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# CONTACT PROBLEMS FOR NONLINEARLY ELASTIC MATERIALS: WEAK SOLVABILITY INVOLVING DUAL LAGRANGE MULTIPLIERS

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**Abstract**

A class of problems modelling the contact between nonlinearly elastic materials and rigid foundations is analysed for static processes under the small deformation hypothesis. In the present paper, the contact between the body and the foundation can be frictional bilateral or frictionless unilateral. For every mechanical problem in the class considered, we derive a weak formulation consisting of a nonlinear variational equation and a variational inequality involving dual Lagrange multipliers. The weak solvability of the models is established by using saddle-point theory and a fixed-point technique. This approach is useful for the development of efficient algorithms for approximating weak solutions.

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## 1. Introduction

There is an abundance of literature in engineering, geophysics and biomechanics involving contact models, covering both frictionless and frictional contact. Publications often focus on specific settings, geometries or materials [3–5, 7, 12, 13]. Attempting to establish the existence, uniqueness or nonuniqueness, and stability of solutions is in general a difficult task due to the nonlinear nature of the problems. The nonlinearity arises from the contact conditions or from the nonlinearity of the constitutive laws; many researchers [8, 15, 17–19, 22] give details on modelling and analysis in contact mechanics. Due to the complexity of the contact models, we cannot hope to find classical solutions. Thus there is much interest in weak solvability of the problems and in the approximation of weak solutions.

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Weak formulations of contact problems involve the theory of variational inequalities [8, 17–19]. Once well-posedness is established, approximation of the weak solutions is a challenging task. Often the approximation is performed after a regularization of the contact condition. We skip the regularization by using weak formulations with Lagrange multipliers. If the materials are linearly elastic, such a weak formulation is equivalent to a saddle-point problem [1, 2, 9]. Wohlmuth [20, 21] has reviewed modern numerical techniques that can be used to approximate weak solutions of contact problems via weak formulations with dual Lagrange multipliers.

The purpose of the present paper is to draw attention to the weak solvability of frictionless unilateral and frictional bilateral contact problems for nonlinearly elastic materials by using a technique involving dual Lagrange multipliers. The considered models were already analysed in the framework of elliptic variational inequalities by, for example, Han and Sofonea [8] (see also references therein). The novelty of the present paper lies in a new variational approach to the envisaged models for nonlinear constitutive laws. The weak formulations of the proposed models in this paper are not equivalent to saddle-point problems. However, in the study of well-posedness for the derived variational problems, saddle-point theory plays a crucial role. To complete the study, a fixed-point technique is used. A background in functional analysis involving Sobolev spaces is necessary, together with a background in mechanics of solids.

The rest of the paper is structured as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we state the mechanical models. In Section 4 we state hypotheses and derive weak formulations with dual Lagrange multipliers. In Section 5 we obtain abstract results that are applied in Section 6, where we prove the weak solvability of the models considered.

## 2. Notation and preliminaries

Let us denote by $\mathbb{S}^3$ the space of second-order symmetric tensors on $\mathbb{R}^3$. Every field in $\mathbb{R}^3$ or $\mathbb{S}^3$ is typeset in boldface. By $\cdot$ and $|\cdot|$ we denote the inner product and the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$. Thus

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad |\boldsymbol{v}| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2}, \quad \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^3;$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad |\boldsymbol{\tau}| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}, \quad \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^3.$$

Here and below, the indices $i$ and $j$ run between 1 and 3 and the summation convention over repeated indices is adopted.

Let $\Omega\subset\mathbb{R}^3$ be a bounded domain. We introduce the following functional spaces on $\Omega$:

$$H = \{\boldsymbol{u} = (u_i)\mid u_i\in L^2(\Omega)\}, \quad \mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij})\mid \sigma_{ij} = \sigma_{ji}\in L^2(\Omega)\},$$
$$H_1 = \{\boldsymbol{u}\in H\mid \boldsymbol{\varepsilon}(\boldsymbol{u})\in\mathcal{H}\}, \quad \mathcal{H}_1 = \{\boldsymbol{\sigma}\in\mathcal{H}\mid \operatorname{Div}\boldsymbol{\sigma}\in H\},$$

where

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \quad \operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

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Here an index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \, dx, \quad (\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\, \boldsymbol{\sigma}, \mathrm{Div}\, \boldsymbol{\tau})_H.$$

The associated norms on the spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

We assume that the boundary of $\Omega$, denoted by $\Gamma$, is Lipschitz continuous, and denote by $\boldsymbol{n}$ the unit outward normal vector on the boundary, defined almost everywhere.

Let us denote by $\boldsymbol{\gamma}$ the Sobolev trace operator,

$$\boldsymbol{\gamma} : H_1 \to L^2(\Gamma)^3.$$

We recall that $\boldsymbol{\gamma}$ is a linear, continuous and compact operator. We denote by $H_\Gamma$ the image of $H_1$ under $\boldsymbol{\gamma}$, that is, $H_\Gamma = \boldsymbol{\gamma}(H_1)$. The subspace $H_\Gamma$ is continuously embedded into the space $L^2(\Gamma)^3$; thus, there exists $c > 0$ such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^3} \le c\|\boldsymbol{v}\|_{H_\Gamma} \quad \text{for all } \boldsymbol{v} \in H_\Gamma. \tag{2.1}$$

Moreover, it is known that the space $H_\Gamma$ is a Hilbert space. In addition, we recall that there exists a linear and continuous operator

$$Z : H_\Gamma \to H_1$$

such that

$$\boldsymbol{\gamma}(Z(\boldsymbol{\zeta})) = \boldsymbol{\zeta} \quad \text{for all } \boldsymbol{\zeta} \in H_\Gamma. \tag{2.2}$$

The operator $Z$ is called the *inverse to the right* of the operator $\boldsymbol{\gamma}$.

Let $\Gamma_1$ be a measurable part of $\Gamma$ such that $\mathrm{meas}(\Gamma_1) > 0$. We consider the Hilbert space

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{\gamma v} = \boldsymbol{0} \text{ almost everywhere on } \Gamma_1\}.$$

Let us recall the Korn inequality: there exists $c_K = c_K(\Omega, \Gamma_1) > 0$ such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \ge c_K\|\boldsymbol{v}\|_{H_1} \quad \text{for all } \boldsymbol{v} \in V.$$

Using this inequality, it follows that $V$ is a Hilbert space endowed with the following scalar product:

$$(\cdot, \cdot)_V : V \times V \to \mathbb{R}, \quad (\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V.$$

We note that

$$\boldsymbol{\gamma}(Z(\boldsymbol{\gamma v})) = \boldsymbol{\gamma v} \quad \text{for all } \boldsymbol{v} \in V.$$

For a vector field $\boldsymbol{v}$, we denote by $v_n$ and $\boldsymbol{v}_\tau$ the *normal* and the *tangential* components on the boundary, defined by

$$v_n = \boldsymbol{\gamma v} \cdot \boldsymbol{n}, \quad \boldsymbol{v}_\tau = \boldsymbol{\gamma v} - v_n\boldsymbol{n} \quad \text{for all } \boldsymbol{v} \in H_1.$$

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Let $\Gamma_3$ be another measurable part of $\Gamma$, such that $\Gamma_3 \cap \Gamma_1 = \emptyset$. We introduce the space

$$V_1 = \{\boldsymbol{v} \in V \mid v_n = 0 \text{ almost everywhere on } \Gamma_3\}.$$

