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ULC PROPERTIES IN NEIGHBOURHOODS OF EMBEDDED SURFACES AND CURVES IN E^3

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1. Introduction. In this paper we derive those properties of topologically embedded curves and surfaces in E^3 which can be obtained without use of Bing's Side Approximation Theorem [**3**] for surfaces. The local homology and homotopy properties studied classically play the largest role in the paper, but the final geometrization of some of the results requires theorems such as the PL Schoenflies Theorem, Dehn's Lemma, the Loop Theorem, the Sphere Theorem, and Waldhausen's generalization of the Loop Theorem (n.b., one application of Waldhausen's theorem (in (3.4)) requires use of the nontrivial normal subgroup in the statement of that theorem). Our major goals are the following:

(1) Lemmas (3.2) and (3.3), which we use together with Dehn's Lemma and related theorems in another paper [**14**] to give a new proof of the Side Approximation Theorem.

(2) A catalogue in Section 2 of the essential *ULC* properties that we have used or seen applied in a study of topologically embedded surfaces in E^3 (including new proofs of a number of theorems proved originally by means of the Side Approximation Theorem).

(3) A new proof of the fact that every topologically embedded disk in E^3 contains numerous tame arcs and tame finite graphs (Sections 2, 3; cf. [**2**]).

(4) A $1 - ULC$ version of the well-known Hosay-Lininger Theorem (Section 6; cf. [**23**; **25**; **15**]).

(5) Global and relative $1 - ULC$ approximation theorems for surfaces in E^3.

The paper is not devoted, however, only to new proofs of old theorems. We also establish the following new theorems:

(6) A finite graph G in E^3 is tame if it has a singular regular neighbourhood in E^3.

(7) An $(n-1)$-sphere S in S^n has $1 - ULC$ complement if S is locally spherical or locally capped.

(8) A 2-sphere S in E^3 is tame if S is locally spherical or locally capped.

Result (3) is a consequence of (6) and (2). Another consequence of (6) is a conjecture communicated to us by O. G. Harrold (cf. [**20**]):

(9) An arc A in E^3 is tame if $E^3 - A$ is $1 - ALG$ at each point of A.

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Result (9) completes a characterization of tame finite graphs begun by McMillan [**27**]:

(10) A finite graph G in E^3 is tame if $E^3 - G$ is $1 - FLG$ at each point of G.

Boyd and Wright [**5**] have also announced (9) and (10) but have found curiously that their proof of (9) is valid for simple closed curves and not for arcs. This puts their proof of (10) in jeopardy. Because of the importance of (9) and (10), we have developed and included independent proofs. Nicholson [**30**] and Detmer [**16**] have used (9) to obtain characterizations of tame topologically embedded complexes in E^3. Detmer [**16**] has extensions of these characterizations to certain subsets of complexes, which subsets need not be topological complexes.

The special case of (6) where G is an arc or simple closed curve settles in the affirmative a conjecture of Gillman [**19**; Conjecture 4 for $k = 1$, $n = 3$].

(11) An arc or simple closed curve in E^3 is tame if it is deformation free.

Result (8) completes, and result (7) generalizes, work begun in papers by Loveland [**26**] and Eaton [**17**].

The idea of writing a paper on surfaces in 3-manifolds which explored in some detail those results which can be obtained without the Side Approximation Theorem and emphasized the $1 - ULC$ property arose in discussions with Eaton in San Antonio in January of 1970. We draw on Eaton's ideas heavily in the proofs of the results mentioned in (4) and (5). We wish to acknowledge his ideas and influence.

We feel some need to apologize to the reader who seeks in our Sections 2 and 3, along with [**14**], the shortest path to the Side Approximation Theorem. We have not included the shortest, most direct proofs that we know for the relevant facts. We have looked, rather, beyond the Side Approximation Theorem to the other main theorems about tame surfaces and tame subsets of surfaces and have chosen those results which lead more directly to these other theorems.

We suggest [**9**] as a basic reference for the topics discussed. We assume as familiar the notions of Euclidean spaces ($E^1, E^2, E^3, \ldots$), n-cells ($B^0, B^1, B^2, \ldots$ and their homeomorphic images), n-spheres ($S^0 = \text{Bd } B^1, S^1 = \text{Bd } B^2, \ldots$ and their homeomorphic images), complexes, manifolds, disks (2-cells), arcs (1-cells), finite graphs (1-complexes), crumpled cubes, tamely (and flatly) embedded complexes, general position and cut-and-paste techniques, and some elementary geometric homology and homotopy theory. We also assume the PL Schoenflies Theorem (see [**9**, p. 277] for references and discussion), Dehn's Lemma [**31**] (cf. also [**9**; 4.5.1 and Addendum to 4.5.1]), the Loop Theorem [**32**; **37**] (including Waldhausen's generalization [**39**]), and the Sphere Theorem [**31**]. In addition, we assume the simpler facts from the homological linking theory (integer coefficient) of simple closed curves in E^3 (cf. [**1**, Chapter 15] and [**34**, § 77]). Namely, we use the notation $L(J, K)$ for the linking number of disjoint oriented simple closed curves or loops in E^3.

We assume the fact that $L(J, K)$ may be calculated, as indicated in [**9**, Section 4.7], by taking nearby polygonal approximations J' and K' to J and K, taking a polyhedral singular oriented surface D in E^3 bounded by J' and in general position with respect to K', and counting the algebraic number of intersections of K' with D. We now list the standard facts of the situation.

1.1. $L(J, K)$ is independent of the choice of J', K' and D.

1.2. $L(J, K) = L(K, J)$.

1.3. If J_1 is homologous (integer coefficients) in $E^3 - K$ to J_2, then

$$L(J_1, K) = L(J_2, K).$$

1.4. If J is an oriented simple closed curve in E^3, then there is an oriented simple closed curve K in $E^3 - J$ such that $L(J, K) = 1$ (cf. paragraphs 2 and 3 of the proof of (3.4)).

1.5. If J and K are disjoint oriented simple closed curves in E^3 and $L(J, K) = 0$, then J is nullhomologous in $E^3 - K$.

1.6. If D is a disk in E^3 (not necessarily polyhedral) and $p \in \operatorname{Int} D$, then there is a simple closed curve J in $(E^3 - D) \cup \{p\}$ for which $L(J, \operatorname{Bd} D) \neq 0$. Furthermore, if J is any such loop and J_1 is any loop in $E^3 - \operatorname{Bd} D$, then J_1 is homologous in $E^3 - \operatorname{Bd} D$ to some multiple of J. In particular, $|L(J, \operatorname{Bd} D)| = 1$ by (1.4).

We use ρ for the Euclidean metric, Diam for diameter, Bd for boundary (point set or combinatorial), Int and Ext for interior and exterior, $N(X, \epsilon)$ for the (open) ϵ-neighbourhood of X in E^m, where $\epsilon > 0$ is a constant or $\epsilon : X \to [0, \infty)$ is a continuous function, depending on context. An ϵ-set has diameter less than ϵ; an ϵ-map or homeomorphism moves no point x as far as $\epsilon(x)$ (unless $\epsilon(x) = 0$); we also use $(\leqq \epsilon)$-sets and $(\geqq \epsilon)$-sets with obvious interpretations. The symbol Cl denotes closure.

Central to our development are the *ulc* and *ULC* properties. The standard references to these properties are [**42**, Chapter 3, Section 5 and Chapter X] and [**18**]. For the reader's convenience we define these properties here. Let coefficients $G = \mathbf{Z}$ or $\mathbf{Z}_2$ (integers or integers mod 2) be fixed. Let A and A' be subsets of E^m (some m). Let i be a nonnegative integer. We say that A is $i - lc$ (i.e., $i - lc$ with respect to G) in A' at $x \in E^m$ if, for each $\epsilon > 0$, there is a $\delta > 0$ such that each i-cycle (with G coefficients) in $N(x, \delta) \cap A$ bounds homologically in $N(x, \epsilon) \cap A'$. If A is $i - lc$ in A' at each point $x \in \operatorname{Cl} A$, then we say that A is $i - lc$ in A'. If A is $i - lc$ in A' and, for each $\epsilon > 0$, the corresponding δ may be chosen independently of $x \in \operatorname{Cl} A$, then we say that A is $i - ulc$ in A'. If A is $i - lc$ in A' for $i = 0, 1, \ldots, n$, then we say that A is lc^n in A'. If A is $i - ulc$ in A' for $i = 0, 1, \ldots, n$, then we say that A is ulc^n in A'. We say that A is $i - LC$ in A' at $x \in E^m$ if, for each $\epsilon > 0$, there is a $\delta > 0$ such that each map $f : S^i \to N(x, \delta) \cap A$ extends to a map $f^* : B^{i+1} \to N(x, \epsilon) \cap A'$. We leave it to the reader to define $i - ULC$, LC^n, and ULC^n. If A is ulc^n or ULC^n in itself (*i.e.*, $A' = A$), then we say simply that A is ulc^n or ULC^n.

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2. *ULC* **properties in** E^3**.** We collect here the local homology (*ulc*) and local homotopy (*ULC*) properties we have found useful in a study of embedded surfaces. (The $2 - ulc$ and $2 - ULC$ properties are included only for completeness, however; we have never seen them used to real advantage.) The more important results appear in a list for reference at the beginning of this section. Each is labeled by the subsection number (2A, 2B, or 2C) in which its proof appears. Related but less important results, lemmas, and discussions appear in the subsections themselves. We highly recommend that most readers refer only to the list of theorems and not to the proofs until a second or third reading of the paper.

We feel that Theorems (2C.6) and (2C.7) and their proofs are the highlights of this section. The former of these theorems is proved in a most roundabout way in the literature (cf. [**13**]) and both were proved originally in ways that depended in a very essential way on the Side Approximation Theorem [**13**; **8**, Theorem 4.2].

Throughout this section, S will denote a 2-sphere in E^3, U and V the components of $E^3 - S$.

2A. LOCAL HOMOLOGY PROPERTIES.

The set U *is* ulc^2 ($\mathbf{Z}$ *or* $\mathbf{Z}_2$ *coefficients*); *hence, in particular,* U *is* $ULC^0 = ulc^0$.

2B. GLOBAL PRELIMINARIES, UNICOHERENCE, SEPARATION.

2B.1. *The set* Cl U *is an absolute neighbourhood retract, locally contractible, and locally connected.*

2B.2. $H_1(U) = H_1(\mathrm{Cl}\ U) = \pi_1(\mathrm{Cl}\ U) = 1$ ($\mathbf{Z}$ *or* $\mathbf{Z}_2$ *coefficients for homology*).

2B.3. *The sets* U *and* Cl U *are unicoherent. Equivalently, if* $p, q \in U$ (*or* $p, q \in \mathrm{Cl}\ U$) *and* p *and* q *are separated in* U (*or* Cl U) *by a set* X, *then some component of* X *separates* p *from* q *in* U (*or in* Cl U).

2B.4. *If* $\epsilon > 0$, *then there is a* $\delta > 0$ *such that no two sets in* U (*or* Cl U) *of diameter* $\geqq \epsilon$ *are separated in* U (*or* Cl U) *by a* δ*-set.*

2C. LOCAL HOMOTOPY PROPERTIES.

2C.1. Homotopies of maps in crumpled cubes. *Suppose* $\epsilon > 0$. *Then there is a* $\delta > 0$ *such that, for each topological space* X, *maps* $f : X \to \mathrm{Cl}\ U$ *and* $g : X \to \mathrm{Cl}\ U$ *which are* δ*-homotopic in* E^3 *are* ϵ*-homotopic in* Cl U.

2C.2. Adjustments of maps in *ULC* sets. *Suppose* $C \subset E^3$ *and* C *is* ULC^n (n *a nonnegative integer*). *Suppose* P *is an* $(n + 1)$*-complex,* B *is a closed subset of* P, *and* $f : P \to \mathrm{Cl}\ C$ *is a map. Then, for each* $\epsilon > 0$, *there is a map* $f^* : P \to \mathrm{Cl}\ C$ *such that*

$$f^*|B = f|B,$$
$$f^*(P - B) \subset C, \text{ and}$$
$$\rho(f^*(x), f(x)) < \epsilon, \text{ for each } x \in P.$$

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2C.3. Expansions of ULC sets. *If* $C \subset C' \subset \mathrm{Cl}\, C \subset E^3$ *and* C *is* ULC^n, *then* C' *is* ULC^n.

2C.4. Intersections of ULC sets. *If* $C_1, C_2, \ldots$ *are* ULC^n *sets in* E^3, *each with the same closure* C *in* E^3, *and each is open in* C, *then* $\bigcap_{i=1}^{\infty} C_i$ *is also* ULC^n.

2C.5. Equivalence among $1 - ULC$ properties. *If* X *is a subset of* S, *then the following two properties are equivalent:*

(1) $(\mathrm{Cl}\, U) - X$ *is* $1 - ULC$.

(2) U *is* $1 - ULC$ *in* $(\mathrm{Cl}\, U) - X$.

If, in addition, X *is compact and has no degenerate component, then the following condition is also equivalent to the first two*:

(3) U *is* $1 - ULC$ *in* $E^3 - X$.

2C.6. *If* X *is a compact subset of* S *and* X *lies on some tame* 2-*sphere* S' *in* E^3, *then* $E^3 - S$ *is* $1 - ULC$ *in* $E^3 - X$.

Remark. Theorem (2C.6) is a special case of the following theorem which we shall not prove in this section. (But cf. (4.3).)

2C.6′. Invariance of the $1 - ULC$ property. *Suppose* X *is a compact subset of* S *and both* U *and* V *are* $1 - ULC$ *in* $E^3 - X$. *Then, if* S' *is another* 2-*sphere in* E^3 *which contains* X, Int S' *and* Ext S' *are* $1 - ULC$ *in* $E^3 - X$.

2C.7. Existence of ULC sets in Cl U.

(1) *If* $U \subset C \subset \mathrm{Cl}\, U$, *then* C *is* $0 - ULC$ $(= ULC^0)$.

(2) *There is a* 0-*dimensional* F_σ-*set* F *in* S *such that* $U \cup F$ *is* ULC^1.

(3) *There is a* 0-*dimensional* G_δ-*set* G *in* S *such that* $U \cup G$ *is* ULC^2.

Addendum to 2C.7.

(2′) *If* $X_1, X_2, \ldots$ *is a sequence of compact sets in* S *and, for each* i, $(\mathrm{Cl}\, U) - X_i$ *is* $1 - ULC$, *then we may require that* $F \subset G \subset S - \bigcup_{i=1}^{\infty} X_i$.

(3′) *If* $U \subset C \subset \mathrm{Cl}\, U$ *and* C *is open in* Cl U, *then* C *is* $2 - ULC$.

2A. (Continued). That U is *ulc*^{2} is classical (cf. [**42**, p. 66 for $\mathbf{Z}_2$ coefficients, Chapter X in general]).

2A.1. Corollary. *Each point of* S *is arcwise accessible from* U.

Proof. We leave this as on exercise.

2B. (Continued).

Proof of (2B.1). That S is an absolute neighborhood retract (ANR) is a well-known consequence of the fact that S can be embedded as a neighborhood retract of the absolute retract E^3 (use the standard embedding of the 2-sphere in E^3; cf., for example, [**36**, Exercises C, pp. 56-57]). But it is an easy exercise to show that a closed subset of E^n is an ANR if its boundary in E^n is an ANR. Hence Cl U is an ANR. A second easy exercise shows that a neighborhood retract of a locally contractible and locally connected space is in turn locally contractible and locally connected.

Proof of (2B.2). The result that $H_1(U) = 1$ is classical (cf. [**42**, p. 61 for $\mathbf{Z}_2$ coefficients]). In order to see that $H_1(\text{Cl } U) = \pi_1(\text{Cl } U) = 1$, take any family $J_1, \ldots, J_k$ of loops in Cl U bounding singular disks $D_1, \ldots, D_k$ in E^3. After slight adjustment by a homotopy that keeps $J_1, \ldots, J_k$ in Cl U (cf. (2C.2.1)), we find that we may assume that $S \not\subset \bigcup_{i=1}^k D_i$. Let $p \in S - \bigcup_{i=1}^k D_i$. Then $S - \{p\}$ is an absolute retract (AR). Thus there is a map $f : \bigcup D_i \rightarrow S - \{p\}$ that does not move $\bigcup D_i \cap S$. Define $g : \bigcup D_i \rightarrow \text{Cl } U$ by $g(x) = f(x)$ for $x \notin \text{Cl } U$ and $g(x) = x$ for $x \in \text{Cl } U$. Then $J_1, \ldots, J_k$ bound the singular disks $g(D_1), \ldots, g(D_k)$ in Cl U.

Proof of (2B.3). The equivalence in locally connected spaces of unicoherence and the second condition stated in (2B.3) is well-known (cf. [**42**, p. 51]) and an easy exercise. If U or Cl U were not unicoherent, then one could construct an essential map of the relevant space onto the circle S^1 (cf. [**41**, p. 227]). But this is clearly impossible because of (2B.2).

Proof of (2B.4). We need one lemma.

2B.4.1. LEMMA. *If P is an open subset of* Cl U, *then there is an open subset Q of P such that*
(1) $Q \cap S = P \cap S$,
(2) (Cl U) $- Q$ *is connected, and*
(3) $U - Q$ *is connected.*

Proof. Since U is arcwise connected, there is a connected finite graph G_1 in U such that $(\text{Cl } U) - P \subset N(G_1, 1)$. Proceeding inductively, we find that there is a connected finite graph G_i in U such that $G_{i-1} \subset G_i$ and $(\text{Cl } U) - P \subset N(G_i, 1/i)$ for each i. Since U is $0 - ULC$ (2A), we may require further that $G_i - G_{i-1} \subset N((\text{Cl } U) - P, 1/i)$. Let $X = [(\text{Cl } U) - P] \cup (\bigcup G_i)$. Then X is closed and $Q = (\text{Cl } U) - X$ satisfies the requirements of (2B.4.1).

We now complete the proof of (2B.4). There is by the lemma an open cover $\{U_\alpha\}$ of Cl U in Cl U such that, for each α, both $(\text{Cl } U) - U_\alpha$ and $U - U_\alpha$ are connected and such that, for each α, Diam $U_\alpha < \epsilon$. There is, by Lebesgue's theorem, a positive number δ such that each δ-set in Cl U lies in some U_α. Let X and Y be two sets in U (or Cl U) each of diameter $\geqq \epsilon$ and Z a δ-set in $U - (X \cup Y)$ (or in $(\text{Cl } U) - (X \cup Y)$). Then $Z \subset U_\alpha$ for some α. But each of X and Y intersects the connected subset $U - U_\alpha$ (or $(\text{Cl } U) - U_\alpha$) of $U - Z$ (or of $(\text{Cl } U) - Z$) since U_α is an ϵ-set. Therefore Z does not separate X and Y in U (or in Cl U).

