A GENERALIZED MODEL FOR THE RISK PROCESS AND ITS APPLICATION TO A TENTATIVE EVALUATION OF OUTSTANDING LIABILITIES

Based on a report to the Astin Colloquium 1963

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I. Review of the Theory of Compound Poisson Processes

1. *Definitions*

A *compound Poisson process*, in this context abbreviated to *cPp*, is defined by a probability distribution of the number m of events in the interval $(0, \tau)$ of the original scale of the process parameter, assumed to be one-dimensional, in the following form.

$$P_m(\tau) = \int_0^\infty e^{-vt} (vt)^m \, d_v \, U(v, \tau) \, / \, m!, \tag{1}$$

where $\int_0^\tau \lambda_u \, du$ shall be inserted for t, λ_τ being the intensity function of a Poisson process with the expected number t of events in the interval $(0, \tau)$ and $U(v, \tau)$ is the distribution function of v for every fixed value of τ, here called the *risk distribution*. If the inverse of $\int_0^\tau \lambda_u \, du = t$ is substituted for τ, in the right membrum of (1), the function obtained is a function of t.

If the *risk distribution* is defined by the general form $U(v, \tau)$ the process defined by (1) is called a *cPp in the wide sense* (*i.w.s.*). In the sequel two particular cases for $U(v, \tau)$ shall be considered, namely when it has the form of distribution functions, which define a primary process being stationary (in the weak sense) or non-stationary, and when it is equal to $U_1(v)$ independently of τ. The process defined by (1) is in these cases called a *stationary or non-stationary* (*s. or n.s.*) *cPp* and a *cPp in the narrow sense* (*i.n.s.*) respectively. If a process is *non-elementary* i.e. the size of one change in the random function constituting the process is a random variable, the distribution of this variable conditioned by the hypothesis that such a change has occurred at τ is here called the *change distribution* and denoted by $V(x, \tau)$, or, if it is independent of τ, by $V_1(x)$. In an *elementary process* the size of one change is a constant, so that, in this case, the change distribution reduces to the *unity distribution* $E(x - k)$, where $E(\xi)$ is equal to 1, 0, if ξ is non-negative, negative respectively, and k is a given constant.

2. *Characteristic functions of a non-elementary s. or n.s. cPp subject to certain conditions*

The following is a brief summary of a proposition given by the present author (in its original form in a report to the Rättvik Colloquium of ASTIN and in a slightly more general form in a report to the International Congress of Actuaries in London, the first-mentioned report has been published in Skand. Akt. Tidskr. 1961 and the latter report in the Transactions of the Congress).

Suppose that the primary process is a *cPp* i.n.s. with the change distribution $H(v)$, and that the change distribution of the s. or n.s. *cPp* is $V_1(x)$, both being independent of τ. Then the probability distribution of the number n of events in the primary process is obtained by the substitution of $Q_n(\tau)$, $G(u)$, u, s, n for $P_m(\tau)$, $U(v, \tau)$, v, t, m respectively in (1). Without restricting the generality, we may assume that both $G(u)$ and $H(v)$ are of mean 1, then s, st represent the expected number of events in the interval $(0, \tau)$ for the primary process and for the s. or n.s. *cPp* respectively. If $\sum_{n=0}^{\infty} Q_n(\tau)\, H^{n*}(v)$ is inserted for $U(v, \tau)$ in (1), here and in the sequel the n^{th} asterisk power of any distribution function denotes the n^{th} convolution of the function with itself for $n > 0$ and unity for $n = 0$, $P_m(\tau)$ of the s. or n.s. *cPp* is obtained. By introducing the parameters s, t into the functions $P_m(\tau)$, $Q_n(\tau)$, these are transformed to functions of s, t and s, which are designated $\bar{P}_m(s, t)$, $\bar{Q}_n(s)$ respectively. By these transformations and by the reversion of the order of integration and summation (permissible on account of the asymptotic properties of the integral and of the sum) the following relation, where $\bar{R}_m(t, n)$ has been written for $\int_0^{\infty} e^{-vt}(vt)^m\, dH^{n*}(v)/m!$, is obtained.

$$\bar{P}_m(s, t) = \sum_{n=0}^{\infty} \bar{Q}_n(s) \bar{R}_m(t, n), \tag{2}$$

where by the properties by the Laplace transform $\bar{R}_0(t, n) = \bar{R}^n(t, 1)$. This leads to the following relation for the characteristic functions defining the s. or n.s. cPp subject to the conditions mentioned above, $\bar{\varphi}(\eta; s, t)$, where η is a real variable.

$$\bar{\varphi}(\eta; s, t) = \bar{Q}_0 \{s [1 - \bar{R}_0 [t (1 - \int_0^{\infty} e^{\eta i x} dV_1(x)), 1]]\} \tag{3}$$

The functions $\bar{R}_0 [t(1 - \int_0^{\infty} e^{\eta i x} dV_1(x)), 1]$ define a process, which may be called the *secondary process*, the expected number of events in this process in the interval $(0, \tau)$ is equal to t.

3. *Characteristic functions defining processes with τ-dependent change distribution*

Esscher extended the well-known characteristic functions defining a Poisson process with the change distribution $V_1(x)$ to the case, where the change distribution is τ-dependent, or after transformation of the parameter to the operational scale, equal to $\bar{V}(x, t)$ say, (Skand. Akt. Tidskr., 1932) by the proposition, that in this case the characteristic function corresponding to $V_1(x)$ should be replaced by the mean of the characteristic functions corresponding to $\bar{V}(x, t)$ with respect to t. The present author extended this deduction to include the so called generalized Hofmann-processes, for which the intensity function for $n = 0$ could be written $\sum_{j=1}^{l} \frac{q_j}{s_j} \left(1 + \frac{t}{s_j}\right)^{-a_j}$, $a_j \geq 0, q_j > 0, s_j > 0, l \geq 1$, (the report to the Rättvik Colloquium, quoted above) by using a transform of the characteristic functions defining such processes with change distribution $V_1(x)$, in the form of such functions for a Poisson process. This transform, also deducted in the paper quoted, is a generalization of such a transform of the Polya process introduced by Ammeter (Skand. Akt. Tidskr., 1948). Recently, Thyrion has generalized the Poisson transform to a much wider class of cPp (Bull. de l'Ass.

Roy. Act. Belges, 1959 and a manuscript published in 1963). By using Thyrion's results, briefly reviewed in the next section, a similar generalization of Esscher's theorem to include this wider class of *cPp* can be obtained. As this leads to rather unhandy relations, a generalization leading to a relation in the same form as for the Poisson process shall be given in section 5.

