9
Finite temperatures

Finite-temperature quantum field theories at thermodynamic equilibrium are naturally described by Euclidean path integrals. The time-variable in this approach is compactified and varies between 0 and the inverse temperature $1/T$. Periodic boundary conditions are imposed on Bose fields, while antiperiodic ones are imposed on Fermi fields in order to reproduce the standard Bose or Fermi statistics, respectively.

The lattice formulation of QCD at finite temperature is especially simple, since the Euclidean lattice has a finite extent in the temporal direction. The Wilson criterion of confinement is not applicable at finite temperatures and is replaced by another one based on the thermal Wilson lines passing through the lattice in the temporal direction. They are closed owing to the periodic boundary condition for the gauge field.

When the temperature increases, QCD undergoes [Pol78, Sus79] a deconfining phase transition which is associated with a liberation of quarks. At low temperatures below the phase transition, thermodynamical properties of the hadron matter are well described by a gas of noninteracting hadrons while at high temperatures above the phase transition these are well described by an ideal gas of quarks and gluons.

The situation with the deconfining phase transition becomes less definite when the effects of virtual quarks are taken into account. The deconfining phase transition makes strict sense only for large values of the quark mass. For light quarks, a phase transition associated with the chiral symmetry restoration at high temperatures occurs with increasing temperature. It makes strict sense only for massless quarks.

In this chapter we first derive a path-integral representation of finite-temperature quantum field theories starting from the Boltzmann distribution. Then we apply this technique to QCD and discuss the confinement criterion at finite temperatures as well as the deconfining and chiral symmetry restoration phase transitions.

9.1 Feynman–Kac formula

Thermodynamic properties of an equilibrium system in $3+1$ dimensions are determined by the thermal partition function

$$Z(T,V) \;=\; \sum_n \mathrm{e}^{-E_n/T} \;\equiv\; \mathrm{Tr}\,\mathrm{e}^{-\boldsymbol{H}/T} \tag{9.1}$$

which is associated with the Boltzmann distribution at the temperature T. Here $\boldsymbol{H}$ is a Hamiltonian of the system and Tr is calculated over any complete set of states, say, over eigenstates of the Hamiltonian, eigenvalues of which are characterized by the energy levels E_n.

For a quantum theory of a single scalar field $\varphi(\vec{x},t)$, the (Schrödinger) states are described by the bra- and ket-vectors $\langle g|$ and $|f\rangle$:

$$\langle g|\vec{x}\rangle \;=\; g(\vec{x})\,, \qquad \langle \vec{x}|f\rangle \;=\; f(\vec{x})\,, \tag{9.2}$$

as is explained in Sect. 1.1. A matrix element of the evolution operator $\exp(-\boldsymbol{H}/T)$ is given by the formula

$$\langle g|\,\mathrm{e}^{-\boldsymbol{H}/T}\,|f\rangle \;=\; \int\limits_{\substack{\varphi(\vec{x},0)=f(\vec{x})\\ \varphi(\vec{x},1/T)=g(\vec{x})}} \mathcal{D}\varphi(\vec{x},t)\;\mathrm{e}^{-\int_0^{1/T}\mathrm{d}t\,\mathcal{L}[\varphi]}\,, \tag{9.3}$$

where $\mathcal{L}$ is a proper Lagrangian, say for example,

$$\mathcal{L}[\varphi] \;=\; \int\limits_V \mathrm{d}^3\vec{x}\left[\frac{1}{2}\left(\partial_\mu\varphi\right)^2+\frac{1}{2}m^2\varphi^2+\frac{\lambda}{3!}\varphi^3\right] \tag{9.4}$$

for the cubic self-interaction of φ. The derivation is quite analogous to that of Problem 1.9 on p. 22.

In order to calculate the trace over states, one should put $g(\vec{x}) = f(\vec{x})$ and perform the additional integration over $f(\vec{x})$. This yields the Feynman–Kac formula*

$$\begin{aligned}\mathrm{Tr}\,\mathrm{e}^{-\boldsymbol{H}/T} \;&=\; \int \mathcal{D}f(\vec{x})\,\langle f|\,\mathrm{e}^{-\boldsymbol{H}/T}\,|f\rangle \\ &=\; \int\limits_{\varphi(\vec{x},1/T)=\varphi(\vec{x},0)} \mathcal{D}\varphi(\vec{x},t)\;\mathrm{e}^{-\int_0^{1/T}dt\,\mathcal{L}[\varphi]}\,.\end{aligned} \tag{9.5}$$

Note that the path integral in Eq. (9.5) is taken with periodic boundary conditions for the field φ:

$$\varphi(\vec{x},1/T) \;=\; \varphi(\vec{x},0)\,. \tag{9.6}$$

* Its derivation in the modern context of non-Abelian gauge theories, which extends the Feynman derivation [Fey53] for statistical mechanics, is due to Bernard [Ber74].

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As $T \to 0$ it reproduces the standard Euclidean formulation of quantum field theory which is discussed in Chapter 2. The point is that nothing depends on real time for a system at thermodynamic equilibrium. The variable t in Eq. (9.5) is just the proper time of the disentangling procedure. This analogy between the partition functions of statistical systems and the Euclidean formulation of quantum field theory has already been mentioned in the Remark on p. 33.

Remark on thermal density matrix

A statistical-mechanical counterpart of the propagator in Euclidean quantum field theory is the (unnormalized) thermal density matrix

$$\left\langle y \left| \mathrm{e}^{-\boldsymbol{H}/T} \right| x \right\rangle = \sum_n \mathrm{e}^{-E_n/T}\, \Psi_n^*(y) \Psi_n(x)\,, \tag{9.7}$$

where $\Psi_n(x)$ denotes the wave function of the nth eigenstate. This equality can be derived by inserting a complete set of states. The thermal partition function (9.1) is then given by the space integral of the diagonal element:

$$Z(T,V) = \int_V \mathrm{d}^d x \left\langle x \left| \mathrm{e}^{-\boldsymbol{H}/T} \right| x \right\rangle . \tag{9.8}$$

For a quantum particle with the nonrelativistic Hamiltonian (1.107), the path-integral representation of the thermal density matrix (9.7) is given by Eq. (1.118) with $\tau = 1/T$. This pursues the analogy between Euclidean quantum field theory and statistical mechanics.

