors from k-**alg** into sets). If X is a k-functor we write $x \in X$ for $x \in X(R)$, $R \in k$-**alg**. The base extension of X from k to K is denoted by X_K or $X \otimes_k K$. A *sheaf* is a k-functor which is a sheaf in the flat (= fppf) topology. We refer to [5, p. 10, p. 50] for the notion of open (closed) subfunctor. It is easily seen that an open (closed) subfunctor of a sheaf is itself a sheaf. A k-functor X is called *separated* if the diagonal in $X \times X$ is closed. The following result can be proved along the lines of [5, p. 296].

1.2. LEMMA. *Let*

$$\begin{array}{ccc} F & \longrightarrow & G \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{\;p\;} & Y \end{array}$$

be a Cartesian square of sheaves where p is an epimorphism of sheaves. Then u is an open (closed) imbedding if and only if v is.

1.3. COROLLARY. *Let $p: X \to Y$ be an epimorphism of sheaves and $E = X \times_Y X \subset X \times X$ the equivalence relation defined by p. Then Y is separated if and only if E is closed in $X \times X$.*

This follows by considering the Cartesian square

$$\begin{array}{ccc} E & \xrightarrow{\;pr_1\;} & Y \\ \downarrow & & \downarrow{\scriptstyle \text{diag}} \\ X \times X & \xrightarrow[\;p \times p\;]{} & Y \times Y \end{array}$$

1.4. A subfunctor U of a k-functor X is called *dense* if, for all open $V \subset X$, and all closed $Z \subset V$ such that $Z \supset U \cap V$ we have $Z = V$, and this property remains valid in all scalar extensions. If X is a scheme then this notion of dense is the same as "universally schematically dense"

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(cf. EGA IV, 11.10). The following lemma will be used often (cf. SGA3, Exp. XVIII, 1.7, and EGA IV, 11.10.10).

1.5. LEMMA. *Let X be a smooth separated finitely presented k-scheme with connected non-empty fibres, and let U be an open subscheme of X. Then the following conditions are equivalent.*

(i) *U is dense in X.*

(ii) *There exists a fppf extension R of k such that $U(R) \neq \varnothing$.*

(iii) *$U(K) \neq \varnothing$ for all algebraically closed fields $K \in k$-**alg**.*

1.6. LEMMA (Cf. SGA3, Exp. XVIII, Prop. 1.1). *Let U be a dense subfunctor of X.*

(a) *U_R is dense in X_R, for all $R \in k$-**alg**.*

(b) *If $U \subset U' \subset X$ then U' is dense in X.*

(c) *If V is open in X then $U \cap V$ is dense in V.*

(d) *If $V \subset U$ is dense in U then V is dense in X.*

(e) *If $U' \subset X$ is open and dense then $U \cap U'$ is dense in X.*

(f) *$U \times Y$ is dense in $X \times Y$, for any k-functor Y.*

(g) *If $f, g: X \to Y$ are morphisms into a separated k-functor Y which agree on U then $f = g$.*

The proofs are mostly straightforward and are omitted.

1.7. LEMMA. *Let $p: X \to Y$ be an epimorphism of sheaves, and let U be a subfunctor of Y such that $p^{-1}(U)$ is dense in X. Then U is dense.*

Proof. Let $V \subset Y$ be open, $Z \subset V$ closed, and assume $U \cap V \subset Z$. Then $p^{-1}(V)$ is open in $X, p^{-1}(Z) \subset p^{-1}(V)$ is closed, and $p^{-1}(U) \cap p^{-1}(V) \subset p^{-1}(Z)$. Hence $p^{-1}(Z) = p^{-1}(V)$. Now $p: p^{-1}(V) \to V$ is an epimorphism of sheaves, and therefore Z contains the image sheaf of $p^{-1}(V)$ under p which is V.

1.8. A k-group functor is a covariant functor from k-**alg** into the category of groups. If G is a k-group sheaf and A and B are subsheaves of G then $A \cdot B$ denotes the image sheaf of $A \times B$ under multiplication. The multiplicative group of k is denoted by k_m, the additive group of a finitely generated projective k-module $\mathfrak{M}$ by $\mathfrak{M}_a$. Thus $\mathfrak{M}_a(R) = \mathfrak{M}_R = \mathfrak{M} \otimes_k R$, for all $R \in k$-**alg**. The Lie algebra of a k-group functor G is denoted by Lie(G). We follow [7, p. 209] and write $e^{\varepsilon x} \in G(k(\varepsilon))$ for $x \in$ Lie(G) (where $k(\varepsilon)$ is the ring of dual numbers). If $G = \mathfrak{M}_a$ is a

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vector group we identify $\mathrm{Lie}(\mathfrak{M}_a)$ with $\mathfrak{M}$ and set $e^{\varepsilon x} = \varepsilon x$. If $G \subset \mathbf{GL}(\mathfrak{M})$ is linear then we identify $\mathrm{Lie}(G)$ with the set of $x \in \mathrm{End}(\mathfrak{M})$ such that $e^{\varepsilon x} = \mathrm{Id} + \varepsilon x \in G(k(\varepsilon))$. (Here $\mathbf{GL}(\mathfrak{M})$ is the k-group functor $R \mapsto \mathrm{GL}(\mathfrak{M}_R)$). The adjoint representation of $G(k)$ on $\mathrm{Lie}(G)$ is defined by $\mathrm{Int}(g)e^{\varepsilon x} = e^{\varepsilon\, \mathrm{Ad} g \cdot x}$, where $\mathrm{Int}(g)h = ghg^{-1}$.

§ 2. Jordan Pairs

2.1. Let $\mathfrak{V}^+$ and $\mathfrak{V}^-$ be finitely generated and projective k-modules, and let $Q_+ : \mathfrak{V}^+ \to \mathrm{Hom}(\mathfrak{V}^-, \mathfrak{V}^+)$ and $Q_- : \mathfrak{V}^- \to \mathrm{Hom}(\mathfrak{V}^+, \mathfrak{V}^-)$ be quadratic maps. For $\sigma = \pm$ define trilinear compositions $\mathfrak{V}^\sigma \times \mathfrak{V}^{-\sigma} \times \mathfrak{V}^\sigma \to \mathfrak{V}^\sigma$, $(x, y, z) \mapsto \{xyz\}$, and bilinear maps $D_\sigma : \mathfrak{V}^\sigma \times \mathfrak{V}^{-\sigma} \to \mathrm{End}(\mathfrak{V}^\sigma)$ by

$$\{xyz\} = D_\sigma(x, y)z = Q_\sigma(x, z)y = Q_\sigma(x + z)y - Q_\sigma(x)y - Q_\sigma(z)y .$$

The pair $\mathfrak{V} = (\mathfrak{V}^+, \mathfrak{V}^-)$ together with the quadratic maps (Q_+, Q_-) is called a *Jordan pair* if the following identities hold in all base ring extensions.

(1) $$D_\sigma(x, y)Q_\sigma(x) = Q_\sigma(x)D_{-\sigma}(y, x) ,$$

(2) $$D_\sigma(Q_\sigma(x)y, y) = D_\sigma(x, Q_{-\sigma}(y)x) ,$$

(3) $$Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x) .$$

A *homomorphism* $h : \mathfrak{V} \to \mathfrak{W}$ of Jordan pairs is a pair $h = (h_+, h_-)$ of k-linear maps, $h_\sigma : \mathfrak{V}^\sigma \to \mathfrak{W}^\sigma$, such that $h_\sigma Q_\sigma(x) = Q_\sigma(h_\sigma(x))h_{-\sigma}$. The automorphism group of $\mathfrak{V}$ is denoted by $\mathrm{Aut}(\mathfrak{V})$. The *opposite* of $\mathfrak{V}$ is $\mathfrak{V}^{\mathrm{op}} = (\mathfrak{V}^-, \mathfrak{V}^+)$ with quadratic maps (Q_-, Q_+). A standard example of a Jordan pair is $\mathfrak{V}^+ = \mathfrak{V}^- = M_{p,q}(k)$, $p \times q$-matrices, with $Q_\pm(x)y = x \cdot {}^t y \cdot x$. For details see [15].

2.2. The *quasi-inverse* ([15, § 3]). Let $\mathfrak{V}$ be a Jordan pair. Following the convention of [15, 2.0], we omit the index σ in D_σ and Q_σ and write $Q_x y$ for $Q(x)y$. For $x \in \mathfrak{V}^+$, $y \in \mathfrak{V}^-$ define $B(x, y) \in \mathrm{End}(\mathfrak{V}^+)$ and $B(y, x) \in \mathrm{End}(\mathfrak{V}^-)$ by

$$B(x, y) = \mathrm{Id} - D(x, y) + Q_x Q_y , \qquad B(y, x) = \mathrm{Id} - D(y, x) + Q_y Q_x .$$

The pair $(x, y) \in \mathfrak{V}^+ \times \mathfrak{V}^-$ is called *quasi-invertible* if there exists $z \in \mathfrak{V}^+$ such that

(1) $$B(x, y)z = x - Q_x y \quad \text{and} \quad B(x, y)Q_z y = Q_x y .$$

This is the case if and only if $B(x, y)$ is invertible, and thus

$$(2) \qquad z = x^y = B(x, y)^{-1}(x - Q_x y) ,$$

called the *quasi-inverse* of (x, y). We denote by $W \subset \mathfrak{V}_a^+ \times \mathfrak{V}_a^-$ the open dense subscheme of quasi-invertible pairs. In the example above, (x, y) is quasi-invertible if and only if $1 - x \cdot {}^t y$ is invertible, and $x^y = (1 - x \cdot {}^t y)^{-1} \cdot x$. If (x, y) is quasi-invertible then

$$\beta(x, y) = (B(x, y), B(y, x)^{-1})$$

is an automorphism of $\mathfrak{V}$, called the *inner automorphism* defined by (x, y). The subgroup $\mathrm{Inn}(\mathfrak{V})$ of $\mathrm{Aut}(\mathfrak{V})$ generated by all inner automorphisms is normal and called the *inner automorphism group*.

2.3. The automorphism group functor of V is defined by

$$\mathbf{Aut}(\mathfrak{V})(R) = \mathrm{Aut}(\mathfrak{V}_R) ,$$

for all $R \in k$-**alg**. We show that $\mathbf{Aut}(\mathfrak{V})$ is an affine finitely presented group scheme. Let $x_1^\sigma, \cdots, x_n^\sigma$ be a set of generators for the k-module $\mathfrak{V}^\sigma$ $(\sigma = \pm)$. Then $h = (h_+, h_-) \in \mathrm{GL}(\mathfrak{V}^+) \times \mathrm{GL}(\mathfrak{V}^-)$ belongs to $\mathrm{Aut}(\mathfrak{V})$ if and only if

$$h_\sigma Q(x_i^\sigma) = Q(h_\sigma(x_i^\sigma))h_{-\sigma} ,$$
$$h_\sigma D(x_i^\sigma, x_j^{-\sigma}) = D(h_\sigma(x_i^\sigma), h_{-\sigma}(x_j^{-\sigma}))h_\sigma ,$$

for $i, j = 1, \cdots, n$, $\sigma = \pm$. Since this remains true in any base ring extension, $\mathbf{Aut}(\mathfrak{V})$ is the closed subscheme of the affine finitely presented k-scheme $\mathbf{GL}(\mathfrak{V}^+) \times \mathbf{GL}(\mathfrak{V}^-)$ defined by finitely many equations, and is therefore itself an affine finitely presented k-scheme ([5, I, § 3]).

The derivation algebra of $\mathfrak{V}$ is $\mathrm{Der}(\mathfrak{V}) = \mathrm{Lie}(\mathbf{Aut}(\mathfrak{V}))$. From 1.8 it follows easily that $\Delta = (\Delta_+, \Delta_-) \in \mathrm{End}(\mathfrak{V}^+) \times \mathrm{End}(\mathfrak{V}^-)$ is a derivation of $\mathfrak{V}$ if and only if

$$\Delta_\sigma Q_x y = \{\Delta_\sigma x, y, x\} + Q_x \cdot \Delta_{-\sigma} y ,$$

for all $x \in \mathfrak{V}^\sigma$, $y \in \mathfrak{V}^{-\sigma}$, $\sigma = \pm$. Finally we note that there is a central homomorphism $\gamma: k_m \to \mathbf{Aut}(\mathfrak{V})$ given by

$$\gamma(t) = (t \cdot \mathrm{Id}, t^{-1} \cdot \mathrm{Id}) .$$

2.4. Let $\mathbf{Inn}(\mathfrak{V})$ be the k-group sheaf associated with the k-group functor $R \mapsto \mathrm{Inn}(\mathfrak{V}_R)$. Then $\mathbf{Inn}(\mathfrak{V})$ is normal in $\mathbf{Aut}(\mathfrak{V})$. If k is a field then $\mathbf{Inn}(\mathfrak{V})$ is a smooth connected algebraic k-group since it is generated by $\beta(W)$ ([SGA3, Exp. VI_B, No. 7], see also [2, p. 106]). For $(x, y) \in \mathfrak{V}^+ \times \mathfrak{V}^-$,

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$$\delta(x, y) = (D(x, y), -D(y, x))$$

is a derivation of $\mathfrak{B}$, called the inner derivation defined by (x, y). The linear span of the $\delta(x, y)$ is an $\mathrm{Aut}(\mathfrak{B})$-invariant ideal of $\mathrm{Der}(\mathfrak{B})$, the inner derivation algebra $\mathrm{Inder}(\mathfrak{B})$. Clearly $\mathrm{Inder}(\mathfrak{B}) \subset \mathrm{Lie}(\mathrm{Inn}(\mathfrak{B}))$. If k is a field of characteristic zero then it can be shown that equality holds.

2.5. Let $\mathfrak{A}$ be a unital quadratic Jordan algebra over k which is finitely generated and projective as a k-module. Then $\mathfrak{A}$ defines a Jordan pair $\mathfrak{B} = (\mathfrak{A}, \mathfrak{A})$ by setting $\mathfrak{B}^+ = \mathfrak{B}^- = \mathfrak{A}$ and $Q(x)\cdot y = P(x)\cdot y$ where P denotes the quadratic representation of A. This establishes a one-to-one correspondence between Jordan algebras "up to isotopy" and Jordan pairs containing invertible elements ([15, 1.11]). Let $\mathrm{Str}(\mathfrak{A})$ be the *structure group* of $\mathfrak{A}$; i.e., the group of all $g \in \mathrm{GL}(\mathfrak{A})$ such that $P(gx) = gP(x)g^{\#}$ for some $g^{\#} \in \mathrm{GL}(\mathfrak{A})$ and all $x \in A$. Then $g^{\#} \in \mathrm{Str}(\mathfrak{A})$ and the map $g \mapsto g^{\#-1}$ is an automorphism of period 2 of $\mathrm{Str}(\mathfrak{A})$. The *inner structure group* is the subgroup $\mathrm{Instr}(\mathfrak{A})$ of $\mathrm{Str}(\mathfrak{A})$ generated by all $P(x)$, x invertible. The structure group functor $\mathbf{Str}(\mathfrak{A})$ is defined by $\mathbf{Str}(\mathfrak{A})(R) = \mathrm{Str}(\mathfrak{A}_R)$ and we denote by $\mathbf{Instr}(\mathfrak{A})$ the k-group sheaf generated by the k-group functor $R \mapsto \mathrm{Instr}(\mathfrak{A}_R)$.

2.6. PROPOSITION. *The map* $f: g \mapsto (g, g^{\#-1})$ *is an isomorphism of* $\mathbf{Str}(\mathfrak{A})$ *onto* $\mathbf{Aut}(\mathfrak{A}, \mathfrak{A})$ *mapping* $\mathbf{Instr}(\mathfrak{A})$ *onto* $\mathbf{Inn}(\mathfrak{A}, \mathfrak{A})$.

Proof. The first statement follows easily from the definitions (cf. [15, 1.8]). Let G be the image sheaf of $\mathbf{Instr}(\mathfrak{A})$ in $\mathbf{Aut}(\mathfrak{A}, \mathfrak{A})$. Then $G \subset \mathbf{Inn}(\mathfrak{A}, \mathfrak{A})$ since $P(x) = B(x, x^{-1} + e)$ and thus $f(P(x)) = \beta(x, x^{-1} + e)$ (here e is the unit element of $\mathfrak{A}$). Since everything is compatible with base extension, the converse inclusion will follow if we show: for every quasi-invertible $(x, y) \in \mathfrak{A} \times \mathfrak{A}$ there exists a fppf extension R of k such that $B(x, y) \in \mathrm{Instr}(\mathfrak{A}_R)$. If x is invertible then $B(x, y) = P(x)P(x^{-1} - y) \in \mathrm{Instr}(\mathfrak{A})$ (cf. [15, 1.12]). If not, let $U \subset \mathfrak{A}_a$ be the open subscheme defined by

$$U(R) = \{z \in \mathfrak{A}_R \,|\, (x + z, y) \text{ quasi-invertible}\},$$

and let $U' = U \cap I \cap (-x + I)$ where I is the open dense subscheme of invertible elements of $\mathfrak{A}$. Then U is dense since $0 \in U(k)$, and so are I and the translate $-x + I$. By 1.5 and 1.6, U' is dense and hence there exists a fppf extension R of k such that $U'(R) \neq \varnothing$. Picking $z \in U'(R)$ we have by [15, p. 25, JP34] that $B(x, y) = B(z, y^x)^{-1}B(x + z, y) \in \mathrm{Instr}(\mathfrak{A}_R)$.