Then $(V_1, (\cdot,\cdot)_{V_1}, \|\cdot\|_{V_1})$ is a Hilbert space, where

$$(\cdot,\cdot)_{V_1} : V_1 \times V_1 \to \mathbb{R}, \quad (\boldsymbol{u},\boldsymbol{v})_{V_1} = (\boldsymbol{u},\boldsymbol{v})_V \quad \text{for all } \boldsymbol{u},\boldsymbol{v} \in V_1.$$

Keeping in mind (2.2), it is straightforward to verify that

$$Z(\boldsymbol{\zeta}) \in V \quad \text{for all } \boldsymbol{\zeta} \in \boldsymbol{\gamma}(V),$$
$$Z(\boldsymbol{\zeta}) \in V_1 \quad \text{for all } \boldsymbol{\zeta} \in \boldsymbol{\gamma}(V_1).$$

Furthermore,

$$R : \boldsymbol{\gamma}(V) \to V, \quad R(\boldsymbol{\zeta}) = Z(\boldsymbol{\zeta}), \tag{2.3}$$
$$R_1 : \boldsymbol{\gamma}(V_1) \to V_1, \quad R_1(\boldsymbol{\zeta}) = Z(\boldsymbol{\zeta}) \tag{2.4}$$

are linear and continuous operators.

Proposition 2.1. *The spaces $\boldsymbol{\gamma}(V)$ and $\boldsymbol{\gamma}(V_1)$ are closed subspaces of $H_\Gamma$.*

Proof. Let us prove that $\boldsymbol{\gamma}(V)$ is a closed subspace of $H_\Gamma$. To this end, let $(\boldsymbol{\gamma}\boldsymbol{v}_m)_m \subset H_\Gamma$ be a sequence such that

$$\boldsymbol{\gamma}\boldsymbol{v}_m \to \boldsymbol{w} \quad \text{in } H_\Gamma.$$

Due to (2.1),

$$\boldsymbol{\gamma}\boldsymbol{v}_m \to \boldsymbol{w} \quad \text{in } L^2(\Gamma)^3.$$

Since

$$\|R(\boldsymbol{\gamma}\boldsymbol{v}_m)\|_V \le k\|\boldsymbol{\gamma}\boldsymbol{v}_m\|_{H_\Gamma},$$

we deduce that $(R(\boldsymbol{\gamma}\boldsymbol{v}_m))_m$ is a bounded sequence in $V$. Consequently, passing eventually to a subsequence but keeping the notation for simplicity,

$$R(\boldsymbol{\gamma}\boldsymbol{v}_m) \rightharpoonup \boldsymbol{v} \quad \text{in } V.$$

Due to the fact that $\boldsymbol{\gamma}$ is a compact operator, we obtain

$$\boldsymbol{\gamma}(R(\boldsymbol{\gamma}\boldsymbol{v}_m)) \to \boldsymbol{\gamma}\boldsymbol{v} \quad \text{in } L^2(\Gamma)^3.$$

On the other hand,

$$\boldsymbol{\gamma}(R(\boldsymbol{\gamma}\boldsymbol{v}_m)) = \boldsymbol{\gamma}(Z(\boldsymbol{\gamma}\boldsymbol{v}_m)) = \boldsymbol{\gamma}\boldsymbol{v}_m$$

and, due to the uniqueness of the limit, we have $\boldsymbol{w} = \boldsymbol{\gamma}\boldsymbol{v}$. Thus $\boldsymbol{w} \in \boldsymbol{\gamma}(V)$.

With a similar technique, but using the operator $R_1$, it can be proved that $\boldsymbol{\gamma}(V_1)$ is a closed subspace of $H_\Gamma$. □

Remark 2.2. Proposition 2.1 allows us to conclude that $\boldsymbol{\gamma}(V)$ and $\boldsymbol{\gamma}(V_1)$ are Hilbert spaces endowed with the inner products

$$(\cdot,\cdot)_{\boldsymbol{\gamma}(V)} : \boldsymbol{\gamma}(V) \times \boldsymbol{\gamma}(V) \to \mathbb{R}, \quad (\boldsymbol{\zeta},\boldsymbol{\phi})_{\boldsymbol{\gamma}(V)} = (\boldsymbol{\zeta},\boldsymbol{\phi})_{H_\Gamma} \quad \text{for all } \boldsymbol{\zeta},\boldsymbol{\phi} \in \boldsymbol{\gamma}(V),$$
$$(\cdot,\cdot)_{\boldsymbol{\gamma}(V_1)} : \boldsymbol{\gamma}(V_1) \times \boldsymbol{\gamma}(V_1) \to \mathbb{R}, \quad (\boldsymbol{\zeta},\boldsymbol{\phi})_{\boldsymbol{\gamma}(V_1)} = (\boldsymbol{\zeta},\boldsymbol{\phi})_{H_\Gamma} \quad \text{for all } \boldsymbol{\zeta},\boldsymbol{\phi} \in \boldsymbol{\gamma}(V_1).$$

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For a regular (say, $C^1$) stress field $\boldsymbol{\sigma}$, the application to $\boldsymbol{n}$ of its trace on the boundary is the Cauchy stress vector $\boldsymbol{\sigma n}$. Furthermore, we define the *normal* and *tangential* components of the Cauchy vector on the boundary by the formulas

$$\sigma_n = (\boldsymbol{\sigma n}) \cdot \boldsymbol{n}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma n} - \sigma_n \boldsymbol{n},$$

and note that the following identity holds:

$$\boldsymbol{\sigma n} \cdot \gamma \boldsymbol{v} = \sigma_n v_n + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau \quad \text{for all } \boldsymbol{v} \in H_1. \tag{2.5}$$

Finally, we recall Green's useful formula,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\operatorname{Div} \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma n} \cdot \gamma \boldsymbol{v}\, da \quad \text{for all } \boldsymbol{v} \in H_1. \tag{2.6}$$

Han and Sofonea [8] provide a proof of (2.6) and more details related to this section.

We recall some elements of convex analysis.

Definition 2.3. Let $A$ and $B$ be nonempty sets. A pair $(u, \lambda) \in A \times B$ is said to be a *saddle point* of a functional $\mathcal{L} : A \times B \to \mathbb{R}$ if and only if

$$\mathcal{L}(u, \mu) \le \mathcal{L}(u, \lambda) \le \mathcal{L}(v, \lambda) \quad \text{for all } v \in A,\ \mu \in B.$$

We use the following existence result.

Theorem 2.4. *Let $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ be two Hilbert spaces and let $A \subseteq X$, $B \subseteq Y$ be nonempty, closed, convex subsets. Assume that a real functional $\mathcal{L} : A \times B \to \mathbb{R}$ satisfies the following conditions:*

$$v \mapsto \mathcal{L}(v, \mu) \text{ is convex and lower semicontinuous for all } \mu \in B,$$
$$\mu \mapsto \mathcal{L}(v, \mu) \text{ is concave and upper semicontinuous for all } v \in A.$$

*Moreover, assume that*

$$A \text{ is bounded or } \lim_{\|v\|_X \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty \text{ for some } \mu_0 \in B$$

*and*

$$B \text{ is bounded or } \lim_{\|\mu\|_Y \to \infty, \mu \in B} \inf_{v \in A} \mathcal{L}(v, \mu) = -\infty.$$

*Then the functional $\mathcal{L}$ has at least one saddle point.*

More details on saddle-point theory and its applications are given by several researchers [1, 2, 6, 9].

