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On Periodic Solutions to Constrained Lagrangian System

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Abstract. A Lagrangian system is considered. The configuration space is a non-compact manifold that depends on time. A set of periodic solutions has been found.

1 Introduction

Existence problems for periodic solutions to Lagrangian systems have been studied intensively since the beginning of the 20th century and even earlier. An immense number of different results and methods have been developed in this field. We mention only those that are closely related to this article.

In [2], periodic solutions were obtained for the Lagrangian system with Lagrangian

$$L(t,x,\dot x)=\tfrac{1}{2}g_{ij}(x)\dot x^i\dot x^j-W(t,x),\quad x=(x^1,\dots,x^m)\in\mathbb{R}^m.$$

Here and in the sequel, we use the Einstein summation convention. The form g_{ij} is symmetric and positive definite:

$$g_{ij}\xi^i\xi^j\geq \mathrm{const}_1\cdot|\xi|^2.$$

The potential is $W(t,x)=V(x)+g(t)\sum_{i=1}^m x^i$, where V is a bounded function, $|V(x)|\leq \mathrm{const}_2$, and g is an ω-periodic function. The functions V, g_{ij} are even.

Under these assumptions we prove that there exists a nontrivial ω-periodic solution.

Our main tools to obtain periodic solutions are variation techniques. Variational problems and Hamiltonian systems have been studied extensively. Classic references for these subjects are found in [4, 6, 7].

2 The Main Theorem

We introduce some notation. Let $x=(x^1,\dots,x^m)$ and $\varphi=(\varphi^1,\dots,\varphi^n)$ be points of the standard $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Then let z stand for the point $(x,\varphi)\in\mathbb{R}^{m+n}$. Denote by $|\cdot|$ the standard Euclidean norm of $\mathbb{R}^k$, $k=m,m+n$, that is, $|x|^2=\sum_{i=1}^k(x^i)^2$.

The variable z can consist only of x without φ or conversely.

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The main object of our study is the Lagrangian system with Lagrangian

$$L(t,z,\dot z)=\tfrac{1}{2}g_{ij}\dot z^i\dot z^j+a_i\dot z^i-V,\quad z=(z^1,\dots,z^{m+n}). \tag{2.1}$$

Remark 2.1 The term $a_i\dot z^i$ corresponds to the so-called gyroscopic forces. For example, the Coriolis force and the Lorentz force are gyroscopic.

The functions g_{ij}, a_i, V depend on (t,z) and belong to $C^2(\mathbb{R}^{m+n+1})$; moreover, all these functions are 2π-periodic in each variable φ^j and ω-periodic in the variable t, for $\omega>0$. For all $(t,z)\in\mathbb{R}^{m+n+1}$ it follows that $g_{ij}=g_{ji}$.

We also assume that there are positive constants C, M, A, K such that for all (t,z) and $\xi\in\mathbb{R}^{m+n}$, we have

$$|a_i(t,z)|\le C+M|z|,\quad V(t,z)\le A|z|^2,\quad \tfrac{1}{2}g_{ij}(t,z)\xi^i\xi^j\ge K|\xi|^2. \tag{2.2}$$

System (2.1) obeys the following ideal constraints:

$$f_j(t,z)=0,\quad j=1,\dots,l<m+n,\quad f_j\in C^2(\mathbb{R}^{m+n+1}). \tag{2.3}$$

The functions f_j are also $2\pi-$periodic in each variable φ^j and ω-periodic in the variable t.

Introduce a set

$$F=\bigl\{(t,z)\in\mathbb{R}^{m+n+1}\mid f_j(t,z)=0,\quad j=1,\dots,l\bigr\}.$$

Assume that

$$\operatorname{rank}\Bigl(\frac{\partial f_j}{\partial z^k}(t,z)\Bigr)=l \tag{2.4}$$

for all $(t,z)\in F$, so that F is a smooth manifold.

Assume also that all the functions f_j are either odd,

$$f_j(-t,-z)=-f_j(t,z), \tag{2.5}$$

or even,

$$f_j(-t,-z)=f_j(t,z). \tag{2.6}$$

Remark 2.2 Actually, it is sufficient to say that all the functions are defined and have formulated the above properties in some open symmetric vicinity of the manifold F. We believe that this generalization is unimportant and will keep referring to the whole space $\mathbb{R}^{m+n+1}$ just for simplicity of exposition.