2C. (Continued).

Proof of (2C.1). Since S is an absolute neighborhood retract, there is a neighborhood N of S in Cl V and an $\epsilon/2$-retraction $R : N \rightarrow S$. Choose

δ, $0 < \delta < \epsilon/2$, such that $N(S, \delta) \subset N \subset \mathrm{Cl}\ U$. If, then, $f : X \to \mathrm{Cl}\ U$ and $g : X \to \mathrm{Cl}\ U$ are δ-homotopic maps in E^3, say by $H : X \times 1 \to E^3$, $H : f \cong g$, we examine the map $H' : X \times I \to \mathrm{Cl}\ U$ defined by

$$H'(x, t) = \begin{cases} H(x, t) \text{ if } H(x, t) \in \mathrm{Cl}\ U \\ RH(x, t) \text{ if } H(x, t) \notin \mathrm{Cl}\ U. \end{cases}$$

One checks immediately that H' is an ϵ-homotopy from f to g in $\mathrm{Cl}\ U$.

Proof of (2C.2). This is a corollary to [**18**, Theorem 2]. For completeness and to emphasize the elementary nature of the result, we include a detailed proof for the case $n = 1$. Choose $\delta_1 > 0$ such that δ_1-loops in C bound singular $\epsilon/3$-disks in C (C is $1 - ULC$). Choose $\delta_0 > 0$ such that each two points of C which lie within δ_0 of each other lie on a singular $\delta_1/2$-arc in C (C is $0 - ULC$). Triangulate the set $P - B$ with mesh approaching 0 near B such that the image under f of each simplex has diameter less than $\delta_0/3$. Enumerate the vertices of $P - B : v_1, v_2, \ldots$. Define $f^*(v_1) \in C$ such that $\rho(f(v_i), f^*(v_i)) < \min(\delta_0/3, 1/i)$. Enumerate the edges: $e_1, e_2, \ldots$. If e_k is a 1-simplex of $P - B$ with vertices v_i and v_j, define $f^* : e_k \to C$ so as to extend $f^*|\{v_i, v_j\}$ and so that $\mathrm{Diam}\, f^*(e_k) < \delta_1/2$. This is possible since

$$\begin{aligned} \rho(f^*(v_i), f^*(v_j)) &\leqq \rho(f^*(v_i), f(v_i)) + \rho(f(v_i), f(v_j)) \\ &\quad + \rho(f(v_j), f^*(v_j)) \\ &< \delta_0. \end{aligned}$$

Require further that $\mathrm{Diam}\, f^*(e_k) \to 0$ as $k \to \infty$. Enumerate the 2-simplexes of $P - B : t_1, t_2, \ldots$. If t_m is a 2-simplex of $P - B$ with edges e_i, e_j, e_k, define $f^* : t_m \to C$ so as to extend $f^*|e_i \cup e_j \cup e_k$, so that $\mathrm{Diam}\, f^*(t_m) < \epsilon/3$, and so that $\mathrm{Diam}\, f^*(t_m) \to 0$ as $m \to \infty$. Define $f^*|B = f|B$. It is an easy matter to check that f^*, so defined, satisfies the requirements of the theorem.

2C.2.1. Corollary. *If $f : S^1 \to \mathrm{Cl}\ U$ is a loop that bounds a singular disk $D : B^2 \to \mathrm{Cl}\ U$, then f bounds a singular disk $D' : B^2 \to \mathrm{Cl}\ U$ arbitrarily close to D such that $(D')^{-1}(S \cap \mathrm{Int}\, B^2)$ is 0-dimensional.*

Proof. Let $G_1, G_2, \ldots$ be a sequence of triangulations of B^2 with mesh going to zero. The image under D of that part of the 1-skeleton of G_1 which lies in $\mathrm{Int}\, B^2$ may be moved into U by (2C.2) since U is $0 - ULC$ (2A). This adjustment may be realized by a slight homotopy of D in E^3 by the homotopy extension property (cf. [**24**, p. 13]). The homotopy, if small enough, may be retracted into $\mathrm{Cl}\ U$ since $\mathrm{Cl}\ U$ is an ANR (2B.1). Sequential application of this procedure pulls the images under D of the 1-skeletons of $G_2, G_3, \ldots$ into U as well. The map D' may be taken as the limiting map.

Proof of (2C.3). This is essentially [**18**, Theorem 3]. Suppose $\epsilon > 0$ and $0 \leqq k \leqq n$ given. Let $\delta > 0$ be chosen such that any map $f : S^k \to C$ of S^k into a δ-set in C extends to a map $g : B^{k+1} \to C$ taking B^{k+1} into an ϵ-set in C. Let $f_0 : S^k \to C'$ be a map into a δ-set in C'. Extend f_0 over a closed collar neighborhood N

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of S^k in B^{k+1}. Use (2C.2) with $P = N$ and $B = S^k$ to adjust $f_0|N$ so as to take $N - S^k$ into C. If one uses enough care in the application of (2C.2), then $f_0|(\text{Bd } N) - S^k$ will be a map of a k-sphere into a δ-subset of C. Thus f_0 may be extended over the remainder of B^{k+1} to take B^{k+1} into a small subset of $C \cup f_0(S^k) \subset C'$. Result (2C.3) follows.

Proof of (2C.4). This is essentially [**10**, Lemma 2.3]. Suppose $\epsilon > 0$ and $0 \leqq k \leqq n$ given. Since C_1 is $k - ULC$, there is a $\delta > 0$ such that each map f taking S^k into a δ-subset of C_1 extends to a map f^* taking B^{k+1} into an ϵ-subset of C_1. Let $f : S^k \to \bigcap_{i=1}^\infty C_i$ be a map into a δ-subset of $\bigcap_{i=1}^\infty C_i$, and $f_1 : B^{k+1} \to C_1$ an extension of f with $\text{Diam}\, f_1(B^{k+1}) < \epsilon$. We now define positive numbers $\epsilon_1, \epsilon_2, \ldots$ and maps $f_1, f_2, \ldots$ inductively as follows. Let

$$0 < \epsilon_1 < (1/4) \cdot \min\{\epsilon - \text{Diam}\, f_1(B^{k+1}),\ \rho(f_1(B^{k+1}),\ C - C_1)\}.$$

Let $f_2 : B^{k+1} \to C - C_2$ be such that $f_2|S^k = f_1|S^k$ and $\rho(f_1(p),\ f_2(p)) < \epsilon_1$ for each $p \in B^{k+1}$. Such a map exists by (2C.2). For $i > 1$, let ϵ_i be in the range $0 < \epsilon_i < (1/4) \cdot \min\{\epsilon_{i-1},\ \rho[f_i(B^{k+1}),\ C - C_i]\}$. Let $f_{i+1} : B^{k+1} \to C - C_{i+1}$ be such that $f_{i+1}|S^k = f_i|S^k$ and $\rho(f_i(p),\ f_{i+1}(p)) < \epsilon_i$ for each $p \in B^{k+1}$. Again, at each stage such a map exists by (2C.2).

Because C is a complete space, it follows that $f^* = \lim f_i$ exists, is continuous, extends f, and takes B^{k+1} into an ϵ-subset of $\bigcap_{i=1}^\infty C_i$. We conclude that $\bigcap_{i=1}^\infty C_i$ is $k - ULC(0 \leqq k \leqq n)$, hence that $\bigcap_{i=1}^\infty C_i$ is ULC^n.

Proof of (2C.5). We need one lemma and a corollary to that lemma.

2C.5.1. Lemma (Burgess [**8**, Lemma 1]). *If $D_1, D_2, \ldots, D_n$ are disjoint disks in E^3 and f is a map of a disk K into E^3 such that $f(\text{Bd } K) \subset E^3 - \bigcup_{i=1}^n D_i$, then there is a map g of K into E^3 such that*

$$g|\text{Bd } K = f|\text{Bd } K,$$
$$g(K) \subset f(K) \cup \textstyle\bigcup_{i=1}^n \text{Int } D_i, \textit{ and}$$
$$g(K) - \textstyle\bigcup_{i=1}^n D_i \textit{ is connected.}$$

Proof. This is an immediate consequence of Tietze's Extension Theorem (a disk is an absolute retract). In rough outline, one chooses disjoint closed subsets of K whose images under f are to be retracted into $D_1, D_2, \ldots, D_n$, respectively, in order to define g.

2C.5.2. Corollary. Cl U *is* $1 - ULC$.

Proof. Suppose $\epsilon > 0$ given. Let $D_1, \ldots, D_n$ be a collection of $\epsilon/3$-disks in S such that $S \subset \bigcup_{i=1}^n \text{Int } D_i$. Let $\delta > 0$ be such that each δ-subset of S lies in some $\text{Int } D_i$. Let $f : S^1 \to \text{Cl } U$ be a δ-loop. Since U is $0 - ULC$, it follows from (2C.2) that f may be closely approximated by a map $g : S^1 \to U$. By (2C.1), we may assume that f and g are homotopic under a very small homotopy in Cl U. That is, we lose no generality in assuming that $f : S^1 \to U$. Let $f^* : B^2 \to E^3$ be a singular δ-disk bounded by f. Then $f^*(B^2) \cap S \subset \text{Int } D_i$

for some i. By (2C.5.1), there is a map $g : B^2 \to E^3$ such that

$$g|S^1 = f^*|S^1 = f|S^1,$$
$$g(B^2) \subset f^*(B^2) \cup \operatorname{Int} D_i, \text{ and}$$
$$g(B^2) - D_i \text{ is connected.}$$

It follows that $g(B^2) \subset \operatorname{Cl} U$ and $\operatorname{Diam} g(B^2) < \delta + \epsilon/3$. We conclude that $\operatorname{Cl} U$ is $1 - ULC$.

We now complete the proof of (2C.5). Clearly (1) implies (2), and (2) implies (3).

We now show that (2) implies (1). Suppose (2) satisfied and suppose $\epsilon > 0$ given. By (2C.5.2), there is a $\delta > 0$ such that δ-loops in $\operatorname{Cl} U$ bound singular ϵ-disks in $\operatorname{Cl} U$. Let $f : S^1 \to (\operatorname{Cl} U) - X$ be a map into a δ-set. Let $f^* : B^2 \to \operatorname{Cl} U$ be a singular ϵ-disk bounded by f. By (2C.2.1), we may assume that $[(f^*)^{-1}(S)] \cap \operatorname{Int} B^2$ is a 0-dimensional subset of U. Let $D_1, D_2, \ldots$ be a null-sequence of very small disjoint disks in $\operatorname{Int} B^2$ such that $(f^*)^{-1}(S \cap \operatorname{Int} B^2) \subset \bigcup \operatorname{Int} D_i$. The loops $f^*|\operatorname{Bd} D_1, f^*|\operatorname{Bd} D_2, \ldots$ form in turn a null sequence of very small loops in U. By (2), these bound a null sequence of small singular disks $g_i : D_i \to (\operatorname{Cl} U) - X$. Then

$$g = [f|(B^2 - \cup D_i)] \cup \cup g_i : B^2 \to E^3$$

is a map from B^2 into a small subset of $(\operatorname{Cl} U) - X$; and if $D_1, D_2, \ldots$ and the singular disks $g_i(D_i)$ are kept sufficiently small, then $\operatorname{Diam} g(B^2) < \epsilon$. We conclude that (1) is satisfied.

Now assume the additional conditions on X and suppose (3) satisfied. Let X_i be the union of components of X having diameter at least $1/i (i = 1, 2, \ldots)$. Then X_i is a closed set satisfying the hypotheses for (3). Let $\epsilon > 0$ be given. Making ϵ smaller if necessary, we may assume $\epsilon < 1/i$. Choose $\delta_1 > 0$ such that δ_1-subsets of S lie in $\epsilon/2$-disks in S and $\delta > 0$ such that δ-loops in U bound singular δ_1-disks in $E^3 - X_i$. Let $f : S^1 \to U$ be a δ-loop,

$$F : B^2 \to E^3 - X_i \text{ a singular } \delta_1\text{-disk in } E^3 - X_i \text{ bounded by } f.$$

Then $F(B^2) \cap S$ lies in the interior of an $\epsilon/2$-disk E in S. Since X_i has no component of diameter less than $1/i$, $\operatorname{Int} E - X_i$ is simply connected (though not necessarily connected). Thus $f(B^2) \cap S$ may be covered by finitely many disjoint disks $D_1, D_2, \ldots, D_n$ in $E - X_i$. By (2C.5.1), there is a map $g : B^2 \to F(B^2) \cup \bigcup_{i=1}^n D_i$ such that $g|\operatorname{Bd} B^2 = f$ and $g(B^2) - \bigcup_{i=1}^n D_i$ is connected. It follows that $g(B^2) \subset (\operatorname{Cl} U) - X_i$ and that $\operatorname{Diam} g(B^2) \leqq \epsilon/2 + \delta_1$. We conclude that U is $1 - ULC$ in $(\operatorname{Cl} U) - X_i$. By (2A) and (2C.3), $\operatorname{Cl} U - X_i$ is $0 - ULC$. By the equivalence of (1) and (2), $(\operatorname{Cl} U) - X_i$ is $1 - ULC$, hence ULC^1. By (2C.4), $(\operatorname{Cl} U) - X = \bigcap_{i=1} ((\operatorname{Cl} U) - X_i)$ is ULC^1. Thus (3) implies (1), and the proof is complete.

This is the natural point to insert some material for use in Section 6. This material is a simple elaboration of the proof that (3) implies (1) in the theorem just completed.

Suppose $\delta > 0$. We define
$\mathscr{E}(S, \delta) = \{\epsilon > 0 | \delta\text{-sets in } S \text{ lie in simply connected sets in } S \text{ of diameter } \leqq \epsilon\}$.
$\epsilon(s, \delta) = \inf \mathscr{E}(S, \delta)$.

2C.5.3. *The number* $\epsilon(S, \delta)$ *is an element of the set* $\mathscr{E}(S, \delta)$.

Proof. Suppose X is a δ-set in S. Then X lies in the interior of a compact δ-set X_0 in S such that Bd X_0 is a union of finitely many disjoint simple closed curves. For each i, let U_i be a simply connected subset of S which contains X_0 and has diameter less than $\epsilon(S, \delta) + 1/i$. Let V_i be the union of those components of $S - X_0$ which lie entirely in U_i. A quick check shows that $X_0 \cup V_i$ is simply connected; indeed it is either S or a union of finitely many disjoint disks in S. Since $S - X$ has only finitely many components, some one of the sets V_i is repeated for infinitely many i. For such a V_i, $X_0 \cup V_i$ is a simply connected set of diameter $\leqq \epsilon(S, \delta)$ in S and it contains X.

2C.5.4. *A* δ*-loop in* U *bounds a singular* $\epsilon(S, \delta) + \delta$*-disk in* Cl U.

Proof. The proof has already been carried out in the proof of (2C.5): a δ-loop $L : S^1 \to U$ bounds a singular δ-disk $D : B^2 \to E^3$; the set $S \cap D(B^2)$ is a δ-set in S, hence lies in a simply connected subset W of S of diameter $\leqq \epsilon(S, \delta)$; as the proof of (2C.5.3) shows, W may be taken as a union of finitely many disjoint disks in S (or $W = S$). Lemma (2C.5.1) shows how to cut D off on W so as to obtain a singular disk $D' : B^2 \to (D(B^2) \cup W) \cap \text{Cl } U$; but Diam $D' \leqq$ Diam $D(B^2)$ + Diam $W < \delta + \epsilon(S, \delta)$.

2C.5.5. $\epsilon(S, \delta) \to 0$ *as* $\delta \to 0$.

Proof. The proof is clear.

2C.5.6. *If* $h : S \to E^3$ *is an* α*-homeomorphism, then*

$$\epsilon(h(S), \delta) \leqq \epsilon(S, \delta + 2\alpha) + 2\alpha.$$

Proof. Let X be a δ-set in $h(S)$. Then $h^{-1}(X)$ is a $\delta + 2\alpha$-set in S, hence lies in a simply connected set W in S of diameter $\leqq \epsilon(S, \delta + 2\alpha)$. Thus $h(W)$ is a simply connected $\epsilon(S, \delta + 2\alpha) + 2\alpha$-set in $h(S)$ which contains X.

Proof of (2C.6). The proof follows a sequence of definitions and lemmas, (2C.6.1)-(2C.6.5). The proof is constructed so as to use the fact that U is *ulc*^{1} with $\mathbf{Z}_2$ coefficients. An analogous proof could be constructed using $\mathbf{Z}$ coefficients.

2C.6.1. *Definition.* Suppose G is a group. Then the (mod 2)-commutator subgroup G^1 of G is the subgroup of G generated by squares of elements of G. Inductively, we define $G^n = (G^{n-1})^1$ and $G^\omega = \bigcap_{n=1}^\infty G^n$.

2C.6.2. *If* $\phi : G \to H$ *is a homomorphism of groups, then* $\phi(G^n) \subset H^n$ *and* $\phi(G^\omega) \subset H^\omega$.

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Proof. Note that $\phi(g^2) = \phi(g)^2$ $(g \in G)$. The desired result is an easy inductive consequence of this trivial observation.

2C.6.3. *If G is a free group, then $G^\omega = 1$.*

Proof. This result is well-known (cf. [**33**, first paragraph of the proof of 8.4.16]).

2C.6.4. *Definition.* If G is the fundamental group $\pi_1(M)$ of a path-connected space M, then we write $G^n = \pi_1{}^n(M)$ and $G^\omega = \omega(M)$.

The group $\pi_1{}^1(M)$ can be interpreted geometrically as the kernel of the natural homomorphism $\pi_1(M) \to H_1(M, Z_2)$ from homotopy group to homology group. That is, a loop $L : S^1 \to M$ represents an element of $\pi_1{}^1(M)$ if an only if it bounds a singular (possibly nonorientable) surface in M. This follows from the classification theorem for 2-manifolds (which shows that a square corresponds to a Möbius strip, a commutator to a disk with one handle; cf. [**29**, Chapter 1, § 5]). (It is helpful to realize that each commutator in a group is a product of squares.) The groups $\pi_1{}^n(M)$ and $\omega(M)$ have similar geometric interpretations.

2C.6.5. THE ω-THEOREM. *Suppose N is a connected open subset of* Cl U, *p is a point of $N \cap U$, and F is the family of loops in $N \cap U$ based at p and bounding singular disks in N. Then each loop in F represents an element of $\omega(N \cap U, p)$.*

Proof. It suffices to show that each element $L : S^1 \to N \cap U$ of F is homotopic in N-Bd U to a product of squares of elements of F. This we can do by showing that L bounds a singular 2-manifold $h : E \to N$-Bd U such that h is nullhomotopic in N. Indeed, this implies that L is homotopic to a product of squares and commutators (hence of squares) of loops in $h(E)$ by the classification theorem for 2-manifolds. Each such loop is in F because h is nullhomotopic in N.

