## Series summable as Arithmetical Progressions.

We may group the terms of the $A.P.\ a,\ a+d,\ a+2d,\ \ldots\ldots$ as follows: the first $p$ terms, the next $\overline{p+t}$ terms, the next $\overline{p+2t}$ terms, and so on. The sum of the terms comprising the $n^{\text{th}}$ group is

$$\tfrac{1}{2}(p-t+nt)\{2a+d(t-p-1)+2nd(p-t)+n^2td\},$$

which is a rational integral function of $n$ of the third degree.

Conversely, if a cubic in $n$, of the above form, is the $n^{\text{th}}$ term of a series, then the sum to $n$ terms of the series is equivalent to the sum to $\{p+\overline{p+t}+\overline{p+2t}+\ldots+\overline{p+(n-1)t}\}$ or $\dfrac{n}{2}\{2p+(n-1)t\}$ terms of the $A.P.\ a,\ a+d,\ a+2d,\ \ldots$.

To find the condition that the cubic $An^3+3Bn^2+Cn+D$ may be of the above form we have, on equating coefficients of like powers of $n$,

$$\begin{cases} A=\tfrac{1}{2}t^2d & \ldots\ldots(1) \\ B=\tfrac{1}{2}td(p-t) & \ldots\ldots(2) \\ C=at+\tfrac{1}{2}td(t-p-1)+d(p-t)^2 & \ldots\ldots(3) \\ D=(p-t)\{a+\tfrac{1}{2}d(t-p-1)\}. & \ldots\ldots(4) \end{cases}$$

From (1) and (2), $\dfrac{B}{A}=\dfrac{p-t}{t}$.

From (3) and (4), $D=\dfrac{p-t}{t}\{C-d(p-t)^2\}$

$$=\frac{B}{A}\left\{C-\frac{2B^2}{A}\right\}$$

or $\quad 2B^3=A(BC-D).\ \ldots\ldots(5)$

If the condition (5) is satisfied, and $B$ is not zero, then $p\neq t$, and we can write

$$\begin{cases} d=\dfrac{2A}{t^2}, \\ p=t\left(1+\dfrac{B}{A}\right) \\ a=\dfrac{B^2+AD}{Bt}+\dfrac{A}{t^2}, \end{cases}$$

where $t$ may be given any integral value which will make $\dfrac{tB}{A}$ integral.

If $B$ is zero then $p = t$ and $D$ must also be zero. In this case we have

$$\begin{cases} d = \dfrac{2A}{t^2} \\ p = t, \\ a = \dfrac{C}{t} + \dfrac{A}{t^2}. \end{cases}$$

*Example*: Find the sum to $n$ terms of the series whose $n^{\text{th}}$ term is $n(n+1)(2n+1)$

Here $A = 2$, $B = 1$, $C = 1$, $D = 0$, and condition (5) is satisfied.

Taking $t = 2$, we have $d = 1$, $p = 3$, $a = 1$, and

$$\frac{n}{2}\{2p + (n-1)t\} = \frac{n}{2}(2n+4) = n(n+2).$$

The sum to $n$ terms of our series is thus equivalent to the sum of $n(n+2)$ terms of the $A.P.$ 1, 2, 3, ..., and is therefore equal to $\frac{1}{2}n(n+2)(n+1)^2$.

A. A. KRISHNASWAMI AYYANGAR.

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## Kepler's Law of Refraction.

The correct law of refraction, $\sin i = \mu \sin r$, is usually assumed to have been first discovered by Snell in 1621, although he did not express it in this form. The astronomer Kepler laboured hard to discover it, but in vain, and Whewell in the *History of the Inductive Sciences* says that it is strange that he should have failed, when the law is so simple. There is, however, a good reason for his want of success.

Kepler attacks the question in chap. IV. of the *Paralipomena ad Vitellionem* which was printed at Frankfort in 1604. The problem is to get a mathematical expression which will fit Vitellio's table of the refraction from air to water. After a vain attempt to connect it with various properties of the conic sections he gives a practical rule which is equivalent to the formula

$$i = \frac{\mu r}{\mu - (\mu - 1)\sec r},$$



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