## 3. Contact problems

We consider a body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$, with the boundary partitioned into three measurable parts, $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, such that $\text{meas}(\Gamma_1) > 0$. The unit outward normal vector to $\Gamma$ is denoted by $\boldsymbol{n}$ and is defined almost everywhere. The body $\Omega$ is clamped on $\Gamma_1$, body forces of density $\boldsymbol{f}_0$ act on $\Omega$, and surface traction of density $\boldsymbol{f}_2$ acts on $\Gamma_2$. On $\Gamma_3$ the body can be in contact with a rigid foundation. We denote by $\boldsymbol{u} = (u_i)$ the displacement field, by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ the infinitesimal strain tensor, and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the Cauchy stress tensor.

In order to describe the behaviour of the materials, we use the constitutive law

$$\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega, \tag{3.1}$$

where $\mathcal{F}$ denotes an elastic operator. This kind of constitutive law can be found in the literature [8]. As an example, we may consider

$$\boldsymbol{\sigma} = \lambda_0(\operatorname{tr} \boldsymbol{\varepsilon})I_3 + 2\mu_0\boldsymbol{\varepsilon} + \beta(\boldsymbol{\varepsilon} - P_K\boldsymbol{\varepsilon}), \tag{3.2}$$

where $\lambda_0$ and $\mu_0$ denote Lame's constants, $\operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{\varepsilon}_{kk}$, $I_3 = (\delta_{ij})$ is the unit in $\mathbb{S}^3$, $K$ denotes a closed, convex subset of $\mathbb{S}^3$ that contains the zero element $0_{\mathbb{S}^3}$, $P_K : \mathbb{S}^3 \to K$ is the projection operator onto $K$, and $\beta$ is a strictly positive constant. A second example is the constitutive law

$$\boldsymbol{\sigma} = k(\operatorname{tr} \boldsymbol{\varepsilon})I_3 + \psi(|\boldsymbol{\varepsilon}^D|^2)\boldsymbol{\varepsilon}^D, \tag{3.3}$$

where $k > 0$ is a coefficient of the material, $\psi : \mathbb{R} \to \mathbb{R}$ is a constitutive function, and $\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - \frac{1}{3}(\operatorname{tr} \boldsymbol{\varepsilon})I_3$ is the deviator of the tensor $\boldsymbol{\varepsilon}$.

Assuming that on $\Gamma_3$ the body is in frictional bilateral contact with a rigid foundation, we use Tresca's law to state the following mechanical problem. Everywhere below, $\bar{\Omega}$ denotes $\Omega \cup \partial\Omega$.

Problem 3.1. Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that

$$\begin{aligned} \operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 &= \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{\sigma} &= \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_1, \\ \boldsymbol{\sigma}\boldsymbol{n} &= \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \end{aligned}$$

$$\left.\begin{array}{l} u_n = 0, |\boldsymbol{\sigma}_\tau| \le \zeta, \\ \text{if } |\boldsymbol{\sigma}_\tau| < \zeta \text{ then } \boldsymbol{u}_\tau = 0, \\ \text{if } |\boldsymbol{\sigma}_\tau| = \zeta \text{ then there exists } \psi > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\psi\boldsymbol{u}_\tau \end{array}\right\} \text{ on } \Gamma_3, \tag{3.4}$$

where $\zeta > 0$ denotes the friction bound.

If we assume that on $\Gamma_3$ the body can be in frictionless unilateral contact with a rigid foundation, we model the contact by Signorini's condition with zero gap, yielding the second problem.

Problem 3.2. Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{3.5}$$

$$\boldsymbol{\sigma} = \mathcal{F}(\boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \text{in } \Omega, \tag{3.6}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{3.7}$$

$$\boldsymbol{\sigma}\boldsymbol{n} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{3.8}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \sigma_n \le 0, u_n \le 0, \sigma_n u_n = 0 \quad \text{on } \Gamma_3. \tag{3.9}$$

Finally, if we model the contact on $\Gamma_3$ by Signorini's condition with nonzero gap, we have to replace (3.9) with the following contact condition:

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \sigma_n \le 0, u_n - g \le 0, \sigma_n(u_n - g) = 0 \quad \text{on } \Gamma_3, \tag{3.10}$$

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where $g : \Gamma_3 \to \mathbb{R}$ is the gap between the deformable body and the foundation, measured along the outward normal $\boldsymbol{n}$. Thus, we formulate the third problem.

PROBLEM 3.3. Find $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \bar{\Omega} \to \mathbb{S}^3$ such that (3.5)–(3.8) and (3.10) hold.

Additional details on this section, including a description of the physical significance of the contact conditions (3.4), (3.9) and (3.10), are given by Han and Sofonea [8].

Once the displacement field $\boldsymbol{u}$ is determined, the stress tensor $\boldsymbol{\sigma}$ is determined by relation (3.1).

## 4. Hypotheses and weak formulations

In this section we state hypotheses and derive weak formulations with dual Lagrange multipliers for each of the models described in the previous section.

**4.1. Hypotheses** We assume that $\mathcal{F}$ is a given nonlinear function that satisfies the following properties:

$$\mathcal{F} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3; \tag{4.1}$$

$$\begin{gathered}\text{there exists } M > 0 \text{ such that } |\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq M|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| \\ \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^3, \text{ almost everywhere in } \Omega;\end{gathered} \tag{4.2}$$

$$\begin{gathered}\text{there exists } m > 0 \text{ such that for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^3, \text{ almost everywhere in } \Omega, \\ (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2;\end{gathered} \tag{4.3}$$

$$\text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^3, \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable in } \Omega; \tag{4.4}$$

$$\boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^3}) \text{ belongs to } L^2(\Omega)^{3\times 3}. \tag{4.5}$$

Referring to (3.2), we note that, using the nonexpansivity property of the projection map, it can be proved that

$$\mathcal{F} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3, \quad \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = \lambda_0(\operatorname{tr} \boldsymbol{\varepsilon})I_3 + 2\mu_0\boldsymbol{\varepsilon} + \beta(\boldsymbol{\varepsilon} - P_K\boldsymbol{\varepsilon})$$

satisfies (4.1)–(4.5). Moreover, referring to (3.3), under appropriate assumptions on the constitutive function $\psi$ [8, p. 125], the map

$$\mathcal{F} : \Omega \times \mathbb{S}^3 \to \mathbb{S}^3, \quad \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) = k(\operatorname{tr} \boldsymbol{\varepsilon})I_3 + \psi(|\boldsymbol{\varepsilon}^D|^2)\boldsymbol{\varepsilon}^D$$

satisfies (4.1)–(4.5).

