# EQUILIBRIUM PRICING TRANSFORMS: NEW RESULTS USING BUHLMANN'S 1980 ECONOMIC MODEL*

BY

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ABSTRACT

In this paper we revisit an economic model of Buhlmann (ASTIN Bulletin, 1980) and derive equilibrium pricing transforms. We obtain the Esscher Transform and the Wang Transform under different sets of assumptions on the aggregate economic environment. We show that the Esscher Transform and the Wang Transform exhibit very different behaviors when used in pricing insurance risks.

KEYWORDS

Equilibrium Pricing, Esscher Transform, Wang Transform, Distorted Probability, Exponential Tilting, Optimal Exchange.

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## 1. INTRODUCTION

In the actuarial research literature, there have developed many probability transforms for pricing financial and insurance risks. Since the pricing of risk is always done in an economic/market environment, it is theoretically desirable to derive pricing transforms from a sound economic model that reflects the collective risk preferences of the market participants. Dr. Hans Buhlmann, in his milestone paper published in 1980 ASTIN Bulletin, has developed such an economic model.

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* This paper is dedicated to Dr. Hans Buhlmann for his tremendous contributions to the actuarial profession and the international actuarial community.

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Buhlmann argued that in real-life situations premiums are not only depending on the risk to be covered but also on the surrounding market conditions. He defined an economic premium principle as

$$H: (X, Z) \rightarrow \mathrm{Price}[X],$$

where $Z$ represents the market condition (e.g., aggregate risk, collective wealth, correlation, etc).

With the goal of developing a sound economic premium principle, Buhlmann considered a risk-exchange model where all individual agents are acting to maximize his/her own expected utility. Buhlmann's risk-exchange model has roots in mathematical economics.

Under a set of assumptions on the aggregate economic environment, Buhlmann derived equilibrium premiums as those obtained from the Esscher Transform, which is a simple exponential tilting of the probability density: $f^*(x) = c \cdot f(x) \cdot \exp(\lambda x)$, where $c$ is a re-scaling constant.

In another major line of research, Venter (1991) made an observation that insurance prices by (excess-of-loss) layer imply a transformed distribution. This inspired Wang (1995, 1996) to propose premium calculation by applying a distortion to the cumulative distribution function:

$$F^*(x) = g[F(x)],$$

where $g: [0, 1] \rightarrow [0, 1]$ is an increasing function with $g(0) = 0$ and $g(1) = 1$. The proportional hazards (PH) transform, an elementary example of a distortion function, is familiar to most actuaries. A newly emerged distortion, the Wang Transform, extends CAPM for underlying assets and Black-Scholes formula for options, which has brought distortion function research to a new territory bordering with financial economics. In this paper we shall discover how the distortion approach is related to Buhlmann's equilibrium pricing model.

In sections 2, we revisit the economic model of Buhlmann and derive equilibrium pricing transforms. We obtain the Esscher Transform and the Wang Transform from the equilibrium model, but under distinct sets of assumptions regarding the aggregate economic environment. By focusing on assumptions underlying these pricing transforms, we gain insights about their differences and connections.

In section 3, we compare the properties of the Esscher Transform and the Wang Transform in pricing insurance risks.

In Appendix A, we discuss "general exponential tilting" to further explore Buhlmann's results. We show that distortion functions are special cases of exponential tilting.

In Appendix B, built upon Buhlmann's results, we discuss how systematic risks can be reflected by the distortion pricing approach.

In Appendix C we give interpretations for the general economic model of Buhlmann (1984).

## 2. Buhlmann's Equilibrium-Pricing Model

Consider risk exchanges among a collective of agents $j = 1, 2, \ldots, n$, (typically reinsurers, insurers, buyers of direct insurance, etc).

Each agent is characterized by his/her

(i) utility function $u_j(x)$, with $u_j'(x) > 0$, and $u_j''(x) \le 0$;

(ii) initial wealth $W_j$.

Each agent $j$ is facing a risk of potential loss $X_j(\omega)$ and is buying a risk-exchange $Y_j(\omega)$, where $\omega$ represents a state in a probability space ($\Omega$, P). If agent $j$ is an insurance company, we can think of $Y_j(\omega)$ as the sum of all (re)insurance policies bought and sold by $j$ as if it were "one" contract.