More concerning the thermal density matrix (9.7) can be found in the book [Fey72].

Problem 9.1 Derive the Feynman–Kac formula for a quantum particle with the nonrelativistic Hamiltonian (1.107).

Solution The matrix element $\langle x \,|\, \exp(-\boldsymbol{H}/T) |\, x\rangle$ is determined by Eq. (1.118) to be

$$\left\langle x \left| \mathrm{e}^{-\boldsymbol{H}/T} \right| x \right\rangle = \int\limits_{\substack{z_\mu(0)=x_\mu \\ z_\mu(1/T)=x_\mu}} \mathcal{D}z_\mu(t)\; \mathrm{e}^{-\int_0^{1/T} \mathrm{d}t\, \mathcal{L}(t)}, \tag{9.9}$$

where the Lagrangian $\mathcal{L}(t)$ is given by Eq. (1.119). Using Eq. (9.8), we obtain [Fey53]

$$\begin{aligned} \mathrm{Tr}\, \mathrm{e}^{-\boldsymbol{H}/T} &= \int_V \mathrm{d}^d x \left\langle x \left| \mathrm{e}^{-\boldsymbol{H}/T} \right| x \right\rangle \\ &= \int\limits_{z_\mu(0)=z_\mu(1/T)} \mathcal{D}z_\mu(t)\; \mathrm{e}^{-\int_0^{1/T} \mathrm{d}t\, \mathcal{L}(t)}. \end{aligned} \tag{9.10}$$

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This integral is over the trajectories with periodic boundary conditions

$$z_\mu(0) \;=\; z_\mu(1/T)\,. \tag{9.11}$$

Problem 9.2 Calculate the partition function (9.10) for the free case.

Solution The Gaussian path integral with the boundary conditions

$$z_\mu(0) \;=\; z_\mu(1/T) \;=\; x_\mu \tag{9.12}$$

is calculated in Sect. 1.5 with the result given by Eq. (1.90). In order to calculate the partition function (9.10), we need to integrate this expression over x_μ which yields [Fey53]

$$Z(T,V) \;=\; \int_V \mathrm{d}^d x\, \mathcal{F}\,(1/mT) \;=\; V\left(\frac{mT}{2\pi}\right)^{d/2}. \tag{9.13}$$

The formula (9.13) is to be compared with that given by the Boltzmann distribution in classical statistics. Since the energy of a free nonrelativistic particle is

$$E(\vec{p}) \;=\; \frac{\vec{p}^{\,2}}{2m}\,, \tag{9.14}$$

the Boltzmann distribution is given by the sum over positions of the particle in a box of volume V and the integration over its momentum $\vec{p}$:

$$Z(T,V) \;=\; V\int \frac{\mathrm{d}^d\vec{p}}{(2\pi)^d}\,\mathrm{e}^{-E(\vec{p})/T} \;=\; V\left(\frac{mT}{2\pi}\right)^{d/2}, \tag{9.15}$$

which coincides with Eq. (9.13) derived from the path integral.

Problem 9.3 Calculate the thermal density matrix (9.7) for the free case.

Solution The calculation is the same as in Sect. 1.5 for $\tau = 1/mT$. The result is

$$\left\langle y \left| \mathrm{e}^{-\boldsymbol{H}/T} \right| x \right\rangle \;=\; \left(\frac{mT}{2\pi}\right)^{d/2} \mathrm{e}^{-mT(x-y)^2/2}\,. \tag{9.16}$$

This formula can alternatively be derived using Eq. (9.7) for the wave functions associated with the plane waves

$$\Psi_{\vec{p}}(x) \;=\; \frac{1}{\sqrt{V}}\,\mathrm{e}^{-\mathrm{i}\vec{p}\vec{x}}\,. \tag{9.17}$$

Then we obtain

$$\begin{aligned}\sum_n \mathrm{e}^{-E_n/T}\,\Psi_n^*(y)\Psi_n(x) \;&=\; \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\mathrm{i}\vec{p}(\vec{y}-\vec{x})-p^2/2mT} \\ &=\; \left(\frac{mT}{2\pi}\right)^{d/2} \mathrm{e}^{-mT(x-y)^2/2}\end{aligned} \tag{9.18}$$

which reproduces Eq. (9.16).

Problem 9.4 Calculate the partition function (9.10) for a harmonic oscillator with $V(x) = m\,\omega^2 x^2/2$.

Solution The path integral in Eq. (9.10) can be calculated using the mode expansion

$$z(t) \quad = \quad a_0 + \sqrt{2}\sum_{n=1}^{\infty}\Big[a_n \cos(2\pi n t T) + b_n \sin(2\pi n t T)\Big], \qquad (9.19)$$

where the sin and cos functions form a set of orthogonal basis functions on the interval $[0, 1/T]$ and satisfy the boundary condition (9.11). The expansion (9.19) is of the same type as Eq. (1.82).

Substituting (9.19) into the action, we have

$$\frac{m}{2}\int\limits_0^{1/T} \mathrm{d}t\,\left(\dot{z}^2 + \omega^2 z^2\right) \quad = \quad \frac{m\omega^2}{2T}a_0^2 + \frac{m}{2T}\sum_{n=1}^{\infty}\Big[(2\pi n T)^2 + \omega^2\Big]\left(a_n^2 + b_n^2\right). \qquad (9.20)$$

Representing the measure as

$$\mathcal{D}z(t) \quad = \quad \frac{\mathrm{d}^d a_0}{(2\pi)^{d/2}}\prod_{n=1}^{\infty}\frac{\mathrm{d}^d a_n}{(2\pi)^{d/2}}\frac{\mathrm{d}^d b_n}{(2\pi)^{d/2}}, \qquad (9.21)$$

which is of the same type as Eq. (1.83), and performing the Gaussian integral over the a_n and b_n, we obtain for the partition function (9.10)

$$Z(T) \quad = \quad \left[\frac{\sqrt{T}}{\sqrt{m}\omega}\prod_{n=1}^{\infty}\frac{T/m}{(2\pi n T)^2 + \omega^2}\right]^d. \qquad (9.22)$$

The infinite product can be calculated by virtue of the formula

$$\prod_{n=1}^{\infty}\left(A + \frac{n^2}{B}\right) \quad = \quad \frac{2}{\sqrt{A}}\sinh(\pi\sqrt{AB}) \qquad (9.23)$$

which implies a zeta-function regularization. Finally, we obtain

$$Z(T) \quad = \quad \left[\frac{1}{2\sinh(\omega/2T)}\right]^d \qquad (9.24)$$

for the thermal partition function of a nonrelativistic harmonic oscillator with frequency ω. Equation (9.24) can be derived alternatively by simply substituting the oscillator spectrum $E_n = \omega\left(n + \frac{1}{2}\right)$ into the Boltzmann formula (9.1).