Definition 2.3 We say that a function $z(t)\in C^2(\mathbb{R},\mathbb{R}^{n+m})$ is a solution to system (2.1), (2.3) if there exists a set of functions

$$\{\alpha^1,\dots,\alpha^l\}\subset C(\mathbb{R})$$

such that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot z^i}\bigl(t,z(t),\dot z(t)\bigr)-\frac{\partial L}{\partial z^i}\bigl(t,z(t),\dot z(t)\bigr)=\alpha^j(t)\frac{\partial f_j}{\partial z^i}\bigl(t,z(t)\bigr), \tag{2.7}$$

$$\bigl(t,z(t)\bigr)\in F \tag{2.8}$$

for all real t.

In the absence of constraint (2.3), the right-hand side of (2.7) is equal to zero, and condition (2.8) is dropped.

The functions α^i are defined from equations (2.7) and (2.3) uniquely.

Theorem 2.4 *Assume that*

(i) *all the following functions are even:*

$$g_{ij}(-t,-z) = g_{ij}(t,z), \quad a_i(-t,-z) = a_i(t,z), \quad V(-t,-z) = V(t,z);$$

(ii) *the following inequality holds:*

$$K - \frac{M\omega}{\sqrt{2}} - \frac{A\omega^2}{2} > 0; \tag{2.9}$$

(iii) *for some* $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$, *there is a function* $\widetilde{z}(t) = (\widetilde{x}(t), \widetilde{\varphi}(t)) \in C^1(\mathbb{R}, \mathbb{R}^{m+n})$ *such that*

$$\widetilde{z}(-t) = -\widetilde{z}(t), \quad \widetilde{x}(t+\omega) = \widetilde{x}(t), \quad \widetilde{\varphi}(t+\omega) = \widetilde{\varphi}(t) + 2\pi v$$

and

$$(t, \widetilde{z}(t)) \in F, \quad t \in \mathbb{R}. \tag{2.10}$$

Then system (2.1), (2.3) *has a solution* $z(t) = (x(t), \varphi(t))$ *such that*

(a) *the function* z *is odd:* $z(-t) = -z(t)$;

(b) $x(t+\omega) = x(t), \quad \varphi(t+\omega) = \varphi(t) + 2\pi v$;

(c) *the functions* $\{\alpha^i\}$ *are* ω*-periodic and odd provided the constraints are odd, and the* α^j *are even for even constraints.*

This assertion remains valid in the absence of constraints (2.3).

Actually, the solution stated in this theorem is as smooth as is allowed by smoothness of the Lagrangian L and the functions f_j up to C^∞.

Remark 2.5 (i) If all the functions do not depend on t, then we can choose ω to be arbitrary small, and inequality (2.9) is satisfied. Taking a vanishing sequence of ω, we obtain infinitely many periodic solutions of the same homotopic type.

(ii) Suppose that $M = 0$ and $V(t,z) \le A_1|x|^\alpha$, $\alpha < 2$. Choose a small constant $A > 0$ such that inequality (2.9) is satisfied. Then choose a constant $c_1 > 0$ such that for all $|x|$, one has $A_1|x|^\alpha \le A|x|^2 + c_1$. Now the second condition of the theorem is satisfied for the new potential

$$V_1 = V - c_1 \le A|x|^2.$$

Remark 2.6 Theorem 2.4(ii) is essential for a non-compact manifold F or when the constraints are absent. Indeed, the system

$$L(t,x,\dot{x}) = \tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}(x - \sin t)^2$$

obeys all the conditions except inequality (2.9). It is easy to see that the corresponding equation $\ddot{x} + x = \sin t$ does not have periodic solutions.

When F is compact then condition (ii) can be dropped and the theorem follows from the standard results.

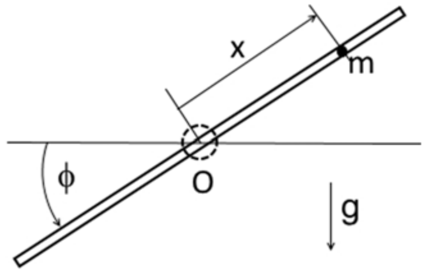

Figure 1: The tube and the ball.

Examples

Our first example is as follows.

A thin tube can rotate freely in the vertical plane about a fixed horizontal axis passing through its centre O. A moment of inertia of the tube about this axis is equal to J. The mass of the tube is distributed symmetrically such that the tube's centre of mass is placed at the point O.

Inside the tube there is a small ball that can slide without friction. The mass of the ball is m. The ball can pass by the point O and fall out of the ends of the tube.

The system undergoes the standard gravity field $\boldsymbol{g}$.

It seems evident that for typical motion, the ball reaches an end of the tube and falls out of the tube. It is surprising, at least at first glance, that this system has very many periodic solutions such that the tube turns around several times during the period.

The sense of generalized coordinates ϕ, x is clear from Figure 1.