Whereas the original risk $X_j$ belongs to agent $j$, the risk exchange $Y_j$ can be freely bought/sold by agent $j$ in the market. Buhlmann introduced the concept of a pricing density $\phi(\omega)$ such that

$$\mathrm{Price}\left[Y_j\right] = \int_\Omega Y_j(\omega)\phi(\omega)d\mathrm{P}(\omega), \tag{2.1}$$

Buhlmann pointed out that the pricing density $\phi(\omega)$ could be understood as an alteration of the actuarially objective probabilities.

**Definition 2.1:** The pair $\{Y_{e,j}$ and $\phi_e\}$ are called in *equilibrium* if

(C-1). For all $j$, $E\left[u_j(W_j - X_j + Y_{e,j} - \mathrm{Price}[Y_{e,j}])\right]$ is maximum among all possible choices of the exchange variables $Y_j$.

(C-2). $\sum_{j=1}^{n} Y_{e,j}(\omega) = 0$ for all $\omega$ in $\Omega$.

In the equilibrium, $Y_{e,j}$ is called the *equilibrium exchange*, and $\phi_e$ the *equilibrium price density*.

**Theorem 2.1 [Buhlmann, 1980]** Assume that each agent $j$ has an exponential utility function $u_j(x) = 1 - \exp(-\lambda_j x)$, the equilibrium price density satisfies:

$$\phi_e(w) = \frac{\exp(\lambda Z(\omega))}{E[\exp(\lambda Z)]}, \tag{2.2}$$

where

$$Z(\omega) = \sum_{j=1}^{n} X_j(\omega) \tag{2.3}$$

is the aggregate risk, and $\lambda$ satisfies

$$\frac{1}{\lambda} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n}. \tag{2.4}$$

From Theorem 2.1, the equilibrium price for any risk $X$ is

$$H_{Buhlmann}[X, \lambda] = \frac{E[X \cdot \exp(\lambda Z)]}{E[\exp(\lambda Z)]}, \tag{2.5}$$

with Z in equation (2.3) and $\lambda$ in equation (2.4).

Buhlmann (1980) further assumed that $X$ and $Z - X$ are independent, and derived that

$$H_{Buhlmann}[X,\lambda] = \frac{E[X \cdot \exp(\lambda X)] \cdot E[\exp(\lambda(Z-X))]}{E[\exp(\lambda X)] \cdot E[\exp(\lambda(Z-X))]} = \frac{E[X \cdot \exp(\lambda X)]}{E[\exp(\lambda X)]}. \quad (2.6)$$

**Theorem 2.2** Under the set of assumptions:

(AS-1a): The insurance market contains a small number of agents, and
(AS-1b): Individual risk $X$ is independent from $Z - X$, where $Z$ is the aggregate risk,

the equilibrium price in equation (2.6) is the same as that obtained from the Esscher Transform:

$$f^*(x) = \frac{f(x)\exp(\lambda x)}{E[\exp(\lambda X)]}.$$

Now we examine more carefully the assumptions underlying the derivation of the Esscher Premium in equation (2.6).

For an insurance market with a large number of agents (policy-holders, insurers and reinsurers), the size of an individual risk $X$ is negligible relative the industry aggregate loss $Z$. According to equation (2.4), the parameter $\lambda$ will be close to zero. Using equation (2.6) we get

$$\lim_{\lambda \to 0+} H_{Buhlmann}[X,\lambda] = \lim_{\lambda \to 0+} \frac{E[X \cdot \exp(\lambda X)]}{E[\exp(\lambda X)]} = E[X].$$

For an insurance market in which any individual risk is negligible relative to the size of the aggregate risk, under the assumption that $X$ and $Z - X$ are independent, the equilibrium premium for risk $X$ equals the expected loss without risk loading.

To avoid the complexity of dealing with infinitely large $Z$ and infinitely small $\lambda$, it is useful to re-scale $Z$ to $Z_0 = (Z - E[Z])/\sigma[Z]$ and rewrite (2.5) into the following:

$$H_{Buhlmann}[X,\lambda_0] = \frac{E[X \cdot \exp(\lambda_0 Z_0)]}{E[\exp(\lambda_0 Z_0)]}. \quad (2.7)$$

Note that $Z_0$ has mean = 0 and variance = 1. For the re-scaled aggregate risk $Z_0$, the parameter $\lambda_0$ represents the market price per unit of risk.