Let $D : B^2 \to N$ be a singular disk bounded by L. There is a positive distance between $D(B^2)$ and (Cl U) $- N$. Although we shall ignore the details of the epsilontics, it is to be understood that all things chosen to be small are to be small with respect to the distance between $D(B^2)$ and (Cl U) $- N$.

There is a triangulation of B^2 with 2-simplexes $B_1, \ldots, B_k$ so small that the image of each under D is very small. By (2A), (2C.2), and the homotopy extension property (cf. the proof of 2C.2.1), we may assume that the image under D of the 1-skeleton lies in U. Using the fact that U is $1 -$ *ulc* (2A), we replace each $D|B_i$ by a map $h_i : E_i \to N \cap U$ of a compact, connected 2-manifold E_i having one boundary component Bd B_i such that $h_i(E_i)$ is a very small subset of $N \cap U$ and $h_i|\text{Bd } E_i = D|\text{Bd } B_i$. We require that $E_1, \ldots, E_k$ have disjoint interiors so that $E = \bigcup_{i=1}^k E_i$ is a compact, connected 2-manifold with one boundary component $S^1 = \text{Bd } B^2$. We therefore see that $h = \bigcup_{i=1}^k h_i : E \to N \cap U$ defines a singular 2-manifold in $N \cap U$. This is the singular 2-manifold promised in the outline given in the first paragraph of the proof. It remains only to show that h is contractible in N.

To this end, let $r_i : E_i \to B_i$ be a map which fixes $\operatorname{Bd} E_i = \operatorname{Bd} B_i$. Define $r : E \to B^2$ by $r = \bigcup r_i$. Then if appropriate care has been taken, the maps $Dr : E \to N$ and $h : E \to N$ will be homotopic in N (by (2C.1)). But D is contractible in N, hence also are Dr and h. This completes the proof.

We now complete the proof of (2C.6). We may assume that S' is a round 2-sphere in E^3. Suppose $\epsilon > 0$ given. For each $p \in X$, let S_p be a round 2-sphere in E^3 which contains p in its interior, has diameter less than ϵ, and intersects S' in a single simple closed curve. Let D_p be a disk in $\operatorname{Int} S_p$ such that

$$p \in \operatorname{Int} D_p \subset D_p \subset S.$$

Let U_p be a spherical neighborhood of p in $\operatorname{Int} S_p$ such that $U_p \cap S \subset \operatorname{Int} D_p$. Let δ, $0 < \delta < \epsilon$, be so small that any δ-subset of E^3 which intersects X lies in some U_p. We prove now that any δ-loop $J : S^1 \to E^3 - S$ bounds a singular ϵ-disk in $E^3 - X$.

If the convex hull of $J(S^1)$ misses X, then J certainly bounds a singular ϵ-disk in $E^3 - X$. Otherwise $J(S^1)$ lies in some U_p, $p \in X$. Then J bounds a singular disk D in U_p and $D(B^2) \cap S \subset D_p$. By (2C.5.1), $D(B^2)$ may be cut off on D_p in such a manner that the singular disk $E(B^2)$ thus obtained lies in $D(B^2) \cup D_p$ and does not intersect both U and V, say $E(B^2) \subset \operatorname{Cl} U$. By the ω-theorem (2C.6.5), J represents an element of $\omega(N)$, where N is the component of $U \cap \operatorname{Int} S_p$ whose closure contains $E(B^2)$. By the homomorphism lemma (2C.6.2), J represents an element of $\omega(\operatorname{Int} S_p - X)$ since $N \subset \operatorname{Int} S_p - X$ and inclusions of spaces induce homomorphisms on fundamental groups. But $\pi_1(\operatorname{Int} S_p - X)$ is clearly a free group since S_p and S' are round 2-spheres. Thus J represents the trivial element of $\pi_1(\operatorname{Int} S_p - X)$ by (2C.6.3), and therefore J shrinks in the ϵ-subset $\operatorname{Int} S_p - X$ of $E^3 - X$. This completes the proof of (2C.6).

2C.6.6. Corollary. *If X is a compact subset of S which has no degenerate components and if X lies on a tame 2-sphere in E^3, then* $(\operatorname{Cl} U) - X$ *and* $(\operatorname{Cl} V) - X$ *are* $1 - ULC$.

Proof. By (2C.6), U is $1 - ULC$ in $E^3 - X$. By (2C.5), $(\operatorname{Cl} U) - X$ is $1 - ULC$. Similarly $(\operatorname{Cl} V) - X$ is $1 - ULC$.

Proof of (2C.7.(1)). This is a consequence of (2A) and (2C.3).

Proof of (2C.7.(2)) *and the inclusion* $F \subset S - \bigcup_{i=1}^\infty X_i$ *of Addendum* (2′) *to* (2C.7). It is easy, using (2C.6), to find a sequence $H_1, H_2, \ldots$ of compact subsets of S, each satisfying the requirement that $(\operatorname{Cl} U) - H_i$ and $(\operatorname{Cl} V) - H_i$ be $1 - ULC$, such that $S - \bigcup_{i=1}^\infty H_i$ is totally disconnected (a 0-dimensional G_δ-set). One simply intersects small 2-spheres, that are tame, with S and takes small continua from the intersection for the H_i's. We may also assume that the X_i's appear among the H_j's. By the countable intersection theorem (2C.4) (take $C_i = (\operatorname{Cl} U) - H_i$), the sets $(\operatorname{Cl} U) - \bigcup_{i=1}^\infty H_i$ and $(\operatorname{Cl} V) - \bigcup_{i=1}^\infty H_i$ are $1 - ULC$. Let $J_1, J_2, \ldots$ be an enumeration of all

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polygonal simple closed curves in $E^3 - S$ having vertices with rational coordinates. Suppose for concreteness that $J_i \subset U$. Let $D_i : B^2 \to (\mathrm{Cl}\ U) - \bigcup_{i=1}^{\infty} H_i$ be a singular disk bounded by J_i such that

$$\mathrm{Diam}\, D_i(B^2) \leqq 2 \cdot \inf\{\mathrm{Diam}\, D(B^2) | D : B^2 \to (\mathrm{Cl}\ U) - \bigcup_{i=1}^{\infty} H_i, \mathrm{Bd}\, D = J_i\}.$$

Let $F_i = D_i(B^2) \cap S$ and $F = \bigcup_{i=1}^{\infty} F_i$. We suffice ourselves with showing that $U \cup F$ is $1 - ULC$. By (2C.5), we need only show that U is $1 - ULC$ in $U \cup F$.

Suppose $\epsilon > 0$ given. Choose $\delta > 0$ such that δ-loops in U bound singular $\epsilon/2$-disks in $(\mathrm{Cl}\ U) - \bigcup_{i=1}^{\infty} H_i$. Let $J : B^2 \to U$ be a δ-loop. We may assume after a slight homotopy that $J = J_i$ for some i. Then J_i bounds an $\epsilon/2$-disk in $(\mathrm{Cl}\ U) - \bigcup_{i=1}^{\infty} H_i$. Hence $\mathrm{Diam}\, D_i(B^2) < \epsilon$. Thus J can be shrunk in an ϵ-subset of $U \cup F_i \subset U \cup F$. This completes the proof.

The usefulness of (2C.7) depends in large measure on the following lemmas.

2C.7(2).1. *Suppose G is a compact 1-dimensional set in S, $f : G \to [0, \infty)$ is a real-valued continuous function, and F is a 0-dimensional F_σ-set in S. Then there is a homeomorphism $h : S \to S$ such that*

$$\rho(x, h(x)) \leqq f(x) \qquad (x \in G), \quad \textit{and}$$
$$h(x) \notin F \qquad (x \in G, f(x) > 0).$$

Proof. We leave the proof as an exercise.

2C.7(2).2. *If G is any graph in S and $\epsilon > 0$, then there is an ϵ-homeomorphism $h : S \to S$ such that* $(\mathrm{Cl}\ U) - h(G)$ *and* $(\mathrm{Cl}\ V) - h(G)$ *are* $1 - ULC$ *sets.*

Proof. This is an immediate consequence of (2C.7(2).1), (2C.7) Addendum (2′), and (2C.3).

Proof of (2C.7(3)), *the inclusions $F \subset G \subset S - \bigcup_{i=1}^{\infty} X_i$ and of* (3′) *from the Addendum.*

Proof of (3): This follows in two steps.

2C.7(3).1. *If M is a closed, connected, unbounded subset of E^3, then $E^3 - M$ is $2 - ULC$.*

Proof. Let $f : S^2 \to E^3 - M$ be a singular 2-sphere of diameter less than ϵ. Then there is a 3-cell B in E^3 of diameter less than ϵ such that $f(S^2) \subset \mathrm{Int}\, B$. Since $M \cup (E^3 - \mathrm{Int}\, B)$ is closed and connected, $\pi_2[(E^3 - M) \cap \mathrm{Int}\, B] = \pi_2\{E^3 - [M \cup (E^3 - \mathrm{Int}\, B)]\} = 0$ by the Sphere Theorem [**31**]. Hence f is nullhomotopic in the ϵ-set $(E^3 - M) \cap \mathrm{Int}\, B$.

2C.7(3).2. *If C is a crumpled cube in E^3 and M is an open subset of C which contains* Int C, *then M is $2 - ULC$.*

Proof. Let $f : S^2 \to M$ be a singular 2-sphere of diameter less than ϵ. Let $\delta = (1/3) \cdot [\epsilon - \mathrm{Diam}\, f(S^2)]$. Although M need not be open in E^3, it is nevertheless an ANR; thus, for some open set W of E^3 which contains M, there is a δ-retraction $r : W \to M$. By (2B.4.1), we may assume that $E^3 - W$ is connected. Hence by (2C.7(3).1) and its proof, f is nullhomotopic in a subset of W of diameter less than Diam $f(S^2) + \delta/3$. Let $f^* : B^3 \to W$ be a singular 3-cell in W that is bounded by f and has diameter less than ϵ-$(2\delta/3)$. Then $r : f^*(B^3) \to M$ is a singular 3-cell in M bounded by f that has diameter less than ϵ.

We now complete the proof of 2C.7(3) and its addendum. Choose a sequence sequence $H_1, H_2, \ldots$ of compact subsets of S, exactly as in the proof of 2C.7(2). Then $(\text{Cl } U) - H_i$ is $0 - ULC$ by (2A) and (2C.3), $(\text{Cl } U) - H_i$ is $1 - ULC$ by choice of H_i, and $(\text{Cl } U) - H_i$ is $2 - ULC$ by (2C.7(3).2). Thus $(\text{Cl } U) - H_i$ is ULC^2 by definition. Hence $\text{Cl } U - \bigcup H_i$ is ULC^2 by (2C.4). Let $G = S - \bigcup H_i$. The inclusions $F \subset G \subset S - \bigcup X_i$ are then obvious. This completes the proof.

3. Taming arcs and finite graphs in E^3. Our main result is the following:

3.1. *An arc A in E^3 is tame if it has a singular regular neighborhood. The taming homeomorphism may be chosen to be locally PL except at the points of A.*

A connected finite graph G in E^3 is said to have a singular regular neighborhood in E^3 if there is a polygonal finite graph G' in E^3, a regular neighborhood P of G' in E^3, and a map $f : P \to E^3$ which takes G' homeomorphically onto G, takes $P - G'$ into $E^3 - G$, and has nonzero degree $\deg(f, G)$ with respect to G.

We define $\deg(f, G)$ as follows. Let A be any arc in G. Let B denote a path in $E^3 - \text{Int } A$ which joins the endpoints of A. Let D be a polyhedral disk in P transverse to $f^{-1}(A)$ whose image under f misses B. Then $\deg(f, G) = |L(B \cup A, f|\text{Bd } D)|$. Note that $\deg(f, G)$ depends only on f and G and not on the choice of A, B, and D (cf. (1.1) and (1.3)).

If G is not connected, we require that $\deg(f, G_0)$ be nonzero for each component G_0 of G.

Our methods are capable of proving more than (3.1) (as we show in (3.10), (3.16), (3.22), and (3.28)), but (3.1) seems to be the optimum result as far as simplicity of statement and proof are concerned. In the latter part of this section we generalize (3.1) to finite graphs and prove that $1 - ALG$ arcs are tame and that an arc is tame if it satisfies singular analogues of Harrold's local unknottedness and local peripheral unknottedness condition [**22**]. We have chosen to isolate the singular regular neighborhood condition for arcs for first consideration partly because it is geometrically the most intuitive of the conditions but mainly because it leads most directly to the following two lemmas, the only two results from this section which we need in our proof of the Side Approximation Theorem [**14**].

3.2. *If S is a 2-sphere in E^3 and A is an arc in S such that $(S \cup \text{Int } S) - A$ and $(S \cup \text{Ext } S) - A$ are $1 - ULC$ (cf. (2C.7(2).2)), then A has a singular regular neighborhood.*

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3.3. *If an arc A in E^3 has a singular regular neighborhood and $p \in \operatorname{Int} A$, then A pierces an almost-polyhedral disk at p.* (A 2-manifold is said to be almost-polyhedral if it is polyhedral except at finitely many points.)

In summary, although (3.1) is our main result, we need only (3.2) and (3.3) for use in [**14**]. We therefore prove these latter results first: first (3.2), then a trio of lemmas (3.4)-(3.6) culminating in a proof of (3.3), then another pair of lemmas (3.7)-(3.8) culminating in a proof of (3.1).

Proof of (3.2). This is an immediate consequence of the *ULC* mapping theorems of Section 2. Note first that $(S \cup \operatorname{Int} S) - A$ and $(S \cup \operatorname{Ext} S) - A$ are 0- and 2-*ULC* by (2A), (2C.3), and (2C.7(3).2). Let D be a disk in S such that $A \subset \operatorname{Int} D$. Let A' be the straight line segment in E^3 joining $(0, 0, 0)$ and $(1, 0, 0)$. Let D' be a closed rectangular neighborhood of A' in the xy-plane $\{(x, y, z) | z = 0\}$. Let $h : D' \to D$ be a homeomorphism which takes A' onto A. Let $P = \{(x, y, z) | (x, y, 0) \in D', \ z \in [-1, 1]\}$, $P^+ = \{(x, y, z) \in P | z > 0\}$, and $P^- = \{(x, y, z) \in P | z < 0\}$. Finally, let $pr : P \to D'$ be the orthogonal projection defined by $(x, y, z) \to (x, y, 0)$. The composite $h \cdot pr$ takes P onto D. By (2C.2), there is a map $f : P \to E^3$ such that $f \mid D' = h \cdot pr | D' = h$ and such that $f(P^-) \subset (S \cup \operatorname{Int} S) - A$ and $f(P^+) \subset (S \cup \operatorname{Ext} S) - A$. One sees immediately that f is a singular regular neighborhood of A with $\deg(f, A) = 1$ (cf. (1.1)-(1.6)).

We now proceed to the lemmas needed in the proof of (3.3). We assume in this section until (3.9) that A is an arc in E^3 with a singular regular neighborhood $f : P \to E^3$. We denote the arc $f^{-1}(A)$ by A' and assume for convenience of description that A' is a straight line segment. We adjust f so that it is piecewise linear at each point of $P - A'$. We choose a simple closed curve J in E^3 such that $A \subset J$. We choose linear orderings on A and A' compatible with the topology and preserved by f. If $x \in A$, then we write $x' = f^{-1}(x)$. If $x < y$ in A, then we use the interval notation $[x, y]$ and $[x', y']$ for the arcs in A and A', respectively, joining x to y and x' to y'.

3.4. *If $x_1 < x_2$ in A and $\epsilon > 0$, then there is a polyhedral annulus B in $N([x_1, x_2], \epsilon) - J$ such that $|L(B, J)| = 1$ and such that the boundary components J_1 and J_2 of B lie in $N(x_1, \epsilon)$ and $N(x_2, \epsilon)$, respectively.*

Proof of (3.4). The idea is to choose small polyhedral 2-spheres about x_1 and x_2, span a singular annulus between these 2-spheres, then use Waldhausen's form of the Loop Theorem [**39**] to change the singular annulus into a real annulus. The principal difficulty is that of spanning the singular annulus between the 2-spheres. The details are as follows.

We may assume that $x_1, x_2 \in \operatorname{Int} A$, that $N(x_1, \epsilon) \cap N(x_2, \epsilon) = \phi$, and that $N([x_1, x_2], \epsilon) \cap J \subset \operatorname{Int} A$. We choose points $t_1, t_2 \in A$ such that $x_1 < t_1 < t_2 < x_2$. We choose polyhedral 2-spheres R_1 and R_2 about x_1 and x_2, respectively, such that R_i separates x_i from t_i in E^3. We require that

(1) $R_1 \cup \operatorname{Int} R_1 \cup [x_1, t_1] \subset N(x_1, \epsilon)$,
(2) $R_2 \cup \operatorname{Int} R_2 \cup [t_2, x_2] \subset N(x_2, \epsilon)$, and
(3) $R_1 \cap [t_1, x_2] = R_2 \cap [x_1, t_2] = \phi$.

For each i ($i = 1, 2$), there is a polygonal simple closed curve K_i in $R_i - J$ which separates $[x_1, x_2] \cap R_i$ from $(J - [x_1, x_2]) \cap R_i$ in R_i. Note that $|L(J, K_i)| = 1$. Let D_i and E_i be the two disks in R_i bounded by K_i, with notation chosen so that $E_i \cap [x_1, x_2| = \phi$.

Consider $U_i = f^{-1}[N(x_i, \epsilon) - E_i]$. Then there is a polyhedral 2-sphere S_i in U_i whose intersection with A' is the two-point set $\{x_i', t_i'\}$ and whose interior lies in U_i and intersects A' precisely in the open arc between x_i' and t_i'. (Since we are assuming A' is a straight line segment, we could take S_i to be the boundary of some rectangular solid in U_i of which the appropriate segment of A' is a spanning arc.) Since $\deg(f, A) \neq 0$, any simple closed curve in S_i which separates the two points of $A' \cap S_i$ in S_i has image under f which links J.

We shall first cut D_i off near $f(S_i)$ where this can be done without introducing new intersections with J. We shall then cut $f(S_i)$ off near the new D_i where this can be done without introducing new intersections with J. We shall obtain thereby a new polyhedral 2-sphere (still denoted by R_i) and a new map $g_i : S_i \to E^3$ (an adjustment of $f\,|S_i$) such that $g_i^{-1}(R_i)$ is a union of finitely many disjoint simple closed curves in S_i, each separating x_i' from t_i' in S_i.