4. *Brief summary of Thyrion's results*

Let a parametric space Ω_α in one or more dimensions define the parameter α and another such space with the same number of dimensions Ω_β define the parameter β. If Ω_α can be mapped with one-to-one correspondence on Ω_β and vice verse, by the transformation $F_1(z, \alpha) = F_2(z, \beta)$, where $F_1(x, \alpha)$ and $F_2(y, \beta)$ are the distribution functions of the variables X and Y respectively, then X and Y are said to be *equivalent*.

Let $g_\mu(z)$, $\mu = 1, 2, 3$, $h_\nu(z)$, $\nu = 1, 2$, $j(z) = g_1[h_1(z)]$, where z is a real variable, be the generating functions of X_μ, Y_ν and Z respectively. The necessary and sufficient condition for Z being equivalent to a variable with the generating function $g_2[h_2(z)]$ is that X_1 is equivalent to a variable with the generating function $g_2[g_3(z)]$. If X_1 and Y_1 both assume only non-negative integral values, the distribution of Z with the generating function $g_1[h_1(z)]$ is called a *bunch distribution*, where X_1 and Y_1 represent the number of bunches and the number of elements in one bunch respectively. Every bunch distribution, which can be interpreted as a *cPp* i.n.s., has a generating function of the form exp $\{-\theta(t)[h_1(z, t) - 1]\}$, where $\theta(t) < 0$ for $t > 0$, $\theta(0) = 0$. If, in addition, lim $\theta(t)$, for t tending to infinity, is equal to $-a$, $0 < a < \infty$, the probability of non-occurrence of an event in the interval $(0, \tau)$, $\bar{P}_0(t)$ say, can be written $\int_0^\infty e^{-vt} d_v \left[\sum_{m=0}^{\infty} e^{-a} \frac{a^m}{m!} K_a^{m*}(v)\right]$, where $\int_0^\infty e^{\eta i\xi} dK(\xi) = 1 + \frac{\theta(-\eta i)}{a}$. Thyrion's result involves that a *cPp* i.n.s. subject to the condition mentioned can be considered as a s. or n.s. *cPp* for which the primary process has been reduced to a Poisson process with a constant expected number of events equal to a. The necessary and sufficient condition for the existence of a Poisson transform of a *cPp* i.n.s. with τ-independent change distribution is that the probability of non-occurrence of an event in (0, τ), $\bar{P}_0(t)$, tends to a positive limit less than unity.

The characteristic functions defining a *cPp* i.n.s. fulfilling this condition and having a τ-independent change distribution $V_1(x)$ can be written in the form

$$\bar{\Psi}(\eta, t) = \exp\left\{-\theta(t)\left(\frac{-1}{\theta(t)}\sum_{\nu=1}^{\infty} q_\nu(t)\, z^\nu - 1\right)\right\};\ \theta(t) = \log\ \bar{P}_0(t),$$

$$z = \int_0^\infty e^{\eta i x}\, dV_1(x), \text{ where } \eta \text{ is a real variable}; \tag{4a}$$

$$q_\nu(t) = \frac{(-t)^\nu}{\nu!}\,\theta^{(\nu)}(t);\ \sum_{\nu=1}^{\infty} q_\nu(t) = -\theta(t).$$

For a *cPp* i.n.s. defined by the distribution functions

$$\bar{F}(x, t) = \sum_{n=0}^{\infty} \bar{P}_n(t)\ {}_2\bar{W}^{n*}(x, t)$$

where ${}_2\bar{W}(x, t)$ is the transformed change distribution and $\bar{P}_n(t)$ the probability distribution of the number n of changes in the interval (0, t) on the operational scale, a relation between ${}_2\bar{W}(x, t)$ and ${}_2\bar{V}(x, t)$ can be defined by the following relation between the corresponding characteristic functions ${}_2\zeta_t$ and ${}_2z_t$ respectively by using (4a) and Esscher's deduction

$$\theta(t)\,.\frac{d\theta\,[t\,(1 - {}_2\zeta_t)]}{dt} = \theta\,[t\,(1 - {}_2z_t)] \tag{4b}$$

5. *A direct extension of Esscher's theorem*

Consider a *cPp* i.n.s. for which the probability distribution and the change distribution are denoted as in the end of the previous section. Let $\bar{\Psi}(\eta, t)$ represent the characteristic functions corresponding to $\bar{F}(x, t)$. The left indices of $\bar{W}$, $\bar{V}$, ζ, z shall be omitted in this section. By the definition of $\overline{W}(x, t)$ we obtain $\overline{\Psi}(\eta, t) = \overline{P}_0 [t(1 - \zeta_t)]$.

The probability of the combined event that n changes has occurred in the interval $(0, t + \Delta t)$ and the random function attached to the process does not exceed nx at $t + \Delta t$ is by the basic forward differential equation given by the following expression (5a). This is based on the fact that the occurrence of n changes in the interval $(0, t + \Delta t)$, $n > 0$, implies the occurrence of either n or $n - 1$ changes in the interval $(0, t)$. The parameter is here measured on the operational scale.

$$\left.\begin{aligned} &\overline{P}_n(t + \Delta t)\, \overline{W}^{n*}(x, t + \Delta t) = \overline{p}_{n-1}(t)\Delta t\, \overline{P}_{n-1}(t)\, \overline{W}^{(n-1)*}(x, t) \\ &* V(x, t) + (1 - \overline{p}_n(t)\Delta t)\, \overline{P}_n(t)\, \overline{W}^{n*}(x, t) \quad \text{for } n > 0 \\ &\overline{p}_0(t + \Delta t) = (1 - \overline{p}_0(t)\Delta t)\, \overline{P}_0(t) \quad \text{for } n = 0 \end{aligned}\right\} \qquad (5a)$$

Here $\overline{p}_n(t)$ is the intensity function of the process.