To carry on the analysis of Buhlmann (1980), we make the following set of revised assumptions:

(AS-2a). In aggregate, the total loss $Z$ has a normal distribution, thus the re-scaled variable $Z_0 = (Z - E[Z])/\sigma[Z]$ has a standard normal distribution $\Phi$.
(AS-2b). For risk $X$ with cdf $F(x)$, there exists a standard normal variable $V$ such that $X = F^{-1}(\Phi(V))$, and $\{V, Z_0\}$ have a bivariate normal distribution with correlation coefficient $\rho$.

**Remarks**:

- Assumption (AS-2a) is reasonable for an insurance market in which (i) there are a large number of agents and uncorrelated risks, and (ii) each individual risk is negligible in size relative to the aggregate industry risk.
- Assumption (AS-2b) is a direct extension of the multivariate normal assumption used in the derivation of CAPM. For risks with general marginal distributions, here we are assuming a normal-copula correlation structure between $X$ and $Z$ (see Wang, 1998; Frees and Valdez, 1998; Embrechts et al. 2002).

Based on the assumption (AS-2b) that $\{V, Z_0\}$ have a bivariate normal distribution with correlation coefficient $\rho$, the variable $Y = Z_0 - \rho \cdot V$ is independent of $X$. Taking $Z_0 = \rho \cdot V + Y$ into equation (2.7), and using the independence between $X$ and $Y$, we have

$$H_{Buhlmann}[X, \lambda] = \frac{E\left[X \cdot \exp(\lambda_0 \rho \cdot V)\right] \cdot E\left[\exp(\lambda_0 Y)\right]}{E\left[\exp(\lambda_0 \rho \cdot V)\right] \cdot E\left[\exp(\lambda_0 Y)\right]},$$

which further leads to

$$H_{Buhlmann}[X, \lambda] = \frac{E\left[X \cdot \exp(\lambda V)\right]}{E\left[\exp(\lambda V)\right]}, \quad \text{where } \lambda = \rho\lambda_0. \tag{2.8}$$

**Theorem 2.3** Under the set of assumptions in (AS-2a) and (AS-2b), the equilibrium premium in equation (2.8) is identical to that obtained by the Wang Transform

$$F^*(x) = \Phi[\Phi^{-1}(F(x)) - \lambda],$$

with $\lambda = \rho\lambda_0$.

**Proof:** See Appendix A, Example A.1.

Recall that $\lambda_0$ represents the aggregate market price per unit of risk and $\rho$ is the correlation coefficient between the normalized variables $V$ and $Z_0$. The relation $\lambda = \rho\lambda_0$ is a generalization of the classic CAPM to risks with general probability distributions (see Wang, 2000, 2002).

**Remark:** As noted in Buhlmann (1984), the main result in Theorem 2.1 is still valid under general utility function assumptions for the participants. Therefore, under the assumptions (AS-2a) & (AS-2b), Theorem 2.3 effectively gives an independent derivation of CAPM.

The correlation between risk $X$ and the aggregate portfolio risk $Z$ is the main driver for risk load. The relation $\lambda = \rho\lambda_0$ is rather intuitive since highly correlated risks demand higher risk loading, such as natural or man-made catastrophe risks. In practice, the meaning of correlation should be interpreted more broadly than the statistical association in the claim generating process. From an insurer's perspective, the correlation in profitability between insurance contracts is as important as the correlation in insurance claims. Parameter uncertainty, pricing cycle, and regulatory capital requirements all contribute to the correlation in profitability between insurance contracts.

## 3. Esscher Transform versus Wang Transform

Consider a variable $X$ with a probability (density) function $f(x)$ and cumulative distribution function $F(x)$. The Esscher Transform applies an exponential tilting to the probability (density) function:

$$f^*(x) = \frac{f(x)\exp(\lambda x)}{E[\exp(\lambda X)]}. \tag{3.1}$$

The Wang Transform is directly applied to the cumulative distribution:

$$F^*(x) = \Phi\left(\Phi^{-1}(F(x)) - \lambda\right). \tag{3.2}$$

The Esscher Transform has received considerable attention, thanks to Buhlmann's economic model, and subsequently the work of Gerber and Shiu (1994) in pricing options. The Esscher Transform has become the subject for numerous recent papers and doctorial thesis. Buhlmann et al. (1998) advocate that Esscher Transform qualifies as a general formula for pricing financial risks.