In contrast with Eq. (9.13), there is no volume-dependence in Eq. (9.24), which comes usually from the translational zero mode, since now the particle oscillates near the origin. It is clear from the integral over a_0 that the volume factor is reproduced as $V \sim (T/m\omega^2)^{d/2}$ when $\omega \to 0$. Then Eq. (9.13) is reproduced as $\omega \to 0$.

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Problem 9.5 Calculate the thermal density matrix (9.7) of the harmonic oscillator.

Solution It is convenient to use the mode expansion

$$z(t) = z_{\rm cl}(t) + \sqrt{2}\sum_{n=1}^{\infty} c_n \sin(\pi n t T)\,, \tag{9.25}$$

where

$$z_{\rm cl}(t) = x\,\frac{\sinh[\omega(1/T - t)]}{\sinh(\omega/T)} + y\,\frac{\sinh(\omega t)}{\sinh(\omega/T)} \tag{9.26}$$

obeys the classical equation of motion

$$\ddot{z}_{\rm cl} - \omega^2 z_{\rm cl} = 0 \tag{9.27}$$

with the boundary condition $z(0) = x$, $z(1/T) = y$. This reproduces Eq. (1.84) with $\tau = 1/T$ as $\omega \to 0$. The sin functions form an appropriate set of orthogonal basis functions for the interval $[0, 1/T]$.

Inserting the mode expansion (9.25) into the action, we obtain

$$\frac{m}{2}\int_0^{1/T} {\rm d}t\,\left(\dot{z}^2 + \omega^2 z^2\right) = S_{\rm cl}(x,y) + \frac{m}{2T}\sum_{n=1}^{\infty}\left[(\pi n T)^2 + \omega^2\right] c_n^2\,, \tag{9.28}$$

where

$$S_{\rm cl}(x,y) = \frac{m\omega}{2}\left[(x^2+y^2)\coth(\omega/T) - 2xy\,\frac{1}{\sinh(\omega/T)}\right]. \tag{9.29}$$

Substituting the measure as in Eq. (9.21) and performing the Gaussian integration over c_n, we have

$$\left\langle y \left| {\rm e}^{-\boldsymbol{H}/T} \right| x \right\rangle \propto \prod_{n=1}^{\infty}\left[\frac{T/m}{(\pi n T)^2 + \omega^2}\right]^{d/2} {\rm e}^{-S_{\rm cl}(x,y)}. \tag{9.30}$$

Finally, using Eq. (9.23), we obtain

$$\left\langle y \left| {\rm e}^{-\boldsymbol{H}/T} \right| x \right\rangle = \left[\frac{m\omega}{2\pi\sinh(\omega/T)}\right]^{d/2} {\rm e}^{-S_{\rm cl}(x,y)} \tag{9.31}$$

for the thermal density matrix of a nonrelativistic harmonic oscillator with frequency ω. The formulas of Sect. 1.5 are reproduced as $\omega \to 0$ which fixes an ω-independent normalization factor in Eq. (9.30). The partition function (9.24) is reproduced when we set $y = x$ in Eq. (9.31) and integrate over x.

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Problem 9.6 Calculate the partition function (9.5) for the free case.

Solution Since the path integral over $\varphi(\vec{x},t)$ is Gaussian, it can be represented as

$$\begin{aligned}\ln Z(T,V) &= -\frac{1}{2}\ln\det\left(-\partial_\mu^2+m^2\right) = -\frac{1}{2}\mathrm{Tr}\,\ln\left(-\partial_\mu^2+m^2\right)\\ &= -\frac{1}{2}V\int\frac{\mathrm{d}^d\vec{p}}{(2\pi)^d}\mathrm{Tr}_t\,\ln\left(-D^2+\omega^2\right),\end{aligned}\tag{9.32}$$

where

$$\omega = \sqrt{\vec{p}^{\,2}+m^2}\,. \tag{9.33}$$

We have used the fact that the $\vec{x}$ variable is not restricted, while the remaining trace of the one-dimensional operator is to be calculated with periodic boundary conditions.

We shall perform the calculation by expressing the trace via the diagonal resolvent of the same operator as has already been done in Problem 4.4 on p. 73. The Green function $G_\omega(t-t')$ is no longer given by Eq. (1.38) because of the periodic boundary conditions. Instead, we obtain the sum over even Matsubara frequencies:

$$G_\omega(t-t') = T\sum_{n=-\infty}^{+\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}nT(t'-t)}}{(2\pi nT)^2+\omega^2}\,, \tag{9.34}$$

which satisfies $G_\omega(1/T)=G_\omega(0)$, as it should for periodic boundary conditions, and reproduces Eq. (1.38) as $T\to 0$. The diagonal resolvent is given by

$$\begin{aligned}G_\omega(0) &= T\sum_{n=-\infty}^{+\infty}\frac{1}{(2\pi nT)^2+\omega^2}\\ &= \frac{1}{2\omega}\coth\frac{\omega}{2T}\,.\end{aligned}\tag{9.35}$$

Therefore,

$$\begin{aligned}\mathrm{Tr}_t\,\ln\left(-D^2+\omega^2\right) &= \int^{\omega^2}d\omega^2\int_0^{1/T}dt\,G_\omega(0)\\ &= \int^{\omega}d\omega\frac{1}{T}\coth\frac{\omega}{2T}\\ &= \frac{\omega}{T}+2\ln\left(1-\mathrm{e}^{-\omega/T}\right)\end{aligned}\tag{9.36}$$

modulo an ω-independent constant. Substituting into Eq. (9.32), we obtain

$$\ln Z(T,V) = -V\int\frac{\mathrm{d}^d\vec{p}}{(2\pi)^d}\left[\frac{\omega}{2T}+\ln\left(1-\mathrm{e}^{-\omega/T}\right)\right], \tag{9.37}$$

which is the standard result for an ideal Bose gas in quantum statistics modulo the first term on the RHS associated with the zero-point energy of the vacuum.