Furthermore, we assume that the density of the volume forces and the density of the traction have the following regularity:

$$\boldsymbol{f}_0 \in L^2(\Omega)^3, \quad \boldsymbol{f}_2 \in L^2(\Gamma_2)^3. \tag{4.6}$$

In addition, we assume that there exists $g_{\text{ext}} : \Omega \to \mathbb{R}$ such that

$$\begin{gathered}g_{\text{ext}} \in H^1(\Omega), \gamma g_{\text{ext}} = 0 \text{ almost everywhere on } \Gamma_1, \\ \gamma g_{\text{ext}} \geq 0 \text{ almost everywhere on } \Gamma \setminus \Gamma_1, g = \gamma g_{\text{ext}} \text{ almost everywhere on } \Gamma_3.\end{gathered} \tag{4.7}$$

Here $\gamma$ denotes the well-known Sobolev trace operator, $\gamma : H^1(\Omega) \to L^2(\Gamma)$.

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Finally, we assume that

$$\text{the unit outward normal to } \Gamma_3,\ \boldsymbol{n}_3, \text{ is a constant vector.} \tag{4.8}$$

**4.2. Weak formulation of Problem 3.1** We define an operator $A : V_1 \to V_1$ such that, for each $\boldsymbol{u} \in V_1$, $A\boldsymbol{u}$ is the element of $V_1$ that satisfies

$$(A\boldsymbol{u}, \boldsymbol{v})_{V_1} = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \text{for all } \boldsymbol{v} \in V_1. \tag{4.9}$$

Using Riesz's representation theorem, we define $\boldsymbol{f} \in V_1$ such that

$$(\boldsymbol{f}, \boldsymbol{v})_{V_1} = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v}\, da \quad \text{for all } \boldsymbol{v} \in V_1.$$

Assuming that the functions $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ that solve Problem 3.1 are sufficiently regular, using (2.5) and (2.6) we obtain

$$(A\boldsymbol{u}, \boldsymbol{v})_{V_1} = (\boldsymbol{f}, \boldsymbol{v})_{V_1} + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\, da \quad \text{for all } \boldsymbol{v} \in V_1.$$

Let $D^T$ be the dual of the space $\boldsymbol{\gamma}(V_1)$. We define $\boldsymbol{\lambda} \in D^T$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{\gamma v} \rangle_T = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau\, da \quad \text{for all } \boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V_1),$$

where $\langle \cdot, \cdot \rangle_T$ denotes the duality pairing between $D^T$ and $\boldsymbol{\gamma}(V_1)$. Furthermore, we define a bilinear form

$$b : V_1 \times D^T \to \mathbb{R}, \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v} \rangle_T \quad \text{for all } \boldsymbol{v} \in V_1, \boldsymbol{\mu} \in D^T. \tag{4.10}$$

Let us introduce the following subset of $D^T$:

$$\Lambda = \left\{ \boldsymbol{\mu} \in D^T \mid \langle \boldsymbol{\mu}, \boldsymbol{\gamma v} \rangle_T \le \int_{\Gamma_3} \zeta |\boldsymbol{\gamma v}|\, da \text{ for all } \boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V_1) \right\}. \tag{4.11}$$

We observe that $\boldsymbol{\lambda} \in \Lambda$. Moreover, by (3.4) we deduce that

$$b(\boldsymbol{u}, \boldsymbol{\lambda}) = \int_{\Gamma_3} \zeta |\boldsymbol{\gamma u}|\, da,$$

and by (4.11),

$$b(\boldsymbol{u}, \boldsymbol{\mu}) \le \int_{\Gamma_3} \zeta |\boldsymbol{\gamma u}|\, da \quad \text{for all } \boldsymbol{\mu} \in \Lambda.$$

Consequently, we are led to the following weak formulation of Problem 3.1.

PROBLEM 4.1. Find $\boldsymbol{u} \in V_1$ and $\boldsymbol{\lambda} \in \Lambda$ such that

$$(A\boldsymbol{u}, \boldsymbol{v})_{V_1} + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_{V_1} \quad \text{for all } \boldsymbol{v} \in V_1,$$
$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \le 0 \quad \text{for all } \boldsymbol{\mu} \in \Lambda.$$

A solution of Problem 4.1 is called a *weak solution* of Problem 3.1.

**4.3. Weak formulation of Problem 3.2** We define an operator $A : V \to V$ such that, for each $\boldsymbol{u} \in V$, $A\boldsymbol{u}$ is the element of $V$ that satisfies

$$(A\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx \quad \text{for all } \boldsymbol{v} \in V. \tag{4.12}$$

Next, we define $\boldsymbol{f} \in V$ such that

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{\gamma v}\, da \quad \text{for all } \boldsymbol{v} \in V. \tag{4.13}$$

For sufficiently regular functions $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ that solve Problem 3.2,

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{f}, \boldsymbol{v})_V + \int_{\Gamma_3} \sigma_n v_n\, da \quad \text{for all } \boldsymbol{v} \in V. \tag{4.14}$$

Let $D^S$ be the dual of the space $\boldsymbol{\gamma}(V)$ and let us denote by $\langle \cdot, \cdot \rangle_S$ the duality pairing between $D^S$ and $\boldsymbol{\gamma}(V)$. We define $\boldsymbol{\lambda} \in D^S$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{\gamma v} \rangle_S = -\int_{\Gamma_3} \sigma_n v_n\, da \quad \text{for all } \boldsymbol{\gamma v} \in \gamma(V). \tag{4.15}$$

In addition, we define a bilinear form

$$b : V \times D^S \to \mathbb{R}, \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v} \rangle_S \quad \text{for all } \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in D^S. \tag{4.16}$$

We introduce the following subset of $D^S$:

$$\Lambda = \{\boldsymbol{\mu} \in D^S \mid \langle \boldsymbol{\mu}, \boldsymbol{\gamma v} \rangle_S \le 0 \text{ for all } \boldsymbol{\gamma v} \in \mathcal{K}\}, \tag{4.17}$$

where

$$\mathcal{K} = \{\boldsymbol{\gamma v} \in \boldsymbol{\gamma}(V) \mid v_n \le 0 \text{ almost everywhere on } \Gamma_3\}. \tag{4.18}$$

Then $\boldsymbol{\lambda} \in \Lambda$. Moreover, by (3.9) we obtain

$$b(\boldsymbol{u}, \boldsymbol{\lambda}) = 0,$$

and, using (4.17),

$$b(\boldsymbol{u}, \boldsymbol{\mu}) \le 0 \quad \text{for all } \boldsymbol{\mu} \in \Lambda.$$

We arrive at the following weak formulation of Problem 3.2.

PROBLEM 4.2. Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$ such that

$$(A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V,$$
$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \le 0 \quad \text{for all } \boldsymbol{\mu} \in \Lambda.$$

A solution of Problem 4.2 is called a *weak solution* of Problem 3.2.

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**4.4. Weak formulation of Problem 3.3** We keep (4.12)–(4.18). By (3.10), we deduce that

$$b(\boldsymbol{u}, \boldsymbol{\lambda}) = b(g_{\text{ext}}\boldsymbol{n}_3, \boldsymbol{\lambda}),$$

and, by (4.17), we obtain

$$b(\boldsymbol{u}, \boldsymbol{\mu}) \leq b(g_{\text{ext}}\boldsymbol{n}_3, \boldsymbol{\mu}) \quad \text{for all } \boldsymbol{\mu} \in \Lambda.$$

Thus, we obtain the following weak formulation of Problem 3.3.