By the theory of cPp i.n.s.

$$\text{(a)} \qquad t\, \overline{p}_{n-1}(t)\, \overline{P}_{n-1}(t) = n\, \overline{P}_n(t)$$

$$\text{(b)} \qquad \frac{n}{t}\, \overline{P}_n(t)\, \zeta_t \frac{n}{t} = \frac{1}{t} \int_0^\infty e^{-vt}\, n\, (v\zeta_t t)^n\, dU_1(v)/n!$$

Using the identity (a) and multiplying (5a) by $e^{\eta i x}$ and integrating over x from zero to infinity the following relation (5b) is easily obtained.

$$\frac{\overline{P}_n(t + \Delta t) - \overline{P}_n(t)}{\Delta t} = \frac{n}{t}\, \overline{P}_n(t)\zeta_t^{n-1}\, z_t - \frac{n+1}{t}\, \overline{P}_{n+1}(t)\zeta_t^n \qquad (5b)$$

Using the identity (b) we may write

$$\frac{1}{t} \sum_{n=0}^{\infty} n\, \overline{P}_n(t)\, \zeta_t^n = \zeta_t \int_0^\infty e^{-vt}(1 - \zeta_t)\, v\, dU_1(v) = \frac{\zeta_t}{1 - \zeta_t - t\zeta_t'}\, \frac{\partial \overline{P}_0[t(1 - \zeta_t)]}{\partial t}$$

and inserting this expression into both terms of (5b), summing from 0 to ∞, the limit passage for Δt tending against zero leads to the differential equation

$$t\zeta_t' = z_t - \zeta_t \qquad (5c)$$

A solution of (5c) is $\zeta_t = \frac{1}{t} \int_0^t z_u \, du$. Thus, the distribution functions defining the *cPp* i.n.s. with ${}_2\overline{V}(x, t)$ as change distribution can be written

$$\overline{P}_0 \, [t(1 - {}_2\zeta_t)]. \tag{6}$$

or by the substitution of ${}_2\overline{W}(x, t)$ for $V_1(x)$ in (3), this relation is generalized to include also a change distribution dependent of t, where ${}_2\overline{W}(x, t)$ is the mean of ${}_2\overline{V}(x, t)$ in the interval $(0, t)$. It shall be remarked that this result was earlier derived by the present author for the special case, where the *cPp* i.n.s. is a Polya process (Trans. XV[th] Int. Congr. Act., New York 1957, II pp. 268-269). Further, (6) is a particular case of a general theorem given by Jung (in a paper read at the ASTIN Colloquium 1963) not known to the present author, when his report to the Colloquium was prepared.

If we observe that both s and t in section 2 have a one-to-one correspondence to τ, it is evident that one-to-one correspondence exists also between s and t, and $\overline{R}_0 \, [t \, (1 - \int_0^\infty e^{\eta i x} \, dV_1(x) \,), \, 1]$ can be regarded also as a function of s, ζ_s say. If in a particular case there exists a $z_s = \int_0^\infty e^{\eta i x} \, d\overline{V}(x, s)$ such that it has the relation (5c), then, (3) can be written in the form $\overline{Q}_0 \, \{s(1 - \zeta_s)\}$ and be interpreted as the characteristic functions for a *cPp* i.n.s. with the change distribution $\overline{V}(x, s)$ dependent on s. Further, if in a particular case ${}_2\zeta_t$ in (6) can be written in the form $\overline{S}_0[q(1 - \hat{\zeta}_q)]$, where t and q are similarly related as s, t, the insertion of the latter expression for the integral in (3) leads to a generalization which involves an iteration of the compounding of the primary and secondary process as described in section 2.

6. *Further generalizations of* (3)

In the sequel the following considerations will be useful. Let each event in the primary process, defined by $\overline{Q}_n(s)$ in section 2, be associated with a change in two random functions instead of only one, and assume that the changes in these functions are mutually independent. Let, to begin with, the change distribution for the first random function be $H(v)$ of mean unity as defined in section 2, independently of τ, and let the change distribution of the other random function be ${}_1\overline{V}(y, s)$ with the corresponding transformed change distribution ${}_1\overline{W}(y, s)$, defined by the relations given in the previous section. Let, further, the expected number of events and the change distribution of the secondary process depend on y, t_y and ${}_2V(x, \tau, y)$ say. The transformed change distribution of the secondary process ${}_2\overline{W}(x; t, y)$ will, then, have the characteristic function ${}_2\overline{\zeta}_{t_y}(\eta, y) = \int_0^\infty e^{\eta i x} d_x\, {}_2\overline{W}(x; t_y, y)$ equal to $\frac{1}{t}\int_0^t du \int_0^\infty e^{\eta i x} d_x\, {}_2\overline{V}(x, u, y)$. The characteristic functions of the s. or n.s. cPp, considered in section 2 but with these wider assumptions, can, then, be written in the following form

$$\overline{\varphi}(\eta; s, t) = \sum_{n=0}^{\infty} Q_n(s) \int_0^\infty \overline{R}_0^n [t_y (1 - {}_2\overline{\zeta}_{t_y}(\eta, y)), 1]\, d_y\, {}_1\overline{W}^{n*}(y, s) \quad (7)$$

The remark at the end of the previous section with respect to the extension of the compounding procedure described in section 2, is valid also for (7). The last-mentioned extension can be iterated ad infinitum (cf. Thyrion, l.c., who calls this type of distributions for three components "distributions par grappes de grappes" and for more than three components "distributions par cascades de grappes"). In this paper only processes with two components are considered. The remark with respect to the double interpretation of (3) generalized by (6), does not, however, generally hold for (7), as was pointed out by Jung.

II. An approach to a generalized model of the risk process

1. *General principles for the choice of stochastic models*

In order to give certain view-points on the applicability of (7) to a model for the risk process, the application of (3) in many other cases shall be reviewed in this section. The applications refer to composite stochastic phenomena more or less related to the mechanism of the risk process, which shall be discussed in the subsequent sections.

Bartlett (Probability and Statistics, The Harald Cramér Volume, New York, Uppsala, 1959, pp. 45-47) has given two examples of the application of models based on s. or n.s. cPp, one example refers to the distribution of bacteria in a culture created by a group of parental bacteria and the other example to the spatial distribution of the progeny of randomly distributed parental plants in a plant association. In both examples the model for the parental distribution is a primary process and the model for distribution of the progeny of one parent is a secondary process. In the second example it is assumed, that the plants of the progeny are independent apart from the mutual dependence within the same family arising from the position of the maternal plant. Bartlett has, further, specified the assumptions with respect to the form of the probability functions involved, so that in both cases the marginal distribution i.e. the distribution of the progeny from all parental plants reduces to a negative binomial, characteristic of the Polya process. From these results he concludes "that little information about the structure of a process can be expected from the marginal distribution and it is usually advisable to study the interrelations in detail". He says, further, that the specification problem of the probabilities concerned in processes for individuals labelled by one or more continuous parameters is now solved "and this should lead to a better grasp of the statistical analysis problem than previously available". The title of the paper quoted is "The Impact of Stochastic Process Theory on Statistics". It contains many valuable comments. In one of these comments he underlines the importance of the much wider scope and outlook that the stochastic process theory has given to the statistician. He says, (l.c. p. 48), "Thus even 'stock' examples and problems may be affected by this wider approach; though more striking developments have naturally occurred in time series analysis and in other fields, where classical methods are quite inappropriate.". The statistician's greater breadth of outlook "will warn him to be rather wary of empirical analysis at least on non-experimental material, not based on a complete, and sometimes necessarily extensive, theoretical appraisal".