§ 3. Elementary torus actions

3.1. Let G be a separated k-group sheaf, and let ψ be an action of the multiplicative group k_m on G by automorphisms. Thus for every invertible $t \in R$, $R \in k\text{-}\mathbf{alg}$ we have an automorphism ψ_t of G_R varying functorially with R and satisfying $\psi_s \psi_t = \psi_{st}$. The action ψ is called *elementary*, and the pair (G, ψ) is called an *elementary system* if there exist subgroup sheaves H, U^+, U^- of G with the following properties.

(i) H is fixed under ψ.

(ii) U^+ and U^- are vector groups on which ψ acts by scalar multiplication (resp. the inverse of scalar multiplication).

More precisely: there exist finitely generated and projective k-modules $\mathfrak{M}^{\pm}$ and isomorphisms $f_{\pm}: \mathfrak{M}^{\pm}_a \to U^{\pm}$ such that $f_+(tx) = \psi_t \cdot f_+(x)$, $f_-(t^{-1}y) = \psi_t \cdot f_-(y)$, for all $t \in k_m$, $x \in \mathfrak{M}^+_a$, $y \in \mathfrak{M}^-_a$.

(iii) $\Omega = U^- \cdot H \cdot U^+$ is open in G.

(iv) G is generated (as a k-group sheaf) by H, U^+, U^-.

We will show later that H, U^+, U^- are uniquely determined by these conditions. Clearly, if (G, ψ) is an elementary system so is any base extension (G_K, ψ_K). A *homomorphism* $f: (G, \psi) \to (G', \psi')$ is a group homomorphism $f: G \to G'$ compatible with ψ and ψ'. If ψ is an elementary action so is ψ^{-1} defined by $\psi_t^{-1} = \psi_{t^{-1}}$. This just amounts to interchanging U^+ and U^- and replacing Ω by Ω^{-1}.

3.2. Example. Let $G = \mathbf{GL}_n$ and divide a $n \times n$-matrix g into 4 blocks:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

of size $p \times p$, $p \times q$, $q \times p$, $q \times q$, where $p + q = n$. Let

$$\psi_t(g) = \begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix}. \tag{2}$$

Then ψ_t is an action which is elementary. Indeed, let H (resp. U^+, U^-) consist of all matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \left(\text{resp. } \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right),$$

then Ω consists of all matrices (1) where a is invertible, and is therefore open. Since G is a smooth separated finitely presented group scheme with connected fibres, Ω is dense in G. Let $g \in G(R)$. Then $\Omega_R \cap g \cdot \Omega_R^{-1}$ is open and dense by 1.6, and by 1.5 there exists a fppf extension S of R such that $(\Omega_R \cap g \cdot \Omega_R^{-1})(S) \neq \varnothing$; i.e., $g \in \Omega(S) \cdot \Omega(S)$. This proves (iv).

Remark. By the same argument, (iv) follows from (iii) whenever G is a smooth finitely presented group scheme with connected fibres.

Along the same lines, one has elementary actions on the orthogonal and symplectic groups $\mathbf{O}_{2n} \subset \mathbf{GL}_{2n}$ and $\mathbf{Sp}_n \subset \mathbf{GL}_{2n}$. Again, ψ is defined by (2), and H is isomorphic with $\mathbf{GL}_n$ imbedded into $\mathbf{GL}_{2n}$ via

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & {}^t a - 1 \end{pmatrix}.$$

The groups $U^{\pm}$ are isomorphic with the additive group of alternating (symmetric) $n \times n$-matrices.

3.3. LEMMA. *Let ψ be an elementary action, and let $\mathfrak{V}^{\pm}$ be the Lie algebra of $U^{\pm}$. Then there exist unique isomorphisms*

$$\exp : \mathfrak{V}_a^{\pm} \to U^{\pm}$$

such that

$$\text{(1)} \qquad \operatorname{Lie}(\exp) = \mathrm{Id}; \quad \text{i.e.,} \quad \exp(\varepsilon x) = e^{\varepsilon x},$$

$$\text{(2)} \qquad \psi_t(\exp x) = \exp(t^{\pm 1} x),$$

$$\text{(3)} \qquad \operatorname{Lie}(\psi_t) \cdot x = t^{\pm 1} \cdot x,$$

for all $t \in k_m$, $x \in \mathfrak{V}_a^{\pm}$.

Proof. Let $f_+ : \mathfrak{M}_a^+ \to U^+$ be as in 3.1 (ii). By the convention of 1.8, $\operatorname{Lie}(f_+) : \mathfrak{M}^+ \to \mathfrak{V}^+$ is an isomorphism of k-modules. Define $\exp(x) = f_+(\operatorname{Lie}(f_+)^{-1} x)$. One checks easily that (1) and (2) hold, and (3) follows from (1) and (2). To prove unicity, assume that $\exp'$ has the same properties. Then $h = (\exp')^{-1} \circ \exp : \mathfrak{V}_a^+ \to \mathfrak{V}_a^+$ is an isomorphism which is by (2) homogeneous of degree one, and therefore induced from a linear isomorphism of k-modules. By (1), this isomorphism is the identity. For $\exp : \mathfrak{V}_a^- \to U^-$ the proof is analogous.

3.4. LEMMA. *The map $U^- \times H \times U^+ \to G$ given by multiplication is an open imbedding.*

Proof. We only have to show that this map is a monomorphism.

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Using the fact that U^+ and U^- are subgroups and everything is compatible with base extension, this reduces to: if

$$(1) \qquad \exp(y)\cdot h\cdot \exp(x) = h'$$

for $x \in \mathfrak{V}^+$, $y \in \mathfrak{V}^-$, $h, h' \in H(k)$ then $h = h'$ and $x = y = 0$. Applying ψ_t to (1) we get

$$(2) \qquad \exp(t^{-1}y)\cdot h\cdot \exp(tx) = h' ,$$

for all $t \in k_m$. In particular, for $t = 1 + \varepsilon \in k(\varepsilon)$ (dual numbers) (2) implies, in view of 3.3,

$$\begin{aligned} h' &= \exp((1-\varepsilon)y)\cdot h\cdot \exp((1+\varepsilon)x) \\ &= \exp(-\varepsilon y)\cdot \exp(y)\cdot h\cdot \exp(x)\cdot \exp(\varepsilon x) = e^{-\varepsilon y}\cdot h'\cdot e^{\varepsilon x} , \end{aligned}$$

and hence $e^{\varepsilon y} = h' e^{\varepsilon x} h'^{-1} = e^{\varepsilon \operatorname{Ad} h'\cdot x}$; i.e., $y = \operatorname{Ad} h'\cdot x$. Applying $\operatorname{Lie}(\psi_t)$ to this we get, by 3.3.3, $t^{-1}y = \operatorname{Ad} h'\cdot tx = t\cdot \operatorname{Ad} h'\cdot x = ty$, for all $t \in k_m$. This implies $y = 0$. Hence also $x = 0$ and $h = h'$.

3.5. LEMMA. *Let $f\colon (G, \psi) \to (G', \psi')$ be a homomorphism of elementary systems. Then f maps $U^\pm$ into $U'^\pm$, and*

$$(1) \qquad f(\exp(x)) = \exp'(\operatorname{Lie}(f)(x)) ,$$

for all $x \in \mathfrak{V}_\pm$.

Proof. Define a morphism $\varphi\colon \mathfrak{V}_a^+ \to G'$ by $\varphi(x) = f(\exp x)$. Then

$$(2) \qquad \varphi(tx) = \psi'_t\cdot(\varphi(x)) ,$$

for $t \in k_m$. Let $X = \varphi^{-1}(\Omega') \subset \mathfrak{V}_a^+$. Then X contains the zero section, is open, and (2) implies that it is invariant under k_m acting on $\mathfrak{V}_a^+$ by scalar multiplication. It follows that $X = \mathfrak{V}_a^+$. By 3.4, we can write

$$(3) \qquad \varphi(x) = \exp'(g_-(x))\cdot g_0(x)\cdot \exp'(g_+(x))$$

with unique morphisms $g_\pm\colon \mathfrak{V}_a^+ \to \mathfrak{V}'^\pm_a$ and $g_0\colon \mathfrak{V}_a^+ \to H'$. From (2) and (3) we get the formulas

$$(4) \qquad g_\pm(tx) = t^{\pm 1} g_\pm(x) ,$$

$$(5) \qquad g_0(tx) = g_0(x) ,$$

for all $t \in k_m$. Hence g_0 and g_- are constant equal to 1 resp. 0, and g_+ is linear (cf. [19]). By 3.3.1, $g_+ = \operatorname{Lie}(f)\,|\,\mathfrak{V}^+$. The argument for $\mathfrak{V}^-$ is similar.

3.6. PROPOSITION. (a) *The subgroups* U^+ *and* U^- *are uniquely determined.*

(b) *H normalizes* U^+ *and* U^-, *and* exp *is H-equivariant*:

$$(1) \qquad h\cdot \exp(x)h^{-1} = \exp(\operatorname{Ad} h\cdot x)\,,$$

for all $x \in \mathfrak{B}_a^{\pm}$, $h \in H$.

(c) $$H \cap U^+\cdot U^-\cdot U^+ = \{1\}\,.$$

Proof. (a) This follows from 3.5, applied to $f = \mathrm{Id}$. (b) Apply 3.5 to the homomorphism $\operatorname{Int}(h)\colon (G_R, \psi_R) \to (G_R, \psi_R)$, for any $h \in H(R)$, $R \in k$-**alg**. (c) Let $h \in (H \cap U^+U^-U^+)(R)$. Then $h = uvw$ with $u, w \in U^+(S)$, $v \in U^-(S)$ for some fppf extension S of R. Hence $h^{-1}uh \in U^+(S)$ and $u^{-1}h = h(h^{-1}u^{-1}h) = vw$. By 3.4, $h = 1$.

3.7. LEMMA. *Let* $u, w \in U^-(k)$ *and* $v \in U^+(k)$. *Then there exists an fppf extension R of k and* $x \in U^+(R)$ *such that* xu^{-1} *and* xvw *belong to* $\Omega(R)$.

Proof. Let $X = U^+ \cap (\Omega\cdot u)$ and $Y = U^+ \cap (\Omega w^{-1}v^{-1})$. Then X and Y are open in U^+, and they are dense since $1 \in X(k)$ and $v^{-1} \in Y(k)$. Therefore $X \cap Y$ is dense, and there exists an fppf extension R of k such that $X(R) \cap Y(R) \neq \varnothing$ (cf. 1.5, 1.6). The lemma follows.

3.8. PROPOSITION. $G = H\cdot U^+\cdot U^-\cdot U^+ = U^+\cdot\Omega$, *and* Ω *is dense in* G.

Proof. Let $G' = U^+\cdot\Omega = H\cdot U^+\cdot U^-\cdot U^+$. Then G' is a subgroup sheaf of G. Indeed, in view of 3.6(b), this amounts to showing that $U^-\cdot U^+\cdot U^- \subset G'$. After a base extension, it suffices to show that, for all $u, w \in U^-(k)$, $v \in U^+(k)$, there exists a fppf extension R of k such that $uvw \in G'(R)$. Picking x as in 3.7, we have $uvw = (ux^{-1})(xvw) \in \Omega^{-1}(R)\Omega(R) = U^+(R)\Omega(R) \subset G'(R)$. Since G is generated by H, U^+, and U^-, we have $G = G'$. Let $Y \subset U^+ \times U^-$ be the inverse image of Ω under the multiplication map $U^+ \times U^- \to G$. Then Y is open and dense, and the inverse image of Ω under the epimorphism of sheaves $U^+ \times U^- \times H \times U^+ \to G$ is $Y \times H \times U^+$. By 1.6(f) and 1.7, Ω is dense in G.

3.9. Let G_{in} be the subgroup sheaf of G generated by U^+ and U^-. Then by 3.6(*b*), G_{in} is normal in G, and clearly stable under ψ. Since $\Omega \cap G_{in} = U^-\cdot(H \cap G_{in})\cdot U^+$ is open in G_{in} we see that ψ induces an elementary action on G_{in}. By 3.8 we have $G = H\cdot G_{in}$. If k is a field and G is an algebraic k-group then by SGA3, Exp $\mathrm{VI_B}$, No. 7, G_{in} is a smooth connected algebraic k-group. Finally, we give the following criterion for a k_m-action to be elementary.

3.10. PROPOSITION. *Let G be an affine finitely presented group scheme with connected fibres, and assume that either G is reductive or k is a field of characteristic 0. Then an action ψ of k_m on G by automorphisms is elementary if (and only if) ψ has at most the weights 0, ± 1 on $\mathfrak{g} = \mathrm{Lie}(G)$.*

Proof. The "only if" is immediate from 3.4. Let $\hat{G} = G \rtimes k_m$ be the semidirect product with k_m acting on G via ψ, and let $S = \{1\} \times k_m \subset \hat{G}$. Identify the character group of S with Z, and let H be the centralizer of S in G. We have the decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ into weight spaces of ψ, and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Also $\mathfrak{g}_0 = \mathrm{Lie}(H)$, and $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G}) = \mathfrak{g} \oplus k \cdot e$ (where e is the canonical generator of $\mathrm{Lie}(S) \cong k$) with $[e, x] = ix$ for $x \in \mathfrak{g}_i$.

(a) Suppose G is reductive and splits over k. Then the same holds for $\hat{G}$. Choose a maximal torus T of $\hat{G}$ containing S, let Φ be the root system of T and $\hat{\mathfrak{g}} = \sum \mathfrak{g}^\alpha$ the root space decomposition. Then $\mathfrak{g}_i = \sum_{\alpha \in \Phi_i} \mathfrak{g}^\alpha$ $(i = \pm 1)$ where Φ_i is the set of all $\alpha \in \Phi$ such that $\alpha | S = i$. Also $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0$ for $\alpha, \beta \in \Phi_i$ since $(\alpha + \beta) | S = 2i$ is not a weight of S in $\mathfrak{g}$. It follows that the root subgroups U^α, U^β corresponding to $\mathfrak{g}^\alpha, \mathfrak{g}^\beta$ commute, and $U^\pm = \prod_{\alpha \in \Phi_{\pm 1}} U^\alpha$ is a vector group, isomorphic with the additive group of $\mathfrak{g}_{\pm 1}$. The exponential maps $\exp_\alpha \colon \mathfrak{g}^\alpha \to U^\alpha$ satisfy $\psi_t \cdot \exp_\alpha(x) = \mathrm{Int}(s) \cdot \exp_\alpha(x) = \exp_\alpha(\mathrm{Ad}\, s \cdot x) = \exp_\alpha(tx)$ for all $t \in k_m$, where we set $s = (1, t) \in S$. Now $\hat{\Omega} = U^- \cdot H \cdot S \cdot U^+$ contains the big cell of $\hat{G}$ defined by an ordering of Φ with the property that Φ_1 consists of positive roots. Hence $\hat{\Omega}$ is open and dense in $\hat{G}$, and therefore $U^- \cdot H \cdot U^+ = \hat{\Omega} \cap G$ is open and dense in G. By the remark in 3.2, ψ is an elementary action.

(b) If G is reductive but not split over k it splits over an fppf extension K of k. Thus we have subgroups $\tilde{U}^\pm$ of G_K such that $H_K, \tilde{U}^\pm$ satisfy the conditions of 3.1, and we have to show that $\tilde{U}^\pm$ is defined over k. By faithfully flat descent, it suffices to show that

$$\tilde{U}^\pm \otimes_{i_1} (K \otimes_k K) = \tilde{U}^\pm \otimes_{i_2} (K \otimes_k K) \tag{1}$$

where $i_1, i_2 \colon K \to K \otimes_k K$ are the maps $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$. Since ψ is defined over k the two base extensions of ψ_K induced by i_1 and i_2 are the same, and thus (1) follows from 3.6(a).

(c) Let k be a field of characteristic 0. Then $\mathfrak{g}_{\pm 1}$ is an algebraic Lie algebra since it is the derived algebra of $k \cdot e \oplus \mathfrak{g}_{\pm 1}$ ([5, p. 262, 2.6]). For

$x \in \mathfrak{g}_{\pm 1}$ we have $(\mathrm{ad}_{\hat{\mathfrak{g}}}(x))^3 = 0$ and $\mathrm{ad}\, x \cdot e = -[e, x] = \mp x$.

Hence x is nilpotent. Therefore $\mathfrak{g}_{\pm 1}$ is the Lie algebra of a unique subgroup $U^{\pm}$ of G which is isomorphic with the additive group of $\mathfrak{g}_{\pm 1}$ under exp. Now it follows easily that $H, U^{\pm}$ satisfy the conditions of 3.1.

§ 4. The Jordan pair associated with an elementary action

The notations of §§ 2, 3 will be used throughout.

4.1. THEOREM. *Let ψ be an elementary action. There exists a unique Jordan pair structure on the pair $\mathfrak{V} = (\mathfrak{V}^+, \mathfrak{V}^-)$ of k-modules, and a unique morphism $b\colon W \to H$ with the following property: For all $(x, y) \in \mathfrak{V}^+_R \times \mathfrak{V}^-_R$ ($R \in k$-**alg**) we have* $\exp(x)\exp(y) \in \Omega(R) \Leftrightarrow (x, y)$ *quasi-invertible and in this case,*

$$\exp(x)\exp(y) = \exp(y^x) b(x, y) \exp(x^y) . \tag{$*$}$$

(b) *H acts on $\mathfrak{V}$ by automorphisms via the adjoint representation* (cf. 3.6(b)).