For notational simplicity in making these adjustments, we drop the subscript i and consider R, S, t, x, $N(x, \epsilon)$, D, E, and g without subscripts. We shall work only in $N(x, \epsilon) - (J \cup E)$, with the single exception that we may remove some intersections of D with J. We shall take care to have g agree with f in some neighborhood of $\{x', t'\}$. With these conventions in mind, we proceed to the adjustment of D.

We delete a small polyhedral (open) neighborhood N of

$$(D \cap [x_1, x_2]) \cup \operatorname{Int} E \text{ from } E^3,\ N \cap f(S) = \phi.$$

We then take a small (relative) regular neighborhood N' of the (noncompact) polyhedron $(D - N) \cup (f(S) - J)$ in $E^3 - [J \cup N]$. Then $D' = D - N$ is a union of finitely many disjoint properly embedded disks-with-holes in N'. It is an easy exercise with the Loop Theorem [**37**] to show that we may adjust D' so that $\pi_1(D') \to \pi_1(N')$ is 1-1. (If D' is not connected, then the condition is to be satisfied for each component of D'.) Indeed, if $\pi_1(D') \to \pi_1(N')$ is not 1-1, then there is, by the Loop Theorem, a disk F in $\operatorname{Int} N'$ whose intersection with D' is $\operatorname{Bd} F$. Replace with F the disk in R which is bounded by $\operatorname{Bd} F$ and which does not contain E. This also changes D'. An iteration of the procedure yields the desired property for D'. Since E is not affected and no new intersections with J are introduced, the new 2-sphere still separates x from t in E^3. Since all adjustments are made in $N(x, \epsilon)$, the appropriate condition (1) or (2) is still satisfied.

We now adjust $f(S)$ slightly near $f(S) \cap R$ so that $f(S)$ and R are in general position. This may be done by a slight adjustment of $f : P \to E^3$. Then

$S \cap f^{-1}(R)$ is a union of finitely many disjoint simple closed curves. Suppose one of these does not separate the two-point set $\{x', t'\}$ in S. Then its image under f is a loop in D' which is trivial in N'. Thus this loop is trivial in D' ($\pi_1(D') \to \pi_1(N')$ is 1-1) and this intersection of $f(S)$ and R may be removed by changing $f(S)$. An iteration of this simplifying procedure yields a map $g : S \to E^3$ such that $g^{-1}(R)$ is a union of finitely many disjoint simple closed curves in S, each of which separates $f^{-1}(x)$ from $f^{-1}(t)$ in S. (The intersection is nonempty since R separates x from t in E^3.)

We are finally in a position to describe a singular annulus which has one boundary component on R_1 (adjusted) and one on R_2 (adjusted). (We resume use of subscripts.) There is in S_i a disk F_i that contains $f^{-1}(t_i) = t_i'$ in its interior, is bounded by one of the curves in $g_i^{-1}(R_i)$, and intersects no other of those curves. Delete from each F_i a very small open subdisk G_i near to, and containing, $f^{-1}(t_i) = t_i'$. We require that G_i be so near t_i' that $g_i|G_i = f|G_i$. There is an annulus B_0 in $P - A'$ which is very near the arc $[t_1', t_2']$ and joins $\operatorname{Bd} G_1$ and $\operatorname{Bd} G_2$. We require that $f(B_0) \cap (R_1 \cup R_2) = \emptyset$ and that $f(B_0) \subset N([x_1, x_2], \epsilon)$. Then $B' = f|(B_0) \cup g_1|(F_1 - G_1) \cup g_2|(F_2 - G_2)$ is a singular annulus spanned between R_1 and R_2 in $N([x_1, x_2], \epsilon)$.

We note that $|L(B', J)| = \deg(f, A) \neq 0$. Thus Waldhausen's form of the Loop Theorem [**39**] applies to the singular annulus B' in the manifold $M = N([x_1, x_2]; \epsilon) - [J \cup \operatorname{Int} R_1 \cup \operatorname{Int} R_2]$ relative to the normal subgroup $\operatorname{kernel}[\pi_1(M) \to \pi_1(E^3 - J) \to H_1(E^3 - J)] \subset \pi_1(M)$. We conclude that there is a nonsingular polyhedral annulus B in M such that $L(B, J) \neq 0$ and such that one boundary component of B lies in R_1, the other in R_2.

It remains only to show that $|L(B, J)| = 1$ if ϵ is sufficiently small. Let p be the image under f of the midpoint of $f^{-1}[x_1, x_2]$. Exactly as we found the disks D_i and E_i early in this proof, we find a small polyhedral disk D near p such that $(\operatorname{Bd} D) \cap J = \emptyset$ and $|L(J, \operatorname{Bd} D)| = 1$. If ϵ is small enough, then B must separate $\operatorname{Bd} D$ from J in D. (Otherwise, one could construct a simple closed curve in $E^3 - B$ that links one boundary component of B and not the other.) We put D in general position with respect to B and find that $\operatorname{Bd} D$ is homologous in $D - J$ to a family of simple closed curves in B. Hence $1 = |L(D, J)| = m \cdot |L(B, J)|$ for some positive integer m. We conclude that $|L(B, J)| = 1$, as desired. This completes the proof of (3.4).

We need some descriptive apparatus before we proceed to our next lemma. For $A \subset E^3$ and $t > 0$, let $tA = \{(tx, ty, tz) | (x, y, z) \in A\}$.

Definition. A Dehn-annulus in E^3 is a PL map $f : B^2 - \operatorname{Int}(\frac{1}{2}B^2) \to E^3$ such that $S(f) \cap (S^1 \cup \frac{1}{2}S^1) = \emptyset$. (Recall that $S(f) = \operatorname{Cl}\{x | f^{-1}f(x) \neq x\}$.) Then $f | S^1 \cup f | \frac{1}{2}S^1$ is called the boundary of f (denoted $\operatorname{Bd} f$), and $\operatorname{Bd} f$ is said to bound f. We write $|\operatorname{Bd} f| = f(S^1 \cup \frac{1}{2}S^1)$. A half-open Dehn-annulus in E^3 is a locally PL map $f : B^2 - \{0\} \to E^3$ such that

(1) $S(f) \cap (1/i)S^1 = \emptyset$ ($i = 1, 2, \ldots$), and

(2) $\limsup_{n \to \infty} f((1/i)B^2 - \{0\})$ is a compact subset of $E^3 - f(B^2 - \{0\})$.

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The restriction $f \mid S^1$ is called the boundary of f (denoted $\mathrm{Bd}\, f$) and is said to bound f. We write $|\mathrm{Bd}\, f| = f(S^1)$.

We recall [**35**] (alternately, use [**9**, Addendum to 4.5.1]):

3.5. *If f is a Dehn-annulus in E^3 and U is a neighborhood in E^3 of $|f| - |Bd\, f|$, then* Bd *f bounds a real polyhedral annulus in* $|\mathrm{Bd}\, f| \cup U$.

An immediate application of (3.5) is the following.

3.6. *If f is a half-open Dehn-annulus in E^3, U is a neighborhood in E^3 of $|f| - |\mathrm{Bd}\, f|$, and f is not contractible in $f(S^1) \cup U$, then* Bd *f bounds a nonsingular, locally-polyhedral, half-open annulus g in $f(S^1) \cup U$ such that*

$$\limsup_{n\to\infty} g((1/i)B^2 - \{0\}) \subset \limsup_{n\to\infty} f((1/i)B^2 - \{0\}).$$

Proof of (3.6). Define $A_i = (1/i)B^2 - \mathrm{Int}((1/i + 1)B^2)$. It follows from condition (2) in the definition of half-open Dehn-annulus that, by consolidating several consecutive $f(A_i)$'s into a single new $f(A_i)$ and adjusting f accordingly, we may assume that $f(A_i) \cap f(A_j) = \emptyset$ for $|i - j| > 1$. By using (3.5) (which applies because of condition (1) in the definition of half-open Dehn-annulus) infinitely many times, we may assume that $f|A_i$ is a PL embedding for each i. By a general position argument, we may assume that $f(A_i) \cap f(A_{i+1})$ is the disjoint union of the simple closed curve $f((1/i + 1)S^1)$ and finitely many other simple closed curves in $f(\mathrm{Int}\, A_i) \cap f(\mathrm{Int}\, A_{i+1})$. Since f is not contractible in $f(S^1) \cup U$, it follows that a curve J of the intersection between $f(A_i)$ and $f(A_{i+1})$ is contractible in one of the two annuli if and only if it is contractible in the other. Thus we have a well-defined notion of trivial and nontrivial intersections. Changing each annulus $f(A_{2i})$ with even subscript, but changing it only near $f(A_{2i-1}) \cup f(A_{2i}) \cup f(A_{2i+1})$, we may remove all trivial intersections by cut and paste. We maintain the requirement that $f(A_i) \cap f(A_j) = \emptyset$ if $|i - j| > 1$. We now piece together a real locally-polyhedral half-open annulus B. Let B_2 be an annulus in $f(A_2)$ which is bounded by one curve of intersection with $f(A_1)$ and one with $f(A_3)$ but which otherwise misses $f(A_j)$ $(j \neq 2)$. Define similarly $B_4, B_6, \ldots$. Let B_1 be the annulus in $f(A_1)$ which is bounded by $B_2 \cap f(A_1)$ and $f(S^1)$. In general, define B_{2i+1} to be the annulus in $f(A_{2i+1})$ which is bounded by

$$[f(A_{2i+1}) \cap B_{2i}] \cup [f(A_{2i+1}) \cap B_{2i+2}].$$

Then $B = \bigcup_{i=1}^{\infty} B_i$ is the desired half-open annulus. This completes the proof of (3.6).

Proof of (3.3). The idea is to use (3.4) to obtain a null sequence of polyhedral annuli running along A and converging to p, to piece these annuli together to form a half-open Dehn-annulus in E^3, and then to apply (3.6) to obtain a nonsingular, locally-polyhedral, half-open annulus which, together with $\{p\}$, forms an almost-polyhedral disk pierced by A at p. The details are as follows.

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We choose a sequence $q_1, q_2, \ldots$ of points in A converging monotonically to $p(q_1 < q_2 < \ldots)$. We choose polyhedral disks $D_1, D_2, \ldots$ in P transverse to A' at $q_1', q_2', \ldots$ such that $f(D_1), f(D_2), \ldots$ is a null sequence of disjoint singular disks in $(E^3 - J) \cup A$. These singular disks will be used to piece together the annuli mentioned in the previous paragraph.

We choose the aforementioned annuli as follows. We first pick points $p_1, r_1, p_2, r_2, \ldots$ in A such that $p_1 < q_1 < r_1 < p_2 < q_2 < r_2 < \ldots$. By (3.4), there are polyhedral annuli $B_i (i = 1, 2, \ldots)$ in $N([p_i, r_{i+1}], 1/i) - J$, respectively, such that $|L(B_i, J)| = 1$ and such that the two boundary components of B_i lie in $N(p_i, 1/i)$ and $N(r_{i+1}, 1/i)$, respectively. By choosing these annuli iteratively and by putting even more stringent conditions on how close they are to be to $[p_i, r_{i+1}]$ and their ends to p_i and r_{i+1}, we may require that the B_i be pairwise disjoint, that $f(D_i)$ intersect only $B_{i-1} \cup B_i$ among the B_k, and (after adjustment for general position) that $f^{-1}(B_{i-1}) \cap D_i$ and $f^{-1}(B_i) \cap D_i$ each contain a simple closed curve which separates $f^{-1}(q_i) = q_i'$ from Bd D_i in D_i (cf. the last paragraph of the proof of (3.4)).

We now piece the annuli together. By the restrictions of the preceding paragraph, there is an annulus A_i in $D_i - \{q_i'\}$ which separates Bd D_i from q_i' in D_i, has one boundary component in $f^{-1}(B_{i-1})$, one in $f^{-1}(B_i)$. We require that A_i be minimal with respect to this property so that if K is a component of $(\text{Int } A_i) \cap f^{-1}(B_{i-1} \cap B_i)$, then K bounds a disk in A_i. For such a curve K, $f \mid K : K \to B_{i-1} \cup B_i$ is trivial since B_{i-1} and B_i link J while $f \mid K$ does not. We remove these trivial intersections of A_i with $B_{i-1} \cup B_i$ by standard cut and paste techniques: redefine f on the disk bounded by K in A_i so as to take this disk into $B_{i-1} \cup B_i$; then push $f(A_i)$ slightly to one side of $B_{i-1} \cup B_i$ near this new singular disk on $B_{i-1} \cup B_i$ which is bounded by $f \mid K$.

This puts us in a position of being able to apply Waldhausen's Loop Theorem once more. It follows that, in an arbitrary neighborhood of the adjusted $f(A_i)$, there is a nonsingular annulus C_i, one boundary component a subset of and nontrivial in B_{i-1}, the other a subset of and nontrivial in B_i, $C_i \cap (\bigcup B_k) =$ Bd C_i. We note that some C_i may intersect some C_j $(i \neq j)$, but this intersection must be a subset of Int $C_i \cup$ Int $C_j \subset E^3 - (\bigcup B_k)$. Let B_i' be the annulus in B_i joining C_i and C_{i+1}. Then

$$B_0 = B_1' \cup C_2 \cup B_2' \cup C_3 \cup B_3' \cup C_3 \cup \ldots$$

is a half-open Dehn-annulus in E^3 (nonsingular on $B_1' \cup B_2' \cup \ldots$). Note that $\{p\} = \lim_{i \to \infty} (C_i \cup B_i' \cup C_{i+1} \cup B_{i+1}' \cup \ldots)$.

We apply (3.6) to find a real, locally-polyhedral, half-open annulus $g : B^2 - \{0\} \to E^3 - J$ such that $\lim \sup_{i \to \infty} g((1/i)B^2 - \{0\}) = \{p\}$. Then

$$g(B^2 - \{0\}) \cup \{p\}$$

is the desired disk. This completes the proof of (3.3).

3.7. *If* $p \in$ Bd A, $q \in$ Int A, *and* $\epsilon > 0$, *then there is a* 2-*sphere* S *in* E^3 *which*

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separates the endpoints of A *in* E^3, *intersects* A *precisely at* q, *is locally polyhedral except at* q, *and lies in an* ϵ*-neighborhood of the arc in* A *from* p *to* q.

Proof of (3.7). For convenience, we assume $p < q$. Let $r \in \operatorname{Int} A$, $p < q < r$, be such that $[p, r] \subset N([p, q], \epsilon)$. Let s be the other endpoint of A. By (3.3), there is a disk D in $N([p, q], \epsilon)$ which intersects J only at q, is pierced there by J, and is locally polyhedral except at q. In the open subset $f^{-1}(N[p, q], \epsilon))$ of P there is a polyhedral 2-sphere S' such that $S' \cap A' = \{r'\}$ and S' separates p' from s' in E^3. By putting a possibly more stringent condition on how close $f(S')$ is to be to $[p, r]$ and by adjusting $f(S^2)$ and D for general position, we may require further that $f(S') \cap D \subset \operatorname{Int} D$ and that, for at least one component K of $f^{-1}(D) \cap S'$, $f|K$ is a loop in D which is not nullhomotopic in $D - \{q\}$. For otherwise one could prove that $\deg(f, A) = 0$. We shall piece together the desired 2-sphere S from D and $f(S^2)$.

Let K be a component of $f^{-1}(D) \cap S'$ such that $f|K : K \to D - \{q\}$ is not nullhomotopic in $D - \{q\}$. Among all such components, we assume K chosen so that the disk E bounded by K in $S' - \{r'\}$ contains no other such component. If $f(\operatorname{Int} E) \cap D$ is nevertheless nonempty, we can remove such intersections by cut and paste: indeed, let K' be a component of $f^{-1}(D) \cap \operatorname{Int} E$; then $f|K'$ is a loop in $D - \{q\}$ that is nullhomotopic in $D - \{q\}$; if E' is the disk in E bounded by K', then $f|E'$ may be redefined to take E' into $D - \{q\}$; this intersection with D may then be removed by pushing the new singular disk $f(E')$ to one side of D. That is, after a cut-and-paste adjustment, we may assume that $f^{-1}(D) \cap E = \operatorname{Bd} E = K$. Thus the Loop Theorem [**37**] is applicable.

By the Loop Theorem [**37**], there is a disk F in an arbitrary neighborhood of $f(E)$ such that $F \cap D = \operatorname{Bd} F \subset \operatorname{Int} D - \{q\}$ and such that $\operatorname{Bd} F$ is not nullhomotopic in $D - \{q\}$. Then S may be taken as the union of F and the disk in D bounded by $\operatorname{Bd} F$.

3.8. *If* $p \in \operatorname{Int} A$, *then there are a singular disk* $f : B^2 \to E^3$ *and a subarc* B *of* A *with* $p \in \operatorname{Int} B$ *such that* f *takes a subarc* B' *of* $\operatorname{Bd} B^2$ *onto* B, *takes the endpoints of* B' *onto the endpoints of* B, *and takes* $B^2 - B'$ *into* $E^3 - A$.

Proof of (3.8). This is an immediate consequence, of the fact that A has a singular regular neighborhood.

Proof of (3.1). We have established that an arc A in E^3 has which has a singular regular neighborhood satisfies the conclusions of (3.3), (3.4), (3.7), and (3.8). Thus (3.1) will be complete when we have established the following.

3.9. *An arc* A *in* E^3 (*not assumed to have a singular regular neighborhood*) *is tame if it satisfies the conclusions of* (3.3), (3.4), (3.7), *and* (3.8). *The taming homeomorphism may be chosen to be locally PL except at the points of* A.

Proof. Result (3.9) is almost the theorem that an arc is tame if it is locally peripherally unknotted (l.p.u.) and locally unknotted (l.u.) [**22**, Theorem 7].

(Indeed, (3.3), (3.4), and (3.7) imply that A is l.p.u. and (3.8) is a singular substitute for the l.u. property.) For this reason, and since (3.9) is not needed in [**14**], we omit a few details.

Step 1. Definition of the compact set F. Let A' be the straight line segment in E^3 joining $(0, 0, 0)$ and $(1, 0, 0)$. For each positive integer i, let $S_i = \mathrm{Bd}\, N(A', 1/i)$. Let $C = \mathrm{Cl}\, N(A', 1) = S_1 \cup \mathrm{Int}\, S_1$. For each positive integer j and odd positive $k < 2^j$, let $D(k|j)$ be the circular disk of radius $1/j$ perpendicular to A' and centered at $(k/2^j, 0, 0)$. Let F denote the compact set $A' \cup \bigcup S_i \cup \bigcup D(k|j)$. Note that the closure of a component K of $C - F$ is either a 3-cell or a solid torus of which K is the interior. Note also that $\mathrm{Bd}\, K \subset F - A'$ and that $F - \mathrm{Bd}\, K$ is connected and intersects S_1.