The Lagrangian of this system is

$$L(x, \phi, \dot{x}, \dot{\phi}) = \tfrac{1}{2}(mx^2 + J)\dot{\phi}^2 + \tfrac{1}{2}m\dot{x}^2 - mgx\sin\phi. \tag{2.11}$$

From Theorem 2.4, it follows that for any constant $\omega > 0$, system (2.11) has a solution $\phi(t), x(t), t \in \mathbb{R}$ such that

(a) $x(t) = -x(-t), \quad \phi(t) = -\phi(-t);$
(b) $x(t+\omega) = x(t), \quad \phi(t+\omega) = \phi(t) + 2\pi.$

This result shows that for any $\omega > 0$ the system has an ω-periodic motion such that the tube turns around once during the period. The length of the tube should be chosen properly.

Our second example is a counterexample. Let us show that the first condition of Theorem 2.4 cannot be omitted.

Consider a mass point m that slides on a right circular cylinder of radius r. The surface of the cylinder is perfectly smooth. The axis x of the cylinder is parallel to the force of gravity $\boldsymbol{g}$ and directed upwards.

The Lagrangian of this system is

$$L(x, \varphi, \dot{x}, \dot{\varphi}) = \frac{m}{2}(r^2\dot{\varphi}^2 + \dot{x}^2) - mgx. \tag{2.12}$$

All the conditions except the evenness are satisfied, but it is clear this system does not have periodic solutions.

3 Proof of Theorem 2.4

In this section, we use several standard facts from functional analysis and Sobolev space theory [1, 3].

3.1 Recall that the Sobolev space $H^1_{\mathrm{loc}}(\mathbb{R})$ consists of functions $u(t), t \in \mathbb{R}$ such that $u, \dot u \in L^2_{\mathrm{loc}}(\mathbb{R})$. The following embedding holds: $H^1_{\mathrm{loc}}(\mathbb{R}) \subset C(\mathbb{R})$.

Lemma 3.1 *Let $u \in H^1_{\mathrm{loc}}(\mathbb{R})$ and $u(0) = 0$. Then for any $a > 0$, we have*

$$\|u\|^2_{L^2(0,a)} \le \frac{a^2}{2}\|\dot u\|^2_{L^2(0,a)}, \quad \|u\|^2_{C[0,a]} \le a\|\dot u\|^2_{L^2(0,a)}.$$

Here and below, the notation $\|\dot u\|_{L^2(0,a)}$ implies that

$$\big\|\dot u|_{(0,a)}\big\|_{L^2(0,a)},$$

which also applies to $\|u\|_{C[0,a]}$, etc.

This lemma is absolutely standard; nevertheless, for completeness of exposition, we provide a sketch of its proof.

Proof of Lemma 3.1 From formula

$$u(t) = \int_0^t \dot u(s)\, ds, \tag{3.1}$$

it follows that

$$\int_0^a u^2(s)\, ds = \int_0^a \Big(\int_0^t \dot u(s)\, ds\Big)^2 dt.$$

It remains to observe that by the Cauchy inequality,

$$\Big|\int_0^t \dot u(s)\, ds\Big| \le \int_0^t |\dot u(s)|\, ds \le \|\dot u\|_{L^2(0,a)}\Big(\int_0^t ds\Big)^{1/2}, \quad t \in [0, a].$$

This implies the first inequality of the lemma. The second inequality also follows from formula (3.1) and the Cauchy inequality.

The lemma is proved. ∎

3.2 Here we collect several spaces that are needed in the sequel.

Definition 3.2 By X denote a space of functions $u \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that for all $t \in \mathbb{R}$, the following conditions hold:

$$u(-t) = -u(t), \quad u(t + \omega) = u(t).$$

By virtue of Lemma 3.1, the mapping $u \mapsto \|\dot u\|_{L^2(0,\omega)}$ determines a norm in X. This norm is denoted by $\|u\|$. The norm $\|\cdot\|$ is equivalent to the standard norm of $H^1[0, \omega]$.

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The space $(X, \|\cdot\|)$ is a Banach space. Since the norm $\|\cdot\|$ is generated by an inner product

$$(u, v)_X = \int_0^\omega \dot{u}(t)\dot{v}(t)\, dt,$$

the space X is also a real Hilbert space; in particular, this implies that X is a reflexive Banach space.

Definition 3.3 Let Φ stand for the space $\{ct + u(t) \mid c \in \mathbb{R},\ u \in X\}$.

By the same argument, $(\Phi, \|\cdot\|)$ is a reflexive Banach space. Observe also that $\Phi = \mathbb{R} \oplus X$, and by direct calculation, we get

$$\|\psi\|^2 = \omega c^2 + \|u\|^2, \quad \psi(t) = ct + u(t) \in \Phi.$$

Observe that $X \subset \Phi$.