(c) *The morphism b satisfies*

$$\mathrm{Ad}\, b(x, y) \cdot z = B(x, y) z , \qquad \mathrm{Ad}\, b(x, y) \cdot w = B(y, x)^{-1} w , \tag{1}$$

$$h b(x, y) h^{-1} = b(\mathrm{Ad}\, h \cdot x, \mathrm{Ad}\, h \cdot y) , \tag{2}$$

$$b(tx, t^{-1}y) = b(x, y) , \tag{3}$$

$$b(x, y) b(x^y, w) = b(x, y + w) , \tag{4}$$

$$b(z, y^x) b(x, y) = b(x + z, y) , \tag{5}$$

for all $t \in k_m$, $(x, y) \in W$, $h \in H$, $(z, w) \in \mathfrak{V}^+_a \times \mathfrak{V}^-_a$ such that $(x, y + w)$ and $(x + z, y)$ are quasi-invertible.

In the examples of 3.2 one checks easily that the associated Jordan pair is isomorphic with the Jordan pair of rectangular (alternating, symmetric) matrices with quadratic maps $Q(x)y = x \cdot {}^t y \cdot x$.

4.2. Let $X \subset \mathfrak{V}^+_a \times \mathfrak{V}^-_a$ be the inverse image of Ω under the map $(x, y) \mapsto \exp(x)\exp(y)$. Then X is open and dense. By 3.4, we can write

$$\exp(x)\exp(y) = \exp(f_-(x, y)) b(x, y) \exp(f_+(x, y)) \tag{1}$$

for all $(x, y) \in X$, with unique morphisms $f_{\pm}\colon X \to \mathfrak{V}^{\pm}_a$ and $b\colon X \to H$. By applying ψ_t to (1) and comparing terms in U^-, H, U^+ we get that $(x, y) \in X$ if and only if $(tx, t^{-1}y) \in X$ and then

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(2) $$b(tx, t^{-1}y) = b(x, y)\ ,$$

(3) $$t^{-1}f_-(x, y) = f_-(tx, t^{-1}y)\ , \qquad tf_+(x, y) = f_+(tx, t^{-1}y)\ .$$

Consider now the elementary action $\psi' = \psi^{-1}$ (cf. 3.1). Then the formula analogous to (1) is

(4) $$\exp(y)\exp(x) = \exp(f'_-(y, x))b'(y, x)\exp(f'_+(y, x))\ .$$

By taking inverses in (1) and using (2) and (3) for $t = -1$ we get

(5) $$\exp(y)\exp(x) = \exp(f_+(x, y))b(x, y)^{-1}\exp(f_-(x, y))\ .$$

Comparison with (4) yields $f'_\mp(y, x) = f_\pm(x, y)$ and

(6) $$b'(y, x) = b(x, y)^{-1}\ .$$

To shorten notation, write

(7) $$f_+(x, y) = f'_-(y, x) = x^y\ , \qquad f_-(x, y) = f'_+(y, x) = y^x\ .$$

Now (1) and (3) read

(8) $$\exp(x)\exp(y) = \exp(y^x)b(x, y)\exp(x^y)\ ,$$

(9) $$t(x^y) = (tx)^{t^{-1}y}\ , \qquad t^{-1}(y^x) = (t^{-1}y)^{tx}\ .$$

If we apply $\mathrm{Int}(h)$ to (8), use 3.6(b) and compare terms we get 4.1.2 and

(10) $$\mathrm{Ad}\,h\cdot(x^y) = (\mathrm{Ad}\,h\cdot x)^{\mathrm{Ad}\,h\cdot y}\ .$$

4.3. LEMMA. *Let* $(x, y) \in X$, $(z, w) \in \mathfrak{V}_a^+ \times \mathfrak{V}_a^-$. *Then* $(x + z, y) \in X$ *if and only if* $(z, y^x) \in X$, *and then*

(1) $$(x + z)^y = x^y + \mathrm{Ad}\,b(x, y)^{-1}\cdot z^{(y^x)}\ ,$$

(2) $$y^{x+z} = (y^x)^z\ ,$$

(3) $$b(z, y^x)b(x, y) = b(x + z, y)\ .$$

Similarly, $(x, y + w) \in X$ *if and only if* $(x^y, w) \in X$, *and then*

(4) $$(y + w)^x = y^x + \mathrm{Ad}\,b(x, y)\cdot w^{(x^y)}\ ,$$

(5) $$x^{y+w} = (x^y)^w\ ,$$

(6) $$b(x, y + w) = b(x, y)b(x^y, w)\ .$$

Proof. We have $\exp(x + z) = \exp(z)\exp(y^x)b(x, y)\exp(x^y)$. This shows that $(x + z, y) \in X$ if and only if $(z, y^x) \in X$. Assuming this to be the case, we get, by 3.6,

$$\begin{aligned}\exp(x+z)\exp(y) &= \exp(y^{x+z})b(x+z,y)\exp((x+z)^y)\\ &= \exp((y^x)^z)b(z,y^x)\exp(z^{(y^x)})b(x,y)\exp(x^y)\\ &= \exp((y^x)^z)b(z,y^x)b(x,y)\exp(\mathrm{Ad}\, b(x,y)^{-1}\cdot z^{(y^x)} + x^y)\,,\end{aligned}$$

and comparing terms in U^-, H, U^+ yields (1)–(3). Now (4)–(6) follow from this by passing to (G, ψ^{-1}), in view of 4.2.

4.4. LEMMA. Let $(x, y) \in \mathfrak{V}_R^+ \times \mathfrak{V}_R^-$, $t \in R$, $R \in k$-**alg**. *Then* $(tx, y) \in X(R)$ *if and only if* $(x, ty) \in X(R)$, *and in this case,*

(1) $$b(tx, y) = b(x, ty)\,,$$

(2) $$(tx)^y = t(x^{ty})\,, \qquad (ty)^x = t(y^{tx})\,.$$

Proof. We may assume $R = k$ after a base extension. Let Y (resp. Y') be the inverse image of X under the morphism $t \mapsto (tx, y)$ (resp. $t \mapsto (x, ty)$) from k_a to $\mathfrak{V}_a^+ \times \mathfrak{V}_a^-$. Then Y and Y' are open subschemes of k_a. To show that they are equal it suffices to show that $Y(K) = Y'(K)$ for all *fields* $K \in k$-**alg**. Thus we have to show that $(tx, y) \in X(K)$ if and only if $(x, ty) \in X(K)$, for all $t \in K$. If $t = 0$ this is trivial. If $t \neq 0$ then $(x, ty) \in X(K)$ if and only if $(tx, t^{-1}ty) = (tx, y) \in X(K)$ by 4.2. Now define morphisms $\varphi_1, \varphi_2\colon Y \to \mathfrak{V}_a^+$ by $\varphi_1(t) = (tx)^y$, $\varphi_2(t) = t(x^{ty})$. By 4.2, φ_1 and φ_2 coincide on $Y \cap k_m$. Since k_m is dense in k_a and $\mathfrak{V}_a^+$ is separated we have $\varphi_1 = \varphi_2$. The proof of the other two formulas is similar.

4.5. *Definition of the quadratic maps* Q_+, Q_-. Let $(x, y) \in \mathfrak{V}_R^+ \times \mathfrak{V}_R^-$, $R \in k$-**alg**. Since X is open and contains $\{0\} \times \mathfrak{V}_a^-$ and $\mathfrak{V}_a^+ \times \{0\}$, it follows that $(\varepsilon x, y)$ and $(x, \varepsilon y)$ are in $X(R(\varepsilon))$ (where $R(\varepsilon)$ is the ring of dual numbers). Thus we can write

$$x^{\varepsilon y} = f(x, y) + \varepsilon g(x, y)$$

with well-defined morphisms $f, g\colon \mathfrak{V}_a^+ \times \mathfrak{V}_a^- \to \mathfrak{V}_a^+$. Since $x^0 = x$ it follows for $\varepsilon \to 0$ that $f(x, y) = x$. We claim that $g(x, y)$ is linear in y and quadratic in x. The first statement is clear since $g(x, y)$ is the derivative of the map $x \mapsto x^w$ at $w = 0$ in direction y. By 4.4, we have

$$\begin{aligned}(tx)^{\varepsilon y} &= tx + \varepsilon g(tx, y) = t(x^{t\varepsilon y})\\ &= t(x + \varepsilon g(x, ty)) = tx + \varepsilon t^2 g(x, y)\end{aligned}$$

for all $t \in R$, and hence $g(x, y)$ is homogeneous of degree 2 in x. By [19], it is quadratic in x. Thus there is a unique quadratic map $Q_+ : \mathfrak{V}^+ \to \operatorname{Hom}(\mathfrak{V}^-, \mathfrak{V}^+)$ such that $g(x, y) = Q_+(x) \cdot y$; i.e.,

$$(1) \qquad x^{\varepsilon y} = x + \varepsilon Q_+(x) \cdot y \, .$$

Similarly, we define $Q_- : \mathfrak{V}^- \to \operatorname{Hom}(\mathfrak{V}^+, \mathfrak{V}^-)$ by

$$(2) \qquad y^{\varepsilon x} = y + \varepsilon Q_-(y) \cdot x \, .$$

Note that

$$(3) \qquad (\varepsilon x)^y = \varepsilon(x^{\varepsilon y}) = \varepsilon x$$

by 4.4. Also, from 4.2.10 it is clear that H acts via Ad by automorphisms of (Q_+, Q_-) on $(\mathfrak{V}^+, \mathfrak{V}^-)$. As in 2.1, 2.2, we drop the subscripts $\pm$ on $Q_\pm$ and define $\{xyz\}$ and $B(x, y)$.

4.6. LEMMA. *For* (x, y), $(z, w) \in \mathfrak{V}^+ \times \mathfrak{V}^-$, $h \in \operatorname{Lie}(H)$ *the following formulae hold* (here $[a, b] = aba^{-1}b^{-1}$).

$$(1) \qquad \operatorname{Int}(\exp x) \cdot \exp(\varepsilon y) = \exp(\varepsilon y) b(x, \varepsilon y) \exp(\varepsilon Q_x y) \, .$$

$$(2) \qquad [\exp(x), \exp(\varepsilon y)] = b(x, \varepsilon y) \exp(\varepsilon Q_x y) \, ,$$

$$(3) \qquad [\exp(y), \exp(\varepsilon z)] = \exp(\varepsilon Q_y z) b(\varepsilon z, -y) \, ,$$

$$(4) \qquad [\exp(\varepsilon x), \exp(\varepsilon y)] = [h, \exp(\varepsilon x)] = [h, \exp(\varepsilon y)] = 1 \, ,$$

$$(5) \qquad \begin{aligned} &[\exp(z), b(x, \varepsilon y)] \\ &\quad = [\exp(z), [\exp(-\varepsilon y), \exp(x)]] = \exp(\varepsilon\{xyz\}) \, , \end{aligned}$$

$$(6) \qquad \begin{aligned} &\operatorname{Int}(\exp(x)\exp(y)) \cdot \exp(\varepsilon z) \\ &\quad = \exp(\varepsilon Q_y z) b(x, \varepsilon Q_y z) b(\varepsilon z, -y) \exp(\varepsilon B(x, y) z) \, . \end{aligned}$$

$$(7) \qquad \operatorname{Ad} b(x, y) \cdot z = B(x, y) z \, , \qquad \operatorname{Ad} b(x, y) \cdot w = B(y, x)^{-1} w \, .$$

Proof. (1)–(3) are immediate from 4.2.8 and 4.5. Replacing x by εx in (2) we get the first formula of (4) since $b(\varepsilon x, \varepsilon y) = b(\varepsilon^2 x, y) = b(0, y) = 1$ 4.4. By 3.6, $[h, \exp(\varepsilon x)] = \exp(\varepsilon \operatorname{Ad} h \cdot x - \varepsilon x)$. Since $h \in \operatorname{Lie}(H)$, $\operatorname{Ad} h \cdot x$ is of the form $x + \varepsilon x'$ and hence $\varepsilon \operatorname{Ad} h \cdot x = \varepsilon x + \varepsilon^2 x' = \varepsilon x$. Similarly, one proves the third formula. To prove (5), we use (1), (4), 4.5, and 4.3.3:

$$\begin{aligned} &\operatorname{Int}(\exp(z)) \cdot b(x, \varepsilon y) \\ &\quad = \operatorname{Int}(\exp(z)) \cdot (\exp(-\varepsilon y) \exp(x) \cdot \exp(\varepsilon y) \exp(-x^{\varepsilon y})) \end{aligned}$$

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$$= \exp(z)\exp(-\varepsilon y)\exp(x)\exp(\varepsilon y)\exp(-x^{\varepsilon y} - z)$$
$$= \exp(-\varepsilon y)b(z, -\varepsilon y)\exp(x + z - \varepsilon Q_z y)\exp(\varepsilon y)\exp(-x - z - \varepsilon Q_x y)$$
$$= \exp(-\varepsilon y)b(z, -\varepsilon y)\operatorname{Int}(\exp(x + z))\cdot(\exp(\varepsilon y)\exp(-\varepsilon Q_x y - \varepsilon Q_z y))$$
$$= \exp(-\varepsilon y)b(z, -\varepsilon y)\exp(\varepsilon y)b(x + z, \varepsilon y)\exp(\varepsilon(Q_{x+z}y - Q_x y - Q_z y))$$
$$= b(-z, \varepsilon y)b(x + z, \varepsilon y)\exp(\varepsilon\{xyz\}) = b(x, \varepsilon y)\cdot\exp(\varepsilon\{xyz\}) .$$

Now (5) follows from (1), (2) and (4). For (6) we use the commutator formula $[ab, c] = (\operatorname{Int}(a)\cdot[b, c])[a, c]$, for $a = \exp(x)$, $b = \exp(y)$, $c = \exp(\varepsilon z)$. Then $[a, c] = 1$ and hence

$$[ab, c] = \operatorname{Int}(a)\cdot[b, c] = \operatorname{Int}(a)\cdot(\exp(\varepsilon Q_y z)b(\varepsilon z, -y))$$
$$= \exp(\varepsilon Q_y z)b(x, \varepsilon Q_y z)\exp(\varepsilon Q_x Q_y z)\exp(-\varepsilon\{xyz\})b(\varepsilon z, -y)$$
$$= \exp(\varepsilon Q_y z)b(x, \varepsilon Q_y z)b(\varepsilon z, -y)\exp(-\varepsilon\{xyz\} + \varepsilon Q_x Q_y z) .$$

Multiplying by $\exp(\varepsilon z)$ on the right we get (6). Finally, if $(x, y) \in X$ then

$$\operatorname{Int}(\exp(x)\exp(y))\cdot\exp(\varepsilon z)$$
$$= \operatorname{Int}(\exp(y^x)b(x, y)\exp(x^y))\cdot\exp(\varepsilon z)$$
$$= \operatorname{Int}(\exp(y^x))\operatorname{Int}(b(x, y))\cdot\exp(\varepsilon z)$$
$$= \exp(y^x)\exp(\varepsilon\operatorname{Ad}b(x, y)\cdot z)\exp(-y^x)$$
$$= \exp(u)\cdot h\cdot\exp(\varepsilon\operatorname{Ad}b(x, y)\cdot z)$$

where the exact form of $u \in \mathfrak{V}^-$ and $h \in H$ is not important. Comparing terms in U^+ with (6) we get the first formula of (7), and the second one follows by passing to ψ^{-1}.

4.7. Lemma. *If $(x, y) \in X$ then $B(x, y)$ and $B(x, y)$ are invertible, and*

$$x^y = B(x, y)^{-1}(x - Q_x y) , \qquad y^x = B(y, x)^{-1}(y - Q_y x) .$$

Proof. By 4.6.7, $B(x, y)$ is invertible. By 4.5.3, 4.3.1, 4.6.7,

$$((1 + \varepsilon)x)^y = (x + \varepsilon x)^y = x^y + \varepsilon B(x, y)^{-1}x .$$

On the other hand, by 4.4, 4.3,

$$((1 + \varepsilon)x)^y = (1 + \varepsilon)(x^{(1+\varepsilon)y}) = (1 + \varepsilon)((x^{\varepsilon y})^y)$$
$$= (1 + \varepsilon)(x + \varepsilon Q_x y)^y = (1 + \varepsilon)(x^y + \varepsilon B(x, y)^{-1}Q_x y) .$$

Comparing coefficients at ε we get the first formula, and the second one follows by passing to ψ^{-1}.

4.8. Lemma. *The quadratic maps Q_+, Q_- satisfy the Jordan identities.*

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Proof. From 4.5.4 and 4.6.7, $B(x, y)Q_z = Q(B(x, y)z)B(y, x)^{-1}$ for all $(x, y) \in X$, $z \in \mathfrak{V}_a^+$, and since X is dense this implies

$$B(x, y)Q_z B(y, x) = Q(B(x, y)z) \tag{1}$$

for all $x, z \in \mathfrak{V}_a^+$, $y \in \mathfrak{V}_a^-$. By expanding and comparing terms of equal degree we get (among other identities)

$$D(x, y)Q_z + Q_z D(y, x) = Q(z, \{xyz\}) . \tag{2}$$

Now compute $x^{y+\varepsilon w}$ for $(x, y) \in X$ in two ways, using 4.3:

$$\begin{aligned} x^{x+\varepsilon w} &= (x^y)^{\varepsilon w} = x^y + \varepsilon Q(x^y)\cdot w \\ &= (x^{\varepsilon w})^y = (x + \varepsilon Q_x w)^y = x^y + \varepsilon B(x, y)^{-1}\cdot Q_x w . \end{aligned}$$

By (1) and 4.7, this implies $Q_x = B(x, y)Q\ (x^y) = Q(x - Q_x y)B(y, x)^{-1}$, and therefore

$$Q(x - Q_x y) = Q_x B(y, x) . \tag{3}$$

Again by density of X, this holds for all $(x, y) \in \mathfrak{V}_a^+ \times \mathfrak{V}_a^-$. By expanding and comparing terms of equal degree we get

$$Q(x, Q_x y) = Q_x D(y, x) , \tag{4}$$

$$Q(Q_x y) = Q_x Q_y Q_x . \tag{5}$$

Setting $z = x$ in (2) and comparing with (4) we have, since $\{xyx\} = 2Q_x y$,

$$D(x, y)Q_x = Q_x D(y, x) . \tag{6}$$

Let $(x, y) \in X$. Then by 4.3 and 4.6.7, $B(x, -\varepsilon y)B(x^{-\varepsilon y}, y) = B(x, (1 - \varepsilon)y)$ $= B((1 - \varepsilon)x, y) = B(x, y^{-\varepsilon y})B(-\varepsilon x, y)$. By density, this holds for all (x, y) $\in \mathfrak{V}_a^+ \times \mathfrak{V}_a^-$. If we expand and compare terms of equal degree at ε we get

$$D(Q_x y, y) = D(x, Q_y x) . \tag{7}$$

Passing to ψ^{-1} it follows that (5)–(7) hold with x and y interchanged. This proves 4.8.