Step 2. Mapping F into E^3. We let J be a simple closed curve in E^3 which contains A. It is a fairly standard (though lengthy) exercise in general-position arguments and cut-and-paste to use (3.3), (3.4), and (3.7) to construct a homeomorphism h from F into E^3 which takes A' homeomorphically onto A, takes each $D(k|j)$ to a disk in $(E^3 - J) \cup A$ which is pierced by J and is locally polyhedral modulo J, and takes each S_i to a polyhedral 2-sphere which contains A in its interior. If K is a component of $C - F$, then

$$[\mathrm{Int}\, h(\mathrm{Bd}\, K)] \cap h(F) = \emptyset$$

by the remarks of the previous paragraph on $\mathrm{Bd}\, K$ and the requirement that $A \subset \mathrm{Int}\, h(S_i)$. We write K^* for $\mathrm{Int}\, h(\mathrm{Bd}\, K)$. Then $\mathrm{Cl}\, K^*$ is either a 3-cell or a cube with a (possibly knotted) hole. We would like to extend $h|\mathrm{Bd}\, K$ to take K into K^*. However, if one does not exercise more care than we have done, this need not be possible. Therefore, we first normalize h, throw away portions of certain of the disks $D(k|j)$ near the ends of A', and finally show that the adjusted and restricted h thus obtained does extend to a homeomorphism from E^3 to E^3.

Step 3. Adjusting h. Let π be the half-plane $\{(x, y, z) \in E^3 | y \geqq 0$ and $z = 0\}$. Suppose K is a component of $C - F$ such that $\mathrm{Cl}\, K$ is a solid torus. Then $\mathrm{Bd}\, K$ intersects, say, S_i and S_{i+1}. Let $J(K)$ be the simple closed curve $\pi \cap \mathrm{Bd}\, K$. We redefine h on the boundary of each such K so that $h|J(K)$ does not link J. We work only in $\bigcup S_j - \bigcup D(k|j)$, work first on S_2, then on S_3, and so on. Suppose therefore inductively that h has already been redefined on the annulus $S_i \cap \mathrm{Bd}\, K$. Redefine h on the annulus $S_{i+1} \cap \mathrm{Bd}\, K$ without changing the image of $S_{i+1} \cap \mathrm{Bd}\, K$ so that any twisting about J done by the arc $h[J(K) - S_{i+1}]$ is precisely undone by the twisting of the arc $h[J(K) \cap S_{i+1}]$ about J. This is possible since $|L(h(S_{i+1} \cap \mathrm{Bd}\, K), J)| = 1$. Geometrically, this change may be accomplished in the following way: cut $h[S_{i+1} \cap \mathrm{Bd}\, K]$ apart along a centerline; twist one of the free boundaries produced by the cut the appropriate number of full revolutions around J, and sew the two free boundary curves back together once again.

Step 4. Throwing away certain portions of $\bigcup D(k|j)$. Let C_1 and C_2 be the two halves into which $D(1|1)$ separates C. Let $\mathscr{K}_i$ $(i = 1, 2)$ be the collection of components of $C_i - F$ whose closures are solid tori. If K, $K_0 \in \mathscr{K}_i$, define $K_0 \geqq K$ if the following two conditions are satisfied:

(1) $\rho(K_0, A') \leqq \rho(K, A')$.

(2) $\sup\{\rho(k_0, D(1|1))|k_0 \in K_0\} \leqq \sup\{\rho(k, D(1|1))|k \in K\}$.

We call an element of K of $\mathscr{K}_1 \cup \mathscr{K}_2$ good if, for each $K_0 \geqq K$ (defined only for certain K and K_0 in the same one of $\mathscr{K}_1$ and $\mathscr{K}_2$), $\mathrm{Cl}\, K_0^*$ is a solid torus (as opposed to a cube with a knotted hole). If K is not good, we delete from F the interior of the annulus in $[\bigcup D(k|j)] \cap \mathrm{Bd}\, K$ which is furthest from $D(1|1)$. Let F' be the new compact set formed from F by the deletions described.

Step 5. Extending $h|F'$ to all of E^3. Let K be a component of $C - F'$. Then, just as with the components of $C - F$, $\mathrm{Cl}\, K$ is a 3-cell or solid torus of which K is the interior and $K^* = \mathrm{Int}\, h(\mathrm{Bd}\, K) \subset E^3 - h(F')$.

If $\mathrm{Cl}\, K$ is a 3-cell, then we extend $h|\mathrm{Bd}\, K$ in any fashion so as to take $\mathrm{Cl}\, K$ homeomorphically onto $\mathrm{Cl}\, K^*$ in a PL fashion. This is possible by the PL Schoenflies Theorem.

If $\mathrm{Cl}\, K$ is a solid torus, then K is a good component of $C - F$ and $\mathrm{Cl}\, K^*$ is a solid torus. But $h(J(K))$ is a meridian in $\mathrm{Cl}\, K^*$ since $h|J(K)$ does not link J. Hence $h|\mathrm{Bd}\, K$ can be extended to take $\mathrm{Cl}\, K$ homeomorphically onto $\mathrm{Cl}\, K^*$.

Clearly $h|S_1$ can also be extended to take $E^3 - \mathrm{Int}\, S_1$ onto $E^3 - \mathrm{Int}\, h(S_1)$.

It remains to be shown that the function $h : E^3 \to E^3$ defined piecewise above is actually continuous. This amounts to showing that, for components K of $C - F'$, $\mathrm{Diam}\, K \to 0$ as $\rho(K, A') \to 0$. In order to show this, it suffices to establish that, given a point $p \in \mathrm{Int}\, A'$ and a component K of $C - F$ which is sufficiently close to p, K is good. It is for this step only that we need (3.8). By an easy compactness argument, it follows from (3.8) that for K sufficiently close to p and any $K_0 \geqq K$, there is a singular disk $f : B^2 \to E^3 - J$ such that $f(\mathrm{Bd}\, B^2) \cap \mathrm{Cl}\, K_0^* = \emptyset$ and $f\,|\mathrm{Bd}\, B^2$ is homologous to J in $E^3 - \mathrm{Cl}\, K_0^*$. Since J is a homology centerline of $E^3 - K_0^*$ and $f(B^2) \cap J = \emptyset$, it is a well-known consequence of the Loop Theorem that $\mathrm{Cl}\, K_0^*$ must be a solid torus. Hence K is good. This completes the proof of (3.9) and (3.1).

We now generalize (3.1) to finite graphs.

3.10. THEOREM. *A finite graph G in E^3 is tame if it has a singular regular neighborhood.*

The proof of (3.10) occupies (3.11)-(3.14), with (3.10) being an immediate consequence of (3.13), (3.14) and the following remark: by (3.1) it suffices to consider the case where G is an *n-od* ($n \geqq 3$) with a single branch point p and G is locally polyhedral except at p. We choose a singular regular neighborhood $f : P \to E^3$ for G and require that f be locally piecewise linear except at p and that the branches of $f^{-1}(G)$ be straight line segments emanating from the origin $(0, 0, 0)$ in E^3. We choose in P a (polyhedral convex) cube C about $(0, 0, 0)$ with the endpoints of $f^{-1}(G)$ in $E^3 - C$ and choose a spherical neighborhood N of p in $E^3 - f(\text{Bd } C)$. Since $\deg(f, G) \neq 0$, it follows that $f \mid \text{Bd } C$ cannot be nullhomotopic both in $E^3 - \{p\}$ and in the complement of set of endpoints of G. Since $f \mid \text{Bd } C$ is nullhomotopic in the complement of the set of endpoints of G, it is not nullhomotopic in the complement of any point of N.

3.11. $\deg(f, G) = 1$.

Proof of (3.11). Fact (3.11) depends only on the fact that $n \geqq 3$. We delete all but three branches of G so that G is a triod.

Consider congruent solid cylinders C_1 and C_2 having the same straight line L as axis. Let D_1 and D_2 be the end disks of C_1; let E_1 and E_2 be the end disks of C_2 corresponding respectively to D_1 and D_2 under the translation of E^3 along L which takes C_1 onto C_2. Let $R_i = L \cap C_i$ $(i = 1, 2)$.

It follows from (3.1) that we may assume that R_2 is a maximal arc in G, that $C_1 \subset P_1$ and that $f \mid R_1$ coincides with the restriction to R_1 of the aforementioned translation which takes R_1 onto R_2.

We may also assume that $f^{-1}[f(C_1 \cap \text{Bd } C_2)] = D_1 \cup D_2$ and that $f \mid D_i$ $(i = 1, 2)$ is a cyclic covering of E_i branched over the point $E_i \cap L$ and (necessarily) of degree $\deg(f, G)$.

We set $B = \text{Cl}(G - R_2)$ and note that we may assume that $C_1 \cap f^{-1}(B)$ is a straight line segment perpendicular to L from the center point of R_1 to a point of Bd C_1 and that $C_2 \cap B$ is an arc missing $E_1 \cup E_2$ and irreducibly joining L to Bd C_2. (Incidentally, note that $(f \mid C_1)(f^{-1}(B)) \neq B$. Also note if $C_2 \cap B$ were not an arc irreducibly joining L to Bd C_2, one could simply throw away part of B.)

Let C_3 be a third congruent solid cylinder with axis L. Let $g : C_3 \to C_2$ be a cyclic covering of C_2, branched over R_2, of degree $\deg(f, G)$. Then f factors through g, say by $f^* : C_1 \to C_3$, by the lifting criterion for coverings. A simple argument shows that $f^*(\text{Bd } C_1)$ must intersect each preimage of Int B in C_3, hence that $f^{-1}(B) \cap C_1$ contains at least $\deg(f, G)$ points. Since $f^{-1}(B) \cap \text{Bd } C_1$ contains exactly one point and $\deg(f, G)$ is positive by hypothesis, we conclude that $\deg(f, G) = 1$.

3.12. *We may assume that f is a piecewise linear homeomorphism on the inverse image of some neighborhood of $G - \{p\}$.*

Proof of (3.12). This is an immediate consequence of (3.11) and the fact that G is already polyhedral except at p.

3.13. *The point p has arbitrarily small neighborhoods in E^3 bounded by polyhedral 2-spheres that intersect G transversely and in precisely n points.*

Proof of (3.13). Let S be a polyhedral 2-sphere in $f^{-1}(N) \cap \text{Int } C$ that bounds a small neighborhood of $(0, 0, 0) = f^{-1}(p)$ in E^3 and intersects $f^{-1}(G)$ transversely and in precisely n points. We may assume, by (3.12), that $f(S)$ has no singularities near $f(S) \cap G$. An application of Dehn's Lemma **[35]**, or [**9**, Addendum to 4.5.1] yields a polyhedral 2-sphere S' in N which coincides with $f(S)$ near $f(S) \cap G$. This completes the proof since N can be chosen arbitrarily small.

3.14. *Suppose R and S are disjoint polyhedral 2-spheres in N, each meeting G transversely and in precisely n points, and suppose that, for some $t > 0$,*

$p \in (\operatorname{Int} S) \subset (R \cup \operatorname{Int} R) \subset \{f(tC) - f[\operatorname{Bd}(tC)]\} \subset f(tC) \subset N$.
(Recall that $tC = \{(tx, ty, tz) | (x, y, z) \in C\}$.)

Then for each homeomorphism $h : R \to S^2$, there is a homeomorphism

$$h^* : [(R \cup \operatorname{Int} R) - (\operatorname{Int} S)] \to [(S^2 \cup \operatorname{Int} S^2) - \operatorname{Int}(\tfrac{1}{2}S^2)]$$

which extends h and takes each component K of $G \cap [(R \cup \operatorname{Int} R) - (\operatorname{Int} S)]$ into the straight line segment joining $h(K \cap R)$ and the origin $0 = (0, 0, 0)$.

Proof of (3.14). Consider the noncompact 3-manifold-with-boundary $M = (tC) - f^{-1}(G)$. Choose small disjoint polyhedral disks $D_1, \ldots, D_n$ on $\operatorname{Bd}(tC)$ with interiors covering respectively the n points of $f^{-1}(G) \cap \operatorname{Bd}(tC)$. Consider the n cones $0 * D_1, \ldots, 0 * D_n$ ($0 * D_i$ denotes the join of the origin 0 and the disk D_i in E^3; recall that C is convex so that it makes sense to take the join). We may assume by (3.12) that f has no singularities on some neighborhood of $(\bigcup_{i=1}^n 0 * D_i) \cap M$ in M. We may also assume that

$$B_i = f(0 * D_i) \cap [(R \cup \operatorname{Int} R) - (\operatorname{Int} S)] \quad (i = 1, \ldots, n)$$

is a 3-cell meeting each of R and S in a single disk. Define

$$M_0 = M - \bigcup_{i=1}^n [0 * (\operatorname{Int} D_i)]$$

and $N_0 = [(R \cup \operatorname{Int} R) - \operatorname{Int} S] - \bigcup_{i=1}^n f[0 * (\operatorname{Int} D_i)]$. Let $R_0 = N_0 \cap R$ and $S_0 = N_0 \cap S$.

In order to prove (3.14), it clearly suffices to show that the pairs (N_0, R_0) and $(R_0 \times I, R_0 \times \{0\})$ are homeomorphic. To this end we show how to use the map $f|tC$ and the Loop Theorem to find disks which cut N_0 apart to form a 3-cell.

Step 1. In $\operatorname{Bd}(tC)$ we choose polygonal arcs $A_2, \ldots, A_n$ joining the point $f^{-1}(G) \cap D_1$ to the points $f^{-1}(G) \cap D_2, \ldots, f^{-1}(G) \cap D_n$, respectively. We require that $A_i \cap A_j = f^{-1}(G) \cap D_1$ for $i \neq j$ and that $A_i \cap \operatorname{Bd} D_j$ be a single point for $j = 1$ or i and be empty otherwise. Let $E_j = 0 * A_j$ ($j = 2, \ldots, n$).

Step 2. We adjust f slightly so that $f(E_2), \ldots, f(E_n)$, and $\operatorname{Bd} N_0$ are all in general position. As a consequence, $(fE_j)^{-1}(\operatorname{Bd} N_0)$ is, for fixed j, the union of finitely many disjoint simple closed curves in $\operatorname{Int} E_j$. Since $f|tC$ is a homeomorphism on some neighborhood of $(\bigcup_{i=1}^n 0 * D_i) \cap M$ in M, exactly one of those curves, say J_j, intersects $\bigcup_{i=1}^n 0 * D_i$ and necessarily does so in exactly two arcs, one in $0 * \operatorname{Bd} D_1$ and the other in $0 * \operatorname{Bd} D_j$. Let E_j' be the subdisk of E_j bounded by J_j. If J is any other component of $(f|E_j')^{-1}(\operatorname{Bd} N_0)$, then $f(J) \subset R_0 \cup S_0$; we claim that $f|J$ is nullhomotopic in $R_0 \cup S_0$. If not, then there would be such a J such that (i) $f|J$ is not trivial in $R_0 \cup S_0$, but (ii) if $E_j(J)$ is the disk in E_j' bounded by J and K is any other component of $[E_j(J)] \cap [(f|E_j')^{-1}(\text{Bd } N_0)]$, then K is trivial in $R_0 \cup S_0$. We can remove those trivial curves $f(K)$ of intersection by adjusting $f|E_j(J)$. The Loop Theorem applied to the adjusted $f|E_j(J)$ implies the existence of a disk D in $N - G$ whose boundary lies in $R_0 \cup S_0$ and is nontrivial there. It follows from the existence of D that Bd D links no loop in $(E^3 - N) \cup G$; but it follows from the fact that Bd D is a nontrivial simple closed curve in $R_0 \cup S_0$, that Bd D does link a loop in $(E^3 - N) \cup G$. This contradiction establishes our claim.

Step 3. We adjust each $f|E_j'$ $(j = 2, \ldots, n)$ so that $(f|E_j')^{-1}(\text{Bd } N_0) = \text{Bd } E_j'$; that this is possible follows from the claim established in Step 2. We note that $f(E_j')$ (as adjusted) necessarily lies in N_0; we further note that $f|\text{Bd } E_j'$ is nontrivial as a loop in Bd N_0 since it links a loop in Int N_0. Hence we may apply the Loop Theorem to find polyhedral disks F_j $(j = 2, \ldots, n)$ having the following properties:

(1) Bd F_j lies in Bd N_0 and is a nontrivial simple closed curve there.

(2) Int $F_j \subset N_0$.

(3) $F_2, \ldots, F_n$ are in general position.

(4) Bd F_j lies in the union of $f(\text{Bd } E_j')$ and a small neighborhood of the singularities of $f(\text{Bd } E_j')$. (This last requirement follows from the proof of the Loop Theorem [**37**].)

It follows from the known properties of $f|E_j'$ and the linking argument indicated in Step 2 that Bd $F_j \not\subset R_0 \cup S_0$. Hence, it follows from (4) that Bd F_j intersects $f(0 * \text{Bd } D_1)$ and $f(0 * \text{Bd } D_j)$ in a single arc from R_0 to S_0.

Step 4. Standard arguments can be used to remove intersections in $F_2, \ldots, F_n$ so that they may be assumed disjoint. If N_0 is split along $F_2 \cup \ldots \cup F_n$ to form a new manifold-with-boundary N^*, it is easy to see that Bd N^* is a 2-sphere, hence that N^* is a 3-cell. With this much structure on the pair (N_0, R_0) it is an easy task to define a homeomorphism

$$h : (N_0, R_0) \to (R_0 \times I, R_0),$$

and we leave the details to the reader. This completes the proof of (3.14).

We remark that another proof of (3.14) can be given by using the results of [**7**; **38**], or [**39**].

We now consider various other versions of (3.1). We first recall a definition.

3.15. *Definition*. An arc A in E^3 is said to have $1 - ALG$ complement in E^3 if, for each $\epsilon > 0$, there is a $\delta > 0$ such that each loop in $E^3 - A$ which bounds homologically (**Z** coefficients) in a δ-subset of $E^3 - A$ also bounds a singular ϵ-disk in $E^3 - A$.

3.16. Theorem. *An arc A in E^3 is tame if it has $1 - ALG$ complement in E^3.*

Proof of (3.16). There are two lines of attack available; one can show that an arc which has $1 - ALG$ complement has a singular regular neighborhood or one can proceed directly to mimic the key parts of the proof of (3.1), namely the proof of (3.4) and of (3.3). By work of McMillan [**27**], it suffices to show that A satisfies the conclusion of (3.3). Since the details of both approaches are fairly standard and are much like what we have already done elsewhere in this paper, we suffice ourselves with a simple indication of the kinds of constructions involved. These indications we simply state as Lemmas (3.17)-(3.19) and accompanying remarks. In each of these lemmas we assume that J is a simple closed curve in E^3 which contains A.