In other papers on such models the primary process is often said to concern the distribution of "centers" and the secondary process is said to concern a „cluster of satellites" associated with each

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center (cf. e.g. Matérn, Meddel. Sta. Skogsforskn. Inst. — Bull. Swe. Sta. Inst. Forestry Research, 49, 1960). A similar terminology has been used by Rényi *et al* (Act. Math. Acad. Sci. Hung., I-II, 1951-1952) in the theory of *composed* Poisson processes, a remark by Kolmogoroff on these processes (l.c. I, p. 211) included. The connection between *composed* Poisson processes and *cPp* has been established by the present author (ASTIN Bull. 1963, II-3). As we have seen above, also Thyrion has used a similar terminology on the bunch distributions ("distributions par grappes") which on the conditions given in the previous part can be interpreted as probability distributions of the number of events in *cPp* i.n.s.

In the report by the present author to the London congress a Markov process subject to the general conditions for the forward and backward differential equations and defined for a population of $N(\tau)$ units, was analyzed as a result of the processes associated with each individual unit. This analysis lead inter alia to a probability distribution in the form of $\bar{R}_m(t, n)$ as defined in (2), for the number of events in the interval $(0, \tau)$ of the process defined for a population consisting of n groups, the volume of each group being independent of n but may depend on τ. If the ν^{th} semi-invariant of the risk distribution for each such group, i.e. of $H(v)$, is denoted by $\varkappa_\nu$, $\varkappa_1$ by assumption being equal to unity, and the variance of $\bar{R}_m(t, n)$ is denoted by ${}_n\mu_2$, it was proved in the paper that the characteristic function of $(m - nt) / \sqrt{{}_n\mu_2}$ can for large n be asymptotically expanded in the following expansion, valid also for *cPp* i.w.s., provided that the risk distribution of each group fulfils certain conditions.

$$\exp. (-u^2/2) \cdot \left[1 + \frac{1 + 3t\varkappa_2 + t^2\varkappa_3}{3!\,(1 + t\varkappa_2)^{3/2}} \frac{(-ui)^3}{\sqrt{nt}} + o\,(n^{-1})\right] \quad (8)$$

which is a generalization of the well-known expansion for the Poisson process and of Ammeter's limit theorem for the Polya process. In the Poisson case the second term within the bracket reduces to $(-ui)^3/(3!\sqrt{nt})$, and in the Polya case $\varkappa_3 = \varkappa_2^2$. Evidently, the difference between (8) in the general case and the expansion in the Poisson case is of the order of $(nt)^{-1/2}$. The difference between

(8) in the Polya and in the Poisson case is less than $w\,(—ui)^3\,e^{-u^2/2}\sqrt{nt}$, where, for $t\,\varkappa_2 = 1,\ 1/2$, w is equal to 0.128, 0,083 respectively; for more general processes, where $\varkappa_3$ is at most equal to $2\,\varkappa_2^2$, w is at most equal to 0.190, 0.105 respectively. If $t\varkappa_2$ is of small order, $\overline{R}_m(t, n)$ can for large values of n be approximated by a Poisson probability distribution.

2. *The classical model of the risk process*

The risk process in insurance is generally conceived, as if the amount payable on a given claim could be determined only by circumstances, which prevail at the time of the occurrence of the event insured against. Thus, no regard is paid to the dependence of the amounts actually payable on the development during the time period, when the payments are made. Accordingly the distribution functions of the accumulated claims are written in the form $\sum_{n=0}^{\infty} \overline{P}_n(t)\ \overline{W}^{n*}(x, t)$, where $\overline{P}_n(t)$ is defined as the probability distribution of the number n of events in a cPp i.n.s. and the dependence of $\overline{W}(x, t)$ on parameter values exceeding t are not in principle accounted for, as $(0, t)$ refers to the interval of the operational parameter scale during which the events insured against occur, but does not contain the time after t, during which payments for the claims may still be made. In numerical applications, however, to cases, where the group of claims under consideration, at least to the greater part, have been paid before the calculation, more or less conventional corrections on $\overline{V}(x, t)$ are applied with regard to the development during the period of the payments up to the time of calculation. Such an *a posteriori* correction does not account for the true interrelations involved. Bartlett's remarks on the use of a marginal process instead of a more complicated model, which accounts for these interrelations, seems to be a memento for the statisticians to widen their views on the mechanism of the risk process.

3. *An approach to a generalized risk process*

The a posteriori correction mentioned at the end of the previous section can, however, serve as a starting point for an approach to a stochastic model for the part of the risk process for a single claim,

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which refers to the period after the event insured against until all payments are made. Suppose that for a given claim payments occur at the time points $\tau_1, \tau_2 \ldots \tau_r$ on the original parameter scale, measured from the time point of the receipt of the loss advice, and that the claim is finally settled by the payment at τ_r. Then, the sum total of the payments at $\tau_1, \tau_1 \ldots \tau$ (eventually by accounting also for the interest, to be accrued or to be discounted, with respect to a suitable time of reference) or the accumulated amount paid for a given claim up to and inclusive the point τ after the receipt of the loss advice represents a random function, which constitutes a stochastic process, defined by a probability distribution of the number m of payments from 0 to τ after the receipt of the loss advice and by a change distribution, in this case dependent on the time point of the payment. This process may to begin with be called the *secondary process of the risk*, this term will in the sequel be restricted to a modification of the process just defined. The process, defined by the probability distribution for the number n of claims occurring for a given group of insurances in the interval $(\hat{\tau}_0, \hat{\tau})$, where the point $\hat{\tau}_0$ is independent of the time-points for occurrence of the events in this interval, and by the change distribution $H(v)$, independent of τ, shall to begin with be called the *primary process of the risk*, subject to a modified definition in the sequel. The accumulated amounts of the payments made in the interval (τ_0, τ), where $\hat{\tau}_0$ is used as point of zero and $\tau_0 \geq \hat{\tau}$, for all claims having occurred for the interval $(\hat{\tau}_0, \hat{\tau})$ in the primary process, shall be said to constitute the *generalized process of the risk*. The modifications in the definitions for the primary and secondary process imply that the former shall be defined as a process with two change distributions in the sense of section I 6 from which follows a modification also of the secondary process. This will be discussed in detail in the sequel. In terms of the usual terminology in the theory of s. or n.s. *cPp* the occurrence of an event in the primary process may be said to constitute a center. To such a center is associated a "cluster of satellites" here being a sequence of payments representing a series of points in the period after the time point, when the loss advice was received, and up to the time τ_r after this time point. The secondary and the generalized risk processes are defined by the distribution of the cluster associated with one center, and of the clusters associated with all centers respectively.