4.9. Lemma. *H is closed in G; it is the fixed point set of ψ.*

Proof. By definition, H is contained in the fixed point set, say H', of ψ. Conversely let $g \in H'(k)$. After an fppf extension we may assume that $g = \exp(x)\exp(y)h\cdot\exp(z)$ with $x, z \in \mathfrak{V}^+$, $y, \in \mathfrak{V}^-$, $h \in H(k)$ (cf. 3.8). Applying ψ_t we get $g = \exp(tx)\exp(t^{-1}y)h\cdot\exp(tz)$ for all $t \in k_m$. Hence

$$\exp((1-t)x)\exp(y) = \exp(t^{-1}y)h\cdot\exp(tz-z)\cdot h^{-1}\in\Omega\ ,$$

and therefore

$$\exp((1-t)x)\exp(y) = \exp(y^{(1-t)x})b((1-t)x, y)\exp([1-t)x]^{y})\ .$$

Comparing terms in H, U^+, U^- we have $b((1-t)x, y) = 1$ and by 4.6.7 and 4.7, $t^{-1}y = y^{(1-t)x} = y - Q_y(x - tx)$, for all $t \in k_m$. By comparing terms at powers of t we see that $y = 0$. Hence $\exp((1-t)x) = h\cdot\exp((t-1)z)\cdot h^{-1}$, and this implies $\exp(x) = h\cdot\exp(-z)\cdot h^{-1}$ and therefore $g = h \in H(k)$. Now by [5, p. 165, 3.6(*d*)], H is closed in G.

4.10. To complete the proof of 4.1, all that remains to be shown is that $X = W$, the subscheme of quasi-invertible pairs. By 4.7, $X \subset W$, and x^y and y^x are just the quasi-inverse of the Jordan pair $\mathfrak{V}$. Define a morphism $\tilde{b}\colon W \to G$ by

$$\tilde{b}(x, y) = \exp(-y^x)\exp(x)\exp(y)\exp(-x^y)\ .$$

Then $\tilde{b}$ extends b and $X \subset \tilde{b}^{-1}(H) \subset W$. Also, $\tilde{b}^{-1}(H)$ is closed in W since H is closed in G by 4.9. By density of X, $W = \tilde{b}^{-1}(H)$. Therefore

$$\exp(x)\exp(y) = \exp(y^x)\tilde{b}(x, y)\exp(x^y)$$

belongs to Ω for all $(x, y) \in W$. This proves $X = W$.

4.11. Corollary. *Assume that G is a scheme, and let $\mathfrak{g}$ (resp. $\mathfrak{h}$) denote the Lie algebra of G (resp. H). Then*

(1) $$\mathfrak{g} = \mathfrak{V}^- \oplus \mathfrak{h} \oplus \mathfrak{V}^+$$

(direct sum of k-modules), and the following multiplication rules hold:

(2) $$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}\ ,\quad [\mathfrak{V}^+, \mathfrak{V}^-] \subset \mathfrak{h}\ ,\quad [\mathfrak{h}, \mathfrak{V}^\sigma] \subset \mathfrak{V}^\sigma\ ,\quad [\mathfrak{V}^\sigma, \mathfrak{V}^\sigma] = 0\ .$$

For $x, z \in \mathfrak{V}^\sigma$, $y \in \mathfrak{V}^{-\sigma}$, $h \in \mathfrak{h}$, we have the formulas

(3) $$\mathrm{Ad}\exp(x)\cdot z = z\ ,$$

(4) $$\mathrm{Ad}\exp(x)\cdot h = h + [x, h]\ ,$$

(5) $$\mathrm{Ad}\exp(x)\cdot y = y + [x, y] + Q_x y$$

(6) $$e^{\varepsilon[x,y]} = b(x, \varepsilon y)\ ,$$

(7) $$-[z, [y, x]] = \{xyz\}\ .$$

Proof. Since Ω is open in G and contains the unit section, (1)

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follows from 3.4. Let $R = k(\varepsilon, \varepsilon')$. If we replace x by $\varepsilon' x$ in 4.6.2 we get $e^{\varepsilon\varepsilon'[x,y]} = [e^{\varepsilon' x}, e^{\varepsilon y}] = [\exp(\varepsilon' x), \exp(\varepsilon y)] = b(\varepsilon' x, \varepsilon y) = b(x, \varepsilon\varepsilon' y)$ which implies (6) in view of [5, p. 210], and also shows that $[\mathfrak{V}^+, \mathfrak{V}^-] \subset \mathfrak{h}$. Now (5) and (7) follow easily from 4.6, and (4) follows from 3.6, as well as $[\mathfrak{h}, \mathfrak{V}^\sigma] \subset \mathfrak{V}^\sigma$. By commutativity of U^σ we have $[\mathfrak{V}^\sigma, \mathfrak{V}^\sigma] = 0$.

4.12. COROLLARY. *Let $G_{in} \subset G$ be as in 3.9, and let H_{in} be the subgroup sheaf of H generated by $b(W)$; i.e., H_{in} is the sheaf associated with the k-group functor $R \mapsto$ subgroup of $H(R)$ generated by all $b(x, y)$, $(x, y) \in W(R)$.*

(a) *The Jordan pairs associated with (G, ψ) and (G_{in}, ψ) are the same.*

(b) *We have $H_{in} = H \cap G_{in}$. If k is a field and H is an algebraic k-group then H_{in} is a smooth connected algebraic k-group.*

Proof. From (*) of 4.1 it is clear that (G, ψ) and (G_{in}, ψ) define the same Jordan pair, and that $H_{in} \subset H \cap G_{in}$. Let now $F = H_{in} \cdot U^+ \cdot U^- \cdot U^+ \subset G_{in}$. We claim that $F = G_{in}$. Let $g \in G_{in}(R)$. Then there exists a fppf extension S of R such that $g = u_1 \cdots u_n$ where the u_i are alternatingly in $U^+(S)$ and $U^-(S)$. After a base extension, it therefore suffices to show the following: if $u, w \in U^-(k)$, $v \in U^+(k)$ then there exists a fppf extension R of k such that $uvw \in F(R)$. Choosing x as in 3.7, we have by 4.1,

$$uvw = (ux^{-1})(xvw) \in U^+(R)H_{in}(R)U^-(R)U^-(R)H_{in}(R)U^+(R) \subset F(R) .$$

Now

$$H \cap G_{in} = H \cap (H_{in} \cdot U^+ \cdot U^- \cdot U^+) = H_{in} \cdot (H \cap U^+U^-U^+) = H_{in} ,$$

by 3.6. The last assertion follows from SGA3, Exp. VI_B, no. 7.

4.13. LEMMA. *Let $X = \mathfrak{V}_a^+ \times \mathfrak{V}_a^- \times H \times \mathfrak{V}_a^+$ and $\pi: X \to G$ the epimorphism of sheaves given by $\pi(x, y, h, z) = \exp(x)\exp(y)h \cdot \exp(z)$. Then $\pi(x, y, h, z) = \pi(x', y', h', z')$ if and only if $(x - x', y)$ is quasi-invertible, and*

$$y' = y^{x-x'} , \quad h' = b(x - x', y)h , \quad z' = z + \operatorname{Ad} h^{-1} \cdot (x - x')^y .$$

This follows easily from 4.1 and 3.6.

4.14. THEOREM (Generators and Relations). *Let (G, ψ) be an elementary system. Let G' be a k-group sheaf, and let $f_0: H \to G'$, $f_\sigma: U^\sigma \to G'$ ($\sigma = +, -$) be homomorphisms. Then there exist a homomorphism $f: G \to G'$ extending f_0 and f_+, f_- if and only if*

$$(1) \qquad f_0(h)f_\sigma(u)f_0(h)^{-1} = f_\sigma(huh^{-1}) ,$$

$$(2) \qquad f_+(\exp(x))f_-(\exp(y)) = f_-(\exp(y^x))f_0(b(x, y))f_+(\exp(x^y)) ,$$

for all $h \in H$, $u \in U^\sigma$, $(x, y) \in W$. (Since G is generated by H, U^+, and U^-, such an f is necessarily unique).

Proof. By 3.6 and 4.1, the conditions (1) and (2) are clearly necessary for the existence of f. Conversely, assume that they hold. Let $\pi: X \to G$ be as in 4.13. Let $\varphi_\sigma: \mathfrak{V}_a^\sigma \to G'$ be given by $\varphi_\sigma = f_\sigma \circ \exp$ and $\varphi: X \to G'$ by $\varphi(x, y, h, z) = \varphi_+(x)\varphi_-(y)f_0(h)\varphi_+(z)$. If f exists then we have $f \circ \pi = \varphi$. To prove existence, we use the fact that the diagram

$$X \times_G X \underset{\text{pr}_2}{\overset{\text{pr}_1}{\rightrightarrows}} X \xrightarrow{\ \pi\ } G$$

of k-sheaves is exact (cf. [5, p. 292]). Therefore it suffices to show that $\pi(w) = \pi(w')$ implies $\varphi(w) = \varphi(w')$, for all $w, w' \in X$. Thus let $w = (x, y, h, z)$ and $w' = (x', y', h', z')$ and assume that $\pi(w) = \pi(w')$. By (1) and (2) and 4.13 we then get

$$\begin{aligned}\varphi_+(x - x')\varphi_-(y) &= \varphi_-(y^{x-x'})f_0(b(x - x', y))\varphi_+((x - x')^y) \\ &= \varphi_-(y')f_0(h'h^{-1})\varphi_+(\text{Ad}\, h \cdot (z' - z)) \\ &= \varphi_-(y')f_0(h')\varphi_+(z' - z)f_0(h)^{-1} ,\end{aligned}$$

which implies $\varphi(w) = \varphi(w')$. This proves the existence of $f: G \to G'$ satisfying $f \circ \pi = \varphi$. We still have to show that f is multiplicative. If $v \in U^+$, $g \in G$ then

$$(3) \qquad f(gv) = f(g)f(v) , \qquad f(vg) = f(v)f(g) .$$

Indeed, we may assume, by passing if necessary to a fppf base extension, that $g = \exp(x)\exp(y)h \cdot \exp(z)$, $v = \exp(z')$, and then

$$\begin{aligned}f(gv) &= f(\exp(x)\exp(y)h \cdot \exp(z + z')) \\ &= \varphi_+(x)\varphi_-(y)f_0(h)\varphi_+(z + z') \\ &= \varphi_+(x)\varphi_-(y)f_0(h)\varphi_+(z)\varphi_+(z') = f(g)f(v) .\end{aligned}$$

Similarly, one proves the second formula, and also

$$(4) \qquad f(hg) = f(h)f(g) , \qquad f(gh) = f(g)f(h) ,$$

for $g \in G$, $h \in H$. Finally, we show that

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$$f(uvw) = f(u)f(v)f(w)\,, \tag{5}$$

for $u, w \in U^-$, $v \in U^+$. Possibly after passing to an fppf base extension, there exists an element $x \in U^+$ such that $ux^{-1} \in \Omega^{-1}$ and $xvw \in \Omega$ (see 3.7). Let $ux^{-1} = zhy$, $xvw = y'h'z'$ where $z, z' \in U^+$, $h, h' \in H$, $y, y' \in U^-$. Then we have by (3) and (4) that

$$\begin{aligned} f(uvw) &= f(ux^{-1}\cdot xvw) = f(zhy\cdot y'h'z') \\ &= f(z)f(h)f(y)f(y')f(h')f(z') = f(zhy)f(y'h'z') \\ &= f(ux^{-1})f(xvw) = f(u)f(x)^{-1}f(x)f(v)f(w) \\ &= f(u)f(v)f(w)\,. \end{aligned}$$

Now the multiplicativity of f is an easy consequence of (3)–(5), in view of the fact that G is generated (as a k-group sheaf) by H, U^+, and U^-.

4.15. COROLLARY. *Let (G, ψ) and (G', ψ') be elementary systems with associated Jordan pairs $\mathfrak{V}$ and $\mathfrak{V}'$.*

(a) *Let $f: (G, \psi) \to (G', \psi')$ be a homomorphism, let*

$$f_0 = f\,|\,H: H \to H'\,, \tag{1}$$

and define $\varphi_\sigma: \mathfrak{V}_a^\sigma \to \mathfrak{V}_a'^\sigma$ by

$$f\circ \exp = \exp'\circ\varphi_\sigma \qquad \text{(cf. 3.5)}\,. \tag{2}$$

Then $(\varphi_+, \varphi_-): \mathfrak{V} \to \mathfrak{V}'$ is a homomorphism of Jordan pairs, and we have

$$\operatorname{Ad} f_0(h)\cdot\varphi_\sigma(z) = \varphi_\sigma(\operatorname{Ad} h\cdot z)\,, \tag{3}$$

$$f_0(b(x, y)) = b'(\varphi_+(x), \varphi_-(y))\,, \tag{4}$$

for all $z \in \mathfrak{V}_a^\sigma$, $(x, y) \in W$, $\sigma = \pm$.

(b) *Conversely, let $f_0: H \to H'$ be a homomorphism, let $(\varphi_+, \varphi_-): \mathfrak{V} \to \mathfrak{V}'$ be a homomorphism of Jordan pairs, and assume that* (3) *and* (4) *hold. Then there exists a unique homomorphism $f: (G, \psi) \to (G', \psi')$ such that* (1) *and* (2) *hold.*

Proof. (a) By 4.9, f maps H into H'. Thus (3) follows from 3.6. If we apply f to $(*)$ of 4.1 we see that (4) holds and that (φ_+, φ_-) preserves quasi-inverses. By 4.5, it follows that (φ_+, φ_-) is a homomorphism of Jordan pairs.

(b) Define $f_\sigma: U^\sigma \to G'$ by (2). Then one checks that $f_0, f_\pm$ satisfy the conditions of 4.14 and hence extend to a homomorphism $f: G \to G'$ which is easily seen to be compatible with ψ and ψ'.

§ 5. The Existence theorem and applications

5.1. We consider the question of how to reconstruct G, given H and the Jordan pair $\mathfrak{V}$, and introduce the following concept. Let $\mathfrak{V}$ be a Jordan pair over k, let H be a separated k-group sheaf, and let $\rho: H \to$ **Aut** $(\mathfrak{V})$ be a homomorphism. Also let $b: W \to H$ be a morphism from the subscheme of quasi-invertible pairs of $\mathfrak{V}_a^+ \times \mathfrak{V}_a^-$ into H. Then the quadruple $\mathscr{J} = (\mathfrak{V}, H, \rho, b)$ is called a *Jordan system* if the following identities hold for $(x, y) \in W$, $(z, w) \in \mathfrak{V}_a^+ \times \mathfrak{V}_a^-$, $h \in H$, $t \in k_m$.

(1) $$\rho(b(x, y)) = \beta(x, y) ,$$

(2) $$hb(x, y)h^{-1} = b(h_+(x), h_-(y)) ,$$

(3) $$b(tx, t^{-1}y) = b(x, y) ,$$

(4) $$b(x, y)b(x^y, w) = b(x, y + w) ,$$

(5) $$b(z, y^x)b(x, y) = b(x + z, y) .$$

Here β is as in 2.2 and $\rho(h) = (h_+, h_-)$.

A *homomorphism of Jordan systems*, $\Phi: \mathscr{J} \to \mathscr{J}'$, is a pair $\Phi = (f_0, \varphi)$ where $f_0: H \to H'$ is a group homomorphism, and $\varphi = (\varphi_+, \varphi_-): \mathfrak{V} \to \mathfrak{V}'$ is a Jordan pair homomorphism, such that

(6) $$\rho'(f_0(h)) \circ \varphi = \varphi \circ \rho(h) ,$$

(7) $$f_0(b(x, y)) = b'(\varphi_+(x), \varphi_-(y)) ,$$

for all $h \in H$, $(x, y) \in W$. From 4.1 it is clear that every elementary system (G, ψ) defines a Jordan system $J(G, \psi) = (\mathfrak{V}, H, \rho, b)$ where ρ is given by the adjoint representation of H on $\mathfrak{V}^+$ and $\mathfrak{V}^-$. Now 4.15 shows that J is a covariant functor from the category of elementary systems to the category of Jordan systems which is fully faithful. In fact, *J is an equivalence of categories* since we have

5.2. THEOREM. *For every Jordan system $\mathscr{J}$ there exists an elementary system (G, ψ) whose associated Jordan system is $\mathscr{J}$.*

The idea of the proof is simple: let $\mathscr{J} = (\mathfrak{V}, H, \rho, b)$ and let $X = \mathfrak{V}_a^+ \times \mathfrak{V}_a^- \times H \times \mathfrak{V}_a^+$. If G exists then it is the quotient of X by the equivalence relation given in 4.13. Thus we take this as the definition of G and show, rather laboriously, that it has the required properties.