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9.2 QCD at finite temperature

QCD at finite temperatures is described by the partition function

$$Z(T,V) = \int \mathcal{D}A_\mu \, \mathcal{D}\bar\psi \, \mathcal{D}\psi \, \mathrm{e}^{-\int_0^{1/T} \mathrm{d}t \int_V \mathrm{d}^3\vec{x} \, \mathcal{L}[A_\mu,\psi,\bar\psi]}, \tag{9.38}$$

which is the proper analog of Eq. (9.5). The path integral is taken with the boundary conditions

$$A_\mu(\vec{x},1/T) = A_\mu(\vec{x},0), \tag{9.39}$$
$$\psi(\vec{x},1/T) = -\psi(\vec{x},0), \tag{9.40}$$
$$\bar\psi(\vec{x},1/T) = -\bar\psi(\vec{x},0), \tag{9.41}$$

which are periodic for the gauge field (gluon) and antiperiodic for the Fermi fields (quarks). The antiperiodicity of the Fermi fields is related, roughly speaking, with the famous extra minus sign of fermionic loops in the vacuum energy.

Problem 9.7 Calculate the partition function for free massive one-dimensional fermions with antiperiodic boundary conditions

$$\psi(1/T) = -\psi(0), \qquad \bar\psi(1/T) = -\bar\psi(0). \tag{9.42}$$

Solution The calculation is analogous to that of Problem 9.6. We obtain

$$\ln Z(T,V) = \ln\det(D+m) = \mathrm{Tr}\,\ln(D+m). \tag{9.43}$$

The fermion Green function $G_m(t-t')$ is given by the sum over odd Matsubara frequencies:

$$G_m(t-t') = T\sum_{n=-\infty}^{+\infty} \frac{\mathrm{e}^{\pi\mathrm{i}(2n+1)T(t'-t)}}{\mathrm{i}\pi(2n+1)T+m}, \tag{9.44}$$

which satisfies $G_m(1/T) = -G_m(0)$, as it should for antiperiodic boundary conditions.

As $T \to 0$, we obtain

$$G_m(t-t') = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\epsilon}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\epsilon(t'-t)}}{\mathrm{i}\epsilon+m} = \theta(t-t') \tag{9.45}$$

since the contour of integration over ϵ can be closed for $t > t'$ ($t < t'$) in the lower (upper) half-plane. We have thus reproduced the fermionic Green function (5.34) from Problem 5.3 on p. 90.

The diagonal resolvent is given by

$$G_m(0) = T\sum_{n=-\infty}^{+\infty} \frac{1}{\mathrm{i}\pi(2n+1)T+m} = \frac{1}{2}\tanh\frac{m}{2T}, \tag{9.46}$$

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which differs from Eq. (9.35) by the change of the coth for tanh. Therefore,

$$\begin{aligned}\ln Z(T,V) &= \int^{m} \mathrm{d}m\,\frac{1}{T}\,\tanh\frac{m}{2T} \\ &= \frac{m}{2T}+\ln\left(1+\mathrm{e}^{-m/T}\right)\end{aligned} \tag{9.47}$$

modulo an m-independent constant. The second term on the RHS involves a plus sign, which characterizes Fermi statistics (remember that $\omega = m$ if there are no spatial dimensions). If we were choose periodic boundary conditions instead of antiperiodic ones, we would have a minus sign as in Eq. (9.37) which is wrong for fermions. The first term on the RHS is again associated with the zero-point energy of the vacuum.

An extension of Eq. (9.47) to d dimensions can be obtained on substituting m by ω, given by Eq. (9.33), and integrating over the phase space, which results in a formula of the type of Eq. (9.37) but with the plus sign in the second term on the RHS.

The discussion of the previous section concerning the relation between the finite-temperature and Euclidean formulations explains why the latter allows one to calculate only static quantities in QCD, say hadron masses or interaction potentials, which do not depend on time. It is also worth noting that we did not add a gauge-fixing term in Eq. (9.38), having in mind a lattice quantization as before.

The lattice formulation of finite-temperature QCD is especially simple. One should take an asymmetric lattice whose size along the temporal axis is much smaller than that along the spatial ones:

$$L_t \;=\; \frac{1}{Ta} \;\ll\; L\,. \tag{9.48}$$

This guarantees that the system is in the thermodynamic limit. Then the temperature is given by

$$T \;=\; \frac{1}{aL_t}\,, \tag{9.49}$$

i.e. it coincides with the inverse extent of the lattice along the temporal axis. The periodic boundary conditions are usually imposed on the lattice by construction.

Since the lattice spacing a and the bare coupling constant g^2 are related by Eq. (6.85), the temperature (9.49) can be rewritten as

$$T \;=\; \frac{1}{L_t}\Lambda_{\rm QCD}\,\exp\left[\int\frac{\mathrm{d}g^2}{\mathcal{B}(g^2)}\right]. \tag{9.50}$$

Therefore, one can change the temperature on the lattice by varying either the size along the temporal axis, L_t, or g^2.