PROBLEM 4.3. Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$ such that

$$(A\boldsymbol{u}, \boldsymbol{v})_V + b(\boldsymbol{v}, \boldsymbol{\lambda}) = (\boldsymbol{f}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{v} \in V,$$

$$b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) \leq b(g_{\text{ext}}\boldsymbol{n}_3, \boldsymbol{\mu} - \boldsymbol{\lambda}) \quad \text{for all } \boldsymbol{\mu} \in \Lambda.$$

A solution of Problem 4.3 is called a *weak solution* of Problem 3.3.

In the next section we analyse an abstract variational problem that is used to recover each of the weak problems formulated in this section, for appropriate functional spaces.

## 5. Auxiliary results

Let $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ be two Hilbert spaces. In addition, we consider

$$A : X \to X, \text{ a nonlinear operator such that} \tag{5.1}$$

$$\text{there exists } m_A > 0 \text{ such that } (Au - Av, u - v)_X \geq m_A\|u - v\|_X^2 \quad \text{for all } u, v \in X, \tag{5.2}$$

$$\text{there exists } L_A > 0 \text{ such that } \|Au - Av\|_X \leq L_A\|u - v\|_X \quad \text{for all } u, v \in X, \tag{5.3}$$

and

$$b : X \times Y \to \mathbb{R}, \text{ a bilinear form such that} \tag{5.4}$$

$$\text{there exists } M_b > 0 \text{ such that } |b(v, \mu)| \leq M_b\|v\|_X\|\mu\|_Y \quad \text{for all } v \in X, \mu \in Y, \tag{5.5}$$

$$\text{there exists } \alpha > 0 \text{ such that } \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X\|\mu\|_Y} \geq \alpha. \tag{5.6}$$

Finally, in addition to the above, we consider

$$\Lambda, \text{ a closed, convex, unbounded subset of } Y \text{ that contains } 0_Y. \tag{5.7}$$

We are interested in the well-posedness of the following problem.

PROBLEM 5.1. For given $f, h \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that

$$(Au, v)_X + b(v, \lambda) = (f, v)_X \quad \text{for all } v \in X, \tag{5.8}$$

$$b(u, \mu - \lambda) \leq b(h, \mu - \lambda) \quad \text{for all } \mu \in \Lambda. \tag{5.9}$$

The following existence and uniqueness result holds.

THEOREM 5.2. *Assume (5.1)–(5.7). Then there exists a unique solution of Problem 5.1, $(u, \lambda) \in X \times \Lambda$.*

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The proof of this theorem is divided into several steps. Let $\eta \in X$ be arbitrarily fixed. We consider the following auxiliary problem.

PROBLEM 5.3. For given $f, h \in X$, find $u_\eta \in X$ and $\lambda_\eta \in \Lambda$ such that

$$(u_\eta, v)_X + \frac{m_A}{2L_A^2} b(v, \lambda_\eta) = \Big(\frac{m_A}{2L_A^2} f - \frac{m_A}{2L_A^2} A\eta + \eta, v\Big)_X \quad \text{for all } v \in X, \tag{5.10}$$

$$b(u_\eta, \mu - \lambda_\eta) \le b(h, \mu - \lambda_\eta) \quad \text{for all } \mu \in \Lambda. \tag{5.11}$$

We are interested in determining the solvability of Problem 5.3 using saddle-point theory applied to the functional

$$\begin{gathered} \mathcal{L}_\eta : X \times \Lambda \to \mathbb{R}, \\ \mathcal{L}_\eta(v, \mu) = \frac{1}{2}(v, v)_X - \Big(\frac{m_A}{2L_A^2} f - \frac{m_A}{2L_A^2} A\eta + \eta, v\Big)_X + \frac{m_A}{2L_A^2} b(v - h, \mu). \end{gathered} \tag{5.12}$$

We prove three technical lemmas.

LEMMA 5.4. *If Problem 5.3 has a solution $(u_\eta, \lambda_\eta) \in X \times \Lambda$ then this solution is a saddle point of the functional $\mathcal{L}_\eta$. Conversely, if the functional $\mathcal{L}_\eta$ has a saddle point then this saddle point is a solution of Problem 5.3.*

PROOF. Assume that $(u_\eta, \lambda_\eta) \in X \times \Lambda$ is a solution of Problem 5.3. Inequality (5.11) implies that $\mathcal{L}_\eta(u_\eta, \mu) \le \mathcal{L}_\eta(u_\eta, \lambda_\eta)$ for all $\mu \in \Lambda$. Moreover, using (5.12) and (5.10),

$$\begin{aligned} \mathcal{L}_\eta(u_\eta, \lambda_\eta) - \mathcal{L}_\eta(v, \lambda_\eta) &= \frac{1}{2}(u_\eta, u_\eta)_X - \frac{1}{2}(v, v)_X + \frac{m_A}{2L_A^2} b(u_\eta - v, \lambda_\eta) \\ &\quad - \Big(\frac{m_A}{2L_A^2} f - \frac{m_A}{2L_A^2} A\eta + \eta, u_\eta - v\Big)_X \\ &= -\frac{1}{2}\|u_\eta - v\|_X^2 \le 0. \end{aligned}$$

Therefore, $\mathcal{L}_\eta(u_\eta, \lambda_\eta) \le \mathcal{L}_\eta(v, \lambda_\eta)$ for all $v \in X$.

Conversely, let $(u_\eta, \lambda_\eta) \in X \times \Lambda$ be a saddle point of the functional $\mathcal{L}_\eta$. It is straightforward to observe that

$$\mathcal{L}_\eta(u_\eta, \mu) \le \mathcal{L}_\eta(u_\eta, \lambda_\eta), \quad \text{for all } \mu \in \Lambda,$$

implies (5.11). Furthermore, we assume that

$$\mathcal{L}_\eta(u_\eta, \lambda_\eta) \le \mathcal{L}_\eta(w, \lambda_\eta), \quad \text{for all } w \in V,$$

and we prove (5.10). Using again (5.12), we deduce that, for all $w \in X$,

$$\frac{1}{2}(u_\eta, u_\eta)_X - \frac{1}{2}(w, w)_X - \Big(\frac{m_A}{2L_A^2} f - \frac{m_A}{2L_A^2} A\eta + \eta, u_\eta - w\Big)_X + \frac{m_A}{2L_A^2} b(u_\eta - w, \lambda_\eta) \le 0.$$

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Replacing $w$ by $u_\eta \pm tv$, with an arbitrary $v \in X$ and $t > 0$, we obtain the following inequality:

$$\mp t(u_\eta, v)_X - \frac{t^2}{2}(v, v)_X \pm t\left(\frac{m_A}{2L_A^2} f - \frac{m_A}{2L_A^2} A\eta + \eta, v\right)_X \mp \frac{m_A}{2L_A^2} tb(v, \lambda_\eta) \leq 0.$$

Dividing by $t$ and passing to the limit as $t \to 0$, we obtain two inequalities that lead to (5.10). □

LEMMA 5.5. *Problem 5.3 has a unique solution* $(u_\eta, \lambda_\eta) \in X \times \Lambda$.