The Wang Transform is a newly emerged distortion function among the distortion family that includes the PH-transform. Under a set of axioms, Wang, Young and Panjer (1997) showed that all coherent risk measures can be represented by a distortion. Among the family of distortions, only the Wang Transform can recover CAPM for underlying assets and Black-Scholes formula for options.

We have seen the Esscher Transform and the Wang Transform both coming out of Buhlmann's equilibrium pricing model. Despite this connection, the Esscher Transform and the Wang Transform exhibit dramatically different behaviors when used in pricing insurance risks.

For the Esscher transform in (3.1) we denote

$$H_{Esscher}[X;\lambda] = E^*[X] = \frac{E[X \cdot \exp(\lambda X)]}{E[\exp(\lambda X)]}. \tag{3.3}$$

For the Wang Transform in (3.2) we denote

$$H_{Wang}[X;\lambda] = E^*[X] = \int_{-\infty}^{0} F^*(x)dx + \int_{0}^{+\infty} [1 - F^*(x)]\,dx. \tag{3.4}$$

When $X$ has a Normal$(\mu, \sigma^2)$ distribution, the Esscher Transform gives another normal distribution with $\mu^* = \mu + \lambda\sigma^2$, and $\sigma^* = \sigma$. Thus, for normally distributed risks, the Esscher premium recovers the variance loading:

$$H_{Esscher}[X;\lambda] = E[X] + \lambda \cdot Var[X].$$

When $X$ has a Normal$(\mu, \sigma^2)$ distribution, the Wang Transform gives another normal distribution with $\mu^* = \mu + \lambda\sigma$, and $\sigma^* = \sigma$. Thus, for normally distributed risks, the Wang Transform reduces to the standard deviation loading:

$$H_{Wang}[X;\lambda] = E[X] + \lambda \cdot \sigma[X].$$

For any positive constant $b$, it can be shown that

$$H_{Esscher}[bX;\lambda] = b \cdot H_{Esscher}[X;b\lambda]. \tag{3.5}$$

For the Esscher Transform, the pricing parameter $\lambda$ depends on the scale of $X$. Not surprisingly, the Esscher premium principle, which uses the same $\lambda$ to price risks of various sizes, is not coherent in the sense of Artner et al (1999).

For any positive constant $b$, it can be shown that

$$H_{Wang}[bX;\lambda] = b \cdot H_{Wang}[X;\lambda]. \tag{3.6}$$

For the Wang Transform, the parameter $\lambda$ is independent of the scale of $X$.

The Esscher premium does not always preserve stochastic dominance, while the Wang Transform does (see Wang, 2000).

**Example 3.1** Consider two loss variables $X$ and $Y$ with

$P[X = 0,\ Y = 0] = P[X = 0,\ Y = 1] = P[X = 3,\ Y = 3] = 1/3.$

Clearly we have $X \leq Y$. [Taken from Kaas et al., 1994, pp. 17].

(a) For the Esscher Transform with $\lambda = 1$, we get $H_{Esscher}[X; \lambda] = 2.73$, which is greater than $H_{Esscher}[Y; \lambda] = 2.65$. This counter-intuitive result is due to the fact that Var$(X) = 2$ is a greater than Var$(Y) = 1.56$, and the Esscher premium behaves like a variance loading. Note that this violation of stochastic dominance can be avoided if we use different $\lambda$ values for the two variables $X$ and $Y$.

(b) For the Wang Transform with $\lambda = 1$, $H_{Wang}[X; \lambda] = 2.15$, which is less than $H_{Wang}[Y; \lambda] = 2.35$. The Wang Transform preserves the stochastic dominance between $X$ and $Y$.

The Esscher premium is not layer-additive, while the Wang Transform produces additive premiums by layer (see Wang, 1996).

**Example 3.2** Consider a risk $X$ with

$P[X = 0] = P[X = 1] = P[X = 2] = 1/3.$

We divide $X$ into two (excess-of-loss) layers: $X_{(0,1]}$ represents the 1xs0 layer, and $X_{(1,2]}$ represents the 1xs1 layer. Clearly $X = X_{(0,1]} + X_{(1,2]}$.

(a) For the Esscher Transform with $\lambda = 0.4$, we have $H_{Esscher}[X_{(0,1]}; \lambda] + H_{Esscher}[X_{(1,2]}; \lambda] = 0.749 + 0.427 = 1.176$ which is less than $H_{Esscher}[X; \lambda] = 1.260$.