3.17. *If* (1) $L : S^1 \to E^3 - J$ *is a loop bounding a singular disk* $D : B^2 \to E^3$, (2) L *does not link* J, (3) $D(B^2) \cap J$ *is a subset of an arc* A_0 *in* J, *and* (4) $\epsilon > 0$, *then* L *bounds homologically* ($\mathbf{Z}$ *coefficients*) *in* $N(D(B^2) \cup A_0, \epsilon) - J$.

Proof. This is a standard duality result.

3.18. *If* $p \in J$ *and* $\epsilon > 0$, *then there is a loop in* $N(p, \epsilon) - J$ *which has linking number* 1 *with* J.

Proof. This is proved in the second and third paragraphs of the proof of (3.4).

Lemmas (3.17) and (3.18) supply curves with any desired homological entanglement with J. With these lemmas it is easy to prove, for example, the following

3.19. *If* $p, q \in \operatorname{Int} A$ *and* $\epsilon > 0$, *then there is a singular annulus* A_0 *in* $N([p, q], \epsilon) - J$ *which has linking number* 1 *with* J *which has one boundary loop in* $N(p, \epsilon)$, *the other in* $N(q, \epsilon)$. *Furthermore, the boundary loops can be taken to be any prescribed loops sufficiently close to* p *and* q *and having linking number* 1 *with* J.

Proof of (3.19). One takes a finite sequence $J_1, \ldots, J_R$ of small loops near the arc pq, each having linking number 1 with J (3.18). One connects each consecutive pair J_i, J_{i+1} with an arc B_i so chosen (3.18) that $J_i B_i J_{i+1}^{-1} B_i^{-1}$ is a loop L_i that does not link J. By (3.17) and definition (3.15), L_i bounds a small singular disk D_i in $E^3 - J$. Then $A_0 = \bigcup_{i-1}^{k} D_i$.

Just as one constructs singular annuli, so one can construct singular 2-spheres like the singular 2-sphere $f(S_i)$ in the fourth paragraph of the proof of (3.4). All further details of the proof of (3.16) we leave to the reader.

We now consider one final version of (3.1), namely (3.22). We first need two definitions.

3.20. *Definition.* An arc A in E^3 is said to be s-l.p.u. (singularly locally peripherally unknotted) at a point $p \in A$ if, for each $\epsilon > 0$ there is singular 2-sphere $f : S^2 \to N(p, \epsilon)$ not null-homotopic in $E^3 - \{p\}$ such that $f^{-1}(A)$ has [order $_A(p)$] points and such that all sufficiently small curves around a point q of $f^{-1}(A)$ in S^2 have images under f which link A locally in E^3. (Recall that [order $_A(p)$] = 1 or 2 as p is an end or interior point of A, respectively.)

An arc A in E^3 is said to be s-l.p.u. if it is s-l.p.u. at each point. We say that a singular 2-sphere like f above meets A regularly and encloses p.

3.21. *Definition.* An arc A in E^3 is said to be s-l.u. (singularly locally unknotted) at $p \in \operatorname{Int} A$ if the conclusion of (3.8) is satisfied at p. An arc is s-l.u. if it is s-l.u. at each interior point.

3.22. Theorem. *An arc A in E^3 is tame if it is* s-l.p.u. *and* s-l.u. (*Remark.* The hypotheses can be weakened slightly; this will be apparent from the proof.)

Proof. The proof is essentially like that of (3.1) except that Lemma (3.4) becomes more difficult to prove. We shall show the adjustments necessary in a sequence of lemmas, Lemmas (3.23)-(3.27). Our result follows from (3.25), (3.26), (3.27), and (3.9). We shall assume throughout that J is a simple closed curve which contains A.

3.23. *Suppose $p \in \operatorname{Int} A$, $0 < \delta < \epsilon < \operatorname{Diam} J$, and $f : S^2 \to N(p, \epsilon)$, where f is a singular 2-sphere that meets A regularly, encloses p, and misses $J - A$. Suppose further that D is a polyhedral disk in $E^3 - J$ whose boundary misses $f(S^2)$ and can be joined to p by an arc B which misses $f(S^2) \cup \operatorname{Int} D$. Then* Bd D *bounds a polyhedral disk in $[D \cup N(f(S^2), \delta)] \cap N(p, \epsilon) - J$.*

Proof of (3.23). We may assume that D and $f(S^2)$ are in general position. We may cover $D - N(p, \epsilon)$ by finitely many disjoint disks-with-holes $D_1, \ldots, D_n$ in $(\operatorname{Int} D) - f(S^2)$. We change D inductively so as to reduce the number of disks-with-holes needed for the covering to zero. This will complete the proof.

Suppose $n > 0$. The existence of the arc B and a linking argument show that $f^{-1}(D)$ is a nonempty collection of disjoint simple closed curves in $S^2 - f^{-1}(A)$. Each of these bounds a disk in $S^2 - f^{-1}(A)$ since they have images under f which do not link J (since they lie in D). If J_i is one of these curves, let E_i be the disk in $S^2 - f^{-1}(A)$ bounded by J_i. If $f|J_i$ is nullhomotopic in $(\operatorname{Int} D) - \bigcup_j D_j$, then we can remove the intersection $f(J_i)$ from $f(S^2) \cap D$ by mapping E_i into $(\operatorname{Int} D) - \bigcup_j D_j$ and then pushing the image of E_i to one side of D. This adjustment can be made without disturbing the arc B which joins Bd D and p in the complement of $f(S^2) \cup \operatorname{Int} D$. Therefore, after the adjustment, some intersections necessarily remain. We find therefore that we may assume $f|J_i$ is not nullhomotopic in $(\operatorname{Int} D) - \bigcup_j D_j$. Among all such J_i's we choose an i such that the corresponding E_i is innermost on S^2. Then the Loop Theorem [**37**] as applied to a one sided neighborhood of $[D - \bigcup D_j] \cup f(E_i)$ supplies a nonsingular disk E in an arbitrary neighborhood of $f(E_i)$ such that Bd $E \subset \operatorname{Int} D$, $\operatorname{Int} E \subset E^3 - D$, and Bd E is not nullhomotopic in $(\operatorname{Int} D) - \bigcup_j D_j$. We replace the disk D' in D bounded by Bd E with the disk E. Since D' necessarily contains some D_j while E does not intersect $E^3 - N(p, \epsilon)$ we find that, for the new D, $D - N(p, \epsilon)$ may be covered by fewer disks-with-hole in Int D than was true for the old D. Again the process can be carried out without disturbing B; hence the process iterates and the proof is complete.

3.24. *Suppose* $f : S^2 \to E^3$ *is a singular* 2-*sphere which meets* A *regularly in two points* $x_1, x_2 \in \operatorname{Int} A$ *and misses* $J - A$. *Then for each* $\epsilon > 0$, *there is a polyhedral annulus* B *in* $N(f(S^2), \epsilon) - J$, *with one boundary component in* $N(x_1, \epsilon)$, *the other in* $N(x_2, \epsilon)$.

Proof of (3.24). We may assume that $N(x_1, \epsilon)$ and $N(x_2, \epsilon)$ are disjoint from each other and from $(J - \operatorname{Int} A)$. Let $f_i : S^2 \to N(x_i, \epsilon)$ $(i = 1, 2)$ be a singular 2-sphere which meets A regularly and encloses x_i. We now proceed as in the proof of (3.4), which proof the reader should at this point review. Instead of the point t_i of (3.4), we choose two points t_{i1} and t_{i2} very near to and on opposite sides of x_i in A. Instead of the 2-sphere R_i of (3.4), we choose a nonsingular, polyhedral, connected 2-manifold R_i in $E^3 - J$ very close to x_i and having precisely two boundary components, one very near t_{i1}, the other near t_{i2}, each linking J exactly once. (Such an R_i may be obtained by applying cut-and-paste techniques to the singular analogue of R_i supplied by (3.18), the proof of (3.19), and (3.17); we suppress details.)

Just as we simplified D_i in (3.4), so we use $f(S^2)$ to cut off handles from our new R_i. By (3.23), we may require that any new disk F used in the simplified R_i lie in $N(x_i, \epsilon)$.

It may not be immediately clear that the process iterates since the boundary of the next F in the adjustment may intersect the first F, hence possibly intersect $f_i(S^2)$, in which case (3.23) would be inapplicable. However, a moment of reflection shows that the boundary of the next F used may be adjusted to miss the first F and the process can be completed.

The remainder of the proof proceeds in the fashion of the proof of (3.4).

3.25. *The conclusion of* (3.4) *is valid for* A.

Proof. One simply chains together annuli obtained from (3.24) to obtain the desired annulus. The proof is much like Harrold's original proof that l.p.u. arcs satisfy the conclusion of (3.4) (cf. [**21**, proof of Theorem 1]).

3.26. *The conclusions of* (3.3) *is valid for* A.

Proof. We leave it as an exercise. (Work directly with (3.24) or use (3.25) as (3.4) was used in the proof of (3.3).)

3.27. *The conclusions of* (3.7) *and* (3.8) *are valid for* A.

Proof. This is also clear.

3.28. THEOREM. *A simple closed curve* J *in* E^3 *is tame if* J *pierces a singular disk at some point and is homogeneous by isotopy.* (The curve J is homogeneous by isotopy if, for each $p, q \in J$, there is an isotopy $H : E^3 \times I \to E^3$ of E^3 such that $H(A \times \{t\}) = A$ for each $t \in I$, $H_0(p) = p$, and $H_1(p) = q$.)

Proof. The curve J clearly satisfies the hypotheses of (3.21); hence J is tame.

4. *ULC* **properties in** E^m $(m \geqq 3)$**.** We first list the results of Section 2 which generalize without real change in proof to higher dimensions. We then sharpen results (2C.6) and (2C.6′) for use in Section 5.

The following results from Section 2 generalize immediately to the higher dimensional setting where S denotes, instead of a 2-sphere in E^3, an $(m-1)$-sphere in E^m $(m \geqq 3)$, and where U and V denote the complementary domains of S in E^m:

(2A) (ulc^n, each n, coefficients either $\mathbf{Z}$ or $\mathbf{Z}_2$), (2B.1), (2B.2), (2B.3), (2B.4), (2C.1), (2C.2), (2C.3), (2C.4), (2C.5) ((1) and (2)), (2C.6), (2A.1), (2B.4.1), (2C.2.1), (2C.5.1) ($D_1, D_2, \ldots, D_n$ disjoint compact absolute retracts; $g(K) \subset f(K) \cup (\cup_{i=1}^n D_i)$), (2C.5.2), (2C.6.5).

We have been unable to generalize the following results:

(2C.5) ((3)), (2C.7) and addenda.

We have made no attempt to generalize (2C.5.3)-(2C.5.6).

We now proceed toward a proof of a stronger version of (2C.6) (Theorem 4.2)). Suppose that X is a compact subset of the $(m-1)$-sphere S in E^m, $p \in U$, and $h : S \to S^{m-1}$ is a homeomorphism from S onto the standard $(m-1)$-sphere S^{m-1} in E^m. If $\alpha \in \pi_1(E^m - X, p)$, then there is a representative loop $f : (S^1, 1) \to (E^m - X, p)$ for α, called a standard representative for α, such that f is a simple closed curve which pierces S at each point of $f(S^1) \cap S$. There is an embedding $h_f : (S \cup f(S^1), p) \to (E^m, 0)$ which extends $h : S \to S^{m-1}$ such that $h_f \cdot f$ is a simple closed curve with $h_f \cdot f(t) \in \operatorname{Int} S^{m-1}$ if and only if $f(t) \in U$. Define $\varphi : \pi_1(E^m - X, p) \to \pi_1(E^m - h(X), 0)$ by $\varphi(\alpha) = \operatorname{cls}(h_f \cdot f)$.

4.1. Theorem. *The function* $\varphi = \varphi_{S,X}$ *is well-defined and is a surjective group homomorphism with* Ker $\varphi = \omega(E^m - X, p)$. *That is,* φ *induces an isomorphism between* $\pi_1(E^m - X, p)/\omega(E^m - X, p)$ *and* $\pi_1(E^m - h(X), 0)$.

Note that the homomorphism depends upon S and the homeomorphism h; the theorem says however that the kernel of the homomorphism depends only on X itself.

Proof of (4.1). That φ is well-defined and a surjective group homomorphism is immediate once we show that if f is an arbitrary standard representative for the trivial homotopy class α, then $h_f \cdot f$ is nullhomotopic in $E^m - h(X)$. (We retain the notation of the paragraph in which φ is defined.) Let $f^* : B^2 \to E^m - X$ be an extension of $f : (S^1, 1) \to (E^m - X, p)$ (where $S^1 = \operatorname{Bd} B^2$) as promised by the supposition that f represents the trivial class in $\pi_1(E^m - X, p)$. Note that the sets R_1 and R_2, $R_1 = (S - X) \cup (f(S^1) \cap V)$ and $R_2 = (S - X) \cup (f(S^1) \cap U)$, are open subsets of topological polyhedra, hence are absolute neighborhood retracts. It follows that there is a closed neighborhood N of $S^1 \cup (f^*)^{-1}(S)$ in B^2 that is a finite union of disjoint disks-with-holes and an extension $g : N \to f(S^1) \cup (S - X) = R_1 \cup R_2$ of $f^* | [S^1 \cup (f^*)^{-1}(S)]$ such that, for each $x \in N$,

(1) $f^*(x) \in S \cup V$ implies $g(x) \in R_1 \subset S \cup V$, and

(2) $f^*(x) \in S \cup U$ implies $g(x) \in R_2 \subset S \cup U$.

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Let M be the component of N that has S^1 as one boundary component. Then $h_f \cdot g|M : M \to S^{m-1} \cup h_f \cdot f(S^1)$ and each boundary component of M distinct from S^1 is mapped by $h_f \cdot g$ to a loop lying either in $(S^{m-1} \cup \operatorname{Ext} S^{-m1}) - h(X)$ or in $(S^{m-1} \cup \operatorname{Int} S^{m-1}) - h(X)$ (by (1) or (2)). In either case, such a loop bounds a singular disk in $E^m - h(X)$ since S^{m-1} is flat. Thus $h_f \cdot f$ bounds a singular disk in $E^m - h(X)$ as desired. We conclude easily that φ is well-defined and a surjective group homomorphism.

In order to see that $\omega(E^m - X, p) \subset \operatorname{Ker} \varphi$, observe that

$$\varphi(\omega(E^m - X, p)) \subset \omega(E^m - h(X), 0)$$

(by (2C.6.2)) and that $\omega(E^m - h(X), 0) = 1$ since $\pi_1(E^m - h(X), 0)$ is a free group (cf. (2C.6.3)).

Suppose finally that $\alpha \in \operatorname{Ker} \varphi$. That means that if f is a standard representative for α, $h_f \cdot f$ is nullhomotopic in $E^m - h(X)$. Let $f^* : B^2 \to E^m - h(X)$ be an extension of $h_f \cdot f$ such that S^{m-1} and the singular disk $f^*(B^2)$ are in general position. It follows that the components of $(f^*)^{-1}(S^{m-1})$ form a finite collection of mutually disjoint simple closed curves in $\operatorname{Int} B^2$, which curves we ignore, and disjoint spanning arcs $\mathscr{A}_1, \ldots, \mathscr{A}_k$ of B^2. Examine the map

$$g : S^1 \cup \mathscr{A}_1 \cup \ldots \cup \mathscr{A}_k \to E^m - X$$

defined by $g(x) = h_f^{-1} \circ f^*(x)$ for each $x \in S^1 \cup \mathscr{A}_1 \cup \ldots \cup \mathscr{A}_k$. Let U_i be a component of $B^2 - (S^1 \cup \mathscr{A}_1 \cup \ldots \cup \mathscr{A}_k)$. Then $J_i = \operatorname{Bd} U_i$ is a simple closed curve and $g|J_i$ is a loop L_i in either $(S \cup V) - X$ or in $(S \cup U) - X$. It follows that $\alpha = \operatorname{cls} f = \operatorname{cls}(a_1 L_1 a_1^{-1}) \ldots \operatorname{cls}(a_{k+1} L_{k+1} a_{k+1}^{-1})$ where $a_1, \ldots, a_{k+1}$ are paths in $E^m - X$. But it follows from the higher dimensional version of Theorem (2C.6.5) that each loop $a_i L_i a_i^{-1}$ represents an element of $\omega(E^m - X, p)$ since L_i does not intersect both components of $E^m - S$. Hence $\alpha \in \omega(E^m - X, p)$ and $\operatorname{Ker} \varphi \subset \omega(E^m - X, p)$. This completes the proof of Theorem (4.1).

ADDENDUM TO THEOREM (4.1). *If f is a standard representative of a class $\alpha \in \pi_1(E^m - X, p)$, then $\alpha \in \omega(E^m - X, p)$ if and only if there is a partition $\mathscr{P} = [(a_1, b_1), \ldots, (a_r, b_r)]$ of $f(S^1) \cap S$ into disjoint pairs such that for each i and j $(i \neq j)$,*

(1) *a_i and b_i are the endpoints of an arc A_i in $S - X$, and*

(2) *a_i and b_i are not separated by a_j and b_j in $f(S^1)$.*

Proof. If $\alpha \in \omega(E^m - X, p)$, then $\alpha \in \operatorname{Ker} \varphi$ by Theorem (4.1). In this case proceed as in the proof of the inclusion $\operatorname{Ker} \varphi \subset \omega(E^m - X, p)$ to find the spanning arcs $\mathscr{A}_i$ of that proof. Let a_i and b_i be the images under g of the endpoints of $\mathscr{A}_i$ and let the A_i required by conclusion (1) of the addendum be an arc in $g(\mathscr{A}_i)$ from a_i to b_i. The converse is proved in essentially the same manner.

Linking compact subsets of $(m - 1)$-spheres in E^m. Consider the sphere S, compact subset X, loop f, partition $\mathscr{P}$, and arcs A_i of the Addendum to

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Theorem (4.1). If the arcs A_i can be chosen to have diameter less than ϵ, then we say that f does not ϵ-link X on S. We can now state the promised sharpened version of (2C.6).

4.2. THEOREM. *Suppose that S and S' are $(m - 1)$-spheres in E^m and that X is a compact subset of $S \cap (S' \cup \text{Ext}\, S')$. Suppose further that $f : S^1 \to \text{Int}\, S'$ is a standard representative (with respect to S) of an element of $\pi_1(E^m - X)$ and that f bounds a singular ϵ-disk $D : B^2 \to S' \cup \text{Int}\, S'$. Then f does not ϵ-link X on S.*

Proof. Let N be a connected open subset of $S' \cup \text{Int}\, S'$ which contains D and has diameter less than ϵ. By the higher dimensional version of Theorem (2C.6.5), f represents an element of $\omega(N \cap \text{Int}\, S') = \omega(N')$. Let $X' = X \cup (\text{Bd}\, N' \cap S)$. Then there is a natural homomorphism

$$\pi_1(N') \to \pi_1(E^m - X')$$

induced by the inclusion $N' \subset E^m - X'$. It follows, with the help of Proposition (2C.6.2), that f represents an element of $\omega(E^m - X')$. The Addendum to Theorem (4.1) supplies a partition $\mathscr{P} = [(a_1, b_1), \ldots, (a_r, b_r)]$ of $f(S^1) \cap S$ and arcs $A_1, \ldots, A_r$ in $S - X'$ satisfying (1) and (2) of the Addendum. But each arc A_i has its endpoints in N' and misses Bd N', therefore is of diameter less than ϵ. The proof is complete.