Definition 3.4 Let E stand for the space

$$X^m \times \Phi^n = \{z(t) = (x^1, \ldots, x^m, \varphi^1, \ldots, \varphi^n)(t) \mid x^i \in X,\ \varphi^j \in \Phi\}.$$

The space E is also a real Hilbert space with an inner product defined as

$$(z, y)_E = \int_0^\omega \sum_{i=1}^{m+n} \dot{x}^i(t)\dot{y}^i(t)\, dt,$$

where $z = (z^k)$, $y = (y^k) \in E$ for $k = 1, \ldots, m+n$.

We denote the corresponding norm in E by the same symbol and write

$$\|z\|^2 = \big\||z|\big\|^2 = \sum_{k=1}^{m+n} \|z^k\|^2.$$

The space E is also a reflexive Banach space.

We introduce the set

$$E_0 = \left\{(x, \varphi) \in E \mid \varphi^j = \frac{2\pi\nu_j}{\omega}t + u_j, \quad \forall x \in X,\ \forall u_j \in X,\ j = 1, \ldots, n\right\}.$$

This set is a closed plane of codimension n in E.

If $(x, \varphi) \in E_0$, then $\varphi(t + \omega) = \varphi(t) + 2\pi\nu$.

Definition 3.5 Let Y stand for the space

$$\left\{u \in L^2_{\text{loc}}(\mathbb{R}) \mid u(t) = u(-t),\ u(t+\omega) = u(t) \text{ almost everywhere in } \mathbb{R}\right\}.$$

The space Y^{m+n} is a Hilbert space with respect to the inner product

$$(z, y)_Y = \int_0^\omega \sum_{i=1}^{m+n} x^i(t)y^i(t)\, dt,$$

where $z = (z^k)$, $y = (y^k) \in Y^{m+n}$, $k = 1, \ldots, m+n$.

Definition 3.6 Let W stand for the set

$$\{z(\cdot) \in E_0 \mid (t, z(t)) \in F,\ t \in \mathbb{R}\}.$$

3.3 By formula (2.10), the set W is non-void.

We introduce the Action Functional $S\colon W \to \mathbb{R}$,

$$S(z) = \int_0^\omega L(t, z, \dot{z})\, dt.$$

Our next goal is to prove that this functional attains its minimum.

Observe that $|x| \le |z|$; then by using estimates (2.2), we get

$$S(z) \ge \int_0^\omega \left(K|\dot{z}|^2 - |\dot{z}|(C + M|z|) - A|z|^2\right) dt.$$

From the Cauchy inequality and Lemma 3.1, it follows that

$$\int_0^\omega |\dot{z}||z|\, dt \le \frac{\omega}{\sqrt{2}}\|z\|^2, \quad \int_0^\omega |z|^2\, dt \le \frac{\omega^2}{2}\|z\|^2, \quad \int_0^\omega |\dot{z}|\, dt \le \sqrt{\omega}\|z\|.$$

We finally obtain

$$S(z) \ge \left(K - \frac{M\omega}{\sqrt{2}} - \frac{A\omega^2}{2}\right)\|z\|^2 - C\sqrt{\omega}\|z\|. \tag{3.2}$$

Taking into account Remark 2.6, we proceed with the assumption that the manifold F is not compact.

By formula (2.9), the functional S is coercive:

$$S(z) \Longrightarrow \infty, \tag{3.3}$$

as $\|z\| \to \infty$.

Note that the Action Functional that corresponds to system (2.12) is also coercive, but, as we see above, property (3.3) by itself does not imply existence results.

3.4 Let $\{z_k\} \subset W$ be a minimizing sequence,

$$S(z_k) \to \inf_{z \in W} S(z),$$

as $k \to \infty$.

By formula (3.3), the sequence $\{z_k\}$ is bounded: $\sup_k \|z_k\| < \infty$. Since the space E is reflexive, this sequence contains a weakly convergent subsequence. Denote this subsequence in the same way: $z_k \to z_*$ weakly in E.

Moreover, the space $H^1[0, \omega]$ is compactly embedded in $C[0, \omega]$. Thus, extracting a subsequence from the subsequence and keeping the same notation, we also have

$$\max_{t \in [0,\omega]} |z_k(t) - z_*(t)| \to 0, \tag{3.4}$$

as $k \to \infty$.

The set E_0 is convex and strongly closed; therefore, it is weakly closed: $z_* \in E_0$. By continuity (3.4), one also gets $z_* \in W$.

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3.5 Let us show that $\inf_{z\in W} S(z) = S(z_*)$.

