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SOME DIRECT AND INVERSE THEOREMS IN APPROXIMATION OF FUNCTIONS

R. N. MOHAPATRA and D. C. RUSSELL

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Abstract

The paper is concerned with the determination of the degree of convergence of a sequence of linear operators connected with the Fourier series of a function of class L_p $(p > 1)$ to that function and some inverse results in relating the convergence to the classes of functions. In certain cases one can obtain the saturation results too. In all cases L_p norm is used.

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1

Let $f(x)$ be a periodic, Lebesgue integrable function with period 2π. Let the Fourier series for $f(x)$ be given by

$$\tag{1} \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(x).$$

Let $S_n(f;\, x)$ be the nth partial sum of the series (1). The conjugate series of the series (1) is

$$\sum_{n=1}^{\infty} B_n(x) = \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx).$$

The conjugate function $\tilde{f}$ of f, is given by

$$\tag{2} \tilde{f}(x) = (2\pi)^{-1} \int_0^{\pi} \{f(x+t) - f(x-t)\} \cot \frac{t}{2}\, dt$$

the integral being interpreted as a Cauchy integral. It is known that $\tilde{f}$ exists almost everywhere whenever f is integrable.

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The space $L_p[-\pi, \pi]$ when $p = \infty$ will be replaced by the space $c_{2\pi}$ of all continuous functions defined over $[-\pi, \pi]$. Throughout the paper, norms will be taken with respect to the variable x and $\|\cdot\|_p$ will denote the usual L_p norm for $1 < p < \infty$, and the supremum norm when $p = \infty$. For $f \in L_p[-\pi, \pi]$ $(1 < p < \infty)$, the modulus of continuity and the modulus of smoothness $w^{(p)}(\delta, f)$ and $w_2^{(p)}(\delta; f)$ are defined respectively by

(3) $$w^{(p)}(\delta; f) = \sup_{|h| \leqslant \delta} \|f(x+h) - f(x)\|_p, \qquad \text{and}$$

(4) $$w_2^{(p)}(\delta; f) = \sup_{|h| \leqslant \delta} \|f(x+h) + f(x-h) - 2f(x)\|_p.$$

The classes Lip α, Lip(α, p) $(p \geqslant 1)$ will be as usual (see [5], page 612; also see [18], pages 42, 45). The class Lip(α, p) with $p = \infty$ will be taken as Lip α.

Two functions f and g are said to be equivalent if $f(x) = g(x)$ almost everywhere.

Let $\{c_n\}$, $\{d_n\}$ be two non-zero sequences with $c_n, d_n \geqslant 0$. Suppose $C_n = \sum_{k=0}^n c_k$ and $D_n = \sum_{k=0}^n d_k$. Let $R_n = c_0 d_n + c_1 d_{n-1} + \cdots + c_n d_0$ $(n = 0, 1, \ldots)$.

Given f, let us associate with it the operator $t_n(f)$ defined by

(5) $$t_n(f; x) = (R_n)^{-1} \sum_{k=1}^{n} c_{n-k} d_k S_k(x).$$

It should be remarked that $t_n(f; x)$ is the (N, c, d) transform of $\{S_k(f; x)\}$ (see [2]).

We shall write $t_n(f; x) = N_n(f; x)$ or $\bar{N}_n(f; x)$ according as $d_n = 1$ for all n or $c_n = 1$ for all n.

If there exists a positive non-increasing function $\phi(n)$ and a normed linear space K of functions such that

(6) $$\|f(x) - t_n(f; x)\| = o(\phi(n)) \Rightarrow f \text{ is a constant a.e.},$$

(7) $$\|f(x) - t_n(f; x)\| = O(\phi(n)) \Rightarrow f \in K, \qquad \text{and}$$

(8) $$f \in K \Rightarrow \|f(x) - t_n(f; x)\| = O(\phi(n)),$$

then we say that the operator $t_n(f)$ or the corresponding method (N, c, d) is saturated with order $\phi(n)$ and class K.