Define then a subfunctor $E \subset X \times X$ as follows. For $R \in k$-**alg**, $E(R)$

is the set of all $((x, y, h, z), (x', y', h', z'))$ in $X(R) \times X(R)$ for which $(x - x', y)$ is quasi-invertible, and

$$(**) \qquad y' = y^{x-x'}\,, \quad h' = b(x - x', y)h\,, \quad h_+(z' - z) = (x - x')^y\,.$$

By 2.2, E may also be described by the equations

$$\begin{aligned} &B(y, x - x')y' = y - Q_y(x - x')\,, \\ &B(y, x - x')Q_{y'}(x - x') = Q_y(x - x')\,, \\ &h' = b(x - x', y)h\,, \\ &B(x - x', y)h_+(z' - z) = x - x' - Q_{x-x'}\cdot y\,, \end{aligned}$$

with values in separated k-functors, and is therefore closed in $X \times X$.

In the following lemmas, we will use standard properties of the quasi-inverse in a Jordan pair without comment; in particular the formulas $x^{y+w} = (x^y)^w$, $(x + z)^y = x^y + B(x, y)^{-1}\cdot z^{(y^x)}$, $B(x, y)B(x^y, w) = B(x, y + w)$, $B(z, y^x)B(x, y) = B(x + z, y)$. See [15, § 3].

5.3. Lemma. *E is an equivalence relation on X.*

Proof. We have to show that $E(R)$ is (the graph of) an equivalence relation on $X(R)$, for all $R \in k$-**alg**. Since $y^0 = y$, $0^y = 0$, $b(0, y) = 1$ (which follows from 5.1.1 for $z = x = 0$) we have reflexivity. Now assume that $(**)$ holds. Then $(x' - x, y)$ is quasi-invertible and $y = (y')^{x'-x}$,

$$h = b(x - x', y)^{-1}h' = b(-(x - x'), y^{x-x'})h' = b(x' - x, y')h'$$

by 5.1.5, and by 5.1.1,

$$\begin{aligned} h'_+(z - z') &= B(x - x', y)h_+(z - z') = -B(x - x', y)(x - x')^y \\ &= -(x - x' - Q(x - x')y) = (x' - x) + Q(x' - x)y \\ &= x' - x + Q(x' - x)(y')^{x'-x} = (x' - x)^{y'}\,. \end{aligned}$$

This shows that E is symmetric. To prove transitivity, assume $(**)$ and also that $(x' - x'', y')$ is quasi-invertible and

$$y'' = (y')^{x'-x''}\,, \quad h'' = b(x' - x'', y')\,, \quad h'_+(z'' - z') = (x' - x'')^{y'}\,.$$

Then $(x - x'', y)$ is quasi-invertible, and

$$y'' = (y')^{x'-x''} = (y^{x-x'})^{x'-x''} = y^{x-x'+x'-x''} = y^{x-x''}\,.$$

Also,

$$\begin{aligned} h'' &= b(x'-x'', y')b(x-x', y)h = b(x'-x'', y^{x-x'})b(x-x', y)h \\ &= b(x'-x''+x-x', y)h = b(x-x'', y)h\ , \end{aligned}$$

by 5.1.5 and finally,

$$\begin{aligned} h_+(z''-z) &= h_+(z''-z') + h_+(z'-z) \\ &= h_+h_+'^{-1}h_+'(z''-z') + (x-x')^y \\ &= B_+(x-x', y)^{-1}\cdot(x'-x'')^{y'} + (x-x')^y \\ &= (x-x')^y + B_+(x-x', y)^{-1}\cdot((x'-x'')^{(y^{x-x'})}) \\ &= (x-x'+x'-x'')^y \\ &= (x-x'')^y\ , \end{aligned}$$

by 5.1.1.

5.4. We now define $G = X/E$ to be the quotient *sheaf* of X by E, and denote by $\pi\colon X \to E$ the canonical map. For $(x, y, h, z) \in X(R)$, the notation

$$\pi(x, y, h, z) = [x, y, h, z]$$

will be used. By general properties of sheaves (cf. [5, chap. III]), we have the exact sequence of sheaves

$$E \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X \xrightarrow{\ \pi\ } G\ .$$

Since E is closed, 1.3 shows that G is separated.

5.5. LEMMA. *There is a well defined function χ on G such that $\chi([x, y, h, z]) = \det(B(x, y)\cdot h_+)$. The open subfunctor $\Omega = \chi^{-1}(k_m)$ of G is dense and isomorphic with $\mathfrak{V}_a^- \times H \times \mathfrak{V}_a^+$ under the map $i\colon (y, h, z) \to [0, y, h, z]$.*

Proof. χ is well-defined: Suppose $[x, y, h, z] = [x', y', h', z']$. Then $y' = y^{x-x'}$ and $h'_+ = B(x-x', y)h_+$ whence $B(x', y')h'_+ = B(x', y^{x-x'})B(x-x', y)h_+ = B(x, y)h_+$. Clearly $\pi^{-1}(\Omega) = W \times H \times \mathfrak{V}_a^+$ which is dense in X. By 1.11, Ω is dense in G. From (**) it follows easily that i is a monomorphism into Ω. Conversely, if $g = [x, y, h, z] \in \Omega$ then (x, y) is quasi-invertible and hence $g = [0, y^x, b(x, y)h, z + h_+^{-1}\cdot x^y]$ belongs to the image of i.

5.6. Let $\mathfrak{V}_a^+$ act on X on the left by addition on the first factor:

$$u\cdot(x, y, h, z) = (u + x, y, h, z)\ ,$$

and let H act on X via

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$$f\cdot(x, y, h, z) = (f_+(x), f_-(y), fh, z)\,.$$

Then these actions are compatible in the sense that

$$f\cdot(u\cdot(x, y, h, z)) = f_+(u)\cdot(f\cdot(x, y, h, z))$$

and it is easily checked that they are compatible with E. Therefore we have actions of $\mathfrak{V}_a^+$ and H on G, compatible in the sense that $f\cdot(u\cdot g) = f_+(u)\cdot(f\cdot g)$ for $f \in H$, $u \in \mathfrak{V}_a^+$, $g \in G$.

5.7. LEMMA. *There exist a unique action of* $\mathfrak{V}_a^-$ *on* G *on the left such that*

(1) $$v\cdot[x, y, h, z] = [x^v, B(v, x)(v^x + y), b(x, v)^{-1}h, z]$$

whenever (x, y) *is quasi-invertible. This action is compatible with the actions of* H *and* $\mathfrak{V}_a^+$ *in the sense that*

(2) $$f\cdot(v\cdot g) = f_-(v)\cdot(f\cdot g)\,,$$

(3) $$u\cdot(v\cdot g) = v^u\cdot b(u, v)\cdot(u^v\cdot g))\,.$$

for all $f \in H$ *and* $(u, v) \in W$.

Proof. Let $v \in \mathfrak{V}^-$ and $g \in G(k)$. After passing to an fppf extension of k we may assume that $g = [x, y, h, z]$. We claim that, again after an fppf extension, we may even assume that (x, v) is quasi-invertible. Indeed, let $Y_i \subset \mathfrak{V}_a^+$ be defined by

$$Y_1(R) = \{x' \in \mathfrak{V}_R^+ \,|\, (x - x', y) \text{ quasi-invertible}\}\,,$$
$$Y_2(R) = \{x' \in \mathfrak{V}_R^+ \,|\, (x', v) \text{ quasi-invertible}\}\,.$$

Then Y_1 and Y_2 are open and dense since $x \in Y_1(k)$ and $0 \in Y_2(k)$. Hence $Y_1 \cap Y_2$ is dense, and there exists a fppf extension R of k such that $(Y_1 \cap Y_2)(R)$ is not empty. Choosing $x' \in Y_1(R) \cap Y_2(R)$, define y', h', z' by $(**)$. Then $g = [x', y', h', z']$ and (x', v) is quasi-invertible.

Now we show that (1) is well defined. Thus let $[x, y, h, z] = [x', y', h', z']$ and assume, as we may, that both (x, v) and (x', v) are quasi-invertible. By $(**)$, we also have, setting $u = x - x'$, that (u, y) is quasi-invertible, and that

$$y' = y^u\,, \qquad h' = b(u, y)h\,, \qquad h_+(z' - z) = u^y\,.$$

To shorten notation, let

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$$f = b(x, v)^{-1}, \quad w = v^x, \quad s = f_-(w + y) = B_-(v, x)(v^x + y),$$

and define f', w', s' analogously. We have to show: $(x^v - x'^v, s)$ is quasi-invertible, and

$$(4) \qquad s' = s^{(x^v - x'^v)},$$

$$(5) \qquad f'h' = b(x^v - x'^v, s) \cdot f \cdot h,$$

$$(6) \qquad f_+h_+(z' - z) = (x^v - x'^v)^s.$$

Using the formula for $(x + z)^y$, we have

$$x^v - x'^v = B_+(x, v)^{-1}((x - x')^{-v^x}) = f_+(u^{-w}),$$

in particular, $(u, -w)$ is quasi-invertible. Now

$$(x^v - x'^v, s) = (f_+(u^{-w}), f_-(y + w))$$

is quasi-invertible if and only if (since (f_+, f_-) is an automorphism of $\mathfrak{V}$) $(u^{-w}, y + w)$ is quasi-invertible, and this is the case since both $(u, -w)$ and (u, y) are quasi-invertible. Also,

$$(x^v - x'^v)^s = f_+((u^{-w})^{y+w}) = f_+(u^y) = f_+(h_+(z' - z)),$$

which proves (6). For (5), we have, by (2)–(5) of 5.1, that

$$(7) \qquad f' = f \cdot b(u, -w)^{-1},$$

and hence

$$\begin{aligned} b(x^v - x'^v, s) \cdot f \cdot h &= b(f_+(u^{-w}), f_-(y + w)) \cdot f \cdot h \\ &= f \cdot b(u^{-w}, y + w) f^{-1} \cdot f \cdot h = f \cdot b(u^{-w}, y + w) \cdot h \\ &= f \cdot b(u, -w)^{-1} b(u, y) \cdot h = f' \cdot h'. \end{aligned}$$

Finally,

$$\begin{aligned} s' &= f'_-(w' + y') = f'_-(w' + y^u) = f'_-(w^{-u} + y^u) \\ &= f'_-(y^u - (-w)^u) = f'_-(B_-(-w, u)^{-1} \cdot ((y + w)^{(u-w)}) \end{aligned}$$

and

$$s^{(x^v - x'^v)} = (f_-(w + y)^{f+(u-w)}) = f_-((w + y)^{(u-w)}).$$

Now (4) follows from (7).

Next we show that (1) defines an action of $\mathfrak{V}_a^-$ on G. Since Ω is dense in G and G is separated, it suffices to verify the condition $v_1(v_2 \cdot g)$

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$= (v_1 + v_2)\cdot g$ for $g = [0, y, h, z]$ in Ω. But then $v\cdot g = [0, v + y, h, z]$ and the assertion follows. Finally, (2) is easily verified from the definitions, and (3) needs to be checked only for $g = [0, y, h, z]$ in Ω. Then we have

$$u\cdot(v\cdot g) = u\cdot[0, v + y, h, z] = [u, v + y, h, z]\ .$$

Let $d = u - Q_u v$. Then (d, v^u) is quasi-invertible, and we have the formulas

$$b(d, v^u) = b(u, v)\ , \tag{8}$$

$$d^{(v^u)} = u\ . \tag{9}$$

Indeed, by (4) and (5) of 5.1 and the "symmetry principle" ([15, 3.3]),

$$\begin{aligned} b(u, v) &= (b(u, v)^{-1})^{-1} = b(-u, v^u)^{-1} = b((-u)^{(v^u)}, -v^u) \\ &= b(-u + Q_u\cdot(v^u)^{-u}, -v^u) = b(d, v^u)\ , \end{aligned}$$

and

$$u^{(-v^u)} = u + Q_u\cdot(-v^u)^u = u + Q_u\cdot((-v)^{-u})^u = d$$

which implies (9). Now we get

$$\begin{aligned} v^u\cdot(b(u, v)\cdot(u^v\cdot g)) &= v^u\cdot(b(u, v)\cdot[u^v, y, h, z]) \\ &= v^u\cdot[B(u, v)\cdot u^v, B(v, u)^{-1}\cdot y, b(u, v)\cdot h, z] \\ &= v^u\cdot[d, B(v, u)^{-1}y, b(u, v)h, z] \\ &= [d^{(v^u)}, B(v, u)(v^u)^d + y,\ h,\ z] \\ &= [u, v + y, h, z]\ , \end{aligned}$$

since $B(v, u)(v^{u+d}) = (B(v, u)\cdot v^u)^{B(u,v)^{-1}\cdot d} = (v - Q_v u)^{(u^v)} = v$, by (9) applied with u and v interchanged. This completes the proof.

5.8. LEMMA. *There exists a unique structure of a k-group sheaf on G such that*

$$[x, y, h, z]\cdot g = x\cdot(y\cdot(h\cdot(z\cdot g)))\ , \tag{1}$$

for all $(x, y, h, z) \in X$, $g \in G$.

Proof. By 5.6 and 5.7, the right hand side of (1) defines a map $X \times G \to G$. We claim that it depends only on the equivalence class $[x, y, h, z]$. Indeed, let $[x, y, h, z] = [x', y', h', z']$. Then by $(**)$, 5.6.1 and 5.7.3 we have, setting $u = x - x'$,

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$$\begin{aligned} u\cdot(y\cdot(h\cdot(z\cdot g))) &= y^u\cdot(b(u,y)\cdot(u^y\cdot(h\cdot(z\cdot g)))) \\ &= y'\cdot(h'h^{-1}\cdot(h_+(z'-z)\cdot(h\cdot(z\cdot g)))) \\ &= y'\cdot(h'h^{-1}\cdot(h\cdot((z'-z)\cdot(z\cdot g)))) \\ &= y'\cdot(h'\cdot(z'\cdot g)) \end{aligned}$$

and hence $x'\cdot(y'\cdot(h'\cdot(z'\cdot g))) = x\cdot(y\cdot(h\cdot(z\cdot g)))$. Thus there exists a unique morphism $G\times G\to G$ satisfying (1). Clearly, $[0,0,1,0]$ is the neutral element of this multiplication. Now we prove associativity. Let $U\subset\Omega\times\Omega$ be the set of all pairs $(g,g') = ([0,y,h,z],[0,y',h',z'])$ for which (z,y') is quasi-invertible. Then U is open and dense in $\Omega\times\Omega$ and hence $U\times G$ is open and dense in $G\times G\times G$. Hence we only have to prove $(gg')g'' = g(g'g'')$ for $(g,g')\in U$. In that case,

$$\begin{aligned} gg' &= y\cdot(h\cdot(z\cdot[0,y',h',z'])) = y\cdot(h\cdot[z,y',h',z']) \\ &= y\cdot(h\cdot[0,y'^z,b(z,y')h',h_+'^{-1}(z^{y'})+z']) \\ &= [0,y+h_-(y'^z),hb(z,y')h',h_+'^{-1}(z^{y'})+z'] = [0,v,f,w]\,, \end{aligned}$$

and therefore

$$(gg')g'' = v\cdot(f\cdot(w\cdot g''))\,.$$

On the other hand, by 5.6 and 5.7.,

$$\begin{aligned} g(g'g'') &= y\cdot(h\cdot(z\cdot(y'\cdot(h'\cdot(z'\cdot g''))))) \\ &= y\cdot(h\cdot(y'^z\cdot(b(z,y')\cdot(z^{y'}\cdot(h'\cdot(z'\cdot g'')))))) \\ &= y\cdot(h_-(y'^z)\cdot(hb(z,y')h'\cdot(h_+'^{-1}(z^{y'})\cdot(z'\cdot g'')))) \\ &= (y+h_-(y'^z))\cdot(hb(z,y')h'\cdot((h_+'^{-1}(z^{y'})+z')\cdot g'')) \\ &= v\cdot(f\cdot(w\cdot g'')) = (gg')g''\,. \end{aligned}$$

Finally, it is easily checked that the inverse is given by

$$[x,y,h,z]^{-1} = [-z,-h_-^{-1}(y),h^{-1},-x]\,.$$

This completes our proof.

5.9. LEMMA. *The action ψ of k_m on G defined by $\psi_t\cdot[x,y,h,z] = [tx,t^{-1}y,h,tz]$ is elementary and the associated Jordan system is isomorphic with $\mathscr{J} = (\mathfrak{V},H,\rho,b)$.*

Proof. One shows easily that ψ is a well defined action by group automorphisms. Identify H with a subgroup of G via $h\mapsto[0,0,h,0]$ and define homomorphisms $f_\pm:\mathfrak{V}_a^\pm\to G$ by

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$$f_+(x) = [x, 0, 1, 0] = [0, 0, 1, x]\ , \qquad f_-(y) = [0, y, 1, 0]\ .$$

Let $U^\pm$ be the image of $\mathfrak{V}_a^\pm$. Then H, U^+, U^- satisfy (i)–(iv) of 3.1 and therefore ψ is elementary. From 5.6–5.8 it follows that H normalizes $U^\pm$; in fact, $hf_\sigma(x)h^{-1} = f_\sigma(h_\sigma \cdot x)$. Finally by (**), $f_+(x)f_-(y) = [x, y, 1, 0] = [0, y', h', z'] \in \Omega$ if and only if (x, y) is quasi-invertible, and then $y' = y^x$, $h' = b(x, y)$, $z' = x^y$. In view of 4.1, this shows that the Jordan system of (G, ψ) is $\mathscr{J}$ and completes the proof of 5.2.