Lemma 3.7 *Let a sequence $\{u_k\} \subset \Phi$ weakly converge to $u \in \Phi$ (or $u_k, u \in X$ and $u_k \to u$ weakly in X), and also $\max_{t\in[0,\omega]} |u_k(t) - u(t)| \to 0$ as $k \to \infty$.*

Then for any $f \in C(\mathbb{R})$ and for any $v \in L^2(0,\omega)$, it follows that

$$\int_0^\omega f(u_k)\dot{u}_k v\,dt \to \int_0^\omega f(u)\dot{u}v\,dt, \text{ as } k \to \infty.$$

Indeed,

$$\int_0^\omega f(u_k)\dot{u}_k v\,dt = \int_0^\omega \big(f(u_k) - f(u)\big)\,\dot{u}_k v\,dt + \int_0^\omega f(u)\dot{u}_k v\,dt.$$

The function f is uniformly continuous in a compact set

$$\Big[\min_{t\in[0,\omega]}\{u(t)\} - c,\ \max_{t\in[0,\omega]}\{u(t)\} + c\Big]$$

with some constant $c > 0$. Consequently, we obtain

$$\max_{t\in[0,\omega]}\big|f(u_k(t)) - f(u(t))\big| \longrightarrow 0.$$

Since the sequence $\{u_k\}$ is weakly convergent, it is bounded:

$$\sup_k \|u_k\| < \infty.$$

In particular, we get

$$\|\dot{u}_k\|_{L^2(0,\omega)} < \infty,$$

so that as $k \to \infty$,

$$\Big|\int_0^\omega \big(f(u_k) - f(u)\big)\,\dot{u}_k v\,dt\Big| \leq$$

$$\Big\|v\big(f(u_k) - f(u)\big)\Big\|_{L^2(0,\omega)} \cdot \|\dot{u}_k\|_{L^2(0,\omega)} \longrightarrow 0.$$

To finish the proof, it remains to observe that a function

$$w \longmapsto \int_0^\omega f(u)\dot{w}v\,dt$$

belongs to Φ' (or to X'). Indeed,

$$\Big|\int_0^\omega f(u)\dot{w}v\,dt\Big| \leq \max_{t\in[0,\omega]}|f(u(t))| \cdot \|v\|_{L^2(0,\omega)}\|w\|.$$

3.6 The following lemma is proved similarly.

Lemma 3.8 *Let a sequence $\{u_k\} \subset \Phi$ (or $\{u_k\} \subset X$) be such that*

$$\max_{t\in[0,\omega]}|u_k(t) - u(t)| \longrightarrow 0,$$

as $k \to \infty$.

Then for any $f \in C(\mathbb{R})$ and for any $v \in L^1(0,\omega)$, it follows that

$$\int_0^\omega f(u_k)v\,dt \longrightarrow \int_0^\omega f(u)v\,dt,$$

as $k \to \infty$.

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3.7 Introduce a function $p_k(t,\xi) = L(t, z_k, \dot z_* + \xi)$. The function p_k is a quadratic polynomial of $\xi \in \mathbb{R}^{m+n}$, so that

$$p_k(t,\xi) = L(t, z_k, \dot z_*) + \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)\xi^i + \frac{1}{2}\frac{\partial^2 L}{\partial \dot z^j \partial \dot z^i}(t, z_k, \dot z_*)\xi^i\xi^j.$$

The last term in this formula is non-negative:

$$\frac{\partial^2 L}{\partial \dot z^j \partial \dot z^i}(t, z_k, \dot z_*)\xi^i\xi^j = g_{ij}(t, z_k)\xi^i\xi^j \geq 0.$$

We consequently obtain

$$p_k(t,\xi) \geq L(t, z_k, \dot z_*) + \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)\xi^i.$$

It follows that

$$\begin{aligned} S(z_k) &= \int_0^\omega p_k(t, \dot z_k - \dot z_*)\,dt \\ &\geq \int_0^\omega L(t, z_k, \dot z_*)\,dt + \int_0^\omega \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)(\dot z_k^i - \dot z_*^i)\,dt. \end{aligned} \tag{3.5}$$

From Lemmas 3.7 and 3.8, it follows that

$$\int_0^\omega L(t, z_k, \dot z_*)\,dt \longrightarrow \int_0^\omega L(t, z_*, \dot z_*)\,dt, \quad \text{as } k \to \infty,$$

and

$$\int_0^\omega \frac{\partial L}{\partial \dot z^i}(t, z_k, \dot z_*)(\dot z_k^i - \dot z_*^i)\,dt \longrightarrow 0, \quad \text{as } k \to \infty.$$

Passing to the limit as $k \to \infty$ in (3.5), we finally yield

$$\inf_{z \in W} S(z) \geq S(z_*) \Longrightarrow \inf_{z \in W} S(z) = S(z_*).$$

Remark 3.9 Based on these formulas, one can estimate the norm $\|z_*\|$. Indeed, take a function $\hat z \in W$; then due to formula (3.2), one obtains

$$S(\hat z) \geq S(z_*) \geq \Big(K - \frac{M\omega}{\sqrt{2}} - \frac{A\omega^2}{2}\Big)\|z_*\|^2 - C\sqrt{\omega}\|z_*\|,$$

here $S(\hat z)$ is an explicitly calculable number.