2

Ever since the definition of saturation of summability methods was given by Favard [3] many authors have studied the saturation property of operators which are obtained as transforms of the nth partial sum of the Fourier series by summability methods. Sunouchi and Watari [15], [16] have obtained the saturation order and class for Cesàro, Abel and the Riesz method $(R, n^{\zeta}, 1)$ ($\zeta = 1, 2, \ldots$). Mohapatra and Sahney [11] have obtained results on saturation for a general class of summability methods in the supremum norm. Sunouchi [14] has studied the local saturation properties of the convolution operator (also see [13], [17]).

Concerning the saturation property of the Nörlund method, Goel, Holland, Nasim and Sahney [4] have proved the following theorem:

THEOREM A ([4], compare [9]). *Let $f \in c_{2\pi}$ and $C_n > 0$ (all n). Then the following hold*:

$$\|f - N_n(f)\|_\infty = o\left(\frac{c_n}{C_n}\right) \Rightarrow f \text{ is a constant} \quad a.e. \tag{9}$$

$$\|f - N_n(f)\|_\infty = O\left(\frac{c_n}{C_n}\right) \Rightarrow f \in \{f \mid \tilde{f} \in \text{Lip } 1\} \tag{10}$$

whenever

$$\lim_{n\to\infty} \frac{c_{n-k}}{c_n} = 1 \qquad (k = 0, 1, \ldots;\ c_n > 0 \text{ for all } n). \tag{11}$$

$$f \in \{f \mid \tilde{f} \in \text{Lip } 1\} \Rightarrow \|f - N_n(f)\|_\infty = O\left(\frac{c_n}{C_n}\right) \tag{12}$$

whenever

$$\sum_{k=0}^{n} |c_k - c_{k-1}| = O(c_n) \qquad (c_{-1} = 0). \tag{13}$$

In Section 3 we obtain the order and class of saturation of the method (N, c, d) or the operator $t_n(f)$ in the L_p $(1 < p \leqslant \infty)$ norm. Special cases of this result extend Theorem A and yield a saturation result for a type of Riesz method.

The other object of this paper is to obtain the degree of convergence of $t_n(f)$ to $f \in L_p$ in terms of the integral modulus of continuity and integral modulus of smoothness with a view to generalizing the following results:

THEOREM B ([12]). *If $f \in \text{Lip}(\alpha, p)$ $(0 < \alpha \leqslant 1,\ p > 1,\ p^{-1} + p'^{-1} = 1)$ and if $C_n \to \infty$ and*

$$\left(\int_1^n \frac{C(y)}{y^{p'\alpha + 2 - p'}}\,dy\right)^{1/p'} = O\left(C_n / n^{\alpha - p^{-1}}\right) \tag{14}$$

(where $C(y) = C_{[y]}$) then

$$\|f - N_n(f)\|_p = O(n^{-\alpha + p^{-1}}).$$

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THEOREM C ([10]). *Let* $C_n \to \infty$ *as* $n \to \infty$, *and* $R(y)/y^\alpha$ *be nondecreasing where* $R(y) = R_{[y]}$. *Then* $f \in \mathrm{Lip}(\alpha, p)$ $(0 < \alpha < 1, p > 1)$ *implies*

$$(16) \qquad \|f - t_n(f)\|_p = O(n^{-\alpha + p^{-1}}).$$

THEOREM D ([8]). *If* $w(t)$ *is the modulus of continuity of* $f \in C[-\pi, \pi]$ *and* $c_n > 0$, $c_n/C_n = O(n^{-1})$,

$$(17) \qquad \|f - N_n(f)\|_\infty = O\left(\frac{1}{C_n} \sum_{k=1}^{n} \frac{w(1/k)}{k} C_k\right).$$

In Section 4 we shall generalize these results and obtain some other special cases.