5.10. Corollary. *Let (G, ψ) be an elementary system. Then there exists a unique function χ on G such that $\chi(g) = \det B(x, y) \cdot \mathrm{Ad}\, h \,|\, \mathfrak{V}^+)$ for $g = \exp(x)\exp(y)h \cdot \exp(z)$, $(x, z \in \mathfrak{V}_a^+, y \in \mathfrak{V}_a^-, h \in H)$, and Ω is the open subfunctor of G defined by χ.*

This is immediate from 5.5. Next, we discuss representability of G.

5.11. Proposition. *Let (G, ψ) be an elementary system, and assume that G is a scheme. Then H is a scheme, and the morphism $\pi\colon X \to G$* (cf. 4.13) *is a finitely presented, smooth, affine, and surjective morphism of k-schemes. The following properties hold for G if and only if they hold for H:*

(i) *finitely presented;*

(ii) *locally finitely presented;*

(iii) *flat and locally finitely presented;*

(iv) *smooth.*

Proof. H is a scheme since it is closed in G ([5, p. 50, 4.1]). We can factor π as follows:

$$X \xrightarrow{\ \cong\ } U^+ \times \Omega \xrightarrow{\mathrm{Id}\times\iota} U^+ \times G \xrightarrow{\ \mu\ } G\ ,$$

where the first map is the isomorphism $(x, y, h, z) \to (\exp(x), \exp(y) \cdot h \cdot \exp(z))$, $\iota\colon \Omega \to G$ is the inclusion, and μ is multiplication. Since ι is an open imbedding it is smooth and locally finitely presented. Let $U \subset G$ be open and affine, and let χ be as in 5.10. Then $\iota^{-1}(U) = U \cap \Omega$ is the open subscheme of U defined by a single function, and is therefore itself affine. This proves that ι is affine and quasicompact (cf. [5, p. 48, 3.8, and p. 41, 2.1]) and therefore also finitely presented. It follows that $\mathrm{Id} \times \iota$ has the same properties. The morphism $\mu\colon U^+ \times G \to G$ is isomorphic with the projection $\mathrm{pr}_2\colon U^+ \times G \to G$ (cf. [5, p. 161, 3.2]). Since U^+ is finitely presented, smooth, and affine over k, pr_2 and hence μ are finitely presented, smooth and affine.

Thus we have shown that π is finitely presented, smooth, and affine, and it is surjective since it is an epimorphism of sheaves.

If one of the properties (i)–(iv) holds for H then it holds for X since $\mathfrak{V}_a^{\pm}$ is smooth and finitely presented. By EGA IV, 11.3.11, 17.7.5, 17.7.7, it holds for G. Conversely, if G has one of these properties then so does $\Omega \cong H \times U^+ \times U^-$ and therefore H, by faithfully flat descent ($U^+ \times U^-$ being faithfully flat and finitely presented over k).

5.12. LEMMA *Let (G, ψ) be an elementary system, and assume that H is a scheme. Let $E = X \times_G X$ be the equivalence relation defined by $\pi: X \to G$. Then $\mathrm{pr}_1: E \to X$ is flat and finitely presented.*

Proof. From 4.13 it is clear that E is isomorphic with the open subscheme Y of $X \times \mathfrak{V}_a^+$ defined by

$$Y(R) = \{(x, y, h, z; x') \in X(R) \times \mathfrak{V}_R^+ \mid (x - x', y) \text{ quasi-invertible}\} ,$$

and that $\mathrm{pr}_1: E \to X$ is isomorphic with $Y \hookrightarrow X \times \mathfrak{V}_a^+ \xrightarrow{p} X$ where $p(x, y, h, z; x') = (x, y, h, z)$. Since $\mathfrak{V}_a^+$ is flat and finitely presented over k, p is flat and finitely presented. Also, Y is the open subscheme of $X \times \mathfrak{V}_a^+$ defined by the function $f(x, y, h, z; x') = \det B(x - x', y)$. Hence the inclusion $Y \hookrightarrow X \times \mathfrak{V}_a^+$ is finitely presented. This finishes the proof. We remark that, in fact, $\mathrm{pr}_1: E \to X$ is even smooth and affine, as follows from the proof.

5.13. PROPOSITION. *Let (G, ψ) be an elementary system over k, and assume that k is noetherian and H is a finitely presented k-scheme.*

(a) *G is a finitely presented algebraic space (in the sense of M. Artin).*

(b) *If k has Krull dimension ≤ 1 then G is a scheme.*

(c) *If k is a Dedekind ring then G is flat and affine if and only if H is.*

Proof. (a) and (b) follow from 5.12 and [1, Th. 3.1.1 and Th. 4B]. If G is affine so is H since it is closed in G. For the converse, we may, by [1, 2.3.1], assume that k is a field. Let $\hat{G} = G \rtimes k_m$ (semidirect) and $S = 1 \times k_m \subset \hat{G}$. Then S is contained in $\hat{G}^0$ and the centralizer of S in $\hat{G}$ is $H \cdot S$. Hence the centre of $\hat{G}^0$ is contained in $H \cdot S \cong H \times k_m$ and is therefore affine. By [5, p. 359, 8.4], $\hat{G}$ is affine, and hence G is affine.

I don't know whether G is a scheme whenever H is.

5.14. For any Jordan pair $\mathfrak{V}$ we can define a canonical Jordan system $\mathscr{J}(\mathfrak{V})$ by setting $H = \mathbf{Aut}(\mathfrak{V})$, $\rho = \mathrm{Id}$, and $b = \beta$. This follows from the

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properties of the inner automorphisms $\beta(x, y)$ ([15, § 3]). The associated group is called the *projective group* of $\mathfrak{V}$ and is denoted by $\mathbf{PG}(\mathfrak{V})$. In the examples of 3.2, $\mathbf{PG}(\mathfrak{V})$ is isomorphic with the projective group $\mathbf{PGL}_n$ $\mathbf{PO}_{2n}$, $\mathbf{PSp}_n$ respectively (except in the first case when $p = q = n/2 > 1$ where $\mathbf{PGL}_n$ is an open subgroup of index 2 in $\mathbf{PG}(\mathfrak{V})$). As a further example, let k be a field, and $\mathfrak{V}$ the Jordan pair of 1×2-matrices over a Cayley division algebra C over k (cf. [15, 8.15]). Then one can show that $\mathbf{PG}(\mathfrak{V})$ is an adjoint group of type E_6 whose k-rational points may be identified with the projective group of the Cayley plane defined by C (cf. ([6]).

5.15. PROPOSITION. *Let (G, ψ) be an elementary system with associated Jordan pair $\mathfrak{V}$, and let $\bar{G} = \mathbf{PG}(\mathfrak{V})$. Let $\bar{\psi}$ be the elementary action on $\bar{G}$ and $\bar{H} = \mathbf{Aut}(\mathfrak{V})$, $\bar{U}^{\pm}$ the corresponding subgroups.*

(a) *There exists a canonical homomorphism $\kappa: (G, \psi) \to (\bar{G}, \bar{\psi})$ given by $\kappa(h) = (\mathrm{Ad}\, h \,|\, \mathfrak{V}^+, \mathrm{Ad}\, h \,|\, \mathfrak{V}^-)$ $(h \in H)$ and $\kappa(\exp(x)) = \overline{\exp}(x)$ $(x \in \mathfrak{V}^{\pm})$. The kernel of κ is the largest normal subgroup of G contained in H; in particular, $\kappa: U^{\pm} \to \bar{U}^{\pm}$ is an isomorphism.*

(b) *Let $\gamma: k_m \to \bar{H}$ be as in 2.3, and $\bar{S}$ the image of γ. Then $\bar{\psi}$ is given by $\bar{\psi}_t(g) = \gamma(t) g \gamma(t)^{-1}$, and hence $\bar{H}$ is the centralizer of $\bar{S}$ in $\bar{G}$.*

(c) *$\bar{H}$ contains no non-trivial normal subgroup of $\bar{G}$. The centre of $\bar{G}$ is trivial.*

Proof. Define $f_0: H \to \bar{H}$ by $h \mapsto (\mathrm{Ad}\, h \,|\, \mathfrak{V}^+, \mathrm{Ad}\, h \,|\, \mathfrak{V}^-)$. Since the Jordan pairs of G and $\bar{G}$ are the same, 4.15 (b) shows the existence of κ, and clearly $\kappa: U^{\pm} \to \bar{U}^{\pm}$ is an isomorphism. Let $\kappa(g) = 1$. After an fppf extension, we may by 3.8 assume that $g = h \cdot \exp(x) \exp(y) \exp(z)$ with $x, z \in \mathfrak{V}_a^+$, $y \in \mathfrak{V}_a^-$, $h \in H$. Then $\kappa(h)^{-1} = \overline{\exp}(x)\, \overline{\exp}(y)\, \overline{\exp}(z) = 1$ by 3.6 (c). Hence $\overline{\exp}(x)\, \overline{\exp}(y) = \overline{\exp}(-z) \in \bar{\Omega}$ and by 4.1, $\overline{\exp}(-z) = \overline{\exp}(y^x) \bar{b}(x, y)$ $\overline{\exp}(x^y)$, which implies $y^x = 0$, and $-z = x^y$. It follows that $y = 0$ and $z = -x$, and therefore $g = h \in H$. Thus the kernel of κ is contained in H. If N is a normal subgroup of G contained in H then N centralizes U^+ and U^- since it normalizes U^+ and U^- and $H \cap U^{\pm} = \{1\}$. By 3.6, it follows that $\mathrm{Ad}\, h \cdot x = x$ for all $h \in N$, $x \in \mathfrak{V}_a^{\pm}$, and therefore N is contained in the kernel of κ. This proves (a). Now (b) is immediate from the definitions, and (c) follows from (a) (applied for $G = \bar{G}$) and (b).

Next we give an application of the existence theorem to the existence of quotients.

5.16. PROPOSITION. *Let* (G, ψ) *be an elementary system, and let* N *be a normal subgroup sheaf of* G *which is stable under* ψ. *Let* $N^{\pm} = N \cap U^{\pm}$ *and* $N_0 = N \cap H$.

(a) *If* $g = uvhw \in U^+(R)U^-(R)H(R)U^+(R)$ *then* $g \in N(R)$ *if and only if* $wu \in N^+(R)$, $v \in N^-(R)$, $h \in N_0(R)$. *In particular.*

$$(1) \qquad N \cap \Omega = N^- \cdot N_0 \cdot N^+ .$$

(b) *Let* $\mathfrak{n}^{\pm} = \mathrm{Lie}\,(N^{\pm}) \subset \mathfrak{V}^{\pm}$. *Then* $(\mathfrak{n}^+, \mathfrak{n}^-)$ *is an outer ideal* (cf. [15, 1.3]) *of the Jordan pair* $\mathfrak{V} = (\mathfrak{V}^+, \mathfrak{V}^-)$ *associated with* (G, ψ), *stable under* $\mathrm{Ad}\, H$.

Proof. (a) By passing to a base extension we may assume that $R = k$. Consider first the case $w = 1$ and let $u = \exp(x)$, $v = \exp(y)$. Applying ψ_t it follows that

$$(2) \qquad \psi_t(g) = \exp(tx)\exp(t^{-1}y)h \in N(R) ,$$

for all invertible $t \in R$, $R \in k$-**alg**. Since N is normal, we have $\exp(-x) \cdot \psi_t(g) \cdot g^{-1} \cdot \exp(x) = \exp((t-1)x)\exp((t^{-1}-1)y) \in N(R)$. If $t-1$ is invertible then by (2), $\psi_{t-1}(g) \in N(R)$, and hence

$$\begin{aligned}(\psi_{t-1}(g))^{-1} &\cdot \exp((t-1)x)\exp((t^{-1}-1)y)\\ &= h^{-1}\cdot\exp((t^{-1}-1-(t-1)^{-1})y)\\ &= h^{-1}\cdot\exp\left(\left(\frac{t^2-t+1}{t(1-t)}\right)y\right) \in N(R) .\end{aligned}$$

Choose now $R = k[T]/(T^2 - T + 1)$ and let t be the image of the indeterminate T in R. Then R is fppf over k, and $t^{-1} = 1 - t$. It follows that $h^{-1} \in N(R)$, and since N is a sheaf, $h \in N(k)$. Similarly, there exists a fppf extension R' of k and an element $t \in R'^*$ such that $1 - t$ and $s = (t^2 - t + 1)/t(1-t)$ are invertible. Then $\exp(sy) \in N(R)$ and hence $\psi_{s^{-1}}(\exp(sy)) = \exp(y) \in N(R')$ which implies $v = \exp(y) \in N(k)$ as before. It follows that $u = \exp(x) = gh^{-1}\exp(y) \in N(k)$.

In the general case, $g = uvhw \in N(k)$ if and only if $wgw^{-1} = (wu)vh \in N(k)$, and by what we just proved, $wu \in N^+(k)$, $v \in N^-(k)$, $h \in N_0(k)$.

(b) Since $N^{\pm}$ is a subgroup functor of $U^{\pm}$, $\mathfrak{n}^{\pm}$ is a k-submodule of $\mathfrak{V}^{\pm} = \mathrm{Lie}\,(U^{\pm})$. From 3.6 it follows that $(\mathfrak{n}^+, \mathfrak{n}^-)$ is stable under $\mathrm{Ad}\, H$. Let $y \in \mathfrak{n}^-$ and $x \in \mathfrak{V}^+$. Then $\exp(\varepsilon y) \in N(k(\varepsilon))$, and by 4.6.2 we have $b(x, \varepsilon y)\exp(\varepsilon Q_x y) \in N(k(\varepsilon))$ which implies $Q_x y \in \mathfrak{n}^+$ by (a). For $z \in \mathfrak{n}^+$ and $x \in \mathfrak{V}^+$, $y \in \mathfrak{V}^-$, a similar argument using 4.6.6 shows that $Q_y z \in \mathfrak{n}^-$ and

$B(x, y)z = z - D(x, y)z + Q_x Q_y z \in \mathfrak{n}^+$. Hence $D(x, y)z \in \mathfrak{n}^+$, and $D(\mathfrak{V}^-, \mathfrak{V}^+)\mathfrak{n}^- \subset \mathfrak{n}^-$ is shown similarly.

5.17. COROLLARY. *Assume that N is a smooth k-scheme. Then N_0, N^+, and N^- are smooth.*

Proof. $N \cap \Omega$ is open in N and therefore smooth. By 5.16 and 3.4, $N \cap \Omega \cong N^- \times N^0 \times N^+$. Since $N^\pm = N \cap U^\pm$ is a fibred product of locally finitely presented k-schemes it is itself locally finitely presented. Hence to show that $N^\pm$ is smooth, we only have to show that, for every $R \in k$-**alg**, and every ideal I of R of square zero, the canonical map $N^\pm(R) \to N^\pm(R/I)$ is surjective (cf. [5, p. 111, 4.6]). Let $n_\pm \in N^\pm(R/I)$. Then $n_- \cdot n_+ \in (N \cap \Omega)(R/I)$, and by smoothness of $N \cap \Omega$, there exists $g = uhv \in N^-(R)N_0(R)N^+(R)$ such that $g_{R/I} = u_{R/I}h_{R/I}v_{R/I} = n_- \cdot n_+$. Hence $u_{R/I} = n_-$, $v_{R/I} = n_+$, and there assertion follows. In particular, $N^+ \times N^-$ is faithfully flat and locally finitely presented over k, and hence the projection $N \cap \Omega \to N_0$ is faithfully flat and locally finitely presented. By EGA IV, 17.7.5 and 17.7.7, N_0 is also smooth.

5.18. LEMMA. *Let $\mathfrak{V}$ and $\mathfrak{V}'$ be finite-dimensional Jordan pairs over a field, and let $f: \mathfrak{V} \to \mathfrak{V}'$ be a surjective homomorphism. Then f induces a surjection of the set of quasi-invertible pairs of $\mathfrak{V}$ onto that of $\mathfrak{V}'$.*

Proof. Let $(\bar{x}, \bar{y})$ be quasi-invertible in $\mathfrak{V}'$ and let $(x, y) \in \mathfrak{V}^+ \times \mathfrak{V}^-$ be such that $f_+(x) = \bar{x}$ and $f_-(y) = \bar{y}$. Replacing $\mathfrak{V}$ and $\mathfrak{V}'$ by the subpairs generated by (x, y) and $(\bar{x}, \bar{y})$, respectively, we may assume that $\mathfrak{V}$ and $\mathfrak{V}'$ are associative (see [15, 15.3]). By [15, 15.8], $\mathfrak{V} \cong \mathfrak{V}_0 \times \mathfrak{V}_1 \times \cdots \times \mathfrak{V}_n$ where $\mathfrak{V}_0$ is nilpotent and the $\mathfrak{V}_1, \cdots, \mathfrak{V}_n$ are local. Let $\mathfrak{K}$ be the kernel of f. Then $\mathfrak{K} \cong \mathfrak{K}_0 \times \mathfrak{K}_1 \times \cdots \times \mathfrak{K}_n$ where either $\mathfrak{K}_i = \mathfrak{V}_i$ or $\mathfrak{K}_i \subset \operatorname{Rad} \mathfrak{V}_i$, for $i = 1, \cdots, n$, since $\mathfrak{V}_i/\operatorname{Rad} \mathfrak{V}_i$ is a division pair and hence the radical is the unique proper maximal ideal of $\mathfrak{V}_i$. Thus $\mathfrak{V} \cong \mathfrak{V}^{(1)} \times \mathfrak{V}^{(2)}$ and $\mathfrak{K} \cong \mathfrak{K}^{(1)} \times \mathfrak{V}^{(2)}$ where $\mathfrak{K}^{(1)} \subset \operatorname{Rad} \mathfrak{V}^{(1)}$. Let $x^{(i)}, y^{(i)}$ be the components of $(x, y) \in \mathfrak{V}^+ \times \mathfrak{V}^-$ in $\mathfrak{V}^{(i)}$. Then $(\bar{x}, \bar{y}) = f_+(x^{(1)}), f_-(y^{(1)}))$ and $(x^{(1)}, y^{(1)})$ is quasi-invertible by [15, 4.3].