3.8 From this point, we begin proving the theorem under the assumption that the constraints are odd (2.5).

Thus, for any $v \in X^{m+n}$ such that

$$\frac{\partial f_j}{\partial z^k}(t, z_*)v^k(t) = 0,$$

it follows that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(z_* + \varepsilon v) = 0.$$

Introduce a linear functional

$$b \colon X^{m+n} \longrightarrow \mathbb{R}, \quad b(v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(z_* + \varepsilon v),$$

and a linear operator

$$A\colon X^{m+n} \longrightarrow X^l, \quad (Av)_j = \frac{\partial f_j}{\partial z^k}(t, z_*)v^k.$$

It is clear that both these mappings are bounded and $\ker A \subset \ker b$.

Lemma 3.10 *The operator A maps X^{m+n} onto X^l that is*

$$A(X^{m+n}) = X^l.$$

Proof Let $\widetilde{A}(t)$ denote the matrix

$$\frac{\partial f_j}{\partial z^k}\big(t, z_*(t)\big).$$

It is convenient to consider our functions to be defined on the circle $t \in \mathbb{S} = \mathbb{R}/(\omega\mathbb{Z})$.

Fix an element $w \in X^l$. Let us cover the circle $\mathbb{S}$ with open intervals U_i, $i = 1, \dots, N$, such that there exists a set of functions

$$v_i \in H^1(U_i), \quad \widetilde{A}(t)v_i(t) = w(t), \quad t \in U_i,\ i = 1, \dots, N.$$

And let ψ_i be a smooth partition of unity subordinated to the covering $\{U_i\}$. A function $\widetilde{v}(t) = \sum_{i=1}^N \psi_i(t)v_i(t)$ belongs to $H^1(\mathbb{S})$, and for each t it follows that $\widetilde{A}(t)\widetilde{v}(t) = w(t)$. But the function $\widetilde{v}$ is not obliged to be odd.

Since $\widetilde{A}(-t) = \widetilde{A}(t)$, we have

$$\widetilde{A}(t)v(t) = w(t), \quad v(t) = \frac{\widetilde{v}(t) - \widetilde{v}(-t)}{2} \in X^{m+n}.$$

The lemma is proved. ∎

3.9 Recall a lemma from functional analysis [5].

Lemma 3.11 *Let E, H, G be Banach spaces and let*

$$A\colon E \to H \quad \textit{and} \quad B\colon E \to G$$

be bounded linear operators; $\ker A \subseteq \ker B$.

If the operator A is onto, then there exists a bounded operator $\Gamma\colon H \to G$ such that $B = \Gamma A$.

Thus, there is a linear function $\Gamma \in (X^l)'$ such that

$$b(v) = \Gamma A(v) \quad \text{and} \quad v \in X^{m+n}.$$

Or by virtue of the Riesz representation theorem, there exists a set of functions $\{\gamma^1, \dots, \gamma^l\} \subset X$ such that

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(z_* + \varepsilon v) = \int_0^\omega \dot{\gamma}^j(t)\frac{d}{dt}\Big(\frac{\partial f_j}{\partial z^k}(t, z_*)v^k(t)\Big)\,dt$$

for all $v \in X^{m+n}$.

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3.10 Every element $v \in X^{m+n}$ is presented as follows:

$$v(t) = \int_0^t y(s)\,ds,$$

where $y \in Y^{m+n}$ is such that

$$\int_0^\omega y(s)\,ds = 0.$$

Introduce a linear operator $h\colon Y^{m+n} \to \mathbb{R}^{m+n}$ by the formula

$$h(y) = \int_0^\omega y(s)\,ds.$$

Define a linear functional $q\colon Y^{m+n} \to \mathbb{R}$ by the formula

$$q(y) = (b - \Gamma A)v, \quad v(t) = \int_0^t y(s)\,ds.$$

Now all our observations lead to $\ker h \subseteq \ker q$. Therefore, there exists a linear functional $\lambda\colon \mathbb{R}^{m+n} \to \mathbb{R}$ such that $q = \lambda h$.