3

Following the method of Sunouchi and Watari [16] we can obtain

THEOREM 1. *Let* $1 \leqslant p \leqslant \infty$. *The following hold*:

$$(18) \qquad \|f - t_n(f)\|_p = o\left(\frac{c_n}{R_n}\right) \Rightarrow f \text{ is equivalent to a constant.}$$

When $c_{n-k}/c_n \to 1$ *as* $n \to \infty$, *k fixed, we have*

$$(19) \quad \|f - t_n(f)\|_p = O\left(\frac{c_n}{R_n}\right) \Rightarrow \left\| \sum_{k=1}^{N} D_{k-1}\left(1 - \frac{k}{N+1}\right) A_k(x) \right\|_p = O(1).$$

Thus $\|f - t_n(f)\|_p = O(c_n/R_n)$ *implies*

(20) $\sum_{k=1}^{\infty} D_{k-1} A_k(x)$ *is the Fourier series of a bounded function, when* $p = \infty$;

(21) $\sum_{k=1}^{\infty} D_{k-1} A_k(x)$ *is the Fourier series of a function of class* L_p, *when* $1 < p < \infty$;

(22) $\sum_{k=1}^{\infty} D_{k-1} A_k(x)$ *is the Fourier-Stieltjes series of a function of bounded variation, when* $p = 1$.

Throughout the paper, we write for $1 \leqslant p < \infty$, $K_p = \{f \in L_p \mid \tilde{f} \in \mathrm{Lip}(1, p)\}$, and $K_\infty = \{f \in c_{2\pi} \mid f \in \mathrm{Lip}\, 1\}$.

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If $d_n = 1$ for all n, then we have, from Theorem 1:

COROLLARY 1. *Let* $C_n > 0$ (*all* n). *Then*

(23) $$\|f - N_n(f)\|_p = o(c_n/C_n) \Rightarrow f \text{ is equivalent to a constant},$$

and if (11) *holds then*

(24) $$\|f - N_n(f)\|_p = O(c_n/C_n) \Rightarrow f \in K_p \qquad (1 \leqslant p \leqslant \infty).$$

PROOF. It is enough to deduce (24). When (11) holds we observe that the conclusion in (19) shows that the $(C, 1)$ mean of $\sum_{k=1}^{\infty} kA_k(x)$ is uniformly bounded in the L_p norm $(1 \leqslant p \leqslant \infty)$. Since $-\sum_k kA_k(x) = \sum_k B_k'(x)$ where $\sum B_k(x)$ is the conjugate series of the Fourier series of $f(x)$, we have $\|\sigma_N'\|_p = O(1)$ where $\sigma_N(x)$ is the first Cesàro mean of $\sum B_k$. This is known to be equivalent to $f \in K_p$.

REMARKS. 1. If $p > 1$, then the conclusion $f \in K_p$ in Corollary 1 can be replaced by $f \in \text{Lip}(1, p)$ (see [6], Lemma 13, page 621).

2. (20), (21) and (22) refer to the Fourier series $\sum_{k=1}^{\infty} D_{k-1} A_k(x)$. Since we do not know much about the behaviour of that series the saturation problem for $(\bar{N}, d)$ turns out to be difficult. However when $p = 2$ we get the following as an easy consequence of Parseval's identity:

COROLLARY 2. *Let* $f \in L_2$. *Corresponding to the order of saturation* $1/D_n$ *the saturation class of the method* $(\bar{N}, d)$ *or of the operator* $\bar{N}_n(f)$ *is the class of all functions* $f \in L_2$ *with Fourier series* $\sum_{k=1}^{\infty} D_{k-1} A_k(x)$.

