Can. J. Math., Vol. XLII, No. 2, 1990, pp. 213–229

TOTAL CATEGORIES AND SOLID FUNCTORS

REINHARD BÖRGER AND WALTER THOLEN

1. Introduction. Totality of a category as introduced by Street and Walters [**17**] is known to be a strong cocompleteness property (cf. also [**21**]) which goes far beyond ordinary (small) cocompleteness. It implies compactness in the sense of Isbell [**11**] and therefore hypercompleteness [**7**], that is: the existence of limits of all those (not necessarily small) diagrams which are not prevented from having a limit merely from size-considerations with respect to the hom-sets. In particular, arbitrary intersections of monomorphisms exist in a total category; which is part of Street's [**16**] characterization of totality and is used in establishing the interrelationship with topoi (cf. also [**15**]).

This article gives solutions to two problems mentioned in Kelly's excellent survey article [**12**] and gives an $\mathcal{E}$-generalization of Day's theorem [**8**] that a cocomplete category with arbitrary cointersections of epimorphisms and generator is total. Day mentions the possibility of replacing epimorphisms by $\mathcal{E}$-morphisms which belong to an $(\mathcal{E}, \mathcal{M})$-factorization system, but gives no generalized statement; in particular, he does not specify how to generalize the notion of generator. In any case, our theorem seems to go beyond what Day had in mind since we do not even require $\mathcal{E}$ to be closed under composition. This is particularly relevant in the case where $\mathcal{E}$ is the class of morphisms which are composites of two regular epimorphisms. We devote special attention to this case, as it gives the theorem that a cocomplete category is total if it has a regular generator; or a small set of objects of which every other object is (somehow) a colimit (which is a strengthening of Kelly's [**12**, Corollary 6.5]).

We give a complete solution to the problem whether the $\mathcal{E}$-generalization of Day's theorem allows for converse statements, and thereby settle the questions raised in Kelly's article: a total category always allows the formation of cointersections of arbitrary families of regular epimorphisms, but not so for strong ones, even when it contains a strong generator; on the other hand, a total category need not have a generator, even when it is cowellpowered and therefore contains cointersections of arbitrary families of epimorphisms.

The paper is self-contained. Although our proof of the $\mathcal{E}$-version of Day's theorem relies heavily on lifting properties of solid functors (formerly called semi-topological [**18**]) we in fact do not require any previous knowledge of these functors, as all relevant facts about them are provided in a new concise form in this paper. In order to keep its length as limited, and the range of potential readers as unlimited, as possible, we have given all definitions and

Received May 2, 1989. The hospitality of Fernuniversität (Germany) and of The University of Sydney (Australia) and partial support under the Australian Research Grant Scheme and by NSERC (Canada) are gratefully acknowledged.

results just for ordinary $\mathcal{S}et$-based categories (with small hom-sets) although we are fully aware of the fact that, even beyond the $\mathcal{V}$-results stated by Kelly [**12**] and Day [**8**], our techniques can be widely established in the $\mathcal{V}$-world: see Anghel [**3**], [**4**].

The main results of this paper were announced in communications given by the authors during the Conference on Category Theory at Louvain-la-Neuve in July 1987 and in the Sydney Category Seminar in January 1988. The authors are grateful for encouraging comments they received. Especially we would like to thank Jiří Adámek who communicated some interesting questions to us which then led to the results exhibited in Section 5.

1. Review on solid functors.

1.1. For a functor $U : \mathcal{A} \longrightarrow \mathcal{H}$, a *$U$-sink* σ is a family of $\mathcal{A}$-objects A_i and of $\mathcal{H}$-morphisms $x_i : UA_i \longrightarrow X$, $i \in I$; here I may be a proper class, or be empty; in which case the family is just given by its codomain, i.e., the $\mathcal{H}$-object X. For such a *U-sink* σ, we can form the full subcategory $(X/U)_\sigma$ of the comma-category X/U whose objects (y, B) with $y : X \longrightarrow UB$ in $\mathcal{H}$ have the property that there are $\mathcal{A}$-morphisms $f_i : A_i \longrightarrow B$ with $Uf_i = yx_i$ for all $i \in I$. The functor U is called *solid* (formerly semi-topological [**18**]) if U is faithful and if $(X/U)_\sigma$ has an initial object for every X and every σ with codomain X; such an initial object (e, C), together with the uniquely determined morphisms $g_i : A_i \longrightarrow C$ with $Ug_i = ex_i$, is also called a *U-semifinal lifting* of σ. The following two important properties follow immediately from the definition.

(1) U has a left adjoint: consider empty U-sinks.

(2) U detects all colimits, i.e., if, for any diagram $H : \mathcal{D} \longrightarrow \mathcal{A}$, colim UH exists in $\mathcal{H}$, then colim H exists in $\mathcal{A}$: indeed, one just has to consider a U-semifinal lifting of the U-sink given by colim UH to obtain colim H.

The following theorem follows from results in [**18**] but we here give a direct proof:

THEOREM 1.2. *A functor $U : \mathcal{A} \longrightarrow \mathcal{H}$ is solid if and only if U has a left adjoint and there is a class $\mathcal{E}$ of $\mathcal{A}$-morphisms such that*

(a) *the co-units of U belong to $\mathcal{E}$,*

(b) *$\mathcal{A}$ is $\mathcal{E}$-cocomplete, i.e., a pushout of an $\mathcal{E}$-morphism along an arbitrary morphism exists in $\mathcal{A}$ and any such belongs to $\mathcal{E}$ and a cointersection of an arbitrary family of $\mathcal{E}$-morphisms exists in $\mathcal{A}$ and any such belongs to $\mathcal{E}$.*

(Note that when $\mathcal{A}$ is $\mathcal{E}$-cocomplete, it follows automatically that $\mathcal{E}$ contains all isomorphims of $\mathcal{A}$ and is closed under composition with them.)

Proof. Let first U be solid; we mentioned before that U then has a left adjoint F, and we let $\mathcal{E}$ be the class of all $\mathcal{A}$-morphisms $e : A \longrightarrow B$ such that (Ue, B) gives a U-semifinal lifting of some U-sink σ with codomain UA. The counits $\epsilon_B : FUB \longrightarrow B$ belong to $\mathcal{E}$ since $(U\epsilon_B, B)$ gives a U-semifinal lifting of the 2-indexed sink consisting of 1_{UFUB} and of the unit $\eta_{UB} : UB \longrightarrow UFUB$. So it remains to show $\mathcal{E}$-cocompleteness.