Let us rewrite the last formula explicitly. There are real constants λ_k such that for any $y^k \in Y$, one has

$$\begin{aligned}\int_0^\omega \Big(\frac{\partial L}{\partial \dot z^k}(t, z_*, \dot z_*) y^k(t) + \frac{\partial L}{\partial z^k}(t, z_*, \dot z_*) \int_0^t y^k(s)\,ds\Big)\,dt \\ = \int_0^\omega \dot y^j(t) \frac{\partial f_j}{\partial z^k}(t, z_*) y^k(t)\,dt + \int_0^\omega \dot y^j(t) \frac{d}{dt}\Big(\frac{\partial f_j}{\partial z^k}(t, z_*)\Big) \int_0^t y^k(s)\,dsdt \\ + \lambda_k \int_0^\omega y^k(s)\,ds.\end{aligned}$$

3.11 By the Fubini theorem, we obtain

$$\begin{aligned}\int_0^\omega \frac{\partial L}{\partial \dot z^k}(t, z_*, \dot z_*) y^k(t)\,dt + \int_0^\omega y^k(s) \int_s^\omega \frac{\partial L}{\partial z^k}(t, z_*, \dot z_*)\,dtds \\ = \int_0^\omega \dot y^j(t) \frac{\partial f_j}{\partial z^k}(t, z_*) y^k(t)\,dt + \int_0^\omega y^k(s) \int_s^\omega \dot y^j(t) \frac{d}{dt}\Big(\frac{\partial f_j}{\partial z^k}(t, z_*)\Big)\,dtds \\ + \lambda_k \int_0^\omega y^k(s)\,ds.\end{aligned} \tag{3.6}$$

In this formula, the functions

$$\frac{\partial L}{\partial \dot z^k}(t, z_*, \dot z_*), \quad \dot y^j(t) \frac{\partial f_j}{\partial z^k}(t, z_*)$$

are even and ω-periodic functions of t.

The functions

$$\dot y^j(t) \frac{d}{dt}\Big(\frac{\partial f_j}{\partial z^k}(t, z_*)\Big), \quad \frac{\partial L}{\partial z^k}(t, z_*, \dot z_*)$$

are odd and ω-periodic, so that the functions

$$\int_s^\omega \dot y^j(t) \frac{d}{dt}\Big(\frac{\partial f_j}{\partial z^k}(t, z_*)\Big)\,ds \quad \text{and} \quad \int_s^\omega \frac{\partial L}{\partial z^k}(t, z_*, \dot z_*)ds$$

are even and ω-periodic in s.

Therefore, equation (3.6) is rewritten as $(y,q)_Y = 0$ for any $y = (y^1, \dots y^{m+n}) \in Y^{m+n}$, and $q = (q_1, \dots, q_{m+n})$ stands for

$$q_k = \frac{\partial L}{\partial \dot z^k}(t, z_*, \dot z_*) + \int_t^\omega \frac{\partial L}{\partial z^k}(s, z_*, \dot z_*)ds - \dot y^j(t)\frac{\partial f_j}{\partial z^k}(t, z_*) - \int_t^\omega \dot y^j(s)\frac{d}{ds}\Big(\frac{\partial f_j}{\partial z^k}(s, z_*)\Big)ds - \lambda_k.$$

Consequently, we obtain the following system:

$$\frac{\partial L}{\partial \dot z^k}(t, z_*, \dot z_*) + \int_t^\omega \frac{\partial L}{\partial z^k}(s, z_*, \dot z_*)ds = \dot y^j(t)\frac{\partial f_j}{\partial z^k}(t, z_*) + \int_t^\omega \dot y^j(s)\frac{d}{ds}\Big(\frac{\partial f_j}{\partial z^k}(s, z_*)\Big)ds + \lambda_k. \tag{3.7}$$

Here, $k = 1, \dots, m+n$.

If we formally differentiate both sides of equations (3.7) in t, then we obtain the Lagrange equations (2.7) with $\alpha^j = \ddot y^j$.

Equations (3.7) hold for almost all $t \in (0, \omega)$ but all the functions contained in (3.7) are defined for all $t \in \mathbb{R}$.

Now we employ the following trivial observation: if $w \in L^1_{\text{loc}}(\mathbb{R})$ is an ω-periodic and odd function, then for any constant $a \in \mathbb{R}$ a function

$$t \longmapsto \int_a^t w(s)ds$$

is also ω-periodic.

Hence the functions

$$\int_t^\omega \dot y^j(s)\frac{d}{ds}\Big(\frac{\partial f_j}{\partial z^k}(s, z_*)\Big)ds, \quad \int_t^\omega \frac{\partial L}{\partial z^k}(s, z_*, \dot z_*)ds$$

are ω-periodic, and equation (3.7) holds for almost all $t \in \mathbb{R}$.