Our next result gives an estimate for the error in approximating a function $f \in K_p$ by $t_n(f)$. Precisely, we prove

THEOREM 2. *Let* $1 < p \leqslant \infty$ *and* $\{c_n\}$ *and* $\{d_n\}$ *satisfy*

(25) $$\sum_{k=0}^{n} |c_{n-k} d_k - c_{n-k-1} d_{k+1}| = O(c_n).$$

Then, for $f \in K_p$,

(26) $$\|f - t_n(f)\|_p = O(c_n/R_n).$$

We shall need the following lemmas for the proof of our theorem:

LEMMA 1 ([5], Theorem 24(i), page 599). *If f belongs to* Lip$(1, p)$ $(1 < p \leqslant \infty)$ *then f is equivalent to the indefinite integral of a function belonging to* L_p. *If* $f \in$ Lip 1 *then f is the indefinite integral of a bounded function.*

LEMMA 2 ([6], Theorem 5, page 627). *Suppose* $f \in \mathrm{Lip}(\alpha, p)$ *where* $p \geqslant 1$, $0 < \alpha \leqslant 1$.

(i) *If* $\alpha p \leqslant 1$ *and* $p < q < p/(1 - \alpha p)$, *then* $f \in \mathrm{Lip}(\alpha - 1/p + 1/q, q)$.

(ii) *If* $\alpha p > 1$ *then* $f \in \mathrm{Lip}(\alpha - 1/p + 1/q, q)$ *for all* $q > p$, *and* f *is equivalent to a function of* $\mathrm{Lip}(\alpha - 1/p)$.

LEMMA 3. *Let*

$$K_n(t) = (R_n)^{-1} \sum_{k=1}^{n} c_{n-k} d_k \frac{\cos(k + 1/2)t}{\sin t/2}, \tag{27}$$

and

$$L_n(t) = \int_t^{\pi} K_n(u)\,du. \tag{28}$$

Then

$$f \in K_p \ (1 < p \leqslant \infty) \text{ implies } \| f - t_n(f) \|_p = O(c_n/R_n) \quad \text{if } \int_0^{\pi} | L_n(t) |\, dt = O(c_n/R_n). \tag{29}$$

PROOF. Let $\tilde{S}_n(\tilde{f}; x)$ denote the partial sums of the conjugate series associated with $\tilde{f}(x)$. We have, from the definition,

$$\begin{aligned} t_n(\tilde{S}_n(\tilde{f}; x)) &= (2\pi R_n)^{-1} \sum_{k=0}^{n} c_{n-k} d_k \int_0^{\pi} [\tilde{f}(x + t) - \tilde{f}(x - t)] \cot \frac{t}{2}\,dt \\ &\quad - (2\pi R_n)^{-1} \sum_{k=0}^{n} c_{n-k} d_k \int_0^{\pi} [\tilde{f}(x + t) - \tilde{f}(x - t)] \cos\left(k + \frac{1}{2}\right) t \csc \frac{t}{2}\,dt \end{aligned} \tag{30}$$

By M. Riesz's theorem (Zygmund [18], Theorem (2.4), page 253) $f \in L_p$ $(1 < p < \infty) \Rightarrow \tilde{f} \in L_p \Rightarrow \tilde{\tilde{f}} \in L_p$ and $\tilde{S}(\tilde{f}) = S(\tilde{\tilde{f}})$. If $p = \infty$, $\tilde{f} \in \mathrm{Lip}\,1$ (by hypothesis) and then $-f + \frac{1}{2}a_0$ is identical to $\tilde{\tilde{f}}$. Thus from (30) and (27),

$$f(x) - t_n(f; x) = (2\pi)^{-1} \int_0^{\pi} \{\tilde{f}(x + t) - \tilde{f}(x - t)\} K_n(t)\,dt \tag{31}$$

almost everywhere.

Since $f \in K_p$, by Lemma 1, we can take $\tilde{f}(u)$ equivalent to the indefinite integral of a function, say $\tilde{f}'(u) \in L_p$ $(p > 1)$. By integration by parts, we have from (31)

$$f(x) - t_n(f; x) = (2\pi)^{-1} \int_0^{\pi} \{\tilde{f}'(x + t) + \tilde{f}'(x - t)\} L_n(t)\,dt.$$

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By using the generalized Minkowski's inequality ([7], page 148, 6.13.9)

$$\| f - t_n(f) \|_p \leqslant (2\pi)^{-1} \int_0^{\pi} \| \tilde{f}'(x+t) + \tilde{f}'(x-t) \|_p \, | L_n(t) | \, dt$$
$$= o\left(\int_0^{\pi} | L_n(t) | \, dt \right) = O(c_n / R_n).$$