5.19. THEOREM. *With the notations of* 5.16, *assume that N is closed in G and that N^+ and N^- are smooth.*

(a) *$\mathfrak{n}^\pm$ is a direct summand of $\mathfrak{V}^\pm$, the restriction* $\exp: \mathfrak{n}^\pm_a \to N^\pm$ *is an isomorphism, and* $(\mathfrak{n}^+, \mathfrak{n}^-)$ *is an ideal of $\mathfrak{V}$, stable under* $\operatorname{Ad} H$.

(b) *Let $G' = G/N$ (quotient sheaf), and let ψ' be the action of k_m on*

G' induced by ψ. Then (G', ψ') is an elementary system, the canonical map $f\colon (G, \psi) \to (G', \psi')$ is a homomorphism, and the Jordan pair associated with (G', ψ') is $\mathfrak{V}' = (\mathfrak{V}^+/\mathfrak{n}^+, \mathfrak{V}^-/\mathfrak{n}^-)$.

Proof. (a) Since N is closed in $G, N^+ = N \cap U^+$ is closed in U^+ and hence is an affine scheme. Let $\omega_{N^+/k}$ be the pullback of the module of differentials $\Omega^1_{N^+/k}$ to k via the unit section. Smoothness of N^+ implies that $\omega_{N^+/k}$ is a finitely generated and projective k-module, and so is $\mathfrak{n}^+ = \operatorname{Lie}(N^+) = \operatorname{Hom}_k(\omega_{N^+/k}, k)$ (cf. [7. p. 208, and p. 215]). From SGA3, Exp. II, 4.11, it follows that $\mathfrak{n}^+$ is a direct summand of $\mathfrak{V}^+$. Let $\exp(x) \in N^+(R)$. Then $\psi_{1+\varepsilon}(\exp(x)) = \exp((1+\varepsilon)x) = \exp(x)\exp(\varepsilon x) \in N^+(R(\varepsilon))$ and hence $\exp(\varepsilon x) \in N^+(R(\varepsilon))$; i,e., $x \in \operatorname{Lie}(N^+_R) = \mathfrak{n}^+_R$. This shows that N^+ is contained in the vector group $U = \exp(\mathfrak{n}^+_a) \cong \mathfrak{n}^+_a$. Since U and N^+ are smooth and have the same Lie algebra (namely $\mathfrak{n}^+$) the fibres of U and N^+ have the same dimension. By EGA IV, 17.11.5, N^+ is open in U. Arguing fibre-wise, it follows easily that $N^+ = U$. The proof for N^- is the same. Let $x \in \mathfrak{n}^+$ and $y \in \mathfrak{V}^-$. Then $\exp(x) \in N^+(k)$, and by 4.6.2, $Q_x y \in \mathfrak{n}^+$. Similarly, one shows that $Q(\mathfrak{n}^-)\mathfrak{V}^+ \subset \mathfrak{n}^-$, and together with 5.16 (b) it follows that $(\mathfrak{n}^+, \mathfrak{n}^-)$ is an ideal of $\mathfrak{V}$.

(b) We will construct a Jordan system $\mathscr{J}' = (\mathfrak{V}', H', \rho', b')$ whose associated elementary system is (G', ψ'). Let $\mathfrak{V}' = (\mathfrak{V}^+/\mathfrak{n}^+, \mathfrak{V}^-/\mathfrak{n}^-)$. Then $\mathfrak{V}'$ is a Jordan pair over k which is finitely generated and projective as a k-module since $\mathfrak{n}^\pm$ is a direct summand of $\mathfrak{V}^\pm$. Let $H' = H/N_0$ (quotient sheaf). Since $N_0 = N \cap H$ is closed in H, the equivalence relation on H defined by N_0 is closed in $H \times H$, and by 1.3, H' is a separated k-group sheaf. It is easily checked that there is a unique homomorphism $\rho'\colon H' \to \mathbf{Aut}(\mathfrak{V}')$ satisfying

$$\rho'_\pm(h')\cdot x' = (\operatorname{Ad} h \cdot x)'\ ,$$

where the $'$ denotes the canonical maps $H \to H'$ and $\mathfrak{V} \to \mathfrak{V}'$. The canonical map $\mathfrak{V} \to \mathfrak{V}'$ is surjective and linear. Hence the induced morphism on the quasi-invertible pairs $W \to W'$ is smooth, and by 5.18, it is surjective. In particular, it is faithfully flat and locally finitely presented and therefore an epimorphism of sheaves ([5, p. 295, 2.10]). This allows us to define a morphism $b'\colon W' \to H'$ by

$$b'(x', y') = b(x, y)'\ ,$$

provided we can show that the right hand side depends only on the equivalence class of $(x, y) \bmod (\mathfrak{n}^+, \mathfrak{n}^-)$. Thus let $(x, y) \in W(R)$, and assume that $x \equiv u(\mathfrak{n}_R^+)$ and $y^x \equiv v(\mathfrak{n}_R^-)$. Since $(\mathfrak{n}_R^+, \mathfrak{n}_R^-)$ is an ideal of $\mathfrak{V}_R$ we have $x^y \equiv u^v(\mathfrak{n}_R^+)$ and $y^x \equiv v^u(\mathfrak{n}_R^-)$, and since $\exp \mathfrak{n}_a^\pm = N^\pm$, it follows from $(*)$ of 4.1 that

$$\begin{aligned} b(x, y) &= \exp(-y^x) \exp(x) \exp(y) \exp(-x^y) \\ &\equiv \exp(-v^u) \exp(u) \exp(v) \exp(-u^v) \\ &= b(u, v) \text{ modulo } N(R) . \end{aligned}$$

Now it is easily verified that $\mathscr{J}' = (\mathfrak{V}', H', \rho', b')$ is a Jordan system, and that the canonical maps $\mathfrak{V} \to \mathfrak{V}'$ and $H \to H'$ define a homomorphism $\Phi: \mathscr{J} \to \mathscr{J}'$ where $\mathscr{J}$ is the Jordan system associated with (G, ψ). Let (G', ψ') be the elementary system defined by $\mathscr{J}'$, and let $f: (G, \psi) \to (G', \psi')$ be the homomorphism induced by Φ (see 4.15 and 5.1). Then $f: G \to G'$ is an epimorphism of group sheaves, and we only have to check that $\operatorname{Ker}(f) = N$. By 5.16 (a), it suffices to show that $\Omega \cap \operatorname{Ker}(f) = \Omega \cap N$, and this is clear from the definitions.

5.20. Example. If N^+ and N^- are not smooth then the induced action ψ' on G/N may not be elementary. For example, let k be a field, let $G = \mathbf{SO}_{2n+1}$ be the special orthogonal group of the quadratic form $q(x_0, \cdots, x_{2n}) = x_0^2 + \sum x_i x_{n+i}$, and let ψ_t be conjugation by the matrix

$$\left(\begin{array}{c|cccc|cccc} 1 & & & & & & & & \\ \hline & t & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ \hline & & & & & t^{-1} & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{array}\right)$$

Then ψ is an elementary action whose associated Jordan pair is the simple Jordan pair defined by the standard quadratic form on k^{2n-1} (cf. [18, p. 196]). Now let $\operatorname{char}(k) = 2$, and let $f: G \to \mathbf{Sp}_{n,k}$ be the exceptional isogeny

$$\begin{pmatrix} 1 & a & b \\ 0 & A & B \\ 0 & C & D \end{pmatrix} \longmapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $a, b \in k^n$ and A, B, C, D are $n \times n$ matrices. Then $N = \operatorname{Ker}(f)$ is an infinitesimal group of height one, and $(\mathfrak{n}^+, \mathfrak{n}^-)$ is the unique proper one-dimensional outer ideal of $\mathfrak{V}$. The induced action ψ' on $G' = \mathbf{Sp}_{n,k}$ is not elementary since it has weights $0, \pm 1, \pm 2$ on $\operatorname{Lie}(G')$.

§ 6. The radical

6.1. In this section, k denotes a field and $\bar{k}$ an algebraic closure of k. A k-group is an affine finitely presented group scheme over k. If (G, ψ) is an elementary system we will always assume that G is a k-group. By 5.13, so is H, and conversely, if $(\mathfrak{V}, H, \rho, b)$ is a Jordan system with H a k-group then the associated group G is a k-group. In particular, the projective group of $\mathfrak{V}$ is a k-group.

Radical and unipotent radical of a smooth connected $\bar{k}$-group are denoted by $R(G)$ and $R_u(G)$. If G is not connected define $R(G) = R(G^0)$ and $R_u(G) = R_u(G^0)$. By definition, a k-group is semisimple (reductive) if it is *smooth* and $R(G_{\bar{k}}) = \{1\}$ ($R_u(G_{\bar{k}}) = \{1\}$). Let S be a torus acting on a k-group G by automorphisms. If G is connected (smooth, reductive) so is the fixed point set G^S; more generally, $R_u(G^S) = R_u(G)^S$ (SGA3, Exp. XIX).

6.2. Let $\mathfrak{V}$ be a Jordan pair over k. An element $x \in \mathfrak{V}^+$ is called *properly quasi-invertible* if (x, y) is quasi-invertible for all $y \in \mathfrak{V}^-$. Similarly one defines properly quasi-invertible for elements of $\mathfrak{V}^-$. The *Jacobson radical* of $\mathfrak{V}$ is $\operatorname{Rad} \mathfrak{V} = (\operatorname{Rad} \mathfrak{V}^+, \operatorname{Rad} \mathfrak{V}^-)$ where $\operatorname{Rad} \mathfrak{V}^\sigma$ is the set of properly quasi-invertible elements of $\mathfrak{V}^\sigma$ (cf. [15, § 4]). We say $\mathfrak{V}$ is *semisimple* if $\operatorname{Rad} \mathfrak{V} = 0$ and $\mathfrak{V}$ is *separable* if $\mathfrak{V}_K$ is semisimple, for all extension fields K of k. An ideal $\mathfrak{J} = (\mathfrak{J}^+, \mathfrak{J}^-)$ of $\mathfrak{V}$ is called *trivial* if $Q(\mathfrak{J}^\pm) \cdot \mathfrak{J}^\mp = 0$. Clearly a trivial ideal is contained in $\operatorname{Rad} \mathfrak{V}$.

6.3. LEMMA. *Let (G, ψ) be an elementary system over k, and let N be a smooth connected abelian normal subgroup of G, invariant under ψ. Then* (with the notation of 5.16) $N = N^- \cdot N_0 \cdot N^+ \subset \Omega$, *and* $(\mathfrak{n}^+, \mathfrak{n}^-)$ *is a trivial ideal of the associated Jordan pair* $\mathfrak{V}$.

Proof. Since N is commutative, $N \cap \Omega = N^- \cdot N_0 \cdot N^+$ is an open subgroup of N. Hence $N = N \cap \Omega$. By 5.17 and 5.19, $N^\pm = \exp(\mathfrak{n}_a^\pm)$ and $(\mathfrak{n}^+, \mathfrak{n}^-)$ is an ideal of $\mathfrak{V}$. If $x \in \mathfrak{n}^+$, $y \in \mathfrak{n}^-$ then $\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(y^x) b(x, y) \exp(x^y)$ implies $y = y^x, b(x, y) = 1$, $x = x^y$. Hence $x = x^y = B(x, y)^{-1} \cdot (x - Q_x y) = x - Q_x y$ and thus $Q_x y = 0$. Similarly $Q_y x = 0$.

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6.4. THEOREM. *The projective group* $G = \mathbf{PG}(\mathfrak{V})$ *of a separable Jordan pair* $\mathfrak{V}$ *is semisimple and its identity component is generated by* U^+ *and* U^- (i.e., $G_{in} = G^0$; cf. 3.9).

Proof. We may assume k algebraically closed. Let S be as in 5.15, and $G' = S \cdot G_{in}$. Then G' is a smooth connected k-group which is normal in G. We claim that G' is semisimple. If this were not so, let N be the last non-trivial term in the derived series of $R(G')$. By 6.3. $(\mathfrak{n}^+, \mathfrak{n}^-)$ is then a trivial ideal of $\mathfrak{V}$ and hence contained in $\mathrm{Rad}(\mathfrak{V}) = 0$. Hence $N^\pm = \{1\}$ and $N = N_0 \subset H$. But H contains no nontrivial normal subgroups of G by 5.15. Hence $N = \{1\}$, a contradiction. Now G_{in} is normal in G' and hence is semisimple as well. Since G'/G_{in} is at most one-dimensional we have $G_{in} = G'$; i.e., $S \subset G_{in}$. Now consider the homomorphism $f: G \to \mathbf{Aut}(G_{in}) = A$ given by $g \mapsto \mathrm{Int}(g)$. Then f is a monomorphism. Indeed, since $S \subset G_{in}$, the kernel of f is contained in $G^S = H$ and is therefore trivial (5.15). By SGA3, Exp. XXIV, A is a semisimple k-group, and $A^0 = f(G_{in}) \subset f(G)$. Hence $f(G) \cong G$ is an open subgroup of A, in particular, it is semisimple. Also, $G^0 \cong f(G^0) = A^0 = f(G_{in})$ and therefore $G^0 = G_{in}$.

6.5. COROLLARY. *Let* $\mathfrak{V}$ *be a separable Jordan pair over* k. *Then* $\mathbf{Aut}(\mathfrak{V})$ *is a reductive* k*-group whose identity component is* $\mathbf{Inn}(\mathfrak{V})$.

Proof. Since $\mathbf{Aut}(\mathfrak{V}) = H = G^S$ and $\mathbf{Inn}(\mathfrak{V}) = H_{in}$ where G is the projective group of $\mathfrak{V}$, this follows from 6.4 and 4.12. From 2.6 we get

6.6. COROLLARY. *Let* $\mathfrak{A}$ *be a finite-dimensional separable unital quadratic Jordan algebra over* k. *Then the structure group* $\mathbf{Str}(\mathfrak{A})$ *is a reductive* k*-group whose identity component is* $\mathbf{Instr}(\mathfrak{A})$.

This was proved by Springer in [20] by a case-by-case verification.

6.7. PROPOSITION. *Let* (G, ψ) *be an elementary system over* k, *and assume that the associated Jordan pair* $\mathfrak{V}$ *is separable. Then the automorphisms* ψ_t *are inner in the following sense: There exists a* k*-split torus* $S \subset H_{in}$ *and a nontrivial character* $\eta: S \to k_m$ *such that*

$$\mathrm{Int}(s) = \psi_{\eta(s)}, \tag{1}$$

for all $s \in S$.

Proof. Let $\kappa: G \to \bar{G} = \mathbf{PG}(\mathfrak{V})$ be the canonical homomorphism, and let $\bar{S} = \gamma(k_m) \subset \bar{G}^0$ as in 5.15. By 6.4, $\bar{G}^0 = \bar{G}_{in}$, and hence the restriction

$\kappa_0\colon G_{in} \to \bar{G}^0$ is an epimorphism. Let $Z = \kappa_0^{-1}(\bar{S}) \subset H_{in}$, and define $\eta\colon Z \to k_m$ by $\kappa_0(s) = \gamma(\eta(s))$. For $x \in \mathfrak{V}_a^{\pm}$ we then have

$$\text{(2)} \qquad \operatorname{Int}(s)\cdot\exp(x) = \exp(\operatorname{Ad} s\cdot x) = \exp(\eta(s)^{\pm 1}\cdot x)\,,$$

since κ induces isomorphisms $U^{\pm} \to \bar{U}^{\pm}$ (cf. 5.15). It follows that $sb(x, y)s^{-1} = b(\operatorname{Ad} s\cdot x, \operatorname{Ad} s\cdot y) = b(x, y)$, and hence Z is central in H_{in}. Now $\kappa(H^0) = \kappa(H_{in}) = \bar{H}^0$ implies that $H^0 \subset \operatorname{Ker}(\kappa)\cdot H_{in}$. But $\operatorname{Ker}(\kappa)$ centralizes G_{in}, and hence Z is central in H^0. Let S be the maximal k-split torus of the centralizer of H in Z. Then S is central in H, and $\kappa\colon S \to \bar{S}$ is an epimorphism. Hence $\eta\colon S \to k_m$ is nontrivial. Now (1) follows from (2) and the fact that S is central in H.

6.8. LEMMA. *Let (G, ψ) be an elementary system over k and assume that G is reductive and connected. Then $P^+ = H\cdot U^+$ and $P^- = H\cdot U^-$ are opposed parabolic subgroups of G.*

Proof. Let $\hat{G} = G \rtimes k_m$ (semidirect product) and let $S = \{1\} \times k_m \subset \hat{G}$. Then $\hat{G}$ is reductive, and S is a one-dimensional split torus of $\hat{G}$ with root system $\Phi = \mathrm{A}_1$. Hence $\hat{G}_{\Phi+} = S\cdot P^+$ and $G_{\Phi-} = S\cdot P^-$ are opposed parabolic subgroups of $\hat{G}$ ([3, 4.15]), and hence $P^+ = G \cap (S\cdot P^+)$ and $P^- = G \cap (S\cdot P^-)$ are opposed parabolic subgroups of G.