For $e : A \longrightarrow B$ in $\mathcal{E}$ and $f : A \longrightarrow C$ in $\mathcal{A}$, one considers a U-sink $\sigma = (x_i : UA_i \longrightarrow UA)$ such that (Ue, B) gives a U-semifinal lifting of σ. A U-semifinal lifting of the U-sink τ which contains all $Uf \cdot x_i$ and 1_{UC} is then actually given by a morphism $e' : C \longrightarrow D$ which, by definition, belongs to $\mathcal{E}$. Since all compositions $Ue' \cdot Uf \cdot x_i$ are U-images of morphisms in $\mathcal{A}$, U-semifinality of (Ue, B) gives a morphism f' such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad e \quad} & B \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ C & \xrightarrow{\quad e' \quad} & D \end{array}$$

commutes; indeed, it is easily shown to be a pushout.

To prove the existence of cointersections, let $e_j : A \longrightarrow B_j$, $j \in J$, be any family of $\mathcal{E}$-morphisms, so that each (Ue_j, A) is the U-semifinal lifting of a U-sink σ_j with codomain UA. Let σ be the U-sink obtained by uniting all σ_j's and adding 1_{UA}. A U-semifinal lifting gives a morphism $e : A \longrightarrow C$ in $\mathcal{E}$ and morphisms $p_j : B_j \longrightarrow C$ such that the diagrams

$$\begin{array}{ccc} A & \xrightarrow{\quad e_j \quad} & B_j \\ & {\scriptstyle e}\searrow & \big\downarrow{\scriptstyle p_j} \\ & & C \end{array}$$

form a generalized pushout.

Vice versa, given a class E with (a) and (b), we first remark that the existence of arbitrary cointersections of $\mathcal{E}$-morphisms makes every morphism in $\mathcal{E}$ an epimorphism (cf. [**5**]). So the co-units of U are epic, whence U is faithful. To show the existence of a U-semifinal lifting of any U-sink $\sigma = (x_i : UA_i \longrightarrow X)$, one first forms the pushout

$$\begin{array}{ccc} FUA_i & \xrightarrow{\quad Fx_i \quad} & FX \\ {\scriptstyle \epsilon_{A_i}}\big\downarrow & & \big\downarrow{\scriptstyle e_i} \\ A_i & \xrightarrow{\quad h_i \quad} & B_i \end{array}$$

with $e_i \in \mathcal{E}$ for every $i \in I$, and then the cointersection

$$\begin{array}{ccc} FX & \xrightarrow{\quad e_i \quad} & B_i \\ & {\scriptstyle e}\searrow & \big\downarrow{\scriptstyle p_i} \\ & & B \end{array}$$

with $e \in \mathcal{E}$. For every i one has

$$Ue \cdot \eta_X \cdot x_i = Up_i \cdot Ue_i \cdot UFx_i \cdot \eta_{UA_i}$$
$$= Up_i \cdot Uh_i \cdot U\epsilon_{A_i} \cdot \eta_{UA_i} = U(p_i h_i),$$

so that $(Ue \cdot \eta_X, B)$ belongs to $(X/U)_\sigma$; that it is actually initial follows easily with the universal properties of the above diagrams.

Remark 1.3. For a solid $U : \mathcal{A} \longrightarrow \mathcal{H}$, there is of course a least $\mathcal{E}$ such that properties (a) and (b) of 1.2 hold; it is the closure of $\{\epsilon_A \mid A \in |A|\}$ under the formation of pushouts and arbitrary cointersections in $\mathcal{A}$. The proof of 1.2 shows that one can give a one-step description of this closure.

COROLLARY 1.4. (cf. [**18**, Corollary 6.8]). *A faithful right adjoint functor U : $\mathcal{A} \longrightarrow \mathcal{H}$ is solid if $\mathcal{A}$ is cocomplete and admits arbitrary cointersections of epimorphisms; in particular if $\mathcal{A}$ is cowellpowered. Cocompleteness of $\mathcal{A}$ is a necessary condition for U to be solid, provided $\mathcal{H}$ is cocomplete.*

There is one other important property of solid functors we shall need: detection of limits. For that we first mention the known result (cf. [**10**], [**20**]):

PROPOSITION 1.5. *For any class $\mathcal{E}$ of $\mathcal{A}$-morphisms that contains all isomorphisms and is closed under composition with them, $\mathcal{A}$ is $\mathcal{E}$-cocomplete if and only if every family $f_i : A \longrightarrow B_i$ ($i \in I$, where I may be a proper class) of $\mathcal{A}$-morphisms has a locally orthogonal $\mathcal{E}$-factorization $f_i = m_i e$, so $e : A \longrightarrow C$ is in $\mathcal{E}$ and $p \perp (e, m_i)$ for every $p \in \mathcal{E}$ (that is: $h_i p = f_i g$ for all i implies the existence of a unique morphism t with $tp = eg$ and $m_i t = h_i$ for all i).*

From [**18**] we recall that a factorization $x_i = Um_i \cdot d$ of a *U-source* (x_i : $X \longrightarrow UA_i$) (dual to U-sink) through a family ($m_i : B \longrightarrow A_i$) of $\mathcal{A}$-morphisms is *U-semi-initial* if, for every family ($g_i : C \longrightarrow A_i$) in $\mathcal{A}$ and every morphism $y : UC \longrightarrow X$ with $x_i y = Ug_i$, there is a unique $t : C \longrightarrow B$ in $\mathcal{A}$ with $Ut = dy$ and $m_i t = g_i$; the factorization is *U-epimorphic* if, for $a, b : B \longrightarrow D$, $Ua \cdot d = Ub \cdot d$ implies $a = b$.

COROLLARY 1.6. *For a solid U, every U-source has a U-epimorphic and U-semi-initial factorization.*

Proof. Let the family ($f_i : FX \longrightarrow A_i$) correspond to ($x_i : X \longrightarrow UA_i$) by adjunction, and let $f_i = em_i$ be a locally orthgonal $\mathcal{E}$-factorization. Then

$$x_i = Um_i \cdot (Ue \cdot \eta_X)$$

is obviously a U-epimorphic factorization. To show its semi-initiality, given y and g_i as above, one considers the commutative diagrams

$$\begin{array}{ccccc} FUC & & \xrightarrow{\epsilon_C} & & C \\ {\scriptstyle Fy}\downarrow & & & & \downarrow{\scriptstyle g_i} \\ Fx & \xrightarrow{e} & B & \xrightarrow{m_i} & A_i \end{array}$$

whose diagonal-fill-in is the needed morphism $t : C \longrightarrow B$.

1.7. The existence of U-epimorphic and U-semi-initial factorization of U-sources is actually equivalent to the solidity of U (cf. [**18**]), but 1.6 suffices to derive the last property of solid functors we shall use:

U detects all limits (and, of course, preserves them); indeed, we can derive the existence of lim H for any $H : \mathcal{D} \longrightarrow \mathcal{A}$ such that lim UH exists from a U-epimorphic and U-semi-initial factorization of the projections of lim UH (cf. [**18**]).