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(b) For the Wang Transform with $\lambda = 0.4$, we have $H_{Wang}[X_{(0,1]}; \lambda] + H_{Wang}[X_{(1,2]}; \lambda] = 0.797 + 0.488$, which is exactly equal to $H_{Wang}[X; \lambda] = 1.285$. The Wang Transform is layer-additive.

When it comes to portfolio diversification, the Esscher Transform is additive for independent risks, and super-additive for positively correlated risks. By contrast, the Wang Transform is sub-additive for independent risks, and additive for co-monotone risks.

**Example 3.3** Consider a portfolio consisting of two risks: $X \sim \text{Normal}(0,1)$ and $Y \sim \text{Normal}(0,1)$. Assume that $X$ and $Y$ have bivariate normal distributions with correlation $\rho$. We have $X + Y \sim \text{Normal}(0, 2(1 + \rho))$.

(a) For the Esscher Transform we have

$$H_{Esscher}[X + Y; \lambda] = \lambda(1 + \rho),$$

and

$$H_{Esscher}[X; \lambda] + H_{Esscher}[Y; \lambda] = \lambda.$$

When $X$ and $Y$ are positively correlated,

$$H_{Esscher}[X + Y; \lambda] > H_{Esscher}[X; \lambda] + H_{Esscher}[Y; \lambda] \text{ for } \lambda > 0.$$

(b) For the Wang Transform we have

$$H_{Wang}[X + Y; \lambda] = \lambda\sqrt{2(1+\rho)},$$

and

$$H_{Wang}[X; \lambda] + H_{Wang}[Y; \lambda] = 2\lambda.$$

For $\rho < 1$ we have sub-additivity:

$$H_{Wang}[X + Y; \lambda] < H_{Wang}[X; \lambda] + H_{Wang}[Y; \lambda],$$

reflecting the benefit of portfolio diversification.

Equality holds if and only if (=1, i.e., when $X$ and $Y$ are perfectly correlated.

Zehnwirth (1981), Gerber (1981), and Van Heerwaarden et al. (1989) expressed similar criticisms of the counter-intuitive properties of the Esscher premium principle.

The Wang Transform satisfies all the desirable properties for a coherent risk measure in Artzner et al. (1999), and is consistent with the first four axioms in Wang, Young and Panjer (1997).

The Wang Transform in (3.2) has close connections with financial economics. It can be used to price assets by switching the sign of $\lambda$ from "minus" to "plus". For a normally distributed asset return $R$ and risk-free interest rate $r$, we get an implied $\lambda = (\mathrm{E}[R] - r)/\sigma[R]$, which is exactly the Sharpe Ratio, a common benchmark for asset risk-return tradeoff. When applied to the aggregate loss distribution for a risk portfolio, the Wang Transform quantifies the diversification benefit, with parameter $\lambda$ extending the traditional Sharpe Ratio used in asset portfolio management.

The Esscher Transform has shown tremendous successes in pricing options, as demonstrated in Gerber and Shiu (1994) and numerous follow-up papers. However, as pointed out by Mildenhall (2000), there are fundamental differences between actuarial-pricing approaches and the option-pricing paradigm. The lack of coherence of the Esscher Transform only reduces its effectiveness in pricing insurance risks. Apparently this does not affect its successful applications in option pricing.

## 4. Summary

In an incomplete market such as insurance, equilibrium prices will generally depend on assumptions about the utility functions of the market participants. Buhlmann's (1980) economic model is very profound in that equilibrium pricing transforms can be derived under general utility functions of the market participants. Other reference papers on equilibrium risk-exchanges include Aase (1993), Taylor (1992), Gerber and Pafumi (1998). In a practical context, Meyers (1996) also takes an equilibrium pricing approach to calculating risk load.

Using Buhlmann's optimal risk-exchange model, we have derived the Esscher Transform and the Wang Transform as two equilibrium-pricing transforms, but under distinct sets of assumptions regarding the aggregate economic environment. We also showed that the Wang Transform satisfies the required properties for a coherent risk measure, while the Esscher Transform does not.

We encourage interested readers to read the Appendices for a more in-depth discussion of Buhlmann's economic model and its intimate connections to the distortion pricing approach.

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## Appendix A.
## EXPONENTIAL TILTING & DISTORTION

To explore further Buhlmann's main results in equation (2.2) and (2.5), we define a general exponential tilting and discuss its connections with the distortion transform.