LEMMA 4 ([4]).

$$(32) \qquad \left| \int_t^{\pi} \frac{\sin(k+1)u}{u^2} \, du \right|$$
$$\leqslant \begin{cases} 2(k+1) \log \dfrac{1}{(k+1)t} & \text{for } 0 < (k+1)t < 1/e; \\ 2/(k+1)t^2 & \text{for any } k \geqslant 0, \, t > 0. \end{cases}$$

The lemma can be proved easily.

PROOF OF THEOREM 2. In view of Lemma 3, it is enough to prove (29). By Abel's transformation

$$(33) \qquad -K_n(t) = \left(2 R_n \sin^2 \frac{t}{2} \right)^{-1} \sum_{k=0}^{n} (c_{n-k} d_k - c_{n-k-1} d_{k+1}) \sin(k+1)t.$$

Since

$$\left(2 \sin^2 \frac{t}{2} \right)^{-1} = \frac{2}{t^2} + O(1),$$

we get, from (33) and (25), that

$$-K_n(t) = (2/R_n t^2) \cdot \sum_{k=0}^{n} (c_{n-k} d_k - c_{n-k-1} d_{k+1}) \sin(k+1)t + O(c_n / R_n).$$

From (33), we observe that (29) holds if

$$(34) \qquad \sum_{k=0}^{n} | (c_{n-k} d_k - c_{n-k-1} d_{k+1}) | \int_0^{\pi} \left| \int_t^{\pi} \frac{\sin(k+1)u}{u^2} \, du \right| dt = O(1).$$

In view of (25), (34) is true whenever

$$(35) \qquad \int_0^{\pi} \left| \int_t^{\pi} \frac{\sin(k+1)u}{u^2} \, du \right| dt = O(1)$$

uniformly in k.

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By Lemma 4, the integral on the left of (35) does not exceed

$$\int_0^{1/e(k+1)}\left|\int_t^{\pi}\frac{\sin(k+1)u}{u^2}\,du\right|dt+\int_{1/e(k+1)}^{\pi}\left|\int_t^{\pi}\frac{\sin(k+1)u}{u^2}\,du\right|dt$$
$$\leqslant\int_0^{1/e(k+1)}2\log(1/(k+1)t)\,dt+\int_{1/e(k+1)}^{\pi}2(k+1)^{-1}t^{-2}\,dt.$$

Since each integral is bounded the result follows.

COROLLARY 3. *Let* $\{d_n\}\in bv$, $d_n\geqslant 0$, $D_n>0$. *If* $f\in K_p$ $(1<p\leqslant\infty)$, *then*

$$\|f-\bar{N}_n(f)\|_p=O\left(D_n^{-1}\right).$$

COROLLARY 4. *Let* $\{c_n\}$ *satisfy* $C_n\geqslant 0$, $C_n>0$, *and*

$$\sum_{k=0}^{n}|c_k-c_{k-1}|=O(c_n)\qquad(c_{-1}=0). \tag{36}$$

Then $f\in K_p$ *implies* $\|f-N_n(f)\|_p=O(c_n/C_n)$ $(1<p\leqslant\infty)$.

The case $p=\infty$ is given in [4, Lemma 2.3].

Combining Corollary 1 and Corollary 4, we get the following:

THEOREM 3. *Let* $\{c_n\}$ *satisfy* (11) *and* (36). *Then the Nörlund method* (N,c_n) *is saturated with order* c_n/C_n *and class* K_p.

REMARK. Lemma 3 shows that (29) is a sufficient condition for $\|f-t_n(f)\|_p=O(c_n/R_n)$ whenever $f\in K_p$ $(1<p\leqslant\infty)$. We do not know if (29) is also necessary.