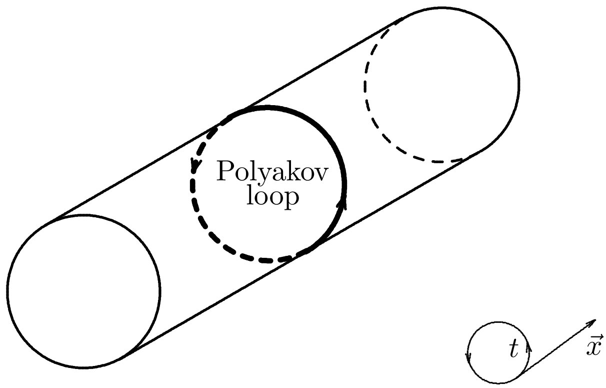

Fig. 9.1. Polyakov loop which winds around compactified temporal direction.

9.3 Confinement criterion at finite temperature

Wilson's confinement criterion, which is discussed in Sect. 6.6, is not applicable at finite temperatures. A proper criterion for confinement at finite temperatures was proposed by Polyakov [Pol78].

The Polyakov criterion of confinement at finite temperature uses the thermal Wilson loop which goes along the temporal direction:

$$L(\vec{x}) \;=\; \mathrm{tr}\, \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}\int_0^{1/T} \mathrm{d}t\, \mathcal{A}_d(\vec{x},t)}. \tag{9.51}$$

It is gauge invariant because of the periodic boundary conditions for the gauge field and is called the *Polyakov loop* or the thermal Wilson line. One can imagine that the time-variable $t \equiv x_d$ is compactified so that the Polyakov loop winds around the temporal direction as shown in Fig. 9.1.

The lattice Polyakov loop

$$L_{\vec{x}} \;=\; \mathrm{tr} \prod_{x_d} U_d(x) \tag{9.52}$$

is just the trace of the product of the link variables along a line which goes in the temporal direction through the lattice with imposed periodic boundary conditions.

Using the lattice gauge transformation (6.13), almost all link variables, associated with links pointing in the temporal direction, can be set equal 1 except for one time slice since the gauge transformation is periodic:

$$\Omega(\vec{x},0) \;=\; \Omega(\vec{x},1/T)\,. \tag{9.53}$$

The average of the Polyakov loop is related to the free energy $F_0(\vec{x})$ of a single quark (minus that of the vacuum) located at the point $\vec{x}$ of a

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three-dimensional space by

$$\langle L(\vec{x})\rangle = \mathrm{e}^{-F_0/T}. \tag{9.54}$$

If F_0 is infinite, which is associated with confinement, then

$$\langle L(\vec{x})\rangle = 0 \qquad \boxed{\text{confinement}}. \tag{9.55}$$

In contrast,

$$\langle L(\vec{x})\rangle \neq 0 \qquad \boxed{\text{deconfinement}} \tag{9.56}$$

is associated with deconfinement. This is the Polyakov criterion of confinement at finite temperature.

This criterion establishes a connection on a lattice between confinement and the $Z(3)$ symmetry – the center of $SU(3)$. The $Z(3)$ transformation of the link variables

$$U_d(x) \rightarrow Z_{x_d} U_d(x) \qquad (Z_{x_d} \in Z(3)) \tag{9.57}$$

leaves the lattice action invariant. This transformation is not of the same type as the local gauge transformation (6.13) since only the temporal link variables are transformed. The parameter Z_{x_d} of the transformation (9.57) depends on x_d, but is independent of the spatial coordinates $\vec{x}$ so the symmetry is a global one.

While the lattice action is invariant under the transformation (9.57), the Polyakov loop transforms as

$$L_{\vec{x}} \rightarrow Z\, L_{\vec{x}} \qquad (Z \in Z(3))\,, \tag{9.58}$$

where

$$Z = \prod_{x_d} Z_{x_d}\,. \tag{9.59}$$

Therefore, Eq. (9.55) holds if the symmetry is unbroken, while Eq. (9.56) signals spontaneous breaking of the symmetry. Thus, confinement or deconfinement are associated with the unbroken or broken global $Z(3)$ symmetry, respectively.

On a lattice of finite volume, the number of degrees of freedom is finite and spontaneous breaking of the $Z(3)$ symmetry is impossible. Then, it is more convenient to use a criterion which is based on the correlator of two Polyakov loops separated by a distance R along a spatial direction. This correlator determines the interaction energy $E(R)$ between a quark and an antiquark by the formula

$$\left\langle L(\vec{x})L^{\dagger}(\vec{y})\right\rangle_{\text{conn}} = \mathrm{e}^{-E(R)/T}. \tag{9.60}$$

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A finite correlation length is now associated with confinement, while an infinite one corresponds to deconfined quarks.

More details concerning the $Z(3)$ symmetry in finite-temperature lattice gauge theories can be found in the review by Svetitsky [Sve86].

Problem 9.8 Calculate the correlator (9.60) to the leading order of the strong-coupling expansion.

Solution The calculation is analogous to that of Sect. 6.5. The group integral is nonvanishing when the plaquettes completely fill a cylinder, spanned by two Polyakov loops, with area equal to R/T. This is analogous to the filling shown in Fig. 6.8. Contracting the indices, we find

$$\left\langle L_{\vec{x}} L^{\dagger}_{\vec{y}} \right\rangle_{\rm conn} \; = \; \left[W(\partial p) \right]^{R/T} , \tag{9.61}$$

where $W(\partial p)$ is given by Eq. (6.72). This yields the same interaction potential $E(R)$ as before (see Eqs. (6.76) and (6.77)).

Remark on high temperatures

At high temperatures $T \to \infty$, the temporal direction shrinks and the partition function (9.38) reduces to a three-dimensional one with the coupling constant

$$g^2_{\rm 3D} \; = \; g^2 T , \tag{9.62}$$

which has the dimension of [mass] in three dimensions. Three-dimensional QCD and QED always confine. If we take a Wilson loop in the form of a rectangle along spatial directions in four-dimensional QCD at high temperature, its average coincides with that in three dimensions and obeys the area law. This does not mean, however, that we are in a confining phase since the confinement criterion at finite temperature is different [Pol78].

9.4 Deconfining transition

The effects of finite temperatures are negligible under normal circumstances in QCD where the typical energy scale is of the order of hundreds of MeV, while a temperature of, say, $T \approx 300$ K is associated with the energy* $kT \approx 3 \times 10^{-8}$ MeV. However, for times of the order of 10^{-4} seconds after the big bang in the very early universe, the energies of thermal fluctuations were ~ 100 MeV, i.e. of the order of the mass of the π-meson. Therefore, π-mesons can be created out of the vacuum at those times, while their density in a unit volume is described by the thermodynamics of an ideal gas. Heavier hadrons are suppressed at these energies by the Boltzmann factor.