We now prove the higher dimensional version of (2C.6′).

4.3. THEOREM. *Suppose that S, S', and X are as in (4.2) and that $E^m - S$ is $1 - ULC$ in $E^m - X$. Then $\text{Int}\, S'$ is $1 - ULC$ in $E^m - X$.*

Proof. Suppose $\epsilon > 0$ given. Choose $\alpha > 0$ such that 3α-loops in $E^m - S$ bound singular $\epsilon/3$-disks in $E^m - X$. Choose β, $0 < \beta < \alpha$, such that β-loops in $\text{Int}\, S'$ bound singular α-disks in $S' \cup \text{Int}\, S'$. Let $f : S^1 \to \text{Int}\, S'$ be a β-loop in $\text{Int}\, S'$. We may assume that f is in standard position with respect to S (i.e., pierces S at each point of $f(S^1) \cap S$ and is a simple closed curve); by our choice of β, f does not α-link X on S (Theorem (4.2)). Let

$$\mathscr{P} = [(a_1, b_1), \ldots, (a_r, b_r)]$$

be a partition of $f(S^1) \cap S$ such that, for each i and j $(i \neq j)$,

(1) a_i and b_i lie in an α-arc A_i in $S - X$, and

(2) a_i and b_i are not separated by a_j and b_j in $f(S^1)$.

Let $a_i' = f^{-1}(a_i)$, $b_i' = f^{-1}(b_i)$, and let A_i' be the straight line interval spanning B^2 $(S^1 = \text{Bd}\, B^2)$ from a_i' to b_i'. Note that $A_i' \cap A_j' = \emptyset$ by (2). Let $f^* : S^1 \cup A_1' \cup \ldots \cup A_r' \to E^m - X$ be an extension of f which takes A_i' to A_i. Let $U_1, \ldots, U_{r+1}$ denote the components of

$$B^2 - (S^1 \cup A_1' \cup \ldots \cup A_r').$$

Note that Bd $U_i = J_i$ is a simple closed curve mapped by f^* to a loop L_i in $E^m - X$ which has diameter less than 3α and which does not intersect both

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complementary domains of S. In order to shrink f in an ϵ-subset of $E^m - X$ it suffices to shrink each L_i in an $\epsilon/3$-subset of $E^m - X$, which we may do by our choice of α (where we here use the fact that 3α-loops in $E^m - S$ bound singular $\epsilon/3$-disks in $E^m - X$).

5. Locally spherical $(m - 1)$-spheres in E^m $(m \geqq 3)$. An $(m - 1)$-sphere S in E^m is said to be *locally spherical* if for each $p \in S$ and each $\epsilon > 0$ there is an $(m - 1)$-sphere S' in E^m of diameter less than ϵ such that $p \in \operatorname{Int} S'$ and $S' - S$ is simply connected. If $m = 3$, then the condition that $S' - S$ be simply connected is equivalent to the requirement that $S \cap S'$ be connected.

5.1. THEOREM. *If S is a locally spherical $(m - 1)$-sphere in E^m, then $E^m - S$ is $1 - ULC$.*

5.1.1. LEMMA. *Suppose the following given:*

(1) S, *an* $(m - 1)$-*sphere in* E^m;

(2) $L : S^1 \to \operatorname{Int} S$, *a loop*;

(3) $D : B^2 \to S \cup \operatorname{Int} S$, *a singular disk bounded by* L;

(4) S', *an* $(m - 1)$-*sphere with* $L \subset \operatorname{Ext} S'$ *such that* $S' - S$ *is simply connected*;

(5) K, *a compact subset of* $(\operatorname{Int} S') \cap S$;

(6) ϵ, *a positive number.*

Then L bounds a singular disk

$$D'' : B^2 \to [(S \cup \operatorname{Int} S) - K] \cap [N(D(B^2), \epsilon) \cup (S' - S)].$$

Proof. We ignore the details of epsilontics but require that all things chosen to be small be chosen very small with respect to both ϵ and to the distance from K to S'.

By the higher dimensional version of (2C.2.1), we may assume that $D^{-1}(S)$ is a 0-dimensional subset of $\operatorname{Int} B^2$. Choose a finite collection $B_1, \ldots, B_k$ of disjoint disks in $\operatorname{Int} B^2$ with boundaries missing $D^{-1}(S)$ such that they form an essential cover for $D^{-1}(S \cap S')$ and such that the image of each under D has very small diameter.

We may assume that $D|\operatorname{Bd} B_i$ is a loop in standard position (i.e., pierces S' at each point of intersection with S' and is a simple closed curve) with respect to S' for each i. Since $D|\operatorname{Bd} B_i$ is a very small loop missing S, by Theorem (4.2) there are disjoint spanning arcs A_{ij} in B_i joining the finitely many points of $(D|\operatorname{Bd} B_i)^{-1}(S')$ into pairs such that $D|B^2 - \bigcup \operatorname{Int} B_i$ can be extended to a map $D' : (B^2 - \bigcup \operatorname{Int} B_i) \cup [\bigcup A_{ij}] \to S \cup \operatorname{Int} S$ such that $D'|A_{ij}$ takes A_{ij} into a very small subset of $S' - S$.

Let U be the component of $B^2 - (D')^{-1}(S')$ which contains $\operatorname{Bd} B^2$. Let $F = \operatorname{Cl} U$ (closure taken in B^2), and let $G = B^2 - U$. We are prepared to define D'', first on F and then by extension to G.

Define $D''|F : F \to S \cup \operatorname{Int} S$ as follows: If $x \in F \cap \operatorname{Domain} D'$, define $D''(x) = D'(x)$. If $x \in [F \cap \operatorname{Int} B_i] - [\bigcup A_{ij}] \subset B^2 - \operatorname{Domain} D'$, then x is in a component V of $\operatorname{Int} B_i - \bigcup A_{ij}$. The set Bd V is a simple closed curve mapped by D' into a small subset of $S \cup \operatorname{Int} S$ and close to a point of

$$D(B^2) \cap S' \cap S.$$

Hence $D'|\mathrm{Bd}\, V : \mathrm{Bd}\, V \to S \cup \operatorname{Int} S$ bounds a small singular disk

$$V' : \mathrm{Cl}\, V \to S \cup \operatorname{Int} S$$

that lies in a small neighborhood of $D(B^2) \cap S' \cap S$, hence misses K. Define $D''|\mathrm{Cl}\, V = V'$.

Extend D'' to $B^2 = F \cup G$ as follows: Since $S' - S$ is an absolute neighborhood retract, there is a neighborhood N of $(D')^{-1}(S') = (D')^{-1}(S' - S)$ in G and a map $r : N \to S' - S$ which extends $D'|(D')^{-1}(S')$. Note that

$$(D''|F) \cup r : F \cup N \to S \cup \operatorname{Int} S$$

and that $N \cup F$ contains a neighborhood N' of $F \cup (D')^{-1}(S')$ in B^2. We may assume that Bd N' is a union of Bd B^2 and disjoint simple closed curves $J_1, \ldots, J_m$ in $\operatorname{Int} B^2$. One of the components N'' of $B^2 - \bigcup J_i$ contains $F - \mathrm{Bd}\, B^2$ in its interior. We may assume $J_1, \ldots, J_p$ are the boundary components of N'' distinct from Bd B^2. Define $D''|N'' = [(D''|F) \cup r]|N''$. Note that D'' as defined thus far takes each J_i $(i = 1, \ldots, p)$ into $S' - S$. Since $S' - S$ is simply connected, $D''|N''$ can be extended across the disk in B^2 bounded by J_i so as to take it into $S' - S$.

The map D'' as defined above satisfies the requirements of (5.1.1).

Proof of Theorem (5.1). Suppose $\epsilon > 0$ given. Choose $\delta > 0$ such that δ-loops in $E^m - S$ bound singular ϵ-disks which do not intersect both complementary domains of S. Let $f : S^1 \to E^m - S$ be a δ-loop. For concreteness, assume $f(S^1) \subset \operatorname{Int} S$. Let $D : B^2 \to S \cup \operatorname{Int} S$ be a singular ϵ-disk bounded by f. Let $S_1, \ldots, S_k$ be $(m - 1)$-spheres and $K_1, \ldots, K_k$ compact sets with

$$K_i \subset (\operatorname{Int} S_i) \cap S \ (i = 1, \ldots, k)$$

such that

(1) $\mathrm{Diam}[D(B^2) \cup S_1 \cup \ldots \cup S_k] < \epsilon$,

(2) $(f(S^1) \subset \operatorname{Ext} S_i$, for each i,

(3) $S_i - S$ is simply connected, for each i,

(4) $D(B^2) \cap S \subset \bigcup_{i=1}^k \operatorname{Int} K_i$.

Choose $\epsilon_1 > 0$ such that

(5) $\mathrm{Diam}[D(B^2) \cup S_1 \cup \ldots \cup S_k] - 2\epsilon_1 < \epsilon$,

(6) $[N(D(B^2), \epsilon_1) \cap S] - K_1 \subset \bigcup_{i=2}^k \operatorname{Int} K_i$.

Let $D_1 : B^2 \to [(S \cup \operatorname{Int} S) - K_1] \cap [N(D(B^2), \epsilon_1) \cup (S' - S)]$ be a map as promised by Lemma (5.1.1). Note that

(1)′ $\mathrm{Diam}[D_1(B^2) \cup S_2 \cup \ldots \cup S_k] < \epsilon$,

(2)′ $f(S^1) \subset \operatorname{Ext} S_i$ $(i = 2, \ldots, k)$,

(3)′ $S_i - S$ is simply connected $(i = 2, \ldots, k)$,

(4)′ $D_1(B^2) \cap S \subset \bigcup_{i=2}^k \operatorname{Int} K_i$.

An iteration of the procedure yields after k steps a singular ϵ-disk

$$D_k : B^2 \rightarrow \operatorname{Int} S$$

bounded by f. We conclude that $E^m - S$ is $1 - ULC$.

Remark. An $(m - 1)$-sphere S in E^m is said to be locally capped if for each $p \in S$, each component U of $E^m - S$, and each $\epsilon > 0$ there is an embedding $f : E^{m-1} \rightarrow U$ of E^{m-1} in an ϵ-subset of U such that if $\{x_i\}$ is an unbounded sequence in E^{m-1}, then $\{f(x_i)\}$ approaches S and such that p is in an ϵ-component of $(S \cup U) - \operatorname{Cl} f(E^{m-1})$. By a method almost identical with the proof of Theorem (5.1), one can prove the following theorem.

5.2. THEOREM. *If S is an $(m - 1)$-sphere in E^m and S is locally capped, then $E^m - S$ is $1 - ULC$.*

5.3. COROLLARY. *If S is a 2-sphere in E^3 and S is locally spherical or locally capped, then S is tame.*

Proof. A 2-sphere in E^3 is tame if $E^3 - S$ is $1 - ULC$ [**44**].

6. *ULC* **approximation theorems for surfaces in** E^3**.** The results of this section were obtained in our attempt to give a proof of the Hosay-Lininger Theorem (cf. [**23**; **25**], and [**15**]) that was independent of Bing's Side Approximation Theorem. We hoped thereby to develop techniques that would be applicable in higher dimensional problems as well. We did essentially free the proof from the Side Approximation Theorem; however, in order that we obtain higher dimensional results, it is still necessary that we find some sort of substitute for (6.4). We have been unable as of yet to find such a substitute. We nevertheless report our partial results here.

Our main results are Theorems (6.1), (6.5), and (6.7). They are special cases of known theorems (cf. [**23**; **25**; **15**; **3**]) originally proved by the Side Approximation Theorem. Each follows rather directly from its own main lemma (respectively, (6.2), (6.6), and (6.8)). Our approach is as follows. In the case of (6.1), we first show how (6.1) follows from (6.2), then prove (6.3) and (6.4), and finally use (6.3) and (6.4) to prove (6.2). We then indicate the slight changes in technique necessary to prove (6.5), (6.7), and related lemmas.

The main ideas involved in the proof of (6.1) are very simple: show that one can reembed a crumpled cube so as to have $1 - ULC$ complement by arranging that certain loops in horizontal planes be freed from entanglement with the crumpled cube (Lemma (6.3)); show how to tear apart linked handles of a crumpled cube so as to free horizontal loops from entanglement (Lemma (6.4)). The technical difficulties in the program, however, are considerable. Not the least of these is showing that, after one has torn apart a number of linked handles, one has really made progress toward making the complement of the crumpled cube $1 - ULC$. One needs some sort of object that "remembers" the progress made; this memory arrangement is built into the notion of n-pair for a crumpled cube (which is the subject of Lemma (6.2)). One needs also to be able to compare $1 - ULC$ properties of crumpled cubes whose boundaries are close homeomorphically. The necessary descriptive apparatus for this measurement comes from Section 2, (2C.5.3)-(2C.5.6). We recall those results. Suppose S is a 2-sphere in E^3 and $\delta > 0$. We defined

$$\epsilon(S, \delta) = \inf\{\epsilon > 0 | \delta\text{-sets in } S \text{ lie in simply connected } (\leqq \epsilon)\text{-sets in } S\}.$$

We proved the following facts.

(i) A δ-loop in $\operatorname{Int} S(\operatorname{Ext} S)$ bounds a singular $\epsilon(S, \delta) + \delta$-disk in $S \cup \operatorname{Int} S$ $(S \cup \operatorname{Ext} S)$.

(ii) $\epsilon(S, \delta) \to 0$ as $\delta \to 0$.

(iii) If $h : S \to E^3$ is an α-homeomorphism, then

$$\epsilon(h(S), \delta) \leqq \epsilon(S, \delta + 2\alpha) + 2\alpha.$$

We now state the main result of this section.

6.1. THEOREM. *Suppose C is a crumpled cube in E^3 and $\epsilon > 0$. Then there is an ϵ-homeomorphism $h : C \to E^3$ such that $E^3 - h(C)$ is $1 - ULC$.*

Proof. The homeomorphism h will be a limit of homeomorphisms obtained by repeated application of (6.2), whose statement must be preceded by a definition.

6.2.0. *Definition.* If C is a crumpled cube in E^3 and n a positive integer, then a pair (h, P) is called an n-pair for C if

(1) $h : C \to E^3$ is a $1/n$-homeomorphism,

(2) $E^3 - N(C, 1/n) \subset P = \operatorname{Cl} P \subset E^3 - h(C)$, and

(3) if $1 \leqq i \leqq n$, then each $1/i$-loop in $E^3 - N(C, 1/n)$ bounds a singular disk in P of diameter less than $3[\epsilon(\operatorname{Bd} C, 1/i + 1/n) + (1/i + 1/n)]$.

6.2. *If C is a crumpled cube in E^3 and n is a positive integer, then there is an n-pair for C.*

We assume (6.2) for the moment and use it to prove (6.1). We choose iteratively an increasing sequence $\{n_i\}$ of positive integers and pairs $\{(h_i, P_i)\}$ so as to satisfy a number of conditions. We first list five of these and some of their consequences:

(4) $\sum 1/n_i < \epsilon$.

(5) $2^i \leqq 2 \cdot n_{i-1} \leqq n_i$.

(6) (h_i, P_i) is an n_i-pair for $h_{i-1} \ldots h_1(C)$.

(7) If $\rho_i = \inf\{\rho(h_{i-1} \ldots h_1(x), h_{i-1} \ldots h_1(y)) | \rho(x, y) \geqq 1/i\}$, then

$$\textstyle\sum_{j=i}^{\infty} 1/n_j < \rho_i/2.$$

(8) $\sum_{j=i}^{\infty} 1/n_j < \rho(P_{i-1}, h_{i-1} \ldots h_1(C))$.

It follows from (4) and (6) (or (5) and (6)) that ($h = \lim_{i\to\infty} h_i \ldots h_1$):

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$C \to E^3$ exists and is continuous; from (4) that h is an ϵ-map; from (7) that h is 1-1; hence a homeomorphism; from (8) that

$$P_i \subset E^3 - N(h_i \dots h_1(C), 1/n_{i+1}) \subset \operatorname{Int} P_{i+1} \subset E^3 - h(C);$$

from (5) and (6) that

$$P_i \supset E^3 - N(h_{i-1} \dots h_1(C), 1/n_i) \supset E^3 - N(h(C), \textstyle\sum_{j=i-1}^{\infty} 1/n_j) \supset E^3 - N(h(C), 1/2^{i-2}).$$

Hence $\bigcup P_i = E^3 - h(C)$.

We now add a condition that forces $E^3 - h(C)$ to be $1 - ULC$. We recommend that the reader review at this point the properties of $\epsilon(S, \delta)$ outlined at the beginning of this section. Note especially property (iii). We may require that, for each i,

(9) $\epsilon[h_{i-1} \dots h_1(\mathrm{Bd}\, C), (2/n_i) + 2(2/n_i)] + 2(2/n_i) < 1/i.$

This is possible by property (ii). In order to see that, under conditions (4)-(9), $E^3 - h(C)$ must be $1 - ULC$, suppose given a positive integer i and a $1/n_i$-loop L in $E^3 - h(C)$. We determined in the previous paragraph that, for each sufficiently large integer $j > i$,

$$|L| \subset P_{j-1} \subset E^3 - N(h_{j-1} \dots h_1(C), 1/n_j) \subset P_j \subset E^3 - h(C).$$

Thus by (6) and (3), L bounds a singular disk E in $P_j \subset E^3 - h(C)$ of diameter less than

$$3 \cdot \{\epsilon[h_{j-1} \dots h_1(\mathrm{Bd}\, C), 1/n_i + 1/n_j) + (1/n_i + 1/n_j)]\}.$$

But $h_{i-1} \dots h_1(\mathrm{Bd}\, C)$ and $h_{j-1} \dots h_1(\mathrm{Bd}\, C)$ are homeomorphically within $2/n_i$ of each other by (5). Thus

$$\epsilon[h_{j-1} \dots h_1(\mathrm{Bd}\, C), 1/n_i + 1/n_j] \leqq \epsilon[h_{i-1} \dots h_1(\mathrm{Bd}\, C), 1/n_i + 1/n_j + 4/n_i] + 4/n_i < 1/i,$$

by (iii) and (9). Thus $\mathrm{Diam}\, E < 3(1/i + 1/n_i + 1/n_j) < 9/i$. We conclude that $E^3 - h(C)$ is $1 - ULC$. This completes the proof of (6.1).