4

Let us write $R(y)=R_{[y]}$. With a view to generalizing Theorem B and Theorem C, and extending Theorem D, we prove the following:

THEOREM 4. *Let* $\{c_n\}$, $\{d_n\}$ *be non-negative, non-increasing sequences and* $R_n>0$. *Let* $f\in L_p[-\pi,\pi]$ $(1<p<\infty)$ *or* $f\in c_{2\pi}$ $(p=\infty)$. *Then*

$$\|f-t_n(f)\|_p=O\left(\frac{1}{R_n}\sum_{k=1}^{n}\frac{w_2^{(p)}(1/k)}{k}R_k\right)+O\left(w_2^{(p)}\left(\frac{1}{n}\right)\right). \tag{42}$$

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REMARK. If in addition to the hypotheses assumed on the sequences $\{c_n\}$ and $\{d_n\}$, we assume that there exists $l > 0$ such that

$$(43) \qquad (R_n)^{-1} \sum_{k=1}^{n} (R_k/k) \geqslant l \qquad (n = 1, 2, \ldots)$$

then

$$(44) \qquad w_2^{(p)}\left(\frac{1}{n}\right) \leqslant (lR_n)^{-1} \sum_{k=1}^{n} \left\{\frac{R_k w_2^{(p)}(1/k)}{k}\right\}.$$

Hence we can get from (42) that

$$\|f - t_n(f)\|_p = O\left(\frac{1}{R_n} \sum_{k=1}^{n} \left(\frac{R_k w_2^{(p)}(1/k)}{k}\right)\right).$$

We shall need the following lemma for the proof of our theorem.

LEMMA 5 ([10]). *If $\{c_n\}$ and $\{d_n\}$ are non-negative, non-increasing sequences and $\tau = [1/t]$ then for $0 \leqslant a < b \leqslant n$ (any n), and $0 < |t| \leqslant \pi$, we have*

$$\left|\sum_{k=1}^{b} c_{n-k} d_k \sin kt\right| = O(R(\tau)) \text{ as } t \to 0.$$

PROOF OF THEOREM 4. We easily get

$$f(x) - t_n(f; x) = \int_0^{\pi} \{f(x+t) + f(x-t) - 2f(x)\} M_n(t)\, dt$$

where

$$M_n(t) = (2\pi R_n)^{-1} \sum_{k=0}^{n} c_{n-k} d_k \frac{\sin(k+\frac{1}{2})t}{\sin t/2}.$$

Hence, by generalized Minkowski's inequality

$$(45) \qquad \|f(x) - t_n(f; x)\| \leqslant I_1 + I_2,$$

where

$$I_1 = \int_0^{\pi/n} w_2^{(p)}(t; f)\, |M_n(t)|\, dt, \quad \text{and} \quad I_2 = \int_{\pi/n}^{\pi} w_2^{(p)}(t; f)\, |M_n(t)|\, dt.$$

Since $0 < \sin(k+\frac{1}{2})t < (2k+1)\sin t/2$ for $0 \leqslant k \leqslant n$, $0 < t < \pi/n$, we have

$$I_1 = O\left((2n+1)\int_0^{\pi/n} w_2^{(p)}(t; f)\, dt\right) = O\left(w_2^{(p)}\left(\frac{\pi}{n}; f\right)\right).$$

By Lemma 5,

$$I_2 = O\left(\frac{1}{R_n}\int_{\pi/n}^{\pi} \frac{R(1/t)}{t} w_2^{(p)}(t; f)\, dt\right)$$

$$= O\left(\frac{1}{R_n}\sum_{k=1}^{n-1}\int_{k/\pi}^{(k+1)/\pi} \frac{w_2^{(p)}\left(\frac{1}{t}, f\right) R(t)}{t}\, dt\right)$$

$$= O\left(\frac{1}{R_n}\sum_{k=1}^{n}\left(\frac{R_k w_2^{(p)}(1/k)}{k}\right)\right).$$

On collecting the estimates, the theorem follows.