PROOF. The map $v \mapsto \mathcal{L}_\eta(v, \mu)$ is convex and lower semi-continuous for all $\mu \in \Lambda$, and $\mu \mapsto \mathcal{L}_\eta(v, \mu)$ is concave and upper semi-continuous for all $v \in X$. In addition, we note that

$$\lim_{\|v\|_X \to \infty, v \in X} \mathcal{L}_\eta(v, 0_Y) = \infty.$$

Let us prove that

$$\lim_{\|\mu\|_Y \to \infty, \mu \in \Lambda} \inf_{v \in X} \mathcal{L}_\eta(v, \mu) = -\infty. \tag{5.13}$$

To this end, let $\mu$ be an arbitrary element in $\Lambda$ and let $u_\mu \in X$ be the unique solution of the equation

$$(u_\mu, v)_X + \frac{m_A}{2L_A^2} b(v, \mu) = (f_\eta, v)_X \quad \text{for all } v \in X, \tag{5.14}$$

where

$$f_\eta = \frac{m_A}{2L_A^2} f - \frac{m_A}{2L_A^2} A\eta + \eta.$$

We have

$$\inf_{v \in X} \mathcal{L}_\eta(v, \mu) = \frac{1}{2}(u_\mu, u_\mu)_X - (f_\eta, u_\mu)_X + \frac{m_A}{2L_A^2} b(u_\mu, \mu) - \frac{m_A}{2L_A^2} b(h, \mu).$$

Let us put $v = u_\mu$ in (5.14). Then

$$-(f_\eta, u_\mu)_X + \frac{m_A}{2L_A^2} b(u_\mu, \mu) = -\|u_\mu\|_X^2.$$

Therefore,

$$\inf_{v \in X} \mathcal{L}_\eta(v, \mu) \leq -\frac{1}{2}\|u_\mu\|_X^2 + \frac{m_A}{2L_A^2} M_b \|h\|_X \|\mu\|_Y.$$

Due to the inf–sup property (5.6), we deduce that there exists $\alpha > 0$ such that

$$\alpha\|\mu\|_Y \leq \frac{m_A}{2L_A^2} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X},$$

and, by (5.14), we obtain

$$\alpha\|\mu\|_Y \leq \sup_{v \in X, v \neq 0_X} \frac{(f_\eta, v)_X - (u_\mu, v)_X}{\|v\|_X} \leq \|f_\eta\|_X + \|u_\mu\|_X.$$

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Therefore, there exists $c > 0$ such that

$$\|\mu\|_Y^2 \leq c(\|f_\eta\|_X^2 + \|u_\mu\|_X^2).$$

Furthermore, there exists $\tilde{c} > 0$ such that

$$\inf_{v\in X} \mathcal{L}_\eta(v, \mu) \leq -\tilde{c}(\|\mu\|^2 - \|f_\eta\|_X^2) + \frac{m_A}{2L_A^2} M_b \|h\|_X \|\mu\|_Y.$$

Since $\mu$ was arbitrarily fixed in $\Lambda$, passing to the limit as $\|\mu\|_Y \to \infty$ yields (5.13).

Consequently, based on Theorem 2.4, we deduce that the functional $\mathcal{L}_\eta$ has at least one saddle point. Then, by Lemma 5.4, we conclude that Problem 5.3 has at least one solution.

In fact, Problem 5.3 has a unique solution $(u_\eta, \lambda_\eta) \in X \times \Lambda$. Indeed, let $(u_\eta^1, \lambda_\eta^1)$ and $(u_\eta^2, \lambda_\eta^2)$ be two solutions of Problem 5.3. Keeping in mind (5.10),

$$(u_\eta^1 - u_\eta^2, u_\eta^2 - u_\eta^1)_X + \frac{m_A}{2L_A^2} b(u_\eta^1 - u_\eta^2, \lambda_\eta^2 - \lambda_\eta^1) = 0. \tag{5.15}$$

Using (5.11), we deduce that

$$b(u_\eta^1 - u_\eta^2, \lambda_\eta^2 - \lambda_\eta^1) \leq 0. \tag{5.16}$$

Combining (5.15) and (5.16), we conclude that $u_\eta^1 = u_\eta^2$. Moreover,

$$b(v, \lambda_\eta^1 - \lambda_\eta^2) = -\frac{2L_A^2}{m_A}(u_\eta^1 - u_\eta^2, v)_X \quad \text{for all } v \in X.$$

By the inf–sup property (5.6), we conclude that $\lambda_\eta^1 = \lambda_\eta^2$. □

Using the unique solution of Problem 5.3, we define an operator

$$T : X \to X, \quad T(\eta) := u_\eta.$$

LEMMA 5.6. *The operator $T$ has a unique fixed point.*

PROOF. Let us take $\eta_1, \eta_2 \in X$. Denoting by $(u_{\eta_1}, \lambda_{\eta_1})$ and $(u_{\eta_2}, \lambda_{\eta_2})$ the corresponding solutions of Problem 5.3 and using (5.10),

$$\begin{aligned}(u_{\eta_1} - u_{\eta_2}, u_{\eta_1} - u_{\eta_2})_X = {} & \frac{m_A}{2L_A^2} b(u_{\eta_2} - u_{\eta_1}, \lambda_{\eta_1} - \lambda_{\eta_2}) \\ & + \Big(\eta_1 - \eta_2 + \frac{m_A}{2L_A^2}(A\eta_2 - A\eta_1), u_{\eta_1} - u_{\eta_2}\Big)_X.\end{aligned}$$

Taking into account (5.11), we obtain

$$\frac{m_A}{2L_A^2} b(u_{\eta_1} - u_{\eta_2}, \lambda_{\eta_2} - \lambda_{\eta_1}) \leq 0.$$

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Then we deduce

$$\|u_{\eta_1} - u_{\eta_2}\|_X \le \left\| \eta_1 - \eta_2 - \frac{m_A}{2L_A^2}(A\eta_1 - A\eta_2) \right\|_X.$$

From this last inequality we obtain

$$\|u_{\eta_1} - u_{\eta_2}\|_X^2 \le \|\eta_1 - \eta_2\|_X^2 - \frac{m_A}{L_A^2}(A\eta_1 - A\eta_2, \eta_1 - \eta_2)_X + \frac{m_A^2}{4L_A^4}\|A\eta_1 - A\eta_2\|_X^2,$$

and, by (5.2) and (5.3), we obtain

$$\|u_{\eta_1} - u_{\eta_2}\|_X^2 \le \left(1 - \frac{3m_A^2}{4L_A^2}\right)\|\eta_1 - \eta_2\|_X^2.$$

Taking into account (5.2) and (5.3), it is straightforward to observe that $m_A \le L_A$. Therefore,

$$0 < 1 - \frac{3m_A^2}{4L_A^2}.$$

Consequently,

$$\|T\eta_1 - T\eta_2\|_X \le \sqrt{1 - \frac{3m_A^2}{4L_A^2}}\|\eta_1 - \eta_2\|_X.$$

Moreover, since $1 - (3m_A^2)/(4L_A^2) < 1$, the operator $T$ is a contraction. The conclusion of Lemma 5.6 follows now from Banach's fixed-point theorem. □

Let us now prove Theorem 5.2.