2. Hypercompleteness, compactness, and totality. All of our categories are assumed to have small hom-sets.

2.1. A category $\mathcal{A}$ is called *total* if colim H exists in $\mathcal{A}$ for every diagram $H : \mathcal{D} \longrightarrow \mathcal{A}$ such that colim $\mathcal{A}(A, H-)$ exists in $\mathcal{S}et$ for all $A \in \mathcal{A}$. So a total category is trivially (small) cocomplete, since colim $\mathcal{A}(A, H-)$ trivially exists in $\mathcal{S}et$ when $\mathcal{D}$ is small. The following is wellknown (cf. [**12**]):

PROPOSITION 2.2. *The following conditions are equivalent:*
(i) *$\mathcal{A}$ is total,*
(ii) colim H *exists in $\mathcal{A}$ whenever, for each $A \in \mathcal{A}$, the comma category A/H has only a small set of connected components,*
(iii) *the Yoneda embedding $Y : \mathcal{A} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{S}et]$ has a left adjoint.*

2.3. When proving (ii) $\Rightarrow$ (iii) above, one has to construct out of a given functor $E : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{S}et$ a suitable H such that A/H satisfies the said smallness condition for all $A \in \mathcal{A}$. This is achieved by the element-construction: the category $e\ell\ E$ has as object pairs (A, a) with $A \in \mathcal{A}$ and $a \in EA$; morphisms $f : (A, a) \longrightarrow (B, b)$ are $\mathcal{A}$-morphisms $f : A \longrightarrow B$ with $(Ef)b = a$; $H : e\ell\ E \longrightarrow \mathcal{A}$ is the canonical functor which, under (ii), has a colimit in $\mathcal{A}$. One can weaken condition (ii) by requiring the existence of colim H only for certain H's built as follows:

$\mathcal{A}$ is *compact* if colim $(e\ell\ E \longrightarrow \mathcal{A})$ exists in $\mathcal{A}$ for every $E : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{S}et$ that preserves all existing limits in $\mathcal{A}^{\mathrm{op}}$. Since, in particular, E will transform the given colimit in $\mathcal{A}$ into a limit in $\mathcal{S}et$, it follows that E is actually represented by that colimit (cf. [**13**]). One therefore has the known:

PROPOSITION 2.4. *The following conditions are equivalent:*
(i) *$\mathcal{A}$ is compact,*
(ii) *every $E : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{S}et$ that preserves all existing limits in $\mathcal{A}^{\mathrm{op}}$ is representable.*

(iii) *every* $F : \mathcal{A} \longrightarrow \mathcal{B}$ *that preserves all existing colimits in* $\mathcal{A}$ *has a right adjoint.*

2.5. It was shown in [7] that compact categories enjoy the strongest completeness property that a category (with small hom-sets) can be expect to have. Here we define this property formally, analagously to totality, and show in 2.6 below the equivalence with the previous definition.

$\mathcal{A}$ is *hypercomplete* if $\lim H$ exists in $\mathcal{A}$ for every diagram $H : \mathcal{D} \longrightarrow \mathcal{A}$ such that $\lim \mathcal{A}(A, H-)$ exists in $\mathcal{S}et$ for all $A \in \mathcal{A}$. (Note that the existence of $\lim \mathcal{A}(A, H-)$ is a necessary condition for $\lim H$ to exist, and that despite the formal analogy, hypercompleteness is not dual to totality.) Of course, hypercompleteness implies small completeness; that it is implied by compactness follows from the following characterization.

PROPOSITION 2.6. *The following conditions are equivalent:*

(i) $\mathcal{A}$ *is hypercomplete,*

(ii) $\lim H$ *exists in* $\mathcal{A}$ *whenever, for each* $A \in \mathcal{A}$, *there is only a small set of cones* $\Delta A \longrightarrow H$ *in* $\mathcal{A}$.

Proof. (i) $\Rightarrow$ (ii). If $H : \mathcal{D} \longrightarrow \mathcal{A}$ is such that $K_A = \mathrm{Nat}(\Delta A, H)$ is small for every $A \in \mathcal{A}$, then the evaluation maps

$$\kappa_d^A : K_A \longrightarrow \mathcal{A}(A, Hd), \quad \alpha \longmapsto \alpha_d \ (d \in \mathcal{D}),$$

present K_A as a limit of $\mathcal{A}(A, H-)$ is $\mathcal{S}et$.

(ii) $\Rightarrow$ (i). One always has one-to-one correspondences

$$\frac{\Delta A \longrightarrow H}{\Delta 1 \longrightarrow \mathcal{A}(A, H-)}$$

and

$$\frac{\Delta 1 \longrightarrow \mathcal{A}(A, H-)}{1 \longrightarrow \lim \mathcal{A}(A, H-)}$$

if the limit exists in $\mathcal{S}et$. So $\mathrm{Nat}\,(\Delta A, H)$ is small in that case.

2.7. A hypercomplete category is $\mathcal{M}$*-complete* (dual to $\mathcal{E}$-cocomplete) for $\mathcal{M}$ the class of all monomorphisms, hence for every conceivable $\mathcal{M}$; indeed the existence of arbitrary intersections of monomorphisms follows directly from the definition of hypercompleteness (since representable functors preserve monomorphisms).

So, summarizing the previous remarks, we have the following system of implications:

$$\begin{array}{lcl} \mathcal{A} \text{ total} & \Longrightarrow & \mathcal{A} \text{ cocomplete} \\ \Downarrow & & \\ \mathcal{A} \text{ compact} & & \\ \Downarrow & & \\ \mathcal{A} \text{ hypercomplete} & \Longrightarrow & \mathcal{A} \text{ complete} \\ \Downarrow & & \\ \mathcal{A}\ \mathcal{M}\text{-complete} & & \end{array}$$

It is known that, in general, there are no other implications in this scheme: the dual of the category of groups is complete, cocomplete and $\mathcal{M}$-complete (for $\mathcal{M}$ = monos), but not hypercomplete, and in particular not total (cf. [**11**], [**7**]). The category of fields is $\mathcal{M}$-complete, but not complete. Isbell [**11**] showed that neither completeness nor cocompleteness implies $\mathcal{M}$-completeness, not even for $\mathcal{M}$ = strong monos (our counterexample 4.3 strengthens this result). Adámek's monadic category [**1**] over a category of graphs is compact but far from being total, since it lacks coequalizers. Finally, Adámek also exhibited [**2**] a hypercomplete category that is not compact.

Problem 2.8. Is a compact cocomplete category total?