Consider variables $X$ and $Z$ in a probability space ($\Omega$, P) with probability distributions $F$ and $Q$, respectively.

**Definition A.1** The transformed probability (density) function

$$f^*(x) = f(x) \cdot \frac{E[\exp(\lambda Z) \mid X = x]}{E[\exp(\lambda Z)]}, \tag{A.1}$$

is called an exponential tilting of $X$, induced by $Z$.

Note that equation (A.1) is very general, allowing for any correlation structure between $X$ and $Z$. One can assume any copula between $X$ and $Z$, and use numerical techniques to calculate $f^*(x)$. Recall that Theorem 2.3 considers the case that $X$ and $Z$ have a normal copula, and $Z$ has a normal distribution. In this Appendix we consider some special cases in which equation (A.1) yields mathematically tractable results.

As a special case of Definition A.1, when $Z = X$ we recover the Esscher Transform. When $Z = h(X)$ is an increasing function of $X$, we recover the generalized Esscher transform by Heilmann (1989), including the special case of $h(X) = 1 - \exp(-\lambda X)$ in Kamps (1998).

Next we consider the cases that $X$ and $Z$ are co-monotone while the probability distribution of Z is unaffected by the distribution of $X$. We say that $X$ and $Z$ are co-monotone if there exists a uniform random variable $U$ such that $X = F^{-1}(U)$ and $Z = Q^{-1}(U)$.

Note that for any probability distribution $F(x)$, the inverse function is defined as

$$F^{-1}(u) = \sup\{x \colon F(x) < u\}, \text{ for } 0 \le u \le 1.$$

Here "co-monotone" means perfect correlation, which extends beyond the concept of perfect linear correlation. There is no diversification benefit between "co-monotone" risks, see Wang and Dhane (1998). When $X$ and $Z$ are co-monotone, it may be impossible to express $Z$ as a direct function of $X$. For example, consider the case that $X$ has a Bernoulli distribution and $Q$ is an exponential distribution.

**Theorem A.1** When $X$ and $Z$ are co-monotone, the exponential tilting in equation (A.1) implies a transform: $F(x) \to F^*(x)$ by

$$F^*(x) = \frac{1}{M_Q(\lambda)} \int_0^{F(x)} \exp\left(\lambda \cdot Q^{-1}(u)\right) du, \tag{A.2}$$

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where

$$M_Q(\lambda) = \int_0^1 \exp\left(\lambda \cdot Q^{-1}(u)\right) du$$

exists for some $\lambda > 0$.

- When $F$ is continuous at $x$, the transformed probability density at $x$ is

$$f^*(x) = \frac{f(x) \cdot \exp\left(\lambda \cdot Q^{-1}(F(x))\right)}{M_Q(\lambda)}. \tag{A.3}$$

- When $F$ is discrete on $\{x_1, x_2, \ldots, x_m\}$, the transformed probability at $x_j$ is

$$f^*\left(x_j\right) = \frac{1}{M_Q(\lambda)} \int_{F(x_{j-1})}^{F(x_j)} \exp\left(\lambda \cdot Q^{-1}(u)\right) du. \tag{A.4}$$

We call equation (A.2) a *co-monotone exponential tilting*, induced by the *kernel Q*.

Consider an important case when the kernel $Q$ is unrelated to the distribution $F$.

**Theorem A.2** When the kernel $Q$ is independent of the distribution $F$, the co-monotone exponential tilting (A.2) is equivalent to a distortion $F^*(x) = g(F(x))$ with

$$g(u) = \frac{\int_0^u \exp\left(\lambda \cdot Q^{-1}(v)\right) dv}{M_Q(\lambda)}. \tag{A.5}$$

**Proof:** The key here is that $Q$ is independent of $F$. It then follows directly from the co-monotone exponential tilting equation (A.2).

□

**Example A.1** When the kernel $Q = \Phi$ is the standard normal distribution, the co-monotone exponential tilting (A.2) recovers the Wang Transform:

$$F^*(y) = \Phi\left(\Phi^{-1}(F(y)) - \lambda\right).$$

**Example A.2** When the kernel $Q(t) = 1 - \exp(-t)$, for $t \geq 0$, is an exponential distribution, the co-monotone exponential tilting (A.2) recovers the proportional hazards (PH) transform as introduced in Wang (1995):

$$F^*(x) = 1 - (1 - F(x))^{1-\lambda}, \text{ with } 0 \leq \lambda < 1,$$

which corresponds to the distortion $g(u) = 1 - (1 - u)^{1-\lambda}$.