* Here $k = 8.6 \times 10^{-11}$ MeV K^{-1} is the Boltzmann constant.

The energy density $\mathcal{E}(T)$ of the hadron matter is given by the standard thermodynamical relation

$$\mathcal{E}(T) \;=\; \frac{1}{V}\frac{\partial}{\partial(1/T)} \ln Z(T,V)\bigg|_V , \tag{9.63}$$

with $Z(T,V)$ being given by Eq. (9.38).

When the density of hadrons is small, $\mathcal{E}(T)$ is given by the formula

$$\mathcal{E}_{\rm h}(T) \;=\; \frac{T}{2\pi^2} \sum_{i=\pi,\rho,\omega,\ldots} g_i \left[m_i^3 {\rm K}_1(m_i/T) + 3m_i^2 {\rm K}_2(m_i/T) \right] , \tag{9.64}$$

where $g_\pi = 3$, $g_\rho = 9$, $g_\omega = 3$, ... are the statistical weights of the π, ρ, ω, ... mesons, while K_1 and K_2 are the modified Bessel functions.

Problem 9.9 Derive Eq. (9.64) starting from the partition function (9.37).

Solution For a dilute gas, the logarithm in Eq. (9.37) can be expanded in $\exp(-E/T)$. Therefore, we find

$$\begin{aligned} \ln Z(T,V) &= \frac{\rm const}{T} + V \int \frac{{\rm d}^3\vec{p}}{(2\pi)^3}\, {\rm e}^{-\sqrt{\vec{p}^{\,2}+m^2}/T} \\ &= \frac{\rm const}{T} + \frac{VTm^2}{2\pi^2} {\rm K}_2(m/T) \,. \end{aligned} \tag{9.65}$$

The second term on the RHS describes the classical statistics of an ideal gas of relativistic particles. Equation (9.64) can now be derived by differentiating this formula with respect to $1/T$ according to Eq. (9.63) and taking into account the statistical weights of the hadron states. The zero-point energy term gives a T-independent contribution to $\mathcal{E}_{\rm h}$, which only changes the energy reference level.

At low temperatures, the hadron matter is in the confinement phase. However, when the temperature is increased, a phase transition associated with deconfinement occurs at some temperature $T = T_{\rm c}$ as was first pointed out by Polyakov [Pol78] and Susskind [Sus79]. For $T < T_{\rm c}$ the interaction potential between static quarks is linear, as is shown in Fig. 6.9a on p. 117, while for $T > T_{\rm c}$ the potential is deconfining, as is shown in Fig. 6.9b. The state of the hadron matter with deconfined quarks and gluons is often called the *quark–gluon plasma*.

There exists a very simple physical argument as to why the deconfining phase transition must occur in QCD when the temperature is increased. It is based on the string picture of confinement which was considered in Sect. 6.6. The string is made of the gluon field between static quarks in the confining phase, which are associated with the string end points. With increasing temperature, condensation of strings of infinite length will inevitably occur owing to the large entropy of such states, which corresponds to a deconfining phase transition.

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Problem 9.10 Derive the temperature of a phase transition for an elastic string by analyzing the temperature dependence of its free energy.

Solution Let us consider the thermodynamics of an elastic string with fixed end points. For low temperatures, thermal fluctuations of the length of the string are suppressed by the Boltzmann factor since the energy is proportional to the length. Therefore, the string is tightened along the shortest distance between the quarks which leads to a linear potential.

When the temperature is increased, entropy effects associated with fluctuations of the shape of the string become essential. An increment of the string length l by Δl increases energy by

$$\Delta E = \frac{\partial E}{\partial l}\Delta l = K\Delta l\,, \tag{9.66}$$

where K is the string tension as before, but causes a gain of the entropy

$$\Delta S = \frac{\partial S}{\partial l}\Delta l\,. \tag{9.67}$$

The change of free energy is given by

$$\Delta F = \Delta E - T\Delta S = \left(K - T\frac{\partial S}{\partial l}\right)\Delta l\,. \tag{9.68}$$

A phase transition occurs at the temperature

$$T_{\mathrm{c}} = K\left(\frac{\partial S}{\partial l}\right)^{-1}, \tag{9.69}$$

when the changes of energy and entropy compensate each other, so that the free energy ceases to depend on Δl. Therefore, the phase transition is associated with a condensation of arbitrarily long strings.

The energy density $\mathcal{E}(T)$ is described by a free gas of hadrons for low temperatures, as has already been mentioned, and by a free gas of quarks and gluons at high temperatures. The latter statement is a result of asymptotic freedom, which says that the effective coupling constant describing a strong interaction at temperature T is given by

$$g^2(T) = \frac{1}{b\ln\left(\frac{\Lambda_{\mathrm{QCD}}}{T}\right)} \tag{9.70}$$

with

$$b = \frac{1}{4\pi^2}\left(-11 + \frac{2}{3}N_{\mathrm{f}}\right) \tag{9.71}$$

and N_{f} being the number of fermion species (or flavors) with mass much less than T. This formula has the same structure as the running constant $g^2(Q)$, which describes the strong interaction at a momentum of Q. Since

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$Q \sim T$ for thermal fluctuations, these two coupling constants coincide with logarithmic accuracy.*

The energy density $\mathcal{E}(T)$ of the quark–gluon plasma is given by Boltzmann's law

$$\mathcal{E}_{\rm p}(T) \;=\; g_{\rm p}\frac{\pi^2}{30}T^4 + B\,, \qquad (9.72)$$

where

$$g_{\rm p} \;=\; 2 \cdot 8 + \frac{7}{8} \cdot 2 \cdot 2 \cdot 3 \cdot N_{\rm f} \qquad (9.73)$$

is the statistical weight, i.e. the number of independent internal degrees of freedom of the particles of the ideal gas. There are two spin and eight color states for gluons, and two spin, two particle–antiparticle, three color and $N_{\rm f}$ flavor states for quarks ($N_{\rm f} = 2$ for the u- and d-quarks). The factor of 7/8 is the usual one for fermions.