REMARK. If $0 < \alpha \leqslant 1$, $p > 1$, $\alpha p > 1$ then, by Lemma 2(ii), $f \in \mathrm{Lip}(\alpha, p)$ implies $w_2^{(p)}(\delta; f) = O(\delta^{\alpha - 1/p})$. In this case

$$w_2^{(p)}(1/n) = O(n^{-\alpha + 1/p})$$

and

$$\frac{1}{R_n}\sum_{k=1}^{n} \frac{R_k w_2^{(p)}(1/k)}{k} = O\left(\frac{1}{R_n}\sum_{k=1}^{n} R_k \frac{1}{k^{\alpha+1-1/p}}\right).$$

Let $\delta \geqslant 0$ and A_n^δ be given by $\sum_{n=0}^{\infty} A_n^\delta x^n = (1-x)^{-\delta-1}$ $(|x| < 1)$. Let $N_n(f)$ be written as $\sigma_n^\delta(f)$ or $H_n(f)$ according as $c_n = A_n^{\delta-1}$ or $c_n = (n+1)^{-1}$ for all n. By putting $d_n = 1$ for all n, we get the following results:

COROLLARY 5. *Let* $f \in \mathrm{Lip}(\alpha, p)$, $1 < p \leqslant \infty$. *Then*

$$\|f - \sigma_n^\delta(f)\|_p = \begin{cases} O(n^{-\delta+1/p}) & (0 < \delta < \alpha \leqslant 1); \\ O\left(\dfrac{\log n}{n^{\delta - 1/p}}\right) & (0 < \delta \leqslant \alpha \leqslant 1). \end{cases}$$

REMARK. The case $p = \infty$ of Corollary 5 was proved by Alexits [1].

COROLLARY 6. *If* $\{c_n\}$ *is a positive non-increasing sequence and* $f \in L_p[-\pi, \pi]$ $(1 < p < \infty)$ *or* $f \in c_{2\pi}$ $(p = \infty)$, *then*

$$\|f - N_n(f)\|_p = O\left(\frac{1}{C_n}\sum_{k=1}^{n} \frac{C_k}{k} w^{(p)}\left(\frac{1}{k}; f\right)\right).$$

REMARK. The case $p = \infty$ of this Corollary is Theorem D.

COROLLARY 7. *If* $f \in \operatorname{Lip}(\alpha, p)$, $\alpha p > 1$, $0 < \alpha \leqslant 1$, $p > 1$, *then*

$$\| f - H_n(f) \|_p = O\big((\log n)^{-1}\big).$$

In what follows, we shall write $\overline{H}_n(f)$ for $\overline{N}_n(f)$ when $d_n = 1/(n+1)$.

COROLLARY 8. *Let* $\{d_n\}$ *be a non-negative, non-increasing sequence. Then for* $f \in L_p[-\pi, \pi]$ $(1 < p < \infty)$, *or* $f \in c_{2\pi}$ $(p = \infty)$,

$$\| f - \overline{N}_n(f) \|_p = O\big(w^{(p)}(1/n)\big) + O\left(\frac{1}{D_n} \sum_{k=1}^{n} \frac{D_k w^{(p)}(1/k)}{k} \right).$$

COROLLARY 9. *If* $f \in \operatorname{Lip}(\alpha, p)$, $\alpha p > 1$, $0 < \alpha \leqslant 1$, $p > 1$, *then*

$$\| f - \overline{H}_n(f) \|_p = O\big((\log n)^{-1}\big).$$

REMARKS. (i) Since $w_2^{(p)}(\delta, f) \leqslant 2w^{(p)}(\delta, f)$, Corollary 6 and Corollary 8 are stated with estimates using modulus of continuity in place of integral modulus of smoothness.

(ii) It can be observed that our corollaries contain assumptions on $\{c_n\}$ and $\{d_n\}$ but we do not use conditions of the type (14) (see Theorem B and Theorem C).

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Department of Mathematics
American University of Beirut
Beirut
Lebanon

Department of Mathematics
York University
Downsview, Ontario M3J 1P3
Canada