3. A generalization of Day's theorem. Solid functors are useful for detecting the (co)complete properties discussed in the previous section:

THEOREM 3.1. (cf. [**7**], [**19**]). *For a solid functor $U : \mathcal{A} \longrightarrow \mathcal{H}$, if $\mathcal{H}$ has any of the properties of being complete, hypercomplete, $\mathcal{M}$-complete* (*for $\mathcal{M}$ all monomorphisms*), *cocomplete, compact, or total, then $\mathcal{A}$ has the same property.*

Proof. For completeness, $\mathcal{M}$-completeness, and cocompleteness, the assertion follows from 1.1(2) and 1.6. (In case of $\mathcal{M}$-completeness, the assertion reads more generally as: if $\mathcal{H}$ is $\mathcal{M}$-complete for some $\mathcal{M}$, then $\mathcal{A}$ is $U^{-1}\mathcal{M}$-complete.) The lifting of hypercompleteness also follows from 1.6 since, if F is left adjoint to U, one has

$$\mathcal{A}(FX, H-) \cong \mathcal{A}(X, UH-)$$

for every $H : \mathcal{D} \longrightarrow \mathcal{A}$ and all $X \in \mathcal{H}$. With the given definitions for totality and compactness, one derives in exactly the same way from 1.1(2) that these properties can be lifted from $\mathcal{H}$ to $\mathcal{A}$.

Examples 3.2. The following categories are total since they admit a solid functor into $\mathcal{S}et$ or a discrete power of $\mathcal{S}et$; these are known to be total (cf. [**17**]): locally presentable categories (in the sense of Gabriel and Ulmer [**9**]), monadic categories over $\mathcal{S}et$ or a discrete power of $\mathcal{S}et$, topological categories over $\mathcal{S}et$, full reflective subcategories of the preceding categories (totality is inherited by a full reflective subcategory $\mathcal{A}$ of any total category $\mathcal{H}$, by 3.1).

We say that a small set $\mathcal{G}$ of objects in $\mathcal{A}$ is an *$\mathcal{E}$-generator* of $\mathcal{A}$ if and only if all coproducts

$$FX = \coprod_{G \in \mathcal{G}} X_G \cdot G \quad (\text{with } (X_G) \in \mathcal{S}et^{\mathcal{G}})$$

exists in $\mathcal{A}$ and the canonical morphism

$$\epsilon_A : \coprod_{G \in \mathcal{G}} \mathcal{A}(A, G) \cdot G \longrightarrow \mathcal{A}$$

belongs to $\mathcal{E}$ for every $A \in \mathcal{A}$. Now the following generalization of Day's Theorem [**8**] is an immediate consequence of 1.2 and 3.1:

THEOREM 3.3. *Every $\mathcal{E}$-cocomplete category with an $\mathcal{E}$-generator is total.*

Proof. By the definition of an $\mathcal{E}$-generator and by 1.2,

$$U : \mathcal{A} \longrightarrow \mathcal{S}et^{\mathcal{G}}, \ A \longmapsto (\mathcal{A}(A, G))$$

is solid, so by 3.1 totality is lifted from $\mathcal{S}et^{\mathcal{G}}$ to $\mathcal{A}$.

COROLLARY 3.4. *A cocomplete category with arbitrary cointersections of* (*strong*) *epimorphisms and a* (*strong*) *generator is total.*

COROLLARY 3.5. *A* (*weakly*) *cowellpowered and cocomplete category with a* (*strong*) *generator is total.*

COROLLARY 3.6. *A total category $\mathcal{A}$ with a cogenerator is cototal* (*i.e.*, $\mathcal{A}^{\mathrm{op}}$ *is total*)*; in particular it is $\mathcal{E}$-cocomplete for $\mathcal{E}$ = all epimorphisms.*

Proof. By 2.7, $\mathcal{A}$ is $\mathcal{M}$-complete with $\mathcal{M}$ = all monomorphisms, so that the assertion follows from the dual of 3.3 (or 3.4) and of 2.7.

Since totality implies compactness, 3.3 yields in particular the following general version of (the dual of) the Special Adjoint Functor Theorem (so that 3.3 is actually a stronger version of it):

THEOREM 3.7. *Let $\mathcal{A}$ be $\mathcal{E}$-cocomplete with an $\mathcal{E}$-generator* (*recall that we suppose $\mathcal{A}$ to have small hom-sets*)*. Then a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ has a right* *adjoint if and only if it preserves small colimits and arbitrary cointersections of $\mathcal{E}$-morphisms.*

Proof. $\mathcal{A}$ is total, hence compact; so every F that preserves all existing colimits has a right adjoint. But by the second part of the proof of 1.2 we actually know how the colimits we need in $\mathcal{A}$ are built from coproducts of objects of the $\mathcal{E}$-generator, from pushouts of $\mathcal{E}$-morphisms, and from arbitrary cointersections of $\mathcal{E}$-morphisms. So it suffices to know that F preserves those colimits.

3.7 was proved differently in [**7**] (in the dual form), but under more restrictive conditions on the class $\mathcal{E}$ (which was assumed to be closed under composition). It was also shown in [**7**] that local smallness (i.e., small hom-sets) is essential in 3.7, not only for formal reasons such as the one that the definition of an $\mathcal{E}$-generator relies on the fact that $\mathcal{A}$ is locally small: 3.7 is false if $\mathcal{B}$ fails to be locally small, even when $\mathcal{A}$ is the category $\mathcal{S}et$ of (small) sets; $\mathcal{B}$ as in 4.2 below is such a category, see [**7**].

4. Converses of the generalization of Day's theorem. In this section we show that a total category is $\mathcal{E}$-cocomplete for $\mathcal{E}$ the class of all regular epimorphisms, but that for any class $\mathcal{E}$ that contains all strong epimorphisms, a total category need neither be $\mathcal{E}$-cocomplete nor contain an $\mathcal{E}$-generator, even when the category enjoys the other of the two properties in question.

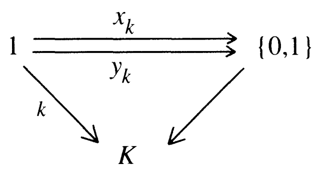
PROPOSITION 4.1. *A total category admits arbitrary cointersections of regular epimorphisms, and is therefore $\mathcal{E}$-cocomplete for $\mathcal{E}$ the class of regular epimorphisms.*

Proof. Considering the kernelpair of a regular epimorphism one obtains that, for every object A, the category $A/\mathcal{E}$ of regular epimorphisms with domain A admits a full and faithful right adjoint functor into the category $\mathcal{M}/(A \times A)$ of regular monomorphisms with codomain $A \times A$. By 2.7, $\mathcal{M}/(A \times A)$ has all infima, hence all suprema, so that the same is true for $A/\mathcal{E}$. In addition, suprema in $A/\mathcal{E}$ are given by coequalizers, so they are actually generalized pushouts in the category.