4. *The probability distribution of the number m of payments*

If the whole or a part of the indemnity for a given claim is in the form of a life annuity or an annuity certain, the time point, when the amount of the annuity is fixed, shall, as is usually done in practice, be considered as the time point for a single payment of the value of annuity. To begin with the following simplifying assumptions are made.

(i) the loss advices are received immediately after the occurrence of the events insured against.

(ii) the expected number of payments is independent of the amount payable.

It is evident that the number of payments for one claim occurring in two non-overlapping adjoining intervals of lengths σ and τ are, necessarily, strongly correlated. That this is so is seen by the following simple example. Suppose that the claim is finally settled by the r^{th} payment occurring at the end of the second interval, then, if 1, 2 . . . r-1 payments occur in the interval of length σ the probability for r, r-1 and r payments, . . ., r to 2 payments respectively in the interval of length τ is equal to zero, provided that r is fixed. If r is not fixed, a smaller probability for r payments in the second interval will be the result of 1 or more payments having occurred in the first interval and similarly for the other cases. One may, however, intuitively expect that the number of payments for a given number n of claims will for large values of n in a time interval be approximately uncorrelated with the number of payments for the n claims in the previous non-overlapping interval. In fact, if the probability distribution of the number of payments for n claims is in the form of $\bar{R}_m(t, n)$ as defined in (2), $\bar{R}_m(t, 1)$ is strongly heterogeneous in time and the secondary process for a given claim has dependent increments, while $\bar{R}_m(t, n)$ should be approximately equal to a distribution defining a homogeneous process with independent increments. It must be understood, that the expected number t of payments for a single claim must necessarily be smaller

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than a given number. It is likely, that this number is less than $\varkappa_2^{-1}$. (For the Polya process Ammeter has used $\varkappa_2^{-1} = 40$ in numerical examples, and Esscher has found values of about 20-40 in material from general insurance). It has been proved above that, in this case, $\overline{R}_m(t, n)$ can be approximated by a Poisson process. Therefore, if the secondary process of each single claim were assumed to be Poisson processes, we should arrive at approximately the same result for all claims of the group. The implications in $\overline{R}_m(t, n)$, due to the delays of the loss advices and to the dependence of the expected number of payments for each claim on the amount payable, will be dealt with further below. Also the influence of τ_r being different for different claims shall be considered.

5. *The change distribution of the generalized risk process*

The change distribution of the secondary process depends on factors, which for motor insurance are the cost of living, of repairs and of spare parts etc. and for fire insurance with reinstatement, the building cost etc. It depends also on the attitude of courts, and lawyers towards damages and the usage of claim settlement. All these factors are subject to a variation in time, which variation should be accounted for in the evaluation of the total amount payable for a claim. In such an evaluation one may also want to pay regard to interest accrued on or discounted from the payments with respect to a certain time of reference. The total amount payable for a claim depends, however, chiefly on the (absolute) extent of the damage determined only by the circumstances prevailing at or before the occurrence of the event insured against. If all other factors, the interest factor included, should be disregarded, the total amount payable for the claim is equal to the (absolute) extent of the damage as measured in money units with reference to the time for the occurrence of the damage. The change in the accumulated extent of the damages represents a random variable, y say, with the distribution function ${}_1V(y, \tau) = {}_1\overline{V}(y, s)$, s being the operational parameter of the primary process, conditioned by the hypothesis that an event has occurred at τ. Still with neglect of other factors, if the number of payments for an actual claim up to and inclusive the final settlement is m_r, the mean of the transformed change distribution of the secondary process should be equal to the mean of y divided by m_r and, consequently, dependent on y. If now all factors are taken into account, the primary process must be considered as having two change distributions in the sense of section I 6. It is, then, natural to assume that the mean of the change distribution of the secondary process (before transformation) shall be equal to y/t_r multiplied by a function of τ, $\chi(\tau)$ say, where $\chi(\tau)$ is assumed to be independent of y, and $1/t_r$ is the expected value of $1/m_r$. $\chi(\tau)$ shall, thus, account for the variation in time due to other factors of influence, the interest inclusive.

If, thus, the assumption of a primary process with only one change distribution is dropped, the change distribution, ${}_1\overline{V}(y, s)$, and its transform, ${}_1\overline{W}(y, s)$ shall together with $H(v)$ be assigned to the primary process, and a change distribution of mean $y\,\chi(\tau)/t_r$ to the secondary process. This latter distribution may be denoted by ${}_2V(t_rx/y, \tau) = {}_2\overline{V}(t_rx/y, t)$ and the transformed change distribution by ${}_2\overline{W}(t_rx/y, t)$.

6. *Discussion of the assumption (i) of section 4*

The secondary process for a given claim is principally defined only for the time interval on the original scale between the occurrence of the event insured against, or, if dropping the assumption (i), from the receipt of the loss advice to the point τ_r for the final settlement. In order to reach more simple formulae than such, where different intervals for different claims are accounted for, the secondary process for each claim shall be extended to include the interval between the absolute point of zero $\hat{\tau}_0$ to the receipt of the loss advice and an interval of sufficient length from the final settlement to contain all the values of τ_r for the claims considered. The intensity function of a secondary process for a single claim is, then, identically equal to zero in these intervals. Thus, the estimate, for a group of n claims incurred in the interval $(\hat{\tau}_0, \hat{\tau})$ of the expected number of payments before the extension, being obtained by the total number of payments divided by the harmonic mean of the number of claims incurred in subintervals in which the number of claims known to be outstanding is constant, will after the extension be obtained by division of the same numerator by the total number of n claims. The expected number of payments for n claims in the orig-