We now return to (6.2). Lemma (6.2) depends on two further lemmas, which we now state and prove.

Our first lemma for (6.2) shows that, in order to shrink a loop in the complement of a crumpled cube, it suffices to shrink certain small "horizontal" loops in the complement of that crumpled cube.

6.3. *Suppose C is a crumpled cube in E^3, $\delta > 0$, $\eta > 0$, and J is a δ-loop in $E^3 - C$. Then J is a boundary loop of some singular $\epsilon(\mathrm{Bd}\, C, \delta) + \delta$-disk-with-holes D_0 in $E^3 - C$ such that each remaining boundary loop*

$$J_j (1 \leqq j \leqq k;\, k \geqq 0) \text{ of } D_0$$

is an η-loop, lies in an η-neighborhood of C, and lies in some horizontal plane.

Proof. By (2C.2.1) and property (i) of this section, J bounds a singular $\epsilon(\text{Bd } C, \delta) + \delta$-disk $D : B^2 \to E^3 - \text{Int } C$ such that $D^{-1}(C)$ is totally disconnected. We may assume, making η smaller if necessary, that

$$(\text{Diam } D(B^2)) + 2\eta < \epsilon(\text{Bd } C, \delta) + \delta.$$

We choose pairs of rectangular or cubical open sets $M_1 \subset N_1, \ldots, M_r \subset N_r$ covering $D(B^2) \cap C$ such that each M_i contains a point of C and each N_i is of diameter less than η. By property (i) of this section, (2C.2.1), and (2C.6) (cf. [**10**, § 2; **11**, proof of Theorem 1]), we may choose these pairs in such a manner that

(*) If $\{\pi_1, \ldots, \pi_s\}$ is a finite family of horizontal planes and $K : S^1 \to M_\rho - C$ is a loop, then K bounds a singular disk in

$$N_\rho - [(\text{Bd } C) \cap (\cup_\sigma \pi_\sigma)] \; (\rho = 1, \ldots, r).$$

Since $D^{-1}(C)$ is totally disconnected, it is possible to cover $D^{-1}(C) \subset B^2$ by the interiors of disjoint disks $D_1, \ldots, D_m$ $(m \geqq 0)$ in $\text{Int } B^2$ such that the image under D of each D_j lies in some one of the sets M_ρ.

The setting which we have just described is precisely the kind of setting to which we can apply the argument of [**11**, Theorem 1]. We conclude from that argument that $D|\text{Bd } D_j$ is a boundary loop of a disk-with-holes E_j in $N_\rho - C$ such that each of the other boundary loops of E_j lies in a horizontal plane. Then $D_0 = [D(B^2 - \cup D(D_j)] \cup (\cup E_j)$ satisfies the requirements of (6.3) (difference and union are taken here in a combinatorial rather than set-theoretic sense).

Our second lemma for (6.2) describes how one might pull apart "linked handles" in a crumpled cube C. This lemma is essentially due to Hosay [**23**] and was described to us verbally by R. H. Bing.

6.4. *Suppose the following given*: D, *a polyhedral disk in* S^3; N, *a regular neighborhood of* $\text{Bd } D$ *in* S^3; H, *a compact and arcwise-connected subset of* $S^3 - (D \cup N)$; C, *a compact, arcwise-connected, and simply connected subset of* $S^3 - (N \cup H)$; C', *a component of* $C - D$. *Then, for each* $\epsilon > 0$, *there is a homeomorphism* $h : H \cup C \to S^3 - D$ *such that*

(1) *h is the identity except on* $N(C - C', \epsilon)$, *and*

(2) *either* $h(x) = x$ *or* $h(x) \in N(D - N, \epsilon)$, *for* $x \in H \cup C$.

Proof. Let $M = S^3 - \text{Int } N$ and $M_0 = M - D$. Let $p : M^* \to M$ be the universal covering of M. There is a lifting $f : H \cup C \to M^*$ (i.e., pf = identity) such that $f(H \cup C')$ lies in a single component M_0^* of $p^{-1}(M_0)$. Let E_1 and E_2 denote the disks which are the components of $p^{-1}(D)$ in $\text{Cl } M_0^*$. Let F_1 and F_2 be two other components of $p^{-1}(D)$ so chosen that the component M_1^* of $M^* - (F_1 \cup F_2)$ which contains both F_1 and F_2 in its boundary also contains $f(H \cup C) \cup \text{Cl } M_0^*$. Choose polyhedral disks E_1' and E_2' in M_0^* very close to E_1 and E_2, respectively, and parallel to them in M^*. There is a homeomorphism $h_0 : \text{Cl } M_1^* \to \text{Cl } M_0^*$ which fixes the cell in M_0^* bounded by E_1'

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and E_2' and which takes each F_i to an E_j $(i, j = 1, 2)$. Then

$$p \cdot h_0 \cdot f : H \cup C \to S^3$$

satisfies (1) and (2) provided only that E_1' and E_2' are chosen close enough to E_1 and E_2, respectively. This completes the proof of (6.4).

Proof of (6.2). We recall that we are given a crumpled cube C in E^3 and a positive integer n and that we seek a pair (h, P) such that

(1) $h : C \to E^3$ is a $1/n$-homeomorphism,

(2) $E^3 - N(C, 1/n) \subset P = \mathrm{Cl}\, P \subset E^3 - h(C)$, and

(3) If $1 \leqq i \leqq n$, then each $1/i$-loop in $E^3 - N(C, 1/n)$ bounds a singular disk in P of diameter less than $3 \cdot [\epsilon(\mathrm{Bd}\, C, 1/i + 1/n) + (1/i + 1/n)]$.

The problem is that of pulling apart (by a homeomorphism $h : C \to E^3$) certain of the linked handles of C so as to remove the obstruction to shrinking (in $E^3 - h(C)$) loops from $E^3 - N(C, 1/n)$. Our proof will eventually depend on a reduction to a finite number the loops that must be considered in satisfying conclusion (3) above (cf. (6.2‴)). However, the notation necessary to describe the reduction would at this point tend to obscure the essential simplicity of the proof. Since even the finite case (cf. (6.2″)) requires more notation than we care to impose upon the reader at this point, we consider first the following case:

6.2′. Special case. *If $J : S^1 \to E^3 - C$ is a single $(1/3n + 1/i)$-loop $(1 \leqq i \leqq n)$, then there is a pair (h, P) satisfying* (1), (2), *and*

(3′) *$J(S^1) \subset P$ and J bounds a singular disk of diameter less than $\epsilon(\mathrm{Bd}\, C, 1/i + 1/n) + (1/i + 1/n)$ in P.*

Proof of (6.2′). Let P_0 be a closed, locally finite, connected polyhedron such that

$$[E^3 - N(C, 1/n)] \cup J(S^1) \subset \mathrm{Int}\, P_0 \subset P_0 \subset E^3 - C.$$

The P required by (6.2′) will be the union of P_0 and a certain disk bounded by J.

Step 1. Application of (6.3). Choose an integer m such that

(4) $6n < m$,

(5) $P_0 \cap N(C, 1/m) = \emptyset$, and

(6) No two sets of diameter $\geqq 1/6n$ in C are separated in C by a $1/m$-set in C. (Such an integer exists by (2B.4).)

By (6.3), J bounds, together with finitely many horizontal $1/m$-loops $J_1, \ldots, J_k$ in $N(C, 1/m)$, a polyhedral, singular disk-with-holes D_0 in $E^3 - C$ such that

(7) Diam $D_0 < \epsilon(\mathrm{Bd}\, C, 1/i + 1/3n) + (1/i + 1/3n)$.

Let $\pi_1, \ldots, \pi_k$ be the horizontal planes containing $J_1, \ldots, J_k$, respectively; U_j $(j = 1, \ldots, k)$ the union of the bounded complementary domains in π_j of the image of the loop J_j. We may clearly require also that

(8) the planes $\pi_1, \ldots, \pi_k$ are distinct, and

(9) $U_j \subset N(C, 1/m)$ $(j = 1, \ldots, k)$.

By a general position and cut-and-paste argument, we may obtain a new D_0 which satisfies the above requirements as well as

(10) $D_0 \cap (J_j \cup U_j)$ is connected $(j = 1, \ldots, k)$.

To obtain (10), one simply cuts part of the original D_0 off near π_j if D_0 protrudes through U_j without hitting J_j. This procedure may possibly increase or decrease the number of curves $J_1, \ldots, J_k$. Conditions (5) and (9) ensure that the process does not affect J. Indeed, $J \subset P_0 \subset E^3 - N(C, 1/m)$ by (5), while $U_j \subset N(C, 1/m)$ by (9). Condition (9) also makes it possible to choose each of the required new curves to be $1/m$-curves in $N(C, 1/m)$.

Step 2. Application of (6.4). By (10), $C \cap U_j$ may be covered by the interiors of finitely many disjoint $1/m$-disks in $U_j - D_0$. Let $\{D_1, \ldots, D_m\}$ be a union of such collections of disks, one collection for each j. By (8), no two of the D_j's intersect. By construction, $C \cap (\bigcup \operatorname{Bd} D_j) = \emptyset$. By (5) and (9), $(P_0 \cup D_0) \cap (\bigcup D_j) = \emptyset$.

Let $M_1, \ldots, M_m$ and $N_1, \ldots, N_m$ be regular neighborhoods in E^3 of $D_1, \ldots, D_m$ and $\operatorname{Bd} D_1, \ldots, \operatorname{Bd} D_m$, respectively. We require that

$$M_1 \cup N_1, \ldots, M_m \cup N_m$$

be disjoint $1/m$-sets, that $\bigcup N_\mu$ be disjoint from C, and that

$$(\bigcup M_\mu) \cup (\bigcup N_\mu)$$

be disjoint from $H = P_0 \cup D_0$. Finally, we require that

$$(M_\mu - N_\mu) \cap (\bigcup \pi_j) \subset \operatorname{Int} D_\mu \text{ for } \mu = 1, \ldots, m.$$

We lose no generality by assuming that $C - \bigcup M_\mu$ has a component C' of diameter $\geqq 1/6n$. Note that each component of $C - \operatorname{Cl} C'$ has diameter less than $1/6n$ by (6) and the unicoherence of C (2B.3).

We apply (6.4) repeatedly to obtain a homeomorphism

$$h : H \cup C \rightarrow E^3 - \bigcup D_i$$

such that h is the identity on $(\operatorname{Cl} C') \cup H$ and either

$$h(x) = x \text{ or } h(x) \in \bigcup (M_\mu - N_\mu) \text{ for } x \in H \cup C.$$

We claim that h moves no point as far as $1/n$. Indeed, let $p \in C - \operatorname{Cl} C'$ and K be the component of $C - \operatorname{Cl} C'$ which contains p. By the remark of the previous paragraph, $\operatorname{Diam} \operatorname{Cl} K < 1/6n$. Furthermore, $\operatorname{Cl} K \cap \operatorname{Cl} C' \neq \emptyset$ and $h(\operatorname{Cl} K) \subset \operatorname{Cl} K \cup (\bigcup M_\mu)$. Therefore, h fixes some point of $\operatorname{Cl} K$ and takes the whole set into a $1/2n$-set. Thus h does not move p any further than $1/6n + 1/2n = 2/3n < 1/n$.

We note that $[P_0 \cup D_0 \cup (\bigcup U_j)] \cap h(C) = \emptyset$. We define

$$P = P_0 \cup D_0 \cup (\bigcup U_j)$$

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and see that (h, P) is a pair satisfying the requirements of (6.2′). Indeed, J can be shrunk to a point in $D_0 \cup (\cup U_j) \subset P \subset E^3 - h(C)$ and

$$\operatorname{Diam}(D_0 \cup \cup U_j) < \epsilon(\operatorname{Bd} C, 1/i + 1/3n) + (1/i + 1/3n) + 2/m.$$

6.2″. Special case. *Suppose* $J_1, \ldots, J_\sigma, \ldots, J_s : S^1 \to E^3 - C$ *is a family of loops to which one may assign integers* $i(1), \ldots, i(\sigma), \ldots, i(s)$ *in the range* $1 \leqq i(\sigma) \leqq n$ *such that, for each* σ $(1 \leqq \sigma \leqq s)$, $\operatorname{Diam} J_\sigma < 1/3n + 1/i(\sigma)$. *Then there is a pair* (h, P) *satisfying conditions* (1), (2), *and*

(3″) $J_\sigma \subset P$ *and* J_σ *bounds a singular disk in* P *of diameter less than* $\epsilon[\operatorname{Bd} C, 1/i(\sigma) + 1/n] + [1/i(\sigma) + 1/n]$.

Proof. The proof is in essentially every detail exactly like the proof of (6.2′). The only real difference is that a great deal more notation is needed to keep track of things. We may safely leave the proof to the reader.

6.2‴. Reduction of (6.2) to (6.2″). Let P_0 be a closed, locally finite, connected polyhedron such that $E^3 - N(C, 1/n) \subset P_0 \subset E^3 - C$. We may assume P_0 endowed with a triangulation T of mesh less than $1/6n$. There are only finitely many connected subcomplexes $T_1, \ldots, T_r$ of T of diameter less than $1/3n + 1$ whose convex hulls do not lie in P_0. Since the fundamental group of each T_ρ is finitely generated, we may choose a finite family $J_1, \ldots, J_s$ of loops in P_0 such that each loop L in each T_ρ bounds, together with a product of J_σ's, a singular disk-with-holes $D(L)$ in that T_ρ. For each σ, let $i(\sigma)$ be the largest integer in the range $1 \leqq i(\sigma) \leqq n$ such that, for some ρ, $J_\sigma \subset T_\rho$ and $\operatorname{Diam} T_\rho < 1/i(\sigma) + 1/3n$.

Apply (6.2″) to obtain a pair (h, P) such that

(1) $h : C \to E^3$ is a $1/n$-homeomorphism,

(2) $[E^3 - N(C, 1/n)] \cup P_0 \subset P = \operatorname{Cl} P \subset E^3 - h(C)$, and

(3″) Each J_σ bounds a singular disk D_σ in P of diameter less than

$$\epsilon[\operatorname{Bd} C, 1/i(\sigma) + 1/n] + [1/i(\sigma) + 1/n].$$

We claim that this is an n-pair for C. In order to see this, let $L : S^1 \to E^3 - N(C, 1/n)$ be a $1/i$-loop $(1 \leqq i \leqq n)$. Since $\operatorname{Diam} L < 1/i$ and mesh $T < 1/6n$, the simplicial neighborhood of L in T is a connected subcomplex of T of diameter less than $1/3n + 1/i$. If the convex hull of this neighborhood lies in P, we are done. Otherwise, this neighborhood is T_ρ for some ρ. Each of the loops J_σ in T is assigned an integer $i(\sigma) \geqq i$. Thus, a J_σ in T_ρ bounds a singular disk D_σ of diameter less than $\epsilon[\operatorname{Bd} C, 1/i + 1/n] + [1/i + 1/n]$ in P. Then $D(L) \cup \cup \{D_\sigma | J_\sigma \subset T_\rho\}$ contains a singular disk D in P which is bounded by L and has the required property,

$$\begin{aligned} \operatorname{Diam} D &\leqq \operatorname{Diam} D(L) + 2 \operatorname{Max}\{\operatorname{Diam} D_\sigma | J_\sigma \subset T_\rho\} \\ &\leqq 3[\epsilon(\operatorname{Bd} C, 1/i + 1/n) + (1/i + 1/n)]. \end{aligned}$$

This completes the proof of (6.2″) and (6.2).

We next describe a relative version of (6.1), i.e., a version which tames part of Bd C and keeps a prescribed closed subset of C fixed.

6.5. THEOREM. *Suppose C is a crumpled cube in E^3 and $\epsilon : C \rightarrow [0, \infty)$ is a continuous nonnegative real-valued function. Then there is an ϵ-homeomorphism h from C into E^3 such that $E^3 - h(C)$ is $1 - LC$ at each point $x \in h(\text{Bd } C)$ for which $\epsilon \cdot h^{-1}(x) > 0$.*

The proof is like that of (6.1) except that one does not attempt to change things very near $X = \epsilon^{-1}(0)$ at any given stage of the proof. More exactly, one uses the following versions of (6.2.0) and (6.2).

6.6.0. *Definition.* Suppose C is a crumpled cube in E^3, X is a closed subset of C, $\delta > 0$, and n is a positive integer. A pair (h, P) is called an n-pair for (C, X, δ) if

(1) $h : C \rightarrow E^3$ is a $1/n$-homeomorphism that fixes the set $N(X, \delta/2) \cap C$,

(2) $E^3 - N(C, 1/n) \subset P = \text{Cl } P \subset E^3 - h(C)$, and

(3) If $1 \leqq i \leqq n$, then each $1/i$-loop in $E^3 - N(C, 1/n)$ which bounds a singular $\epsilon(\text{Bd } C, 1/i)$-disk in $\text{Cl}(E^3 - C) - N(X, \delta)$ bounds a singular disk in P of diameter less than $3 \cdot [\epsilon(\text{Bd } C, 1/i + 1/n) + (1/i + 1/n)]$.

6.6. *Suppose C is a crumpled cube in E^3, X is a closed subset of C, $\delta > 0$, and n is a positive integer. Then there is an n-pair for (C, X, δ).*

When one applies (6.6) in the proof of (6.5), one chooses $n_1, n_2, \ldots$ much as in the proof of (6.1) but also chooses $\delta_1, \delta_2, \ldots$ converging to 0, and uses $\epsilon^{-1}(0)$ as the X of (6.6). We do not go into the details of the proof.

We note one final refinement of the result.

6.7. THEOREM. *Suppose C is a crumpled cube in E^3, X is a compact subset of* Bd C, $[\text{Cl}(E^3 - C)] - X$ *is $1 - ULC$, and $\epsilon > 0$. Then there is an ϵ-homeomorphism h from C into E^3 which fixes X such that $E^3 - h(C)$ is $1 - ULC$.*

This result simply requires the following lemma.

6.8. *Let C, X, and ϵ be as in* (6.7). *Then for each positive integer n, there is an n-pair (h, P) for C* (Definition (6.2.0)) *such that h is fixed on some neighborhood of X in C.*

This lemma is proved in exactly the same way as (6.2) except that one may use the fact that $[\text{Cl}(E^3 - C)] - X$ is $1 - ULC$ in conjunction with (2C.2) to keep the small horizontal loops supplied by (6.3) sufficiently far away from X so that things need not be moved near X. We do not go into more detail except to mention that, since h is fixed on a neighborhood of X in C, $[\text{Cl}(E^3 - h(C))] - X$ is also $1 - ULC$ and one can iterate the procedure.

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