Proof. Denoting by $\eta^*$ the unique fixed point of the operator $T$, we observe that the solution of Problem 5.3 with $\eta = \eta^*$ $(u_{\eta*}, \lambda_{\eta*})$ is a solution of Problem 5.1. In order to prove the uniqueness of the solution, we assume that Problem 5.1 has two solutions $(u_i, \lambda_i)$, $i = 1, 2$. Using (5.8), we obtain

$$\begin{aligned}(Au_1 - Au_2, u_2 - u_1)_X + b(u_2 - u_1, \lambda_1 - \lambda_2) &= 0,\\ b(u_2 - u_1, \lambda_1 - \lambda_2) &\le 0,\end{aligned}$$

and taking into account (5.2), we obtain $u_1 = u_2$. Moreover,

$$(Au_1 - Au_2, v)_X + b(v, \lambda_1 - \lambda_2) = 0 \quad \text{for all } v \in X.$$

By the inf–sup property (5.6) of the form $b(\cdot, \cdot)$,

$$\alpha\|\lambda_1 - \lambda_2\|_Y \le L_A\|u_1 - u_2\|_X,$$

and from this we obtain $\lambda_1 = \lambda_2$. □

Following the techniques used by Matei et al. [11, 16], we establish stability results. More precisely, in the homogeneous case, that is, $h = 0_X$ in Problem 5.1, the following theorem applies.

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Theorem 5.7. *Assume (5.1)–(5.7). If $(u_1, \lambda_1)$ and $(u_2, \lambda_2)$ are two solutions of Problem 5.1 in the homogeneous case, corresponding to the data $f_1, f_2 \in X$, then there exists $C = C(\alpha, L_A, m_A) > 0$ such that*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le C\|f_1 - f_2\|_X. \tag{5.17}$$

Proof. Let us consider the data $f_1, f_2 \in X$ and denote by $(u_1, \lambda_1)$ and $(u_2, \lambda_2)$ the corresponding solutions, respectively. Using (5.8) and (5.9), we find

$$\begin{aligned}(Au_1 - Au_2, u_1 - u_2)_X &= (f_1 - f_2, u_1 - u_2)_X + b(u_1 - u_2, \lambda_2 - \lambda_1),\\ b(u_1 - u_2, \lambda_2 - \lambda_1) &\le 0.\end{aligned}$$

Moreover, using (5.2), we obtain

$$m_A\|u_1 - u_2\|_X^2 \le \|f_1 - f_2\|_X\|u_1 - u_2\|_X,$$

and thus

$$\|u_1 - u_2\|_X \le \frac{1}{m_A}\|f_1 - f_2\|_X. \tag{5.18}$$

Since $b(v, \lambda_1 - \lambda_2) = (f_1 - f_2, v)_X - (Au_1 - Au_2, v)_X$, due to the inf–sup property (5.6) of the form $b(\cdot, \cdot)$ we have

$$\|\lambda_1 - \lambda_2\|_Y \le \frac{L_A}{\alpha}\|u_1 - u_2\|_X + \frac{1}{\alpha}\|f_1 - f_2\|_X. \tag{5.19}$$

Combining (5.18) and (5.19), we deduce (5.17). □

In the nonhomogeneous case, that is, the case $h \neq 0_X$, the following stability result applies.

Theorem 5.8. *Assume (5.1)–(5.7). If $(u_1, \lambda_1)$ and $(u_2, \lambda_2)$ are two solutions of Problem 5.1 in the nonhomogeneous case, corresponding to the data $f_1, h_1 \in X$ and $f_2, h_2 \in X$ ($h_i \neq 0_X$, $i \in \{1, 2\}$), then there exists $C = C(\alpha, L_A, m_A, M_b) > 0$ such that*

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le C(\|f_1 - f_2\|_X + \|h_1 - h_2\|_X). \tag{5.20}$$

Proof. Let us consider $f_1, h_1 \in X$ and $f_2, h_2 \in X$. We denote by $(u_1, \lambda_1)$ and $(u_2, \lambda_2)$ the corresponding solutions. Using (5.8) and (5.9),

$$\begin{aligned}(Au_1 - Au_2, u_1 - u_2)_X &= (f_1 - f_2, u_1 - u_2)_X + b(u_1 - u_2, \lambda_2 - \lambda_1),\\ b(u_1 - u_2, \lambda_2 - \lambda_1) &\le b(h_1 - h_2, \lambda_2 - \lambda_1).\end{aligned}$$

Moreover, using (5.2)–(5.6), we obtain

$$\begin{aligned}m_A\|u_1 - u_2\|_X^2 &\le \|f_1 - f_2\|_X\|u_1 - u_2\|_X + M_b\|h_1 - h_2\|_X\|\lambda_1 - \lambda_2\|_Y,\\ \alpha\|\lambda_1 - \lambda_2\|_Y &\le \|f_1 - f_2\|_X + L_A\|u_1 - u_2\|_X,\end{aligned}$$

and from this we deduce

$$\begin{aligned}m_A\|u_1 - u_2\|_X^2 &\le \frac{\|f_1 - f_2\|_X^2}{2k_1} + \frac{k_1\|u_1 - u_2\|_X^2}{2} + \frac{M_b^2\|h_1 - h_2\|_Y^2}{2k_2} + \frac{k_2\|\lambda_1 - \lambda_2\|_Y^2}{2},\\ \|\lambda_1 - \lambda_2\|_Y^2 &\le \frac{2}{\alpha^2}(\|f_1 - f_2\|_X^2 + L_A^2\|u_1 - u_2\|_X^2),\end{aligned} \tag{5.21}$$

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where $k_1, k_2$ are strictly positive real constants. Let us choose $k_1$ and $k_2$ such that

$$m_A - \frac{k_1}{2} - \frac{k_2 L_A^2}{\alpha^2} > 0.$$

Then

$$\|u_1 - u_2\|_X^2 \leq \mathcal{T}\Big(\frac{\|f_1 - f_2\|_X^2}{2k_1} + \frac{M_b^2\|h_1 - h_2\|_Y^2}{2k_2}\Big) + \frac{2k_2}{2m_A\alpha^2 - k_1\alpha^2 - 2k_2L_A^2}\|f_1 - f_2\|_X^2,$$

where $\mathcal{T} = (m_A - k_1/2 - k_2L_A^2/\alpha^2)^{-1}$. Combining with (5.21), we deduce (5.20). □

We note that in the homogeneous case, $C$ depends on $m_A$, $L_A$ and $\alpha$, while in the nonhomogeneous case, $C$ depends in addition on $M_b$.

## 6. Weak solvability of the models

The purpose of this section is to investigate the well-posedness of the weak problems formulated in Section 4. The well-posedness of Problem 4.1 is given by the following theorem.