The T-independent constant $B > 0$ in Eq. (9.72) is associated with the fact that the vacuum energy in the plasma phase is higher than in the hadron phase. In other words, the energy density of the perturbative vacuum is larger by B than that of a nonperturbative one. It is because of this energy difference that hadrons are stable at low temperatures. Such a shift of energy densities between perturbative and nonperturbative vacua is typical for bag models of hadrons.

Numerical Monte Carlo simulations of lattice gauge theory at finite temperature indicate that the deconfining phase transition is of first order and occurs at $T_{\rm c} \approx 200$ MeV. The actual dependence of the energy density on T, calculated by the Monte Carlo method, is well described by Eq. (9.64) for $T < T_{\rm c}$ and Eq. (9.72) for $T > T_{\rm c}$. This behavior is illustrated in Fig. 9.2.

Problem 9.11 Calculate $T_{\rm c}$ and the latent heat $\Delta\mathcal{E}$, approximating $\mathcal{E}_{\rm h}$ by an ideal gas of massless π-mesons.

Solution It is reasonable to disregard the mass of the π-mesons for $T \gtrsim$ 200 MeV. Then,

$$\mathcal{E}_{\rm h}\,(T) \;=\; g_{\rm h}\frac{\pi^2}{30}T^4\,, \qquad (9.74)$$

where $g_{\rm h} = 3$ as a result of the three isotopic states (π^+, π^-, and π°). $\mathcal{E}_{\rm h}\,(T)$ for the plasma state is given by Eq. (9.72).

* The perturbative calculations in QCD at finite temperature are described in the book by Kapusta [Kap89] and in the more recent review by Smilga [Smi97] and the book by Le Bellac [Bel00].

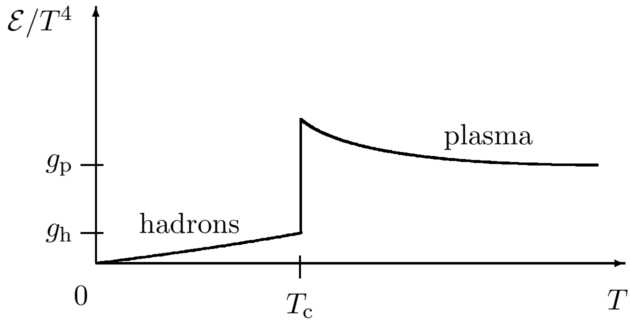

Fig. 9.2. Temperature dependence of the energy density for hadron matter. $\mathcal{E}(T)$ for the hadron and the plasma phases are given by Eqs. (9.64) and (9.72). The difference $\mathcal{E}_p - \mathcal{E}_h$ at the temperature T_c of the deconfining phase transition is equal to the latent heat $\Delta\mathcal{E}$.

The pressure for the relativistic gases with the energy densities (9.74) and (9.72) is given, respectively, by

$$\mathcal{P}_h(T) = g_h \frac{\pi^2}{90} T^4 \tag{9.75}$$

and

$$\mathcal{P}_p(T) = g_p \frac{\pi^2}{90} T^4 - B\,. \tag{9.76}$$

The positive constant B in the energy density (9.72) leads to a negative pressure in the plasma state at low temperatures. Therefore, the hadron phase is preferable at low temperatures. This is in the spirit of the bag model of hadrons. At high energies the pressure is higher for the plasma phase, since

$$g_p = 37 \; > \; g_h = 3\,, \tag{9.77}$$

so that the plasma phase is realized. The behavior of the pressure versus T^4 is shown in Fig. 9.3 for both phases of hadron matter.

The deconfining phase transition occurs when the pressures in both phases coincide. Therefore, we obtain

$$T_c^4 = \frac{B}{\frac{\pi^2}{90}(g_p - g_h)} \tag{9.78}$$

and

$$\Delta\mathcal{E} \equiv \mathcal{E}_p(T) - \mathcal{E}_h(T) = 4B\,. \tag{9.79}$$

If we were set $g_h = 0$ in Eq. (9.78), this would change the value of T_c by a few per cent. This justifies the approximation of massless π-mesons.

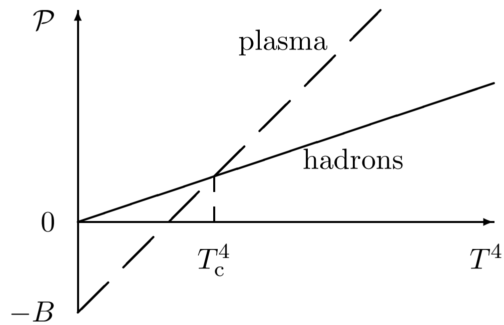

Fig. 9.3. Pressure versus T^4 for the two phases of hadron matter. The solid and dashed lines represent Eqs. (9.75) and (9.76), respectively. The hadron phase is stable for $T < T_c$, while the plasma phase is stable for $T > T_c$.

Remark on the deconfining phase transition in the early universe

The confining phase transition from a quark–gluon plasma to hadrons happened in the early universe when its age was $\approx 10^{-5}$ seconds. The equation of state of the hadron matter is described by Eqs. (9.72) and (9.76) before that time and by Eqs. (9.64) and (9.75) after that time. There are presumably no cosmological consequences of this phase transition, which survive to our time, since it happened too long ago. For instance, fluctuations of the hadron matter density which might have occurred just after the phase transition were washed out by further expansion. The confining phase transition in the early universe is considered in the review [CGS86], Section 6.

9.5 Restoration of chiral symmetry

The chiral symmetry is broken spontaneously in QCD at $T = 0$, as is discussed in Sect. 8.5. With increasing temperature, the chiral symmetry should be restored at some temperature T_{ch} (which does not necessarily coincide with T_c) since perturbation theory is applicable at high T. This restoration occurs as a phase transition with $\langle\bar{\psi}\psi\rangle$ being the proper order parameter. Therefore, the quark condensate is destroyed at $T = T_{ch}$. Monte Carlo simulations indicate that this chiral phase transition is of first order.