3.12 Let g^{ij} stand for the components of the matrix inverse to $(g_{ij}) : g^{ij}g_{ik} = \delta^j_k$. Present equation (3.7) in the form

$$\dot z^j_*(t) = g^{kj}(t, z_*(t)) \cdot \Big(\lambda_k + \dot y^i(t)\frac{\partial f_i}{\partial z^k}(t, z_*(t)) + \int_t^\omega \dot y^i(s)\frac{d}{ds}\Big(\frac{\partial f_i}{\partial z^k}(s, z_*(s))\Big)ds - \int_t^\omega \frac{\partial L}{\partial z^k}(s, z_*(s), \dot z_*(s))ds - a_k(t, z_*(t))\Big). \tag{3.8}$$

Together with equation (3.8), consider the follwing equations:

$$\frac{\partial f_j}{\partial t}\big(t, z_*(t)\big) + \frac{\partial f_j}{\partial z^k}\big(t, z_*(t)\big)\dot z^k_*(t) = 0. \tag{3.9}$$

These equations follow from (2.3).

Recall that by the Sobolev embedding theorem, $z_* \in X \subset C(\mathbb{R})$.

Due to (2.4), we have

$$\det B(t, z_*) \neq 0, \quad B(t, z_*) = \Big(g^{kj}(t, z_*)\frac{\partial f_i}{\partial z^k}(t, z_*)\frac{\partial f_l}{\partial z^j}(t, z_*)\Big)$$

for all t. Substituting $\dot{z}_*$ from (3.8) to (3.9), we can express $\dot{\gamma}^j$ and see $\dot{\gamma}^j \in C(\mathbb{R})$. Thus, from (3.8), it follows that $\dot{z}_* \in C(\mathbb{R})$.

Applying this argument again, we obtain $\ddot{\gamma}^j, \ddot{z}_* \in C(\mathbb{R})$.

This proves the theorem for the case of odd constraints.

3.13 Let us discuss the proof of the theorem under the assumption that the constraints are even (2.6).

Definition 3.12 By Z denote a space of functions $u \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that for all $t \in \mathbb{R}$,

$$u(-t) = u(t) \quad \text{and} \quad u(t+\omega) = u(t).$$

The space Z^l is a real Hilbert space with respect to an inner product

$$(u,v)_Z = \sum_{i=1}^{l} \int_0^{\omega} \big(u_i(t)v_i(t) + \dot{u}_i(t)\dot{v}_i(t)\big)\,dt.$$

This is the standard inner product in $H^1[0,\omega]$.

So what has changed now? The operator A takes the space X^{m+n} onto the space Z^l. The proof of this fact is the same as in Lemma 3.10.

By the Riesz representation theorem, there exists a set of functions $\{\gamma^1, \dots, \gamma^l\} \subset Z$ such that

$$\begin{aligned} b(v) = &\int_0^{\omega} \dot{\gamma}^j(t) \frac{d}{dt}\Big(\frac{\partial f_j}{\partial z^k}(t,z_*)v^k(t)\Big)\,dt \\ &+ \int_0^{\omega} \gamma^j(t) \frac{\partial f_j}{\partial z^k}(t,z_*)v^k(t)\,dt \end{aligned}$$

for all

$$v = \int_0^t y(s)ds, \quad y \in Y^{m+n}, \quad \int_0^{\omega} y(s)ds = 0,$$

so that equation (3.7) is replaced with the following one

$$\begin{aligned} &\frac{\partial L}{\partial \dot{z}^k}(t,z_*,\dot{z}_*) + \int_t^{\omega} \frac{\partial L}{\partial z^k}(s,z_*,\dot{z}_*)ds \\ &\quad = \dot{\gamma}^j(t)\frac{\partial f_j}{\partial z^k}(t,z_*) + \int_t^{\omega} \dot{\gamma}^j(s)\frac{d}{ds}\Big(\frac{\partial f_j}{\partial z^k}(s,z_*)\Big)\,ds \\ &\quad\quad + \int_t^{\omega} \gamma^j(s)\frac{\partial f_j}{\partial z^k}(s,z_*)ds + \lambda_k. \end{aligned}$$

Here, $k = 1, \dots, m+n$. By the same argument, the functions

$$\int_t^{\omega} \gamma^j(s)\frac{\partial f_j}{\partial z^k}(s,z_*)ds \quad \text{and} \quad \int_t^{\omega} \dot{\gamma}^j(s)\frac{d}{ds}\Big(\frac{\partial f_j}{\partial z^k}(s,z_*)\Big)\,ds$$

are ω-periodic, and one can put $\alpha^j = \ddot{\gamma}^j - \gamma^j$.

The other argument is the same as above. The theorem is proved. ∎

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