THEOREM 6.1. *Assume (4.1)–(4.6). Then Problem 4.1 has a unique solution $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V_1 \times \Lambda$. Moreover, if $(\boldsymbol{u}, \boldsymbol{\lambda})$ and $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ are two solutions of Problem 4.1, corresponding to the data $\boldsymbol{f} \in V_1$ and $\boldsymbol{f}^* \in V_1$, then there exists $C_T > 0$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_{V_1} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_{D^T} \leq C_T\|\boldsymbol{f} - \boldsymbol{f}^*\|_{V_1}. \tag{6.1}$$

PROOF. Let us set $X = V_1$, $Y = D^T$ and $f = \boldsymbol{f}$. Then $\Lambda$ defined by (4.11) is an unbounded, closed, convex subset of $D^T$. Moreover, $\mathbf{0}_{D^T} \in \Lambda$. By (4.2) and (4.3), we deduce that the operator $A$ defined in (4.9) has the properties

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{V_1} \geq m_{\mathcal{F}}\|\boldsymbol{u} - \boldsymbol{v}\|_{V_1}^2 \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V_1,$$
$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_{V_1} \leq M_{\mathcal{F}}\|\boldsymbol{u} - \boldsymbol{v}\|_{V_1} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V_1.$$

In addition, the bilinear form $b(\cdot, \cdot) : V_1 \times D^T \to \mathbb{R}$ defined by (4.10) satisfies

$$|b^T(\boldsymbol{v}, \boldsymbol{\mu})| \leq \|\boldsymbol{\mu}\|_{D^T}\|\boldsymbol{v}\|_{\gamma(V_1)},$$

and therefore, due to properties of the Sobolev trace operator, there exists $M_b > 0$ such that (5.5) is satisfied. Using the operator $R_1$ (see (2.4) in Section 2), we prove that there exists $\alpha^T > 0$ such that $b(\cdot, \cdot)$ satisfies (5.6). Indeed, there exists $\tilde{c} > 0$ such that

$$\begin{aligned}
\|\boldsymbol{\mu}\|_{D^T} &= \sup_{\boldsymbol{w}\in\gamma(V_1), \boldsymbol{w}\neq\mathbf{0}_{\gamma(V_1)}} \frac{\langle \boldsymbol{\mu}, \boldsymbol{w}\rangle_T}{\|\boldsymbol{w}\|_{\gamma(V_1)}} \\
&\leq \tilde{c} \sup_{\boldsymbol{w}\in\gamma(V_1), \boldsymbol{w}\neq\mathbf{0}_{\gamma(V_1)}} \frac{b(R_1\boldsymbol{w}, \boldsymbol{\mu})}{\|R_1\boldsymbol{w}\|_{V_1}} \\
&\leq \tilde{c} \sup_{\boldsymbol{v}\in V_1, \boldsymbol{v}\neq\mathbf{0}_{V_1}} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_{V_1}}.
\end{aligned}$$

Thus, we take $\alpha^T = 1/\tilde{c}$.

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Consequently, applying Theorem 5.2, we deduce that Problem 4.1 has a unique solution. In addition, inequality (6.1) is a consequence of Theorem 5.7. □

Let us now prove the well-posedness of Problem 4.2.

THEOREM 6.2. *Assume (4.1)–(4.6). Then Problem 4.2 has a unique solution* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$. *Moreover, if* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem 4.2, corresponding to the data* $\boldsymbol{f} \in V$ *and* $\boldsymbol{f}^* \in V$, *then there exists* $C_1^S > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_V + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_{D^S} \le C_1^S \|\boldsymbol{f} - \boldsymbol{f}^*\|_V. \tag{6.2}$$

PROOF. Let $X = V$, $Y = D^S$, $f = \boldsymbol{f}$. Then $\Lambda$ defined in (4.17) is an unbounded, closed, convex subset of $D^S$. Moreover, $\mathbf{0}_{D^S} \in \Lambda^S$. Keeping in mind (4.2) and (4.3), we deduce that the operator $A$ defined in (4.12) has the properties

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \ge m_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V^2 \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V,$$
$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_V \le M_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V.$$

For the bilinear form $b(\cdot, \cdot) : V \times D^S \to \mathbb{R}$ defined in (4.16), we obtain

$$|b(\boldsymbol{v}, \boldsymbol{\mu})| \le \|\boldsymbol{\mu}\|_{D^S} \|\boldsymbol{v}\|_{\gamma(V)},$$

and therefore, due to the fact that $\boldsymbol{\gamma}$ is a linear and continuous operator, we deduce that there exists $M_b > 0$ such that (5.5) is satisfied. Using the operator $R$ (see (2.3)), we prove that there exists $\alpha^S > 0$ such that $b(\cdot, \cdot)$ satisfies (5.6). Indeed, there exists $\bar{c} > 0$ such that

$$\begin{aligned}
\|\boldsymbol{\mu}\|_{D^S} &= \sup_{\boldsymbol{w} \in \gamma(V), \boldsymbol{w} \ne \mathbf{0}_{\gamma(V)}} \frac{\langle \boldsymbol{\mu}, \boldsymbol{w} \rangle_S}{\|\boldsymbol{w}\|_{\gamma(V)}} \\
&\le \bar{c} \sup_{\boldsymbol{w} \in \gamma(V), \boldsymbol{w} \ne \mathbf{0}_{\gamma(V)}} \frac{b(R\boldsymbol{w}, \boldsymbol{\mu})}{\|R\boldsymbol{w}\|_V} \\
&\le \bar{c} \sup_{\boldsymbol{v} \in V, \boldsymbol{v} \ne \mathbf{0}_V} \frac{b(\boldsymbol{v}, \boldsymbol{\mu})}{\|\boldsymbol{v}\|_V},
\end{aligned}$$

and we take $\alpha^S = 1/\bar{c}$.

The existence and uniqueness of the solution of Problem 4.2 follow again from Theorem 5.2. To obtain (6.2), we apply Theorem 5.7. □

Finally, we prove the well-posedness of Problem 4.3.

THEOREM 6.3. *Assume (4.1)–(4.8). Then Problem 4.3 has a unique solution* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$. *Moreover, if* $(\boldsymbol{u}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem 4.3, corresponding to the data* $(\boldsymbol{f}, g_{\mathrm{ext}}\boldsymbol{n}_3) \in V \times V$ *and* $(\boldsymbol{f}^*, g^*_{\mathrm{ext}}\boldsymbol{n}_3) \in V \times V$, *then there exists* $C_2^S > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_V + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_{D^S} \le C_2^S (\|\boldsymbol{f} - \boldsymbol{f}^*\|_V + \|g_{\mathrm{ext}}\boldsymbol{n}_3 - g^*_{\mathrm{ext}}\boldsymbol{n}_3\|_V). \tag{6.3}$$

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PROOF. Let us take $X = V$, $Y = D^S$, $f = \boldsymbol{f}$, $h = g_{\text{ext}}\boldsymbol{n}_3$ and $\Lambda$ given by (4.17). The conclusion of this theorem follows by similar arguments to those used in the proof of Theorem 6.2. We apply Theorem 5.2 to obtain the existence and uniqueness result. In order to obtain (6.3), we apply Theorem 5.8. □

## 7. Conclusion

To conclude, we point out that by using weak formulations with dual Lagrange multipliers, efficient algorithms can be written to approximate the weak solutions for a class of contact problems. For example, Hüeber et al. [12] analysed the frictional contact between two linearly elastic bodies subjected to antiplane shear deformation. The weak formulation of their mechanical model is a version of the problem (5.8)–(5.9) for appropriate functional spaces; we point out that their operator $A$ is linear and $h = 0$. To discretize the frictional problem, Hüeber et al. [12] used a mortar technique on nonconforming meshes. Under some regularity assumptions on the solution, an optimal a priori error estimate was obtained. In order to solve the discrete nonlinear problem, a primal–dual active set strategy was applied. The strategy can be interpreted as a semi-smooth quasi-Newton method. One advantage of the approach is that the nonlinear friction conditions are applied as Dirichlet or Neumann boundary conditions on the interface, and this makes the implementation simple. Hüeber et al. [12] also presented numerical examples confirming the theoretical result and illustrating the performance of the algorithm. We also refer the reader to other work of Hüeber et al. [10, 11, 13, 14].

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