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inal process and in the extended process will both be equal to nt, where t is the expected number of the extended secondary process. Thus, by ascribing the value t, obtained by using n as denominator, to the secondary process for each claim, the same result for the expected number of payments in the period from $\hat{\tau}_0$ to τ for all the claims will be obtained by the extended approach as by the original approach. In fact, the distribution of the number of payments for each time period from $\hat{\tau}_0$ will by a high degree of accuracy be approximated by the extended distribution. Apart from terms of the order of n^{-1} the difference between the characteristic functions, defining the original and extended distributions, is by (8) equal to $e^{-u^2/2}(-ui)^3/\sqrt{nt}$,invariant under the extension, multiplied by a function, w say, of the expected number of payments in the original approach in the interval $(0, t)$, and of the ratio between the denominators for the estimate of the expected number in the two approaches, f say. It is extremely unlikely that the expected number of payments of a single claim can ever exceed $\varkappa_2^{-1/2}$, as explained here above. Assuming for a Polya process, where $(\varkappa_2 t)^2 = \varkappa_3 t^2$, that the expected number in the original approach is at most equal to $0.5\,\varkappa_2^{-1}$, w will be equal to 0.048, 0.070 for $f = 1/2$, $1/4$ respectively and in the limit, when f tends to zero, equal to 0.083, $\overline{R}_m(t, n)$ in the limit being a Poisson distribution. If $\varkappa_2^{-1} = 40$, even $0.5\,\varkappa_2^{-1}$ is too high for the upper limit of the expected number of payments in the original process. If this number is at most equal to $0.25\,\varkappa_2^{-1}$, w reduces in the limit for $f \to 0$ to 0.031. Furthermore, by the (formal) conversion of (8) the term considered will depend on the third differential quotient of the normal function. The error when using the expansion for the distribution function of the extended process, will, if terms of lower order are neglected, be approximately equal to $w/\sqrt{nt}$ multiplied by this differential quotient. For the standardized variable v being equal to 0, the absolute value of this quotient is equal to about 0.4, decreases to zero for $v = 1$, increases to about 0.18 for $v = 1.7$ and, thereinafter, rapidly decreases. It follows that the relative approximation error in the distribution function of the standardized variable is of the order of $(10^4 nt)^{-1/2}$; it shall also be remarked, that the difference between the variances for $t = 0.5\,\varkappa_2^{-1}$, $\varkappa_2^{-1} = 40$ is about 0.03. The generalization of the numerical example by assuming the original process being a process with

$\varkappa_3 = 2\,\varkappa_2^2$ will not essentially alter the estimations made. If $\overline{R}_m(t, n)$ is associated with a t-independent change distribution, similar assertions hold for the distribution of the random function, as by the theory of cPp i.n.s. the k^{th} semi-invariant of this distribution is a linear function of $t^i\varkappa_i$ up to and inclusive $i = k$.

As, however, by the remarks of the previous section the change distribution of the original secondary process for one claim depends on t, the secondary processes cannot be extended without studying the effect of the extension on the change distribution. The sum total of all payments made in the interval from the absolute point of zero, $\hat{\tau}_0$, to τ for n claims, each of extent y, incurred in the interval $\hat{\tau}_0$ to $\hat{\tau}$ is in the mean equal to $nt \int_0^\infty dJ(u) \int_0^\infty x\, d_x\, {}_2\overline{W}(ux/y, t)$, $J(u)$, being the distribution function defining the distribution of t_r among the different claims of the group, n and t being defined in terms of the extension. This sum, is evidently, equal to the corresponding sum expressed in terms of the original approach. As nt also is invariant under the extension, this applies also to the integral in the expression just given, which is the mean of the transformed change distribution of the secondary process weighted with $dJ(u)$ as weight functions. This mean can be estimated in a sample function by the sum total of the payments with appropriate interest factors. The invariance under the extension holds, thus, both for the expected number of payments and for the mean just defined. The effect of the extension on the semi-invariants of the 2nd and higher order of the transformed change distribution, cannot be ascertained, however, unless specific forms of this function are assumed; such assumptions shall not be made in this paper.

From now on also the assumption (ii) of section 4 is dropped, so that the operational parameter of the secondary process, t_y say, is assumed to depend on y. Let τ_y be the value of τ defined by the inverse of the relation $t_y = \int_0^{\tau_y} \lambda_u\, du$, then also the functions $J(u, y)$ and $\chi(\tau_y)$ shall be substituted for $J(u)$ and $\chi(\tau)$ here above. If $\overline{\chi}(t_y)dt_y = \chi(\tau_y)d\tau_y$ then the increments of $\int_0^{\tau_y} \overline{\chi}(u)du = \int_0^{\tau_y} \chi(u)du$ are needed for the prediction of the payments after the time for calculation.

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The characteristic function of $\bar{\chi}(t_y)$, $\int_0^\infty e^{\eta ix} d_x [\int_0^\infty {}_2\overline{W}(ux/y, t_y)\, d_u J(u, y)]$, shall be denoted ${}_2\bar{\zeta}^*_{t_y}(\eta, y) = \zeta_{t_y}(\eta/y)$. By the assumptions leading to the generalized expression for the characteristic functions defining the s. or n.s. *cPp* in (7), the parameters s, t should correspond to the same interval $(0, \tau)$. The generalization of the deduction of (7) to cases, where s corresponds to $(\hat{\tau}_0, \hat{\tau})$ and t to $(\hat{\tau}_0, \tau)$, can be straight forwardly performed. Even in this general case (7) can be used as it stands, and, then, the characteristic functions defining the extended generalized risk process are given in the form of (7) with ${}_2\bar{\zeta}^*_{t_y}$ instead of ${}_2\bar{\zeta}_{t_y}$. One of the remarks under (7) was based on the one-to-one correspondence to τ for each of the parameters s, t. For the extended generalized risk process this remark is, thus, valid, if τ-$\hat{\tau}$ has a fixed value.

7. *Discussion of the assumption (ii) in section 4*

The dependence of t on the amount payable for a given claim has in the previous section been restricted to a dependence on y, the extent of damage. Such a restriction made it possible to base the characteristic functions defining the extended generalized risk process on (7). Also the following considerations are based on this restriction. These considerations will give a tentative approach for a possible dependence of this type. This approach shall not be needed in the sequel and is meant only as an illustration, in many cases other approaches will be of greater use.