However, there is a subtlety in the above string picture of quark confinement when virtual quarks are taken into account. The effects of virtual quarks are suppressed when their mass m is infinitely large and the picture of confinement is the same as in pure gluodynamics: quarks are permanently confined by strings constructed from the flux tubes of the

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Fig. 9.4. Breaking of the flux tube by creating a quark–antiquark pair (depicted by the open circles) out of the vacuum.

gluon field. This is associated with a linear interaction potential.

For light virtual quarks, the flux tube can break creating a quark–antiquark pair out of the vacuum, as is shown in Fig. 9.4. This happens when the energy saved in the flux tube is large enough to compensate the kinetic energy of the particles produced. Hence, the linear growth of the potential will stop at such distances.

The average of the Polyakov loop (9.51) is no longer a criterion for quark confinement in the presence of virtual quarks. The test static quark can always be screened by an antiquark created out of the vacuum (a quark created at the same time will go to infinity). Therefore, the free energy F_0 in Eq. (9.54) is always finite so that $\langle L(\vec{x})\rangle \neq 0$ in both phases.

The effects of virtual quarks usually weaken a phase transition in a pure gauge theory. If the deconfining phase transition in the $SU(3)$ pure gauge theory was of second order rather than first order, it would presumably disappear for an arbitrarily large but finite value of m. Such a phenomenon happens in the Ising model where an arbitrarily small external magnetic field (which is an analog of the quark mass) destroys the second-order phase transition. A discontinuity of $\langle L(\vec{x})\rangle$ at the first-order deconfining transition continues in the (T, m)-plane as illustrated by Fig. 9.5. It seems to terminate at some value m_{c} of the quark mass.

This situation with the order parameter for the deconfining phase transition is somewhat similar to that for the chiral phase transition. $\left\langle\bar{\psi}\psi\right\rangle$ vanishes in the unbroken phase only for $m = 0$. If $m \neq 0$ but is small, there is a small explicit breaking of chiral symmetry owing to the quark mass. Since the chiral phase transition is of first order for $m = 0$, it is natural to expect that a discontinuity of $\left\langle\bar{\psi}\psi\right\rangle$ continues in the (T, m)-plane up to some value m_{ch} of the quark mass.

If $m_{\mathrm{ch}} < m_{\mathrm{c}}$, the phase diagram in the (T, m)-plane may look like that shown in Fig. 9.5. In the intermediate region $m_{\mathrm{ch}} < m < m_{\mathrm{c}}$, the behavior of neither $\langle L(\vec{x})\rangle$ nor $\left\langle\bar{\psi}\psi\right\rangle$ can answer the question of whether a phase transition (or two separate transitions) occurs. A proper parameter, which signals a phase transition is this region, could be the temperature-dependence of the energy density $\mathcal{E}(T)$ that undergoes discontinuities at the points of first-order phase transitions.

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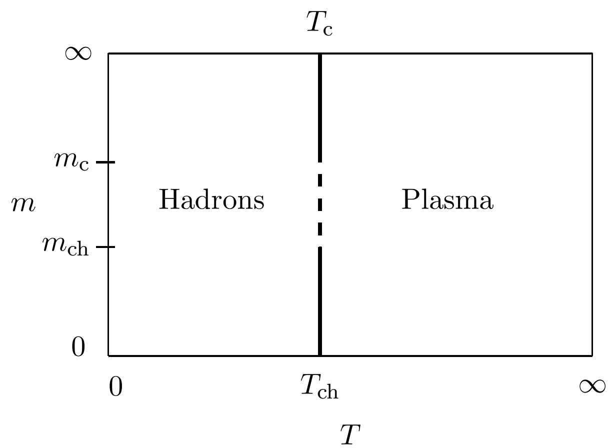

Fig. 9.5. Expected phase diagram of the hadron matter in the (T, m)-plane. The deconfining phase transition starts at $T = T_c$ for $m = \infty$. $\langle L(\vec{x})\rangle$ is its order parameter for $m > m_c$. The chiral phase transition starts at $T = T_{ch}$ for $m = 0$. $\bar{\psi}\psi$ is its order parameter for $m < m_{ch}$.

It is worth noting that an alternative behavior of the phase diagram in the (T, m)-plane, when $m_{ch} > m_c$, is not confirmed by Monte Carlo simulations.

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Bibliography to Part 2

Reference guide

There are many very good introductory lectures/reviews on lattice gauge theory. For a perfect description of motivations and the lattice formulation, I would recommend the original paper [Wil74] and lectures [Wil75] by Wilson. Original papers on lattice gauge theories are collected in the book edited by Rebbi [Reb83]. Various topics within lattice gauge theory are covered in the well-written book by Creutz [Cre83]. The book by Seiler [Sei82] contains some mathematically rigorous results. The more recently published book by Montvay and Münster [MM94] contains a comprehensive look at lattice gauge theory.

I shall also list some of the old reviews on lattice gauge theory which might be useful for deeper studies of the lattice methods. The strong-coupling expansion and the mean-field method are discussed in the review by Drouffe and Zuber [DZ83]. The Monte Carlo method and some results of numerical simulations are considered in [CJR83, Mak84]. The fermion doubling problem and the Wilson fermions are discussed in the lectures [Wil75].

An introduction to quantum field theory at finite temperature is given in the book by Kapusta [Kap89], which contains, in particular, a discussion of perturbation theory in QCD at finite temperature. Lattice gauge theory aspects of the deconfining phase transition at finite temperature are considered in the review by Svetitsky [Sve86]. A description of the thermal phases of hadron matter, a comparison with results of Monte Carlo simulations and a discussion of the deconfining phase transition in the early universe are contained in the review [CGS86]. Various physical aspects of thermal QCD are considered in the recent review by Smilga [Smi97] and book by Le Bellac [Bel00].

Most of the reviews mentioned above were written in the 1980s and

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contain a description of lattice methods as well as early Monte Carlo results. The best way to follow current developments of the subject is via plenary talks at the annual Lattice Conference ([Lat00] and those for the preceding years).

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