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This result gives an interpretation of the parameter $\lambda$ in the PH-transform.

**Example A.3** When the kernel $Q(t) = t$, for $0 \le t \le 1$ is a uniform distribution, the co-monotone exponential tilting (A.2) recovers the exponential distortion:

$$g(u) = \frac{\exp(\lambda u) - 1}{\exp(\lambda) - 1}, \quad \text{for } 0 < u \le 1.$$

**Example A.4** When the kernel $Q(z) = \text{Gamma}(z; \alpha, \beta)$ has a gamma distribution with mean $= \alpha/\beta$, the co-monotone exponential tilting (A.2) corresponds to the following distortion:

$$g(u) = \text{Gamma}\left(Q^{-1}(u); \alpha, \beta - \lambda\right).$$

For the distortion in (A.5) we have

$$g'(u) = \exp\left(\lambda \cdot Q^{-1}(u)\right) > 0. \tag{A.6}$$

When $Q(x)$ is differentiable, we have

$$g''(u) = \lambda \exp\left(\lambda \cdot Q^{-1}(u)\right) / Q'\left(Q^{-1}(u)\right) \ge 0, \quad \text{for } \lambda > 0. \tag{A.7}$$

For the Beta family of distortion (Wirch and Hardy, 1999)

$$F(x) = 1 - \int_0^{1-F(x)} \frac{t^{a-1}(1-t)^{b-1}}{\Gamma(a,b)} dt$$

to be an exponential tilting with $\lambda > 0$, it is necessary that $a \le 1$ and $b \ge 1$.

Finally we offer a comment on the numerical implementations of co-monotone exponential tilting. Computer calculations almost always use discrete data points. Consider a discrete representation $\{x_1 < x_2 < \ldots < x_m\}$ of variable $X$, where $F(x_m) = 1$. One would need to perform numerical integration to carry out the exponential tilting as shown in equation (A.4). The reader needs to be aware of that the simple approximation by

$$f^*(x_j) \approx f(x_j) \exp\left(\lambda \cdot Q^{-1}\left(F(x_j)\right)\right) / M_Q(\lambda) \tag{A.8}$$

is often very poor, especially at the tails of the distribution. For the Wang Transform and PH-transform, we have $Q^{-1}(F(x_m)) = +\infty$, thus the approximation $f^*(x_m)$ in (A.8) is undefined. Such numerical difficulties can be avoided by applying the distortion in (A.5) directly on the (discrete) cumulative distribution function.

## Appendix B.
## SYSTEMATIC RISKS & THE DISTORTION APPROACH

Consider Buhlmann's economic model in section 2, we assume that risk $X$ can be decomposed into two parts

$$X = X_{sys} + X_{non},$$

where

- $X_{sys}$ (being co-monotone with $Z$) represents the systematic portion of $X$, and
- $X_{non}$ (being uncorrelated with $Z$) represents the idiosyncrasy or non-systematic portion.
- By definition, $X_{sys}$ and $X_{non}$ are uncorrelated.

From equations (2.5) & (2.6) we have

$$H_{Buhlmann}[X,\lambda] = E\left[X_{non}\right] + \frac{E\left[X_{sys}\cdot \exp(\lambda Z)\right]}{E\left[\exp(\lambda Z)\right]}. \tag{B.1}$$

In other words, Buhlmann's equilibrium pricing model indicates that only the systematic risk requires risk loading.

For convenience, we assume that the distribution $F$ for a risk $X$ only reflects the systematic risk of $X$. For practically minded reader, this is quite in agreement with reality. For instance, life insurers are generally not too concerned about the volatility of an individual life contract, but rather more concerned about the systematic errors in their estimate of mortality rates, and systematic shocks. As a result, in the pricing exercise by insurers, only systematic risks enter into the distribution $F$, manifested in the modeling of potential variations for a large block of contracts, or for a whole line of business, etc.
In light of equation (B.1) we now make the following simplifying assumptions:

(**AS-3a**). All potential variations that are reflected in the distribution $F_j$ are systematic risk only. As a result, risk $X_j$ is co-monotone with the aggregate risk $Z$.