6.9. THEOREM. *Let (G, ψ) be an elementary system over an algebraically closed field k, and assume that G is smooth and connected. Let* $\operatorname{Rad}(\mathfrak{V}) = (\mathfrak{n}^+, \mathfrak{n}^-)$ *be the radical of the associated Jordan pair $\mathfrak{V}$, and let $N^{\pm} =$* $\exp(\mathfrak{n}_a^{\pm}) \subset U^{\pm}$. *Then $R_u(G) = N^-\cdot R_u(H)\cdot N^+$.*

Proof. Let N_0 be the k-subgroup of H generated by $b(\mathfrak{V}_a^+, \mathfrak{n}_a^-)$ and $b(\mathfrak{n}_a^+, \mathfrak{V}_a^-)$. This is a smooth connected k-group. We claim that

$$N = N^-\cdot N_0\cdot N^+$$

is a smooth connected unipotent normal subgroup of G; in particular, $N \subset R_u(G)$. Clearly, N is smooth and connected. To show that it is a normal subgroup of G it suffices to show that $N(k)$ is normal in $G(k)$, since k is algebraically closed, and all groups involved are smooth. From 4.1 and the definition of $\operatorname{Rad}(\mathfrak{V})$ it follows that $N(k)$ is a subgroup of $G(k)$. The invariance of $\operatorname{Rad}(\mathfrak{V})$ under $\operatorname{Aut}(\mathfrak{V})$ implies that $N(k)$ is stable under conjugation by $H(k)$. If $x \in \mathfrak{V}^+$ and $y \in \mathfrak{n}^-$ then

$$\operatorname{Int}(\exp(x))\cdot\exp(y) = \exp(y^x)\cdot b(x, y)\cdot\exp(x^y - x)$$

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with $y^x = y + Q_y \cdot x^y \in \mathfrak{n}^-$ and $x^y - x = Q_x \cdot y^x \in \mathfrak{n}^+$, since Rad$(\mathfrak{B})$ is an ideal of $\mathfrak{B}$. Also $b(x, y) \in N_0(k)$. If $u \in \mathfrak{B}^+$, $v \in \mathfrak{n}^-$, or $u \in \mathfrak{n}^+$, $v \in \mathfrak{B}$ then by 3.6 and 4.1,

$$\operatorname{Int}(\exp(x)) \cdot b(u, v) = \exp(x - B(u, v) \cdot x) \cdot b(u, v) ,$$

and $x - B(u, x) \cdot x = -\{uvx\} + Q_u Q_v x \in \mathfrak{n}^+$, again since Rad$(\mathfrak{B})$ is an ideal. Thus $N(k)$ is stable under conjugation by $U^+(k)$, and similarly one shows that it is stable under conjugation by $U^-(k)$. This proves that N is normal in G. Now we show that N is unipotent. Replacing G by $G/R_u(G)$ (which is permissible by 5.19), we may assume that G is reductive, and then have to show that $N = \{1\}$. By 6.8, P^+ and P^- are parabolic subgroups of G, and hence $N \cap P^+ = N_0 \cdot N^+$ and $N \cap P^- = N_0 \cdot N^-$ are parabolic subgroups of N. Then $N/N_0 \cdot N^\pm \cong N^\mp$ is both affine and projective which implies $N^\pm = \{1\}$ and therefore also $N_0 = \{1\}$.

Let now $N' = N^- \cdot R_u(H) \cdot N^+$. By 6.2, $N_0 = N \cap H \subset R_u(G) \cap H = R_u(H)$ and hence $N \subset N'$. Since $N^+ \cdot N^- \subset N \subset N'$ it follows that N' is a subgroup of G. Also, N' is unipotent since N and $N'/N = R_u(H)/N_0$ are unipotent. We claim that N' is normal in G. In view of what we proved before, it suffices to show that conjugation by elements of $U^\pm(k)$ maps $R_u(H)$ into N'. Let $x \in \mathfrak{B}^\pm$ and $h \in R_u(H)(k)$. Then

$$\operatorname{Int}(\exp(x)) \cdot h = \exp(x - \operatorname{Ad} h \cdot x) \cdot h . \tag{1}$$

and we have to show that $x - \operatorname{Ad} h \cdot x \in \mathfrak{n}^\pm$. Consider the homomorphism $\varphi: G \to G' = \mathbf{PG}(\mathfrak{B}/\text{Rad}(\mathfrak{B}))^0$ induced by the canonical homomorphisms $\mathfrak{B} \to \mathfrak{B}/\text{Rad}(\mathfrak{B})$ and $H \to \mathbf{Aut}(\mathfrak{B}) \to \mathbf{Aut}(\mathfrak{B}/\text{Rad}(\mathfrak{B}))$ (cf. 4.15). By 6.4, G' is semisimple, and $G' = G'_{in}$. Hence φ is an epimorphism, and we have $R_u(H) \subset R_u(G) \subset \operatorname{Ker}(\varphi)$. Applying φ to (1) it follows that $\operatorname{Lie}(\varphi)(x - \operatorname{Ad} h \cdot x) = 0$, and hence $x - \operatorname{Ad} h \cdot x \in \mathfrak{n}^\pm$. This proves that N' is normal in G and therefore contained in $R_u(G)$. To prove equality, we show that G/N' is reductive. Replacing G by G/N' (which we can do in view of 5.19), we may assume that H is reductive and $\mathfrak{B}$ is separable, and then have to show that G is reductive. Consider the canonical homomorphism $\kappa: G \to \mathbf{PG}(\mathfrak{B})^0$. Then by 6.4 and 5.15, $R_u(G) \subset \operatorname{Ker}(\kappa) \subset H$ which shows $R_u(G) = \{1\}$. This completes the proof.

6.10. Corollary. (a) *$\mathfrak{B}$ is separable if and only if the projective group G of $\mathfrak{B}$ is semisimple.*

(b) $\mathfrak{B} = \text{Rad}\,\mathfrak{B}$ *if and only if the "inner projective group" G_{in} is uni-*

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potent.

6.11. COROLLARY. *Let $\mathfrak{V}$ be a Jordan pair over an algebraically closed field with non-zero radical. Then $\mathfrak{V}$ contains a nonzero trivial ideal, invariant under all automorphisms and antiautomorphisms of $\mathfrak{V}$.*

Proof. Let $G = \mathbf{PG}(\mathfrak{V})$. By 6.9, G_{in} is not semisimple. Let N be the last non-trivial term in the derived series of $R(G_{in})$, and let $N = N^- \cdot N_0 \cdot N^+$ as in 6.3. Then N^+ and N^- are not both equal to $\{1\}$. Otherwise, $N = N_0$ would be a normal subgroup of G contained in H and therefore $N_0 = \{1\}$. Then $(\mathfrak{n}^+, \mathfrak{n}^-)$ has the desired properties.

Remark. This results was used in [14] to prove the nilpotence of the radical of a Jordan triple system. A direct proof can now be found in [15].

6.12. THEOREM. *Let (G, ψ) be an elementary system over k and let K be an extension field of k. Then*

$$G(K) = U^+(k) \cdot \Omega(K) = \Omega(K) \cdot U^-(k) .$$

Proof. (a) *k infinite.* Let $\bar{K}$ be the algebraic closure of K, and let $\Omega^* = U^-(k) \cdot H(K) \cdot U^+(K)$. Then Ω^* is Zariski-dense in $G(\bar{K})$. Indeed, let $\pi: X \to G$ be as in 4.13. Then $\pi: X(\bar{K}) \to G(\bar{K})$ is surjective ([5, p. 291, 1.15]), and $\pi^{-1}(\Omega^*)$ contains $W(k) \times H(\bar{K}) \times \mathfrak{V}^{\pm}_{\bar{K}}$ (where $W \subset \mathfrak{V}^+_a \times \mathfrak{V}^-_a$ is the subscheme of quasi-invertible pairs). Since k is infinite, $W(k)$ is Zariski-dense in $\mathfrak{V}^+_{\bar{K}} \times \mathfrak{V}^-_{\bar{K}}$ and hence $\pi^{-1}(\Omega^*)$ is Zariski-dense in $X(\bar{K})$. It follows that Ω^* is Zariski-dense in $G(\bar{K})$. Now let $g \in G(K)$. Then $(g \cdot \Omega^*) \cap \Omega(\bar{K})$ is not empty since $\Omega(\bar{K})$ is Zariski-open in $G(\bar{K})$. Hence there exist $u \in U^-(k)$, $h \in H(\bar{K})$, $v \in U^+(\bar{K})$, $\omega \in \Omega(\bar{K})$ such that $guhv = \omega$. This implies $gu = \omega v^{-1} h^{-1} \in \Omega(\bar{K}) \cap G(K) = \Omega(K)$, and hence $g = (\omega v^{-1} h^1) u^{-1} \in \Omega(K) \cdot U^-(k)$. By passing to (G, ψ^{-1}), we get the other equation.

(b) *k finite.* Let us first assume that G is reductive and connected. Then by 6.8, P^+ and P^- are opposed parabolic subgroups of G, and by 6.9, $\mathfrak{V}$ is separable. Choose a split torus S as in 6.7, and let S' be a maximal k-split torus of G containing S. Let P'^+ and P'^- be the opposed minimal parabolic subgroups of G defined by S'. Then $P'^\sigma \subset P'^\sigma$, and by [3, 6.25], $G(K) = P'^+(k) \cdot P'^-(K) \cdot P'^+(K) = P^+(k) \cdot P^-(K) \cdot P^+(K) = U^+(k) \cdot \Omega(K)$.

Now let G be arbitrary. We have $G = H \cdot U^+ \cdot U^- \cdot U^+ = H \cdot G_{in} = G_{in} \cdot H$, and the kernel of the homomorphism $G_{in} \rtimes H \to G$ given by multiplication

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is isomorphic with $H \cap G_{in} = H_{in}$ (4.12), in particular, it is connected. Hence if K is finite, $G(K) = G_{in}(K)\cdot H(K)$ by [5, p. 426, 7.6], and if K is infinite, this is still true by (a). Replacing G by G_{in}, we may therefore assume that G is smooth and connected. Let $G' = G/R_u(G)$ and let ψ' be the action induced by ψ on G'. Then G' is reductive, (G', ψ') is an elementary system, and the canonical map $\varphi: (G, \psi) \to (G', \psi')$ is a homomorphism (5.19). By 6.9, $R_u(G) = N^-\cdot R_u(H)\cdot N^+$, and $U'^\sigma = U^\sigma/N$ and $H' = H/R_u(H)$. Since $U^\sigma \to U'^\sigma$ is induced by the surjective linear maps $\mathfrak{V}^\sigma \to \mathfrak{V}^\sigma/\mathrm{Rad}(\mathfrak{V}^\sigma)$, the maps $U^\sigma(k) \to U'^\sigma(k)$ and $U^\sigma(K) \to U'^\sigma(K)$ are surjective. Let now $g \in G(K)$. Then $\varphi(g) \in G'(K) = U'^+(k)\cdot \Omega'(K)$, by what we proved before. Hence there exist $u \in U^+(k)$, $v \in U^-(K)$, $w \in U(K)$ such that $\varphi(g) \equiv \varphi(uvw)$ modulo $H'(K)$, in other words, $w^{-1}v^{-1}u^{-1}g \in \varphi^{-1}(H')(K)$. We have $\varphi^{-1}(H') = N^-\cdot H\cdot N^+$. Indeed, since: $\varphi: H \to H'$ is an epimorphism, $\varphi^{-1}(H')$ $H\cdot R_u(G) = N^-\cdot H\cdot N^+$. It follows that $w^{-1}v^{-1}u^{-1}g = xyh$ where $x \in N^-(K)$, $h \in H(K)$, $y \in N^+(K)$, or $g = uvwxhy$. Since $N^-(K) = \exp(\mathrm{Rad}(\mathfrak{V}_K^-))$ it follows from 4.1 and 6.2 that $wx \in \Omega(K)$. Hence $g \in U^+(k)\cdot U^-(K)\cdot\Omega(K)\cdot H(K)\cdot N^+(K) = U^+(k)\cdot\Omega(K)$. The second formula follows by passing to (G, ψ^{-1}).

6.13. COROLLARY. (Generators and relations for $G(k)$). *Let k be infinite. Let Γ be an abstract group, and let $\varphi_0: H(k) \to \Gamma$, $\varphi_\pm: U^\pm(k) \to \Gamma$ be homomorphisms. Then there exists a homomorphism $\varphi: G(k) \to \Gamma$ extending φ_0 and φ_+, φ_- if and only if*

$$(1) \qquad \varphi_0(h)\varphi_\sigma(u)\varphi_0(h)^{-1} = \varphi_\sigma(huh^{-1}),$$

$$(2) \qquad \varphi_+(\exp(x))\varphi_-(\exp(y)) = \varphi_-(\exp(y^x))\varphi_0(b(x, y))\varphi_+(\exp(x^y)),$$

for all $h \in H(k)$, $u \in U^\sigma(k)$, $(x, y) \in \mathfrak{V}^+ \times \mathfrak{V}^-$ quasi-invertible. Indeed, $G(k) = U^+(k)\cdot\Omega(k)$ by 6.12, and hence $G(k)$ is the quotient of $\mathfrak{V}^+ \times \mathfrak{V}^- \times H(k) \times \mathfrak{V}^+$ by the equivalence relation given in 4.13. We can now copy the proof of 4.14, provided we show: for all $u, w \in U^-(k)$, $v \in U^+(k)$ there exists $x \in U^+(k)$ such that xu^{-1} and xvw are in $\Omega(k)$. In view of 4.1, this amounts to showing that for all $a, c \in \mathfrak{V}^-$, $b \in \mathfrak{V}^+$ there exists $z \in \mathfrak{V}^+$ such that $(z, -a)$ and $(z + b, c)$ are quasi-invertible. Since k is infinite, this is always possible.

6.14. As an application, we determine the relation with the group Ξ introduced by Koecher in [11]. Similar remarks apply to the groups studied in [12, 13]. Let $\mathfrak{A}$ be a finite-dimensional quadratic Jordan algebra with unit element e over an arbitrary field k, and let $\Xi(\mathfrak{A})$ be the group of birational transformations of $\mathfrak{A}$ generated by the structure group $\mathrm{Str}(\mathfrak{A})$ and the maps $x \mapsto t_y(x) = x + y$ $(y \in \mathfrak{A})$ and $x \mapsto j(x) = -x^{-1}$. Let $\mathfrak{B} = (\mathfrak{A}, \mathfrak{A})$ be the Jordan pair defined by $\mathfrak{A}$ and $G = \mathbf{PG}(\mathfrak{B})$ the projective group of $\mathfrak{B}$. Thus $H = \mathbf{Aut}(\mathfrak{B})$. Then $\Xi(\mathfrak{A})$ is isomorphic with $G(k)$. Indeed, let K be an infinite extension field of k, let $\varphi_0\colon H(K) = \mathrm{Aut}(\mathfrak{B}_K) \to \mathrm{Str}(\mathfrak{A}_K)$ be the isomorphism $(g, g^{\#-1}) \mapsto g$ (cf. 2.6), and define $\varphi_\pm\colon U^\pm(K) \to \Xi(\mathfrak{A}_K)$ by $\varphi_+(\exp x) = t_x$ and $\varphi_-(\exp y) = j \circ t_y \circ j = \tilde{t}_y$. For $g \in \mathrm{Str}(\mathfrak{A}_K)$ we have $g \circ t_x \circ g^{-1} = t_{gx}$ and $j \circ g \circ j = g^{\#-1}$. This implies (1) of 6.13. For (2), we have to show that $t_x \circ \tilde{t}_y = \tilde{t}_{y^x} \circ B(x, y) \circ t_{x^y}$. Taking inverses, this is equivalent with $\tilde{t}_y \circ t_x = t_{x^y} \circ B(x, y)^{-1} \circ \tilde{t}_{y^x}$. Now $\tilde{t}_y(z) = (z^{-1} - y)^{-1} = z^y$ ([15, 3.13]), and hence

$$(\tilde{t}_y \circ t_x)(z) = (x + z)^y = x^y + B(x, y)^{-1} \cdot z^{(y^x)} = (t_{x^y} \circ B(x, y)^{-1} \circ \tilde{t}_{y^x})(z)\ .$$

By 6.13, we have a homomorphism $\varphi\colon G(K) \to \Xi(\mathfrak{A}_K)$ extending, $\varphi_0, \varphi_\pm$ which is surjective since $j = t_e \circ \tilde{t}_e \circ t_e$. Assume that $\varphi(\exp(x)\exp(y)h \cdot \exp(z)) = t_x \circ \tilde{t}_y \circ g \circ t_z = \mathrm{Id}$ (where $h = (g, g^{\#-1})$). Then $\tilde{t}_y \circ g = t_{-x-z}$, therefore $0 = \tilde{t}_y(0) = t_{-x-z} \circ g^{-1}(0) = -x - z$, hence $g^{-1} = \tilde{t}_y = j \circ t_y \circ j$, and $t_y = j \circ g^{-1} \circ j = g^\#$ which implies $0 = g^\#(0) = t_y(0) = y$. Thus $g = \mathrm{Id}$, $h = 1$, and

$$\exp(x)\exp(y)h \cdot \exp(z) = \exp(x + z) = 1\ .$$

This shows that φ is an isomorphism. By 6.12, $G(k) = U^+(k)U^-(k)H(k)U^+(k)$. It follows that φ maps $G(k)$ isomorphically onto $\Xi(\mathfrak{A})$. Also we see that every element of $\Xi(\mathfrak{A})$ is of the form $g \circ t_x \circ j \circ t_y \circ j \circ t_z$ with $g \in \mathrm{Str}(\mathfrak{A})$, x, y, $z \in \mathfrak{A}$. This was proved by Koecher in case k is infinite and of charcteristic $\neq 2$.

Added in proof. In the paper O. Loos, Homogeneous algebraic varieties defined by Jordan pairs. Mh. Math. 86 (1978), 107–127, some of the results of § 6 are extended to the case of an arbitrary base ring.

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