It is likely that the intensity function of the original secondary process for a given claim, will for values of τ, which are near to the receipt of the loss advice, be slowly decreasing with y, as a small claim is on an average settled at an earlier time point. For higher values of τ, however, the intensity function is likely to be increasing with y, as the total number of payments for a large claim is on an average higher than for a smaller claim. Defining τ_y as in the last section by the inverse of $t_y = \int_0^{\tau_y} \lambda_u \, du$, then the approach $\tau_y = (a\tau - b)\, {}^{10}\log\, y + c$ seems for Swedish conditions to give satisfactory results. For Swedish Third Party Liability Motor Insurance y ranges from 100 Sw.Cr., values of 100 000 Sw. Cr. or more occur very seldom, the mean is equal to about 1400 Sw.Cr. If $a = 0.7\,b$ and $\tau_y = b\,w_\tau(y) + c$ then, — $w_1(y) = 0.6,\ 0.9,\ 1.2,\ 1.5$ and $w_5(y) = 5,\ 7.5,\ 10.0,\ 12.5$ for $^{10}\log\ y = 2,\ 3,\ 4,\ 5$ respectively. This approach seems flexible enough to permit of graduating any given data.

8. *Remarks on the applicability of the model introduced in the previous section*

In the following part of this paper an illustration for the application of the theory propounded shall be given, which applies to some particular problems in the insurance field. It is evident, that the model can be used for many other problems in this field.

It shall here be remarked that the domaine of application may be much wider. Before the extension of the secondary processes for each individual claim, these were defined for different intervals of the absolute time scale. Thus, the generalized risk process, as defined before this extension, did in this respect not agree with the definition given for *cPp* in the first part of this paper, where the parameter of the secondary processes, t, corresponded to the same interval $(0,\ \tau)$ on the absolute time scale. In the original generalized risk process the secondary processes were defined for different time intervals. After the extension of these secondary processes according to section II 6 characteristic functions could be derived, which were in the form of such functions for s. or n.s. *cPp* as defined in (7). In cases, not necessarily in the insurance field, where the secondary processes for each individual are defined for different intervals on the absolute time scale, and fulfil the conditions with respect to the highest value of t given in section 6, a model of the form of a *cPp* may be used by the application of the extension procedure propounded in section 6, so that by this procedure the domaine of application of *cPp* seems to have been materially enlarged.

III. Illustration of the Applicability of the Theory

1. *Estimation*

The conditioned mean, in terms of the extended process, of the amount of the payment in the interval $(\tau,\ \tau + d\tau)$ for a claim occurred in the interval $(\hat{\tau}_0,\ \hat{\tau})$ with the fixed value y for the extent of the damage will be denoted $\hat{l}_\tau(y)$, where with purpose a suggestive similarity to common notations in life insurance technique has been aimed at.

$$\hat{l}_\tau(y) = \hat{l}_0(y)\, \chi_0(\tau_y), \tag{10}$$

where $\hat{l}_0(y) = y \int\limits_0^\infty \frac{dJ(u,y)}{u}$ and $\chi_0(\tau)$ signifies the function $\chi(\tau)$, defined in section II 5, for the intensity of interest equal to zero and τ_y is defined by the inverse of the relation $t_y = \int\limits_0^{\tau_y} \lambda_u\, du$. Let, further,

$$-\hat{\mu}_\tau(y) = \frac{d\, \hat{l}_\tau(y)}{\hat{l}_\tau(y)\, d\tau_y} = \frac{d\, \chi_0(\tau_y)}{\chi_0(\tau_y)\, d\tau_y} \lesseqgtr 0 \tag{11}$$

and l_τ, μ_τ be the means with respect to ${}_1\overline{W}(y, s)$ of $\hat{l}_\tau(y)$, $\hat{\mu}_\tau(y)$ respectively. Evidently, a table with double entry of the payments made in the period $(\hat{\tau}_0, \tau)$ for n_y claims of extent y, which have incurred in the period $(\hat{\tau}_1, \hat{\tau})$, the rows and columns referring to τ and y in suitable intervals respectively, should correspond to a select life table, if $\hat{\mu}_\tau(y)$ were to be positive. As a rule, however, $\hat{\mu}_\tau(y)$ is negative, in this case the analogy is made complete by reversing the order of the rows. It is, therefore, possible to apply ordinary methods of mortality statistics for estimating $\hat{\mu}_\tau(y)$ from $n_y\, \hat{l}_\tau(y)$, which corresponds to the estimating of the death intensity from the number of persons of a given number at a given age (y), who live to a certain age (τ). It is, evidently, advisable to graduate $\hat{\mu}_\tau(y)$ with a suitable function, if such a function can be found. The expression for $\hat{\mu}_\tau(y)$ is likely to be simpler than the Makeham formula. When $\hat{\mu}_\tau(y)$ has been estimated the estimate of $\hat{l}_0(y)$ is directly calculated by using (10). It is easily seen, that the estimation of μ_τ from l_τ in a table with single entry (τ) corresponding to an aggregate life table can be made in a quite analogous way.

$\mu_\tau(y)$ and μ_τ depend, generally, on the three values $\hat{\tau}_0$, $\hat{\tau}$ and τ, if they are independent of these values but depend only on the length of the intervals $\hat{\theta} = \hat{\tau} - \hat{\tau}_0$, $\theta = \tau - \hat{\tau}_0$ say, they might be called weakly stationary. For the prediction of μ for an interval $(\tau, \tau + \bar{\theta})$, retaining $(\hat{\tau}_0, \hat{\tau})$ for the incurrence of the claims, one can use the estimate based on a table for an interval of length $\bar{\theta}$ beginning at $\hat{\tau}_0 - \bar{\theta}$, provided that μ is weakly stationary. If one cannot assume that this provision is true, a correction must be applied on the estimate when using it for prediction.

Let, further, $\bar{N}_\tau(y)$ and $\bar{N}_\tau$ be defined by

$$\bar{N}_\tau(y) = \int_0^{\tau_y} \hat{l}_v(y)\, e^{-\delta v}\, dv;\ \bar{N}_\tau = \int_0^{\tau} l_v\, e^{-\delta v}\, dv, \qquad (12)$$

where the lower bound of the integral refers to the absolute point of zero $\hat{\tau}_0$. $n\,\bar{N}_\tau$ represents the mean of the total payments in the interval $(\hat{\tau}_0, \tau)$ for all the n claims considered, while $n_y\; N_\tau(y)$ represents the same mean for each value of y. The direct estimates for these means ought to be compared with the corresponding values calculated by using the graduation for μ, $\hat{\mu}$ respectively for periods $(\hat{\tau}_0, \tau)$, where τ is at most equal to the time of calculation.