(**AS-3b**). There are many market participants so that the re-scaled aggregate variable $Z_0 = (Z - E[Z]) / \sigma[Z]$ has a distribution $Q$ which is unrelated to the individual risk distribution $F_j$.

**Theorem B.1** Under the assumptions in (AS-3a) and (AS-3b), the equilibrium price density in equation (2.2) is identical to the distortion $F^*(x) = g(F(x))$ with

$$g(u) = \frac{\int_0^u \exp\left(\lambda\cdot Q^{-1}(v)\right)dv}{M_Q(\lambda)}.$$

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Specially,

- when $Q$ is an Exponential(1) distribution, we recover the PH-transform;
- when $Q$ is standard normal distribution, we recover the Wang Transform.

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## APPENDIX C.
## BUHLMANN'S GENERAL ECONOMIC MODEL

In a follow-up paper, Buhlmann (1984) extended his economic premium principle using general utility functions for each participant. He discovered that all equilibrium prices are locally like the one where agents have exponential utilities. The only difference lies in that risk aversion is no longer constant but depends on the agents' net wealth. His general economic model provides further insights on the relation between risk premium and aggregate market conditions.

Under general utility assumptions, Buhlmann used the notion of absolute risk aversion of Pratt (1964):

$$\lambda_j(x) = -u_j''(x)\,/\,u_j'(x), \tag{C.1}$$

which depends on the amount of net wealth $x$. Note that for an exponential utility function $u_j(x) = 1 - \exp(-\lambda_j x)$ we have $\lambda_j(x) = \lambda_j$ being a constant.

Buhlmann showed that equilibrium exists under only modest theoretical assumptions. He pointed out that this equilibrium also coincides with the Pareto optimal exchange in Borch (1962).

As an important observation, Buhlmann pointed out that in equilibrium $Y_{e,j}$ and $\phi_e$ should depend on ω only through $Z(\omega) = \sum_{j=1}^{n} X_j(\omega)$.

**Theorem C.1** [Buhlmann, 1984] Under general utility assumptions, the equilibrium pricing density satisfies

$$\frac{\phi_e'(Z)}{\phi_e(Z)} = \lambda(W, Z), \tag{C.2}$$

where $\lambda(W, Z)$ is the total risk aversions satisfying

$$\frac{1}{\lambda(W,Z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j\left(W_j - \text{"Net Loss to } j \text{ given total loss Z"}\right)}, \tag{C.3}$$

where "Net Loss to $j$ given total loss Z" $= X_j(Z) - Y_{e,j}(Z) + \mathrm{Price}[Y_{e,j}(Z)]$.

From equation (C.2) we can see that the local behavior for the equilibrium pricing density is the same as that for the exponential utilities.

Theorem C.1 provides additional insights to the parameter $\lambda = \lambda(W, Z)$. As the total risk-aversion of the market, the parameter $\lambda$ depends on the aggregate net wealth (or capital) of market participants. The presence of excessive capital will drive down the parameter $\lambda = \lambda(W, Z)$ and the resulting risk premiums. A shortage of capital can boost the level of $\lambda = \lambda(W, Z)$ and the resulting risk premiums.

The insurance industry has experienced surprises by unexpected catastrophe events, for instance, the huge insurance losses due to 1992 Hurricane Andrew, and the September 11, 2001 Terror Attack on America. Buhlmann's economic model can explain some of the after effects of unexpected catastrophe events:

a) As a Bayesian update, the estimated probability of loss will increase, especially for large loss amounts.

b) A catastrophe may simultaneously impact many lines of business. This will elevate the perceived correlation between lines of business, and have an effect of increasing the systematic risk for $X_j$.

c) The market price of risk, $\lambda = \lambda(W, Z)$ in equation (C.3), will increase because of the depletion of the aggregate wealth after paying for the occurred catastrophe loss. Note that the September 11 event did not increase the likelihood of California earthquake, however the price of earthquake cover increased significantly after September 11, due to an increase in the market price of risk, $\lambda = \lambda(W, Z)$.

d) The combined effect of these factors is a dramatic increase in risk load and premium rates.

e) Because of the increase in the prospective Sharpe ratio $\lambda = \lambda(W, Z)$ in (C.3), "smart" capital may be injected from the outside to take advantage the increased risk-return prospect, as evidenced in new entrants to the insurance market after hurricane Andrew & Terror Attack.

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