4.2. Example of a total category $\mathcal{A}$ which is cowellpowered (hence $\mathcal{E}$-cocomplete for the largest conceivable $\mathcal{E}$, namely $\mathcal{E}$ the class of all epimorphisms) but which does not have a generator:

$\mathcal{A}$ can be chosen to be comma category $\mathcal{B}/K$, with the following category $\mathcal{B}$: the objects of $\mathcal{B}$ are given by all (small) sets and one fixed proper class K; the class of morphisms $A \longrightarrow B$ consists of all mappings $A \longrightarrow B$ for $A \neq K$, is empty for $A = K \neq B$, and contains only id_K for $A = B = K$. Note that $\mathcal{A}$ (as opposed to $\mathcal{B}$) has small hom-sets and is a legitimate category. It was shown in [**7**, 1.12] that $\mathcal{A}$ is compact but does not have a generator. (For the latter statement one simply observes that, for every $k \in K$ (i.e., $k : 1 \longrightarrow K$), the two morphisms

https://doi.org/10.4153/CJM-1990-012-x Published online by Cambridge University Press



must be distinguishable by a morphism $z_k : (a_k : A_k \longrightarrow K) \longrightarrow k$ with a_k belonging to a generator if one exists in $\mathcal{B}$; but then all a_k must be different, so that they cannot all belong to a small set.) Now we show that $\mathcal{A}$ is even total:

First we point out that to give a functor $H : \mathcal{D} \longrightarrow \mathcal{A}$ is to give a functor $J : \mathcal{D} \longrightarrow \mathcal{B}$ and a cocone $\eta : J \longrightarrow \Delta K$; and to give a cocone $\alpha : H \longrightarrow \Delta(a : A \longrightarrow K)$ in $\mathcal{A}$ is to give a cocone $\beta : J \longrightarrow \Delta A$ in $\mathcal{B}$ with $\Delta a \cdot \beta = \eta$. We must show that colim H exists in $\mathcal{A}$ whenever colim $\mathcal{A}(a, H-)$ exists in $\mathcal{S}et$ for all $a : A \longrightarrow K$ in $\mathcal{B}$. Let

$$L = \bigcup_{d \in \mathcal{D}} \mathrm{im}(\eta_d)$$

with η as above. Note that, if there is any cocone $\alpha : H \longrightarrow \Delta a$, we must have $L \subseteq \mathrm{im}(a)$; thus, if L is a proper class, A must be large as well, whence $A = K$. That leads us to consider:

Case 1. L is a proper class. Then the terminal object id_K in $\mathcal{A}$ gives the colimit of H.

Case 2. L is small. By hypothesis, for every $k \in L$ ($k : 1 \longrightarrow K$), the colimit cocone $\lambda^k : \mathcal{A}(k, H-) \longrightarrow \Delta M^k$ exists in $\mathcal{S}et$, so we have natural maps

$$\lambda_d^k : \mathcal{A}(k, Hd) \cong \eta_d^{-1}(\{k\}) \longrightarrow M^k$$

for every $d \in \mathcal{D}$. With M the disjoint union of the $M^k (k \in L)$, and with $p : M \longrightarrow K$ the projection, one has natural maps

$$\lambda_d : Jd \longrightarrow M, \quad x \longmapsto \lambda_d^k(x) \quad \text{with } k = \eta_d(x),$$

with $p\lambda_d = \eta_d$ for all $d \in \mathcal{D}$. So there is actually a cocone $H \longrightarrow \Delta p$ which is easily shown to be a colimit in $\mathcal{A}$.

Cowellpoweredness of $\mathcal{A}$ is trivial (as $f : a \longrightarrow b$ is an epimorphism in $\mathcal{A}$ if and only if f is surjective or $f = a$). Note, however, that $\mathcal{A}$ is not wellpowered, and that it does not have a cogenerator either (by an argument similar to that which shows non-existence of a generator).

4.3. Example of a total category $\mathcal{A}$ with a strong generator which does not admit arbitrary cointersections of strong epimorphisms (and which therefore is not $\mathcal{E}$-cocomplete for any class $\mathcal{E}$ containing the strong epimorphisms, and not weakly cowellpowered).

Let Ord be the class of all (small) ordinals, and let ∞ be a symbol with $\alpha < \infty$ for all $\alpha \in \text{Ord}$. Objects of the category $\mathcal{A}$ are quadruples

$$A = (X, \alpha, (a_\nu), (b_\nu))$$

with a (small) set X, $\alpha \in \text{Ord} \cup \{\infty\}$ and families $(a_\nu)_{\nu \leqq \alpha}$, $(b_\nu)_{\nu \leqq \alpha}$ of X-elements such that one of the following two sets of conditions holds:

1. (A is of type 1) $\alpha < \infty$, $a_\nu = b_\nu$ for all $\nu < \alpha$, and $a_\alpha \neq b_\alpha$,
2. (A is of type 2) $\alpha = \infty$, $a_\nu = b_\nu$ for all $\nu \leqq \infty$, and there is some $\lambda < \infty$ such that $a_\nu \neq a_\mu$ for all $\nu < \mu \leqq \lambda$ and $a_\kappa = a_\lambda$ for all $\kappa \geqq \lambda$.

An $\mathcal{A}$-morphism

$$f : A \longrightarrow C = (Z, \gamma, (c_\nu), (d_\nu))$$

is a mapping $f : A \longrightarrow C$ with $\alpha \leqq \gamma$ and $f(a_\nu) = c_\nu, f(b_\nu) = d_\nu$ for all $\nu \leqq \alpha$. Note that we have $\mathcal{A}(A, C) = \emptyset$ if $\gamma > \alpha$; in particular, there are no morphisms from type 2 to type 1. $\mathcal{A}$ is a legitimate category in the sense that $|\mathcal{A}|$ is codable by a class and that all hom's are small; the former statement follows from the fact that the cardinality of λ as in (2) cannot exceed the cardinality of X. We first show that $\mathcal{A}$ is total. By 3.1 it suffices to show that $U : \mathcal{A} \longrightarrow \mathcal{S}et$ with $A \longmapsto X$ is solid. So we must prove that every U-sink $(x_i : UA_i \longrightarrow Y)_{i \in I}$ admits a U-semifinal lifting. Let

$$A_i = (X_i, \alpha_i, (a^i_\nu), (b^i_\nu)) \quad \text{and}$$
$$\beta = \inf\{\mu \in \text{Ord} \cup \{\infty\} \mid \quad \mu > \alpha_i \text{ for all } i \in I\}$$

(so $\beta = 0$ if $I = \emptyset$, and $\beta = \infty$ if $\sup\,\{\alpha_i \mid i \in I\} = \infty$). Now we distinguish two cases:

Case I. $\beta < \infty$. Then we consider the least equivalence relation $\sim$ on Y subject to the condition

$$(1) \qquad x_i(a^i_\nu) \sim x_j(a^j_\nu) \quad \text{and} \quad x_i(b^i_\nu) \sim x_j(b^j_\nu)$$

for all $i, j \in I$ and $\nu \leqq \min\{\alpha_i, \alpha_j\}$. For the projection $p : Y \longrightarrow Z = Y/\sim$, for $i \in I$ and $\nu \leqq \alpha_i$, let

$$c_\nu = px_i(a^i_\nu) \quad \text{and} \quad d_\nu = px_i(b^i_\nu).$$

By (1), c_ν and d_ν are well defined for all $\nu < \beta$. Moreover, if $\nu + 1 < \beta$, then we get $\nu < \nu + 1 \leqq \alpha_i < \beta$ for some $i \in I$, hence

$$a^i_\nu = b^i_\nu \quad \text{and} \quad c_\nu = px_i(a^i_\nu) = px_i(b^i_\nu) = d_\nu.$$

There are two subcases to be looked at:

Case I.1. $c_\gamma \neq d_\gamma$ for some $\gamma < \beta$. Then we must have $\beta = \gamma + 1$, and all c_ν, $d_\nu (\nu \leqq \gamma)$ are defined. Now $p : Y \longrightarrow UC$ with $C = (Z, \gamma, (c_\nu), (d_\nu))$ serves as the desired U-semifinal lifting.

Case I.2. $c_\nu = d_\nu$ for all $\nu < \beta$. Then we choose two distinct symbols c_β, $d_\beta \notin Z$ and obtain the desired U-semifinal lifting as up: $Y \longrightarrow UC$ with

$$C = (Z \cup \{c_\beta, d_\beta\}, \beta, (c_\nu), (d_\nu))$$

and u the inclusion mapping. Note that, in both subcases, C is an object of type 1.

Case II. $\beta = \infty$. In this case we consider the least equivalence relation $\sim$ on Y satisfying (1) of Case I and

$$(2) \qquad x_i(a^i_\nu) \sim x_i(a^i_\mu) \quad \text{for all } i \in I, \nu \leqq \mu \leqq \alpha_i$$

such that there exists a κ with

$$\nu < \kappa \leqq \alpha_i \quad \text{and} \quad x_i(a^i_\nu) \sim x_i(a^i_\kappa).$$

As above, one has $x_i(a^i_\nu) \sim x_i(b^i_\nu)$ for all $i \in I$ and all $\nu \in \text{Ord}$. Again, we may define

$$c_\nu = px_i(a^i_\nu) = px_i(b^i_\nu)$$

(for suitable $i \in I$) with $p : Y \longrightarrow Z = X/\sim$ the projection. Condition (2) guarantees that $C = (Z, \infty, (c_\nu), (c_\nu))$ is an $\mathcal{A}$-object of type 2 such that p : $Y \longrightarrow UC$ serves as the wanted U-semifinal lifting.

Next we show that

$\mathcal{A}$ has a strong generator. It suffices to show that the solid and therefore faithful functor $U : \mathcal{A} \longrightarrow \mathcal{S}et$ with a left adjoint F is conservative, whereupon

$$G = F1 = (\{a_0, b_0, x\}, 0, (a_0), (b_0))$$

gives a (single-object) strong generator of $\mathcal{A}$. So let the underlying map of the $\mathcal{A}$-morphism $f : A \longrightarrow C$ as above be bijective. By the definition of $\mathcal{A}$-morphisms, one has $\alpha \leqq \gamma$. If we had $\alpha < \gamma$, then A would be of type 1 and $a_\alpha \neq b_\alpha$; on the other hand, $\alpha < \gamma$ would give $f(a_\alpha) = c_\alpha = d_\alpha = f(b_\alpha)$, contradicting the injectivity of f. Therefore, we must have $\alpha = \gamma$, and it is easy to see that $f^{-1} : C \longrightarrow A$ is an $\mathcal{A}$-morphism.

Finally we must show that

$\mathcal{A}$ does not admit arbitrary cointersections of strong epimorphisms. This will be done by considering, for every $\mu < \infty$, $\mathcal{A}$-objects

$$M_\mu = (K_\mu \cup \{\infty\}, \mu, (\nu), (\nu * \mu))$$

of type 1 with $K_\mu = \{\kappa \mid \kappa \leqq \mu\}$ and $\kappa * \mu = \kappa$ for $\kappa < \mu$ and $\mu * \mu = \infty$. Note that $M_0 = F\emptyset$ is an initial object of $\mathcal{A}$, so that one has uniquely determined morphisms $e_\mu : M_0 \longrightarrow M_\mu$ for all μ, and $e_\mu(0) = e_\mu(\infty) = 0$ whenever $\mu \geqq 1$.

We claim that each e_μ is an extremal epimorphism (hence strong in the total $\mathcal{A}$). It is enough to show that each monomorphism

$$f : A = (X, \alpha, (a_\nu), (b_\nu)) \longrightarrow M_\mu$$

is an isomorphism. By injectivity of the mapping f we conclude $\alpha = \mu$ as above. Since $f(a_\kappa) = \kappa$ for all $\kappa \in K_\mu$ and $f(b_\mu) = \infty$ one also has that f is surjective, hence bijective, and therefore f is an isomorphism.

Let us now assume that the family $(e_\mu)_{\mu<\infty}$ admits a cointersection $e_\infty : M_0 \longrightarrow M_\infty$ with canonical projections $p_\mu : M_\mu \longrightarrow M_\infty$. For all $\nu < \infty$,

$$N_\nu = (K_\nu, \infty, (\min\{\kappa, \nu\})_\kappa, (\min\{\kappa, \nu\})_\kappa)$$

is an $\mathcal{A}$-object of type 2, and one has $\mathcal{A}$-morphisms $t_{\mu\nu} : M_\mu \longrightarrow N_\nu$ with

$$t_{\mu\nu}(\kappa) = \min\{\kappa, \mu, \nu\} \quad \text{for all } \mu, \nu < \infty.$$

Since M_0 is initial, $t_{\mu\nu}e_\mu$ is always the only morphism $M_0 \longrightarrow N_\nu$, so one has a morphism $s_\nu : M_\infty \longrightarrow N_\nu$ with $s_\nu p_\mu = t_{\mu\nu}$ for all $\mu < \infty$. Each s_ν is surjective since

$$s_\nu p_\nu = t_{\nu\nu} : K_\nu \cup \{\infty\} \longrightarrow K_\nu$$

maps onto. Therefore the cardinality of no K_ν can exceed the cardinality of UM_∞, which is impossible.