2. *Comments on the estimations*

The choice between select and aggregate tables for the estimation depends of course on whether the amounts of the payments appearing in the compartments of the select table are large enough to permit of a select estimation. Also in other cases the aggregate estimation must be preferable. By such estimation it is possible, without a material increase in the calculation work, to refine the analysis by establishing aggregate tables for different $\hat{\theta}$ and θ. Such a differentiation will make it easier to predict μ_τ for periods after the time of calculation. As will be seen below in section 4 the present author has earlier made a study of this kind for evaluating outstanding liabilities. Further, by using smaller values of $\hat{\theta}$ the errors involved in the extension procedure can be diminished. By the use of different tables for different groups of claims, it is, further, possible to estimate also the variances and, in principle, even semi-invariants of higher order than the 2nd order of the variables implied. If the data are completed with the number of payments, and if the number of payments are grouped with regard to y, the select means and variances can be used also for a study of the dependence on y. If a separate estimate of the integral appearing in $\hat{l}_0(y)$ should be of interest, this might easily be performed in an analysis of the total number of payments for a group of claims finally settled before the time of calculation. Such an estimate seems not to be necessary, if only estimates and predictions for $n\,\bar{\bar{N}}_\tau$ are wanted. Neither the estimation of ${}_1W(y, s)$ is in this case necessary.

3. *Outstanding Liabilities*

If T denotes a point in the τ-scale, such that $\bar{\theta} = T - \tau$ is sufficiently large to ensure that the final settlement of claims, which have occurred in the interval from the absolute point of zero, $\hat{\tau}_0$, to $\hat{\tau}$, shall occur before T, then the value at τ of the liabilities for n claims known to be outstanding at the same time point is defined by the following expression.

$$n e^{\delta\tau}(\bar{N}_T - \bar{N}_\tau) = n e^{\delta\tau} \int_\tau^T l_v e^{-\delta v} dv = n e^{\delta\tau} \int_\tau^T e^{-\delta v} dv \int_0^\infty \hat{l}_v (y) d_y\, {}_1\bar{W}^{n*}(y, s) \tag{13}$$

The corresponding "select value" is obtained by substituting $\bar{N}(y)$ for $\bar{N}$ and $\hat{l}_v\,(y)$ for l_v in the 1st and 2nd membrum. and by omitting the integration with respect to $d_y\, {}_1\bar{W}^{n*}\,(y, s)$ in the last membrum.

The similarity with the technique of life insurance is striking. If regard shall be paid also to unknown claims a correction must be applied to (13). The correction can be made by the addition of a term of the form of (13), provided that the number of unknown claims can be predicted. The prediction can be based on estimates of the delay of loss advices for claims having occurred in the interval of length $\hat{\theta} = \hat{\tau} - \hat{\tau}_0$ beginning at $\hat{\tau}_0 - T + \tau$.

The calculation of outstanding liabilities is needed for the statement in the balance sheet and for the risk statistics. The requirement of grouping is different in the two cases. For the balance sheet the differentiation of the estimate needs only pay regard to the grouping in the system of accounting. It is, for the balance sheet, not aimed at exactitude but rather at safety i.e. that the statements are not to be made lower than the exact outstanding liabilities. For the risk statistics the outstanding liabilities of each statistical group must be estimated and added to a term representing the total payments before τ of all claims incurred in $(\hat{\tau}_0, \hat{\tau})$ including also claims settled before τ.

4. *An intuitive method for the estimation of outstanding liabilities*

If for n claims known to have occurred in the interval $(\hat{\tau}_0, \hat{\tau})$ the value $n\, e^{\delta\tau}\, \bar{N}_\tau$ is estimated by the payments actually made for these claims up to and inclusive the time point τ, A_τ say, the expression $n\, e^{\delta\tau}\, (\bar{N}_T - \bar{N}_\tau)$ can be written in the form

$$A_\tau \cdot \frac{1}{\bar{N}_\tau} \int_\tau^T l_v\, e^{-\delta v}\, dv. \tag{14}$$

By the assumption that the factor of A_τ represents a weakly stationary function (cf. section 1), the prediction of this factor can be based on statistics for claims having occurred in the interval of length $\hat{\theta} = \hat{\tau} - \hat{\tau}_0$ beginning at $\hat{\tau}_0 - T + \tau$.

In fact, about twenty years ago, the present author intuitively introduced this method for the calculation of outstanding liabilities with $\delta = 0$, $\hat{\theta} = 1$. Aggregate tables were established for each value of τ_0 and τ in intervals of 1 year. The method was for several calendar years applied to a large business of Group Accident Insurance. This business was approximately homogeneous with regard to the attitude towards damages, to the technical treatment of claims and also with regard to the sum assured and to the accident risk. The estimates were originally made for statements in the balance sheets, but even for the risk statistics the rough grouping of the estimates was deemed to be appropriate. This business started in 1925 and the first analysis of this kind was made about 1940. Thereinafter, the analysis was made each calendar year with addition of the new experience to the old experience. In 1948 the total number of claims from 1925 was 800 000. Each prediction was tested several times against the actual payments made after the prediction. In the first calendar years of application the agreement was deemed to be very good. Thereinafter a slight time trend could be found in the factor of A_τ in (14), which lead to an introduction of a correction term, which should account for the deviation from weak stationarity of this factor.

5. *Other estimates*

If also s is estimated, the value of outstanding liabilities can be based on s, the expected number of claims in the period concerned, i.e. s shall be substituted for n in (13). In this case no correction shall be made for unknown claims, provided that the interval between $\hat{\tau}$ corresponding to s and the time point of calculation is large enough to permit of the assumption that all advices for claims having occurred before $\hat{\tau}$ have been received before the calculation, which can be tested by the dislocation of the observation period referred to above. This leads to an estimate of the risk premium with regard to the development during the period of payments.

The data in the tables referred to in section 2 may be used for a graduation of the distribution function corresponding to

$\int_0^\infty \overline{R}_0 \left[t_y \left(1 - \zeta_{t_y} (\eta/y)\right), 1\right] d_1 W^{n*}(y, s)$. where ζ_{t_y} has been defined at the end of II 6.

$Q_n(s)$ can be graduated according to usual methods in terms of the classical theory of risk. By using such graduations and the estimations described in the sections 1 to 3 both the distribution of the direct risk premium, and the risk premium for reinsurance against excess of aggregate loss can be analyzed in terms of the generalized risk process.