Remarks 4.4. We list some additional properties of the category $\mathcal{A}$ of 4.3:

(1) The full subcategory $\mathcal{A}_1$ of objects of type 1 in $\mathcal{A}$ is closed under colimits (hence cocomplete) since colimits in $\mathcal{A}$ are constructed as semifinal liftings, and one easily sees that only Case I of 4.3 may occur in its construction, so that one again obtains an object of type 1. Therefore, the colimit-closure of $\{G\}$ in $\mathcal{A}$, for any $G \in \mathcal{A}_1$, is contained in $\mathcal{A}_1$, i.e., properly smaller than $\mathcal{A}$. In particular, this holds for $\{G\}$ a strong generator of $\mathcal{A}$; hence the colimit-closure of a strong generator of a total category $\mathcal{A}$ may be properly smaller than $\mathcal{A}$; something that does not happen when $\mathcal{A}$ is weakly cowellpowered. One can actually show here that $\mathcal{A}_1$ is the colimite-closure of $\{G\}$ in $\mathcal{A}$ (but we omit the proof since this fact has no relevance to us here).

(2) We further observe that the colimit-closure $\mathcal{C}$ of the strong generator $\{G\}$ (as in 4.3) has no terminal object. For that, by transfinite induction, one first shows that the objects M_ν of 4.3 belong to $\mathcal{C}$ for $\nu \in \mathrm{Ord}$. Indeed, M_0 is the colimit of the empty diagram, so it belongs to $\mathcal{C}$. For $\nu < \mu$ there is a unique $\mathcal{A}$-morphism $q_{\nu\mu} : M_\nu \longrightarrow M_\mu$, namely

$$q_{\nu\mu}(\kappa) = \min\{\nu, \kappa\} \quad \text{for all } \kappa \in K_\nu \cup \{\infty\}.$$

For all $\nu \in \mathrm{Ord}$, $q_{\nu\nu+1}$ is the coequalizer of $f_\nu, g_\nu : G \longrightarrow M_\nu$ with $f_\nu(x) = \nu$, $g_\nu(x) = \infty$. Hence $M_\nu \in \mathcal{C}$ implies $M_{\nu+1} \in \mathcal{C}$. For a limit-ordindal ν, $M_\kappa \in \mathcal{C}$ for all $\kappa < \nu$ implies

$$M_\nu \cong \coprod_{\kappa<\nu} M_\kappa \in \mathcal{C}.$$

Let us now assume that $\mathcal{C}$ has a terminal object

$$C = (Z, \gamma, (c_\nu), (d_\nu)).$$

Since $C \in \mathcal{C} \subseteq \mathcal{A}_1$ is of type 1, on the one hand we have $\gamma < \infty$, but on the other hand we have $\mathcal{A}(M_\nu, C) \neq \emptyset$, hence $\gamma \geqq \nu$ for each $\nu \in \mathrm{Ord}$. This is impossible.

(3) From (2) we see in particular that the colimit-closed subcategory $\mathcal{C}$ is not coreflective since it is not complete. *A fortiori*, $\mathcal{C}$ is not total. Therefore, in Theorem 3.3, $\mathcal{E}$-cocompleteness cannot be replaced by (small) cocompleteness, even if the category in question is the colimit-closure of a small set of objects, which is a stronger condition than then existence of a strong generator.

(4) By the same argument as in (2) one has that $\mathcal{A}_1$ is a colimit-closed subcategory without a terminal object. Symmetrically, the full subcategory $\mathcal{A}_2$ of objects of type 2 is closed under all existing limits (even large ones), but $\mathcal{A}_2$ has no initial object: if there is an initial object $A = (X, \infty, (a_\nu), (a_\nu))$, there must exist morphisms $f_\nu : A \longrightarrow N_\nu$ (cf. 4.3) with

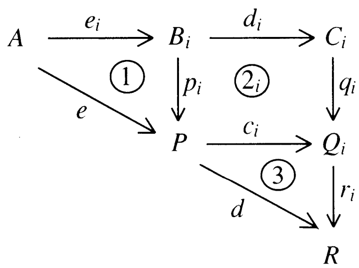
$$f_\nu(a_\mu) = \mu \neq \nu = f_\nu(a_\nu) \quad \text{for all } \mu < \nu < \infty;$$

i.e., all a_ν must be distinct, which is impossible.

5. Improvements of the generalization of Day's theorem. In view of 4.1 it seems necessary to devote special attention to the case where, in 3.3, $\mathcal{E}$ is the class of all regular epimorphisms. It turns out that one gets even stronger results by considering the class $\mathcal{E}^2$ of morphisms that are the composites of two morphisms in $\mathcal{E}$. We may first work with a general class $\mathcal{E}$ and will specialize later.

LEMMA 5.1. *Every $\mathcal{E}$-cocomplete category is $\mathcal{E}^2$-cocomplete.*

Proof. Trivially, pushouts of $\mathcal{E}^2$-morphisms exist and belong to $\mathcal{E}^2$. For an arbitrary family $(d_i e_i)$ of $\mathcal{E}$-morphisms e_i, d_i we may construct their cointersection $de \in \mathcal{E}^2$ as



with pushouts (2_i) for each i and multiple pushouts (1) and (3).

A small set $\mathcal{G}$ of objects in $\mathcal{A}$ is called an *almost-$\mathcal{E}$-generator* if coproducts of small families in $\mathcal{G}$ exist in $\mathcal{A}$ and if every $A \in \mathcal{A}$ admits a presentation

$$e : \coprod G_i \longrightarrow A$$

with a small family (G_i) in $\mathcal{G}$ and $e \in \mathcal{E}$.

THEOREM 5.2. *For any class $\mathcal{E}$ that contains all split-epimorphisms, an $\mathcal{E}$-cocomplete category with an almost-$\mathcal{E}$-generator is total.*

Proof. In view of 3.3 and 5.1, it suffices to show that an almost-$\mathcal{E}$-generator is an $\mathcal{E}^2$-generator. For $A \in \mathcal{A}$, let ϵ_A be the canonical morphism (see 3.3). By 1.5, ϵ_A has a locally orthogonal $\mathcal{E}$-factorization $\epsilon_A = mp$; it then suffices to show that m is split-epic to be able to conclude $\epsilon_A \in \mathcal{E}^2$. But, by hypothesis, there is an $\mathcal{E}$-morphism

$$e : \coprod G_i \longrightarrow A$$

with a family (G_i) in $\mathcal{G}$. One also has a canonical map k with $\epsilon_A k = e$. Local orthogonality gives a morphism t with $mt = 1$.

In case $\mathcal{E}$ = all regular epimorphisms, 5.2 can be simplified further:

COROLLARY 5.3. *For a category $\mathcal{A}$ with an almost-regular generator, the following assertions are equivalent:*
(i) *$\mathcal{A}$ is total,*
(ii) *$\mathcal{A}$ has small cointersections and pushouts of regular epimorphisms,*
(iii) *$\mathcal{A}$ has coequalizers.*

Proof. (iii) $\Rightarrow$ (ii). In a category with a generator $\mathcal{G}$ and coproducts of objects in $\mathcal{G}$, every regular epimorphism is a coequalizer of two morphisms whose domain is a coproduct of objects in $\mathcal{G}$. With this observation it is elementary to construct the needed pushouts and cointersections just with the help of coequalizers.

(ii) $\Rightarrow$ (i) It is well known (cf. [**14**]) that a category with a generator is cowellpowered with respect to regular epimorphisms. So 5.2 yields (i).

(i) $\Rightarrow$ (iii) is trivial.

Remark 5.4. In 5.2, the hypothesis that $\mathcal{E}$ contains all split epimorphisms can be dropped if the category has coequalizers. For that, in the proof of 5.2, one may replace $\mathcal{E}^2$ by the class $\tilde{\mathcal{E}}$ of all morphisms which are composites of an $\mathcal{E}$-morphism followed by a regular epimorphism. The proof of 5.1 works for $\tilde{\mathcal{E}}$ instead of $\mathcal{E}^2$ since the necessary pushouts and cointersections of regular epimorphisms can be constructed from coequalizers (see the proof of 5.3).

COROLLARY 5.5. *If in the cocomplete category $\mathcal{A}$ there is a small set $\mathcal{G}$ of objects such that every object is* (*not necessarily canonically*) *a colimit of objects in $\mathcal{G}$, then $\mathcal{A}$ is total.*

Proof. $\mathcal{G}$ is an almost-regular generator of $\mathcal{A}$.

Remarks 5.6. (1) The hypothesis of 5.5 is satisfied in particular when $\mathcal{G}$ is dense in $\mathcal{A}$ (so that every object is the canonical colimit of objects in $\mathcal{G}$) which, however, is a more restrictive assumption (consider $\{K\}$ in the category of K-vector spaces). On the other hand, 5.5 can be stated with a slightly weaker hypothesis: it suffices that every object in $\mathcal{A}$ is the coequalizer of two morphisms between coproducts of objects in $\mathcal{G}$.

(2) 5.5 becomes false if one weakens the hypothesis so far that $\mathcal{A}$ is just the colimit closure of $\mathcal{G}$ in $\mathcal{A}$, i.e., the least (full and replete) colimit-closed subcategory of $\mathcal{A}$ containing $\mathcal{G}$: see 4.4(3).

(3) 5.3 allows one to produce a total category with a strong generator which fails to have any regular generator even an almost-regular one. Indeed, if $\mathcal{H}$ were an almost-regular generator of the category $\mathcal{A}$ of 4.3, then, in particular, every object A of type 1 admits a regular epimorphism $\coprod H_i \longrightarrow A$ for some coproduct of objects in $\mathcal{H}$. But all H_i must be of type 1 since, in $\mathcal{A}$, there are no morphisms from type 2-objects into type 1-objects. So $\mathcal{H} \cap \mathcal{A}_1$ would have to be an almost-regular generator of the subcategory $\mathcal{A}_1$ as in 4.4, which would have to be total by 5.3. But, in fact, $\mathcal{A}_1$ does not even have a terminal object.

Finally we state the Special Adjoint Functor Theorem 3.7 in the almost-regular case; with 5.3 one obtains:

COROLLARY 5.7. *A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ that preserves small colimits has a right adjoint, provided $\mathcal{A}$ has small colimits and an almost-regular generator; in particular if $\mathcal{A}$ is monadic over $\mathcal{S}et^I$* (I *a small set*).

Note added in proof. Generalizations and improvements of the results in Section 5 will appear in a forthcoming article by the authors, entitled "Factorizations and colimit closures". The paper "Total categories with generators" (to appear in J. of Algebra) by J. Adámek and W. Tholen gives characterization theorems for the categories described by its title.

REFERENCES

1. J. Adámek, *Colimits of algebras revisited*, Bull. Austral. Math. Soc. *17* (1977), 433–450.

2. ——— *How complete are categories in algebras?* Bull. Austral. Math. Soc. *36* (1987), 389–409.
3. C. Anghel, *Factorizations and initiality in enriched categories*, Doctoral dissertation, Fernuniversität Hagen (1987).
4. ——— *Lifting properties of V-functors*, preprint (1987).
5. R. Börger and W. Tholen, *Cantors Diagonalprinzip für Kategorien*, Math. Z. *160* (1978), 135–138.
6. ——— *Strong, regular, and dense generators*, Cahiers Topologie Géom. Différentielle Catégoriques (submitted).
7. R. Börger, W. Tholen, M. B. Wischenwsky and H. Wolff, *Compact and hypercomplete categories*, J. Pure Appl. Algebra *21* (1981), 120–144.
8. B. J. Day, *Further criteria for totality*, Cahiers Topologie Géom. Différentielle Catégoriques *28* (1987), 77–78.
9. P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math. *221* (Springer-Verlag, Berlin-Heidelberg-New York, 1971).
10. R.-E. Hoffmann, *Topological functors admitting generalized Cauchy-completion*, Lecture Notes in Math. *540* (Springer-Verlag, Berlin-Heidelberg-New York, 1976) 286–344.
11. J. R. Isbell, *Small subcategories and completeness*, Math. Systems Theory *2* (1968), 27–50.
12. G. M. Kelly, *A survey of totality for enriched and ordinary categories*, Cahiers Topologie Géom. Différentielle Catégoriques *27* (1986), 109–131.
13. S. Mac Lane, *Categories for the working mathematician* (Springer-Verlag, Berlin-Heidelberg-New York, 1971).
14. B. Pareigis, *Categories and functors* (Academic Press, New York-London, 1970).
15. R. Street, *Notions of topos*, Bull. Austral. Math. Soc. *23* (1981), 199–208.
16. ——— *The family approach to total cocompleteness and toposes*, Trans. Amer. Math. Soc. *284* (1984), 355–369.
17. R. Street and R. Walters, *Yoneda structures on 2-categories*, J. Algebra *50* (1978), 350–379.
18. W. Tholen, *Semi-topological functors I*, J. Pure Appl. Algebra *15* (1979), 53–73.
19. ——— *Note on total categories*, Bull. Austral. Math. Soc. *21* (1980), 169–173.
20. ——— *Factorizations, localizations and the orthogonal subcategory.*
21. R. J. Wood, *Some remarks on total categories*, J. Algebra *75* (1982), 538–545.

Fernuniversität,
Hagen, Federal Republic of Germany;
York University